UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 02-57
Economics Series 21
November 2002

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SOME RESULTS ON STRATEGIC VOTING AND PROPORTIONAL REPRESENTATION WITH MULTIDIMENSIONAL POLICY SPACE *

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Abstract

We study a model of proportional representation, in which the policy space is multidimensional. We first show, via an example, that the assumption of quasi-concavity of the utility function is not sufficient to obtain the result that only the extreme parties get votes, contrary to the unidimensional case. We, then, study two cases in which stronger assumptions on voters' preferences assure that voters essentially vote, in any equilibrium, only for the extreme parties.

Keywords: Strategic Voting, Proportional Rule, Nash Equilibrium.

JEL Classification: C72, D72.

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* We want to thank Jean-François Mertens and Thomas Palfrey for precious discussions.

1 Introduction

In this paper we examine the equilibria of a spatial model of proportional representation, in which the policy space is multidimensional and the policy outcome is a linear combination of parties' positions weighted by the share of votes each party gets in the election. The understanding of such issues is a fundamental step in order to study the relation between strategic voting and the number of parties resulting at equilibrium, as well as the relation between strategic voting and the position of the parties voters decide to vote for.

In a recent paper (De Sinopoli and Iannantuoni 2000) we analyze a completely analogous voting game of proportional representation, in which the policy space is unidimensional. The main result is that, if voters' preferences are single peaked, essentially an unique Nash equilibrium exists, characterized by the fact that any voter on the left/right of the corresponding policy outcome votes for the leftmost/rightmost party. The incentive to vote for an extreme is given by the maximal effect that such a vote has on the outcome.

In this paper, we first show, via an example, how the assumption of quasi-concavity of the utility function (that is the natural extension of single-peakedness to the multidimensional case) is not sufficient to obtain the result that only the *extreme* parties get votes. More precisely, we describe a game in which there are five parties located, respectively, at the four corners and at the center of the square: independently of the number of players, every voter voting for the party located at the center is a Nash equilibrium of the game.

From this example we argue that to obtain an *extreme* result stronger assumptions on voters preferences are needed. If we assume that the policy space is the unit square and there are parties located at the four corners, it is unambiguous that these parties are the *extreme* ones. Under this political situation, we trivially prove that, if voters' preferences are single peaked in each dimension with the peak independent from the other dimension, only the *extreme* parties take a relevant amount of votes. Hence, under the above assumptions, the *extreme* result holds also when strategic voters face a two-dimensional policy space.

Making stronger assumptions on utility function, we are able to drop the assumption that there are four parties located at the corners. We study the case in which each voter has a loss function given by a weighted sum of each issue's distance from his preferred policy. In this case we show that only parties located on the boundary of the convex hull of parties' positions take a relevant amount of votes.

We stress that even if all the results are proven assuming a two-dimensional policy space, the extension to more than two dimensions is straightforward.

Before proceeding to the model, let us mention that Schofied and Sened (2002) present a model of multi-party spatial competition under proportional rule. They model each party as a set of delegates choosing a leader who announces the policy declaration to the electorate. Their main result, supported also by an empirical analysis to the Israeli politics, shows that "the centre is empty in politics".

We describe the model in Section 2, we present the counterexample in which voters' utilities are quasi-concave but there exist an equilibrium where the center takes all the votes in Section 3. Section 4 contains the study of the case where four parties are located at corners, while in Section 5 we analyze the case in which each voter has a loss function given by a weighted sum of each issue's distance from his preferred policy.

2 The Model

Policy Space. The policy space $\mathbb{X} = [0,1]^2$

Parties. Parties are fixed both in number and in their positions, in that there is no strategic role for them: there is an exogenously given set of parties $M = \{1, ..., k, ...m\}$, indexed by k. Each party k is characterized by a policy $\zeta_k \in [0,1]^2$.

Proportional Rule. Given the set of parties M, each voter can cast his vote for any party.[1] The pure strategy space of each player i is $S_i = \{1, ..., k, ..., m\}$ where each $k \in S_i$ is a vector of m components with all zeros except for a one in position k, which represents the vote for party k.

A mixed strategy of player i is a vector $\sigma_i = (\sigma_i^1, ...\sigma_i^k, ..., \sigma_i^m)$ where each σ_i^k represents the probability that player i votes for party k.

The policy outcome. The position of the government, i.e., the policy outcome, is a linear combination of parties' policies each coefficient being equal to the corresponding share of votes. Given a pure strategy combination $s = (s_1, s_2, ..., s_n)$, $v(s) = \frac{1}{n} \sum_{i \in N} s_i$ is the vector representing for each party its share of votes, hence the policy outcome can be written as:

$$X(s) = \sum_{k=1}^{m} \zeta_k v_k(s). \tag{1}$$

Voters. Each strategic voter i is characterized by a bliss point $\theta_i \in \Theta = [0,1]^2$. We assume that it exists a fundamental utility function (à la Harsany) $u : \Re^4 \rightarrow \Re$, which represents the preferences, that is $u_i(X) = u(X, \theta_i)$. In other words, a player is fully characterized by his bliss point.

Given the set of parties and the utility function u, a finite game Γ is characterized by a set of players $N = \{1, ..., i, ..., n\}$ and their bliss points.

The utility that player i gets under the strategy combination s is:

$$U_i(s) = u(X(s), \theta_i)$$

[1] In this model we do not allow for abstention. We cannot claim that this assumption is neutral. In our proof we use the fact that, as the number of players goes to infinity, the weight of each player goes to zero, and this does not hold if a large number of voters abstains.

Given a mixed strategy combination $\sigma = (\sigma_1, ..., \sigma_n)$, because players make their choice independently of each other, the probability that $s = (s_1, s_2, ..., s_n)$ occurs is:

$$\sigma(s) = \prod_{i \in N} \sigma_i^{s_i}.$$

The expected utility that player i gets under the mixed strategy combination σ is:

$$U_i(\sigma) = \sum \sigma(s) U_i(s).$$

In the following, as usual, we shall write $\sigma = (\sigma_{-i}, \sigma_i)$, where $\sigma_{-i} = (\sigma_1, ...\sigma_{i-1}, \sigma_{i+1}, ...\sigma_n)$ denotes the $(n-1)$ −tuple of strategies of the players other than i. Furthermore s_i will denote the mixed strategy σ_i that gives probability one to the pure strategy s_i.

3 A counterexample

In De Sinopoli and Iannantuoni (2000) we have shown that single-peakdness of voters' preferences is the only assumption needed to prove that almost any voter, in any pure strategy equilibrium, votes only for the two extremist parties. Then, it is quite natural to check if the strict-quasi-concavity of the utility function, that is the natural extension of single-peakdness when the policy space is multidimensional, leads to a similar result.

In this section we discuss an example, in which the policy space is two-dimensional, that shows that this is not the case.

More precisely, we describe a game in which there are five parties located, respectively, at the four corners and at the center of the square. Voters have a strictly quasi-concave utility function and are located in precisely defined regions of the square. We show that, independently of the number of players, every voter voting for the party located at the center is a Nash equilibrium of the game.

The policy space is the unit square, i.e. $X = [0,1]^2$.

There are five parties located at the four corners of the square and at the center, i.e. at $\left\{(0,0), (0,1), (1,0), (1,1), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$.

The utility function of voter i, characterized by the bliss point $\theta_i = (\theta_{i1}, \theta_{i2})$, is:

$$\begin{aligned} u_i(X, \theta_i) &= -(X_1 - \theta_{i,1})^2 - (X_2 - \theta_{i,2})^2 - \\ &\quad -10\sqrt{\left[\left(\theta_{i,2} - \frac{1}{2}\right) X_1 + \frac{1}{2}(\theta_{i,1} - \theta_{i,2}) - \left(\theta_{i,1} - \frac{1}{2}\right) X_2\right]^2} \end{aligned}$$

Voters are located in four regions (see figure 1) on $\Theta = \Theta_1 \times \Theta_2 = [0,1]^2$:

$$A = \left\{ \theta_1 \le \min \left\{ \frac{3}{2}\theta_2 - \frac{1}{4}, \frac{5}{4} - \frac{3}{2}\theta_2, \frac{1}{2} \right\} \right\}$$

$$B = \left\{ \theta_1 \ge \max \left\{ \frac{3}{2}\theta_2 - \frac{1}{4}, \frac{5}{4} - \frac{3}{2}\theta_2, \frac{1}{2} \right\} \right\}$$

$$C = \left\{ \theta_2 \le \min \left\{ \frac{3}{2}\theta_1 - \frac{1}{4}, \frac{5}{4} - \frac{3}{2}\theta_1, \frac{1}{2} \right\} \right\}$$

$$D = \left\{ \theta_2 \ge \max \left\{ \frac{3}{2}\theta_1 - \frac{1}{4}, \frac{5}{4} - \frac{3}{2}\theta_1, \frac{1}{2} \right\} \right\}$$

It takes few calculations to check that, independently from the number of voters, everybody voting for the center is a Nash Equilibrium of the game.

We show that for a player i located in region A and with $\theta_{i,2} \le \frac{1}{2}$ (see figure 2), voting for the center is a best reply to everybody voting for the center. By symmetry, it will follow that everybody voting for the center is a Nash equilibrium of the game.

If player i votes for the center the policy outcome is $(X_1, X_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$, hence
$u\left(\left(s_{-i}, \left(\frac{1}{2}, \frac{1}{2}\right)\right), \theta_i\right) = -\left(\frac{1}{2} - \theta_{i,1}\right)^2 - \left(\frac{1}{2} - \theta_{i,2}\right)^2$

If player i votes for the left-bottom corner $(0,0)$ the policy outcome is $(X_1, X_2) = \left(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} - \frac{1}{2n}\right)$, hence
$u\left(\left(s_{-i}, (0,0)\right), \theta_i\right) = -\left(\frac{1}{2} - \frac{1}{2n} - \theta_{i,1}\right)^2 - \left(\frac{1}{2} - \frac{1}{2n} - \theta_{i,2}\right)^2$
$-10\sqrt{\left[\left(\theta_{i,2} - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{2n}\right) + \frac{1}{2}\left(\theta_{i,1} - \theta_{i,2}\right) - \left(\theta_{i,1} - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{2n}\right)\right]^2}$

If player i votes for the left-top corner the policy outcome is $(X_1, X_2) = \left(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}\right)$, hence
$u\left(\left(s_{-i}, (0,1)\right), \theta_i\right) = -\left(\frac{1}{2} - \frac{1}{2n} - \theta_{i,1}\right)^2 - \left(\frac{1}{2} + \frac{1}{2n} - \theta_{i,2}\right)^2$
$-10\sqrt{\left[\left(\theta_{i,2} - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{2n}\right) + \frac{1}{2}\left(\theta_{i,1} - \theta_{i,2}\right) - \left(\theta_{i,1} - \frac{1}{2}\right)\left(\frac{1}{2} + \frac{1}{2n}\right)\right]^2}$

If player i votes for the right-bottom corner the policy outcome is $(X_1, X_2) = \left(\frac{1}{2} + \frac{1}{2n}, \frac{1}{2} - \frac{1}{2n}\right)$, hence
$u\left(\left(s_{-i}, (1,0)\right), \theta_i\right) = -\left(\frac{1}{2} + \frac{1}{2n} - \theta_{i,1}\right)^2 - \left(\frac{1}{2} - \frac{1}{2n} - \theta_{i,2}\right)^2$
$-10\sqrt{\left[\left(\theta_{i,2} - \frac{1}{2}\right)\left(\frac{1}{2} + \frac{1}{2n}\right) + \frac{1}{2}\left(\theta_{i,1} - \theta_{i,2}\right) - \left(\theta_{i,1} - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{2n}\right)\right]^2}$

If player i votes for the right-top corner the policy outcome is $(X_1, X_2) = \left(\frac{1}{2} + \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}\right)$, hence

$$u\left(\left(s_{-i}, (1,1)\right), \theta_i\right) = -\left(\frac{1}{2} + \frac{1}{2n} - \theta_{i,1}\right)^2 - \left(\frac{1}{2} + \frac{1}{2n} - \theta_{i,2}\right)^2$$
$$-10\sqrt{\left[\left(\theta_{i,2} - \frac{1}{2}\right)\left(\frac{1}{2} + \frac{1}{2n}\right) + \frac{1}{2}\left(\theta_{i,1} - \theta_{i,2}\right) - \left(\theta_{i,1} - \frac{1}{2}\right)\left(\frac{1}{2} + \frac{1}{2n}\right)\right]^2}$$

The first easy observation is that voting the left-bottom corner, i.e. for $(0,0)$, is a better reply than voting for any other corner of the square[2].

Hence if $u\left(\left(s_{-i}, \left(\frac{1}{2}, \frac{1}{2}\right)\right), \theta_i\right) \geq u\left(\left(s_{-i}, (0,0)\right), \theta_i\right)$ voting for the party at $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a best reply for player i.

It is easy to calculate that

$$u\left(\left(s_{-i}, \left(\frac{1}{2}, \frac{1}{2}\right)\right), \theta_i\right) - u\left(\left(s_{-i}, (0,0)\right), \theta_i\right) = \frac{1}{n}\left[\frac{1}{2n} - 1 + \theta_{i,1} + \theta_{i,2} + 5\sqrt{\left(\theta_{i,1} - \theta_{i,2}\right)^2}\right],$$

which is strictly positive since $\theta_i \in A$ (i.e. $\theta_{i,1} \leq 3/2\theta_{i,2} - 1/4$).

By symmetry, everybody voting for the party at $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a Nash equilibrium (independently from the number of the players).[3]

4 Parties at the corners of the square

The example above shows that to obtain an extreme result we have to make stronger assumption on the utility function. In this section we furthermore assume that the are four parties located at the four corners of the policy space.

Assumption 1: There exist the four extremists parties, i.e., LB, LT, RB, RT with $\zeta_{LB} = (0,0)$, $\zeta_{LT} = (0,1)$, $\zeta_{RB} = (1,0)$, $\zeta_{RT} = (1,1)$.

Under Assumption 1, if preferences are single peaked in X_1(resp. X_2) with the peak independent from X_2(resp. X_1), in any pure strategy equilibrium, almost all the voters vote for extreme parties. If we add the assumption that the utility function $u(X, \theta)$ is continuously differentiable with respect to the policy, the result holds also for mixed strategies. The two assumptions can be formulated as:

Assumption 2: $\forall X_2, \theta_i$:
$X_1 < X_1' \leq \theta_{i,1}$ or $\theta_{i,1} \leq X_1' < X_1$ implies $u(X_1, X_2, \theta_i) < u(X_1', X_2, \theta_i)$

$\forall X_1, \theta_i$:
$X_2 < X_2' \leq \theta_{i,2}$ or $\theta_{i,2} \leq X_2' < X_2$ implies $u(X_1, X_2, \theta_i) < u(X_1, X_2', \theta_i)$

[2] For example, $u_i\left(\left(s_{-i}, (0,0)\right), \theta_i\right) - u\left(\left(s_{-i}, (0,1)\right), \theta_i\right) = \frac{6}{n}\left(1 - 2\theta_{i2}\right) \geq 0$ for $\theta_{i,2} \leq \frac{1}{2}$.

[3] Note that our proof also implies that this Nash equilibrium is strict, and then it cannot be eliminated by any usual refinement.

Assumption 3: The fundamental utility function $u : \Re^4 \to \Re$ is continuously differentiable with respect to the policy.[4]

We first state a result for pure strategy equilibria. We stress that assumption 3 is not needed to obtain the result.

Proposition 1 *Under Assumptions 1 and 2, let s be a pure strategy equilibrium of the game Γ with n voters, then*
(α) if $\theta_i \leq X(s) - \frac{\vec{1}}{n}$ then $s_i = (0,0)$
(β) if $\theta_i \geq X(s) + \frac{\vec{1}}{n}$ then $s_i = (1,1)$
(γ) if $\theta_{i,1} \leq X_1(s) - \frac{1}{n}$ and $\theta_{i,2} \geq X_2(s) + \frac{1}{n}$ then $s_i = (0,1)$
(δ) if $\theta_{i,1} \geq X_1(s) + \frac{1}{n}$ and $\theta_{i,2} \leq X_2(s) - \frac{1}{n}$ then $s_i = (1,0)$

Proof. (α) First notice that if $\theta_i \leq X(s_{-i},(0,0))$, then by Assumption 2 voting for $(0,0)$ is the only best reply for player i against s_{-i}. Because $X(s_{-i},(0,0)) = X(s) - \frac{1}{n}(\zeta_{s_i} - (0,0)) \geq X(s) - \frac{1}{n}$, then $\theta_i \leq X(s) - \frac{1}{n}$ implies that $(0,0)$ is the unique best reply, for player i, to s_{-i}. (β) (γ) (δ) A symmetric argument holds. ■

We now study the case in which voters are allowed to play mixed strategies. In order to get the result also assumption 3 is needed.

We recall that, given the set of parties M and the utility function u, a game Γ is characterized by the set of players and their bliss points. Let $\sigma = (\sigma_1, ..., \sigma_n)$ and $\bar{\mu}^\sigma = \sum_{i \in N} \frac{\sigma_i}{n}$. With abuse of notation, let $X(\bar{\mu}^\sigma) = \sum_{k=1}^{m} \zeta_k \bar{\mu}_k^\sigma$.

We can state the following proposition:

Proposition 2 *Under assumptions 1, 2 and 3, $\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$ if σ is a Nash equilibrium of a game Γ with n voters, then:*
(α) if $\theta_i \leq X(\bar{\mu}^\sigma) - \vec{\varepsilon}$ then $\sigma_i = LB$
(β) if $\theta_i \geq X(\bar{\mu}^\sigma) + \vec{\varepsilon}$ then $\sigma_i = RT$
(γ) if $\theta_{i,1} \leq X_1(\bar{\mu}^\sigma) - \varepsilon$ and $\theta_{i,2} \geq X_2(\bar{\mu}^\sigma) + \varepsilon$ then $\sigma_i = LT$
(δ) if $\theta_{i,1} \geq X_1(\bar{\mu}^\sigma) + \varepsilon$ and $\theta_{i,2} \leq X_2(\bar{\mu}^\sigma) - \varepsilon$ then $\sigma_i = RB$.

Proof:

Given a mixed strategy σ_j, the player j's vote is a random vector[5] $\tilde{s}_j$ with $Pr(\tilde{s}_j = k) = \sigma_j^k$. Given $\sigma_{-i} = (\sigma_1, ...\sigma_{i-1}, \sigma_{i+1}, ...\sigma_n)$, let $\bar{\tilde{s}}^{-i} = \frac{1}{n-1} \sum_{j \in N/i} \tilde{s}_j$ and $\bar{\mu}^{\sigma_{-i}} = \frac{1}{n-1} \sum_{j \in N/i} \sigma_j$. The first step of the proof consists in proving the next lemma:

[4] With assumption 2, this implies that $\forall x_2, \forall \theta_i$ $\frac{\partial u(X,\theta_i)}{\partial x_1} \gtreqless 0$ if $x_1 \lesseqgtr \theta_{i1}$ as well as $\forall x_1, \forall \theta_i$ $\frac{\partial u(X,\theta_i)}{\partial x_2} \gtreqless 0$ if $x_2 \lesseqgtr \theta_{i2}$.

[5] We remind readers that a vote is a vector with m components.

Lemma 3 *$\forall \phi > 0$ and $\forall \delta > 0$, if $n > \frac{m}{4\phi^2\delta} + 1$, then $\forall \sigma, \forall i$*

$$\Pr\left(\left|\bar{\tilde{\vec{s}}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) > 1 - \delta.$$

Proof. To prove the lemma we can use Chebichev's inequality component by component. Given σ_{-i}, it is easy to verify that $E(\tilde{s}_j^k) = \sigma_j^k$ and $Var(\tilde{s}_j^k) = \sigma_j^k(1-\sigma_j^k) \leq \frac{1}{4}$, hence $E(\bar{\tilde{s}}_k^{-i}) = \bar{\mu}_k^{\sigma_{-i}}$ and $Var(\bar{\tilde{s}}_k^{-i}) \leq \frac{1}{4(n-1)}$. By Chebychev's inequality we know that $\forall k, \forall \phi$:

$$\Pr\left(\left|\bar{\tilde{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}}\right| > \phi\right) \leq \frac{1}{4(n-1)\phi^2}.$$

Hence

$$\Pr\left(\left|\bar{\tilde{\vec{s}}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) \geq 1 - \sum_k \Pr\left(\left|\bar{\tilde{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}}\right| > \phi\right) \geq 1 - \frac{m}{4(n-1)\phi^2},$$

which is strictly greater than $1 - \delta$ for $n > \frac{m}{4\phi^2\delta} + 1$. ■

Lemma 4 *$\forall \varepsilon > 0$, $\exists n_0^{LB}$ such that $\forall n \geq n_0^{LB}$, if the game has n voters and if $\theta_i \leq X(\bar{\mu}^\sigma) - \vec{\varepsilon}$, then LB is the only best reply for player i to σ^{-i}.*

Proof: Fix $\varepsilon > 0$. Define $\forall \theta \in \left[0, 1 - \frac{\varepsilon}{2}\right]^2$

$$M_{\varepsilon,1}(\theta) = \max_{(X_1,X_2)\in\left[\theta+\frac{\varepsilon}{2},1\right]^2} \frac{\partial u(X,\theta)}{\partial X_1}.$$

By assumption 2 we know that $M_{\varepsilon,1}(\theta) < 0$. Moreover, given the continuity of $\frac{\partial u(X,\theta)}{\partial X_1}$ we can apply the theorem of the maximum[6] to deduce that the function $M_{\varepsilon,1}(\theta)$ is continuous, hence it has a maximum on $\left[0, 1 - \frac{\varepsilon}{2}\right]^2$, which is strictly negative. Let

$$M_{\varepsilon,1}^* = \max_{\theta\in\left[0,1-\frac{\varepsilon}{2}\right]^2} M_{\varepsilon,1}(\theta).$$

Define analogously $M_{\varepsilon,2}^*$. Let $M_\varepsilon^* = \max\left\{M_{\varepsilon,1}^*, M_{\varepsilon,2}^*\right\}$, and $\underline{\varsigma}$=$\min_{k\neq(0,0)}\left\{\zeta_{k,1} + \zeta_{k,2}\right\}$. Let $\bar{M}$ denote the upper bound[7] of $\left|\frac{\partial u(X,\theta)}{\partial X}\right|$ on $[0,1]^4$, and let $\delta_\varepsilon^* = \frac{-M_\varepsilon^*\underline{\varsigma}}{2\bar{M} - M_\varepsilon^*\underline{\varsigma}} > 0$

[6]Because there are various versions of the theorem of the maximum, we prefer to state explicitly the version we are using. Let $f : \Psi\times\Phi \to \Re$ be a continuous function and $g : \Phi \to P(\Psi)$ be a compact-valued, continuous correspondence, then $f^*(\phi) := \max\{f(\psi,\phi) \mid \psi \in g(\phi)\}$ is continuous on Φ.

[7]The continuity of $\frac{\partial u(X,\theta)}{\partial X}$ assures that such a bound exists.

and $\phi^* = \frac{(-2+\sqrt{6})\varepsilon}{m}$. We prove that if $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*}+1$, then LB is the only best reply for player i, which, setting n_0^{LB} equal to the smallest integer strictly greater than $\frac{m}{4\phi^{*2}\delta_\varepsilon^*}+1$, directly implies the claim.

Take a party $c \neq LB$. We show that

$$u\left(X\left(\sigma_{-i},c\right),\theta_i\right) - u\left(X\left(\sigma_{-i},LB\right),\theta_i\right) < 0$$

which implies that $c \neq LB$ is not a best reply for player i.

$$\begin{aligned} & u\left(X\left(\sigma_{-i},c\right),\theta_i\right) - u\left(X\left(\sigma_{-i},LB\right),\theta_i\right) \\ = & \sum_{s_{-i}\in S_{-i}} \sigma\left(s_{-i}\right)\left[u\left(X\left(s_{-i},c\right) - \frac{1}{n}(\zeta_c - \zeta_{LB}),\theta_i\right) - u\left(X\left(s_{-i},c\right),\theta_i\right)\right] \end{aligned}$$

Because the outcome function $X\left(s\right)$ depends only upon $v(s)$, denoting with V_n^{-i} the set of all vectors representing the share of votes obtained by each party with $(n-1)$ voters, the above expression can be written as:

$$\sum_{v_n^{-i}\in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i})\left[u\left(X\left(v_n^{-i},c\right) - \frac{1}{n}(\zeta_c - \zeta_{LB}),\theta_i\right) - u\left(X\left(v_n^{-i},c\right),\theta_i\right)\right]$$

where, with abuse of notation, $X\left(v_n^{-i},c\right) = \frac{\zeta_c}{n} + \frac{n-1}{n}\sum_{k=1}^{m}\zeta_k v_{n(k)}^{-i}$.

By the *mean value theorem* we know that $\forall v_n^{-i}$,
$\exists X^*$belonging to the line joining $X\left(v_n^{-i},c\right) - \frac{1}{n}\zeta_c$ and $X\left(v_n^{-i},c\right)$ such that

$$\frac{\left[u\left(X\left(v_n^{-i},c\right),\theta_i\right) - u\left(X\left(v_n^{-i},c\right) - \frac{1}{n}(\zeta_c - \zeta_{LB}),\theta_i\right)\right]}{\frac{1}{n}} = \frac{\partial u(X^*,\theta_i)}{\partial X_1}\zeta_{c,1} + \frac{\partial u(X^*,\theta_i)}{\partial X_2}\zeta_{c,2}.$$

Hence we have:

$$\begin{aligned} & u\left(X\left(\sigma_{-i},c\right),\theta_i\right) - u\left(X\left(\sigma_{-i},LB\right),\theta_i\right) \\ = & \frac{1}{n}\sum_{v_n^{-i}\in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i})\frac{\left[u\left(X\left(v_n^{-i},c\right),\theta_i\right) - u\left(X\left(v_n^{-i},c\right) - \frac{1}{n}(\zeta_c - \zeta_{LB}),\theta_i\right)\right]}{\frac{1}{n}} \leq \end{aligned}$$

$$\begin{aligned} & \frac{1}{n}\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*\right)\left[M_{n,1}^*(\phi^*,\theta_{i1})\zeta_{c,1} + M_{n,2}^*(\phi^*,\theta_{i2})\zeta_{c,2}\right] \\ & +\frac{2}{n}\left(1 - \Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*\right)\right)\bar{M} \end{aligned}$$

where

$$M^*_{n,1}(\phi^*,\theta_{i1}) = \max_{(X_1,X_2)\in\left[X(\bar{\mu}^{\sigma_{-i}}-\vec{\phi}^*,c)-\frac{1}{n}\zeta_c,1\right]^2} \frac{\partial u(X,\theta_i)}{\partial X_1}$$
$$M^*_{n,2}(\phi^*,\theta_{i2}) = \max_{(X_1,X_2)\in\left[X(\bar{\mu}^{\sigma_{-i}}-\vec{\phi}^*,c)-\frac{1}{n}\zeta_c,1\right]^2} \frac{\partial u(X,\theta_i)}{\partial X_2}$$

Now we prove that, for $n > \frac{m}{4\phi^{*2}\delta^*_\varepsilon}+1$, $M^*_{n,1}(\phi^*,\theta_{i,1}) \le M^*_\varepsilon$as well as$M^*_{n,2}(\phi^*,\theta_{i,2}) \le M^*_\varepsilon$.

We show that $M^*_{n,1}(\phi^*,\theta_{i,1}) \le M^*_\varepsilon$, the other one being completely analogous.

From the definition of M^*_ε, it suffices to prove that $M^*_{n,1}(\phi^*,\theta_{i,1}) \le M_{\varepsilon,1}$, which is true if $X_1(\bar{\mu}^{\sigma_{-i}}-\vec{\phi}^*,c)-\frac{1}{n}\zeta_{c,1}$ is greater than $\theta_{i,1}+\frac{\varepsilon}{2}$, and $X_2(\bar{\mu}^{\sigma_{-i}}-\vec{\phi}^*,c)-\frac{1}{n}\zeta_{c,2}$ is greater than $\theta_{i,2}+\frac{\varepsilon}{2}$. We only prove the inequality for the first coordinate because the proof for the second coordinate is completely analogous.

$$X_1(\bar{\mu}^{\sigma_{-i}}-\vec{\phi}^*,c)-\frac{1}{n}\zeta_{c,1} = \frac{n-1}{n}\sum_k \bar{\mu}_k^{\sigma_{-i}}\zeta_{k,1} - \frac{n-1}{n}\sum_k \phi^*\zeta_{k,1} =$$

$$X_1(\bar{\mu}^\sigma) - \frac{1}{n}\sum_k \sigma_i^k\zeta_{k,1} - \frac{n-1}{n}\sum_k \phi^*\zeta_{k,1} >$$

$$X_1(\bar{\mu}^\sigma) - \frac{1}{n} - m\phi^* \ge \theta_{i,1}+\varepsilon-\frac{1}{n}-m\phi^*.$$

Hence this step of the proof is concluded by noticing that δ^*_ε is by definition less than $\frac{1}{2}$, hence[8]

$$\theta_{i,1}+\varepsilon-\frac{1}{n}-m\phi^* > \theta_{i,1}+\varepsilon-m\phi^*-\frac{2\phi^{*2}}{m} =$$

$$\theta_{i,1}+\varepsilon-\frac{(20-8\sqrt{6})\varepsilon^2}{m^3}-\varepsilon\left(-2+\sqrt{6}\right) \ge \theta_{i,1}+\varepsilon(1-\frac{(20-8\sqrt{6})}{8}+2-\sqrt{6}) =$$

$$\theta_{i,1}+\frac{1}{2}\varepsilon.$$

By Lemma 3, we know that, for $n > \frac{m}{4\phi^{*2}\delta^*_\varepsilon}+1$,

$$\Pr(\left|\bar{\tilde{s}}^{-i}-\bar{\mu}^{\sigma_{-i}}\right| \le \vec{\phi}^*)\left[M^*_{n,1}(\phi^*,\theta_{i1})\zeta_{c,1}+M^*_{n,2}(\phi^*,\theta_{i2})\zeta_{c,2}\right]+$$
$$+2(1-\Pr(\left|\bar{\tilde{s}}^{-i}-\bar{\mu}^{\sigma_{-i}}\right| \le \vec{\phi}^*))\bar{M} <$$

[8] In the following we assume that $\varepsilon \le 1$, since otherwise the proposition is trivially true.

$$(1-\delta_\varepsilon^*)M_\varepsilon^*\underline{c}+2\delta_\varepsilon^* M=(1-\frac{-M_\varepsilon^*\underline{c}}{2M-M_\varepsilon^*\underline{c}})M_\varepsilon^*\underline{c}+2\frac{-M_\varepsilon^*\underline{c}}{2M-M_\varepsilon^*\underline{c}}M=0.$$

Summarizing, we have proved that for $n>\frac{m}{4\phi^{*2}\delta_\varepsilon^*}+1$, for every strategy $c\neq LB$

$$u\left(X\left(\sigma_{-i},c\right),\theta_i\right)-u\left(X\left(\sigma_{-i},LB\right),\theta_i\right)\leq$$

$$\frac{1}{n}\Pr(\left|\bar{\tilde{s}}^{-i}-\bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*)\left[M_{n,1}^*(\phi^*,\theta_{i1})\zeta_{c,1}+M_{n,2}^*(\phi^*,\theta_{i2})\zeta_{c,2}\right]+$$

$$+\frac{2}{n}(1-\Pr(\left|\bar{\tilde{s}}^{-i}-\bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*))\bar{M}<$$

$$\frac{1}{n}(1-\delta_\varepsilon^*)M_\varepsilon^*\underline{c}+\frac{2}{n}\delta_\varepsilon^* M=0,$$

which implies that $c\neq LB$ is not a best reply for player i.

Analogously, the following Lemmas can be proved:

Lemma 5 *$\forall\varepsilon>0$, $\exists n_0^{RT}$ such that $\forall n\geq n_0^{RT}$, if the game has n voters and if $\theta_i\geq X\left(\bar{\mu}^\sigma\right)+\vec{\varepsilon}$, then RT is the only best reply for player i to σ^{-i}.*

Lemma 6 *$\forall\varepsilon>0$, $\exists n_0^{LT}$ such that $\forall n\geq n_0^{LT}$, if the game has n voters and if $\theta_{i,1}\leq X_1\left(\bar{\mu}^\sigma\right)-\varepsilon$, and $\theta_{i,2}\geq X_2\left(\bar{\mu}^\sigma\right)+\varepsilon$, then LT is the only best reply for player i to σ^{-i}.*

Lemma 7 *$\forall\varepsilon>0$, $\exists n_0^{RB}$ such that $\forall n\geq n_0^{RB}$, if the game has n voters and if $\theta_{i,1}\geq X_1\left(\bar{\mu}^\sigma\right)+\varepsilon$, and $\theta_{i,2}\leq X_2\left(\bar{\mu}^\sigma\right)-\varepsilon$, then RB is the only best reply for player i to σ^{-i}*

Setting $n_0=\max\left\{n_0^{LB},n_0^{RT},n_0^{LT},n_0^{RB}\right\}$ completes the proof. ■

5 A specific utility function

In the previous section we have made a very strong political assumption, that is there are four parties located at the corners of the policy space. In the following we will drop Assumption 1, at the cost to deal with a much more specific utility function, which is linear and separable in the two dimensions:

Assumption 4: The fundamental utility function $u : \Re^4 \rightarrow \Re$ takes the form:

$$u(X, \theta_i) = -\alpha(\theta_i)|\theta_{i,1} - X_1| - |\theta_{i,2} - X_2|$$

where $\alpha(\theta_i) : [0,1]^2 \rightarrow \Re_{++}$ is a continuous function.

The above utility is simply a normalized weighted sum of each issue's distance from player i's bliss policy.

For any θ, we define the following four sets of parties:

$$LB(\theta) = \left\{ k \in M \text{ s.t. } \arg\min_{k \in M} \left[\alpha(\theta)\zeta_{k,1} + \zeta_{k,2}\right] \right\}$$

$$RT(\theta) = \left\{ k \in M \text{ s.t. } \arg\max_{k \in M} \left[\alpha(\theta)\zeta_{k,1} + \zeta_{k,2}\right] \right\}$$

$$RB(\theta) = \left\{ k \in M \text{ s.t. } \arg\max_{k \in M} \left[\alpha(\theta)\zeta_{k,1} - \zeta_{k,2}\right] \right\}$$

$$LT(\theta) = \left\{ k \in M \text{ s.t. } \arg\min_{k \in M} \left[\alpha(\theta)\zeta_{k,1} - \zeta_{k,2}\right] \right\}.$$

We will show that almost every voter votes, in any equilibrium, for parties belonging to the above sets. Hence, parties that are not located on the boundary of the convex hull of parties' positions will not take a relevant amount of votes. The following proposition presents the result for pure strategy combinations:

Proposition 8 *Under Assumption 4, let s be a pure strategy equilibrium of the game Γ with n voters, then*
(α) *if $\theta_i \leq X(s) - \frac{\vec{1}}{n}$ then $s_i = lb \in LB(\theta_i)$*
(β) *if $\theta_i \geq X(s) + \frac{\vec{1}}{n}$ then $s_i = rt \in RT(\theta_i)$*
(γ) *if $\theta_{i,1} \leq X_1(s) - \frac{1}{n}$ and $\theta_{i,2} \geq X_2(s) + \frac{1}{n}$ then $s_i = lt \in LT(\theta_i)$*
(δ) *if $\theta_{i,1} \geq X_1(s) + \frac{1}{n}$ and $\theta_{i,2} \leq X_2(s) - \frac{1}{n}$ then $s_i = rb \in RB(\theta_i)$*

Proof. (α) Given s_{-i}, take a party $lb \in LB(\theta_i)$ and a party $c \notin LB(\theta_i)$:

$$u(X(s_{-i}, lb), \theta_i) - u(X(s_{-i}, c), \theta_i)$$
$$= \left[-\alpha(\theta_i)\zeta_{lb,1} - \zeta_{lb,2} + \alpha(\theta_i)\zeta_{c,1} + \zeta_{c,2}\right] > 0$$

which implies that $c \notin LB(\theta_i)$ is not a best reply for player i. (β) (γ) (δ) A symmetric argument holds. ■

Analogously to the previous section, we now study the case in which voters are allowed to play mixed strategies:

Proposition 9 *Under Assumption 4, $\forall\varepsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$ if σ is a Nash equilibrium of a game Γ with n voters, then:*
(α) if $\theta_i \leq X(\bar{\mu}^\sigma) - \vec{\varepsilon}$ and $c \notin LB(\theta_i)$ then $\sigma_i^c = 0$
(β) if $\theta_i \geq X(\bar{\mu}^\sigma) + \vec{\varepsilon}$ and $c \notin RT(\theta_i)$ then $\sigma_i^c = 0$
(γ) if $\theta_{i,1} \leq X_1(\bar{\mu}^\sigma) - \varepsilon$, $\theta_{i,2} \geq X_2(\bar{\mu}^\sigma) + \varepsilon$ and $c \notin LT(\theta_i)$ then $\sigma_i^c = 0$
(δ) if $\theta_{i,1} \geq X_1(\bar{\mu}^\sigma) + \varepsilon$, $\theta_{i,2} \leq X_2(\bar{\mu}^\sigma) - \varepsilon$ and $c \notin RB(\theta_i)$ then $\sigma_i^c = 0$.

Proof*:* We first prove the following Lemma:

Lemma 10 *$\forall\varepsilon > 0$, $\exists n_0^{LB}$ such that $\forall n \geq n_0^{LB}$, if the game has n voters, if $\theta_i \leq X(\bar{\mu}^\sigma) - \vec{\varepsilon}$, and $c \notin LB(\theta_i)$ then $\sigma_i^c = 0$.*

Proof: Fix $\varepsilon > 0$. Take a party $lb \in LB(\theta_i)$. Define for $k \notin LB(\theta_i)$:

$$D^k(\theta_i) = \left[\alpha(\theta_i)\zeta_{lb,1} + \zeta_{lb,2} - \alpha(\theta_i)\zeta_{k,1} - \zeta_{k,2}\right] < 0$$

Define also $D(\theta_i) = \max_{k \notin LB(\theta_i)} D^k(\theta_i) < 0$, as well as $D = \max_{\theta_i} D(\theta_i) < 0$. Let $\alpha^* = \max_{\theta} \alpha(\theta)$, $\delta^* = \frac{-D}{1+\alpha^*-D} > 0$, and $\phi^* = \frac{(-2+2\sqrt{2})\varepsilon}{m} > 0$.
Now we prove that if $n > \frac{m}{4\phi^{*2}\delta^*} + 1$, then $\forall c \notin LB(\theta_i)$:

$$u(X(\sigma_{-i}, c), \theta_i) - u(X(\sigma_{-i}, lb), \theta_i) < 0$$

which implies that $c \notin LB(\theta_i)$ cannot be a best reply for player i.

As in the proof of Lemma 4, because the outcome function $X(s)$ depends only upon $v(s)$, denoting with V_n^{-i} the set of all vectors representing the share of votes obtained by each party with $(n-1)$ voters:

$$\begin{aligned} & u(X(\sigma_{-i}, c), \theta_i) - u(X(\sigma_{-i}, lb), \theta_i) \\ = & \sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, lb\right), \theta_i\right)\right] \end{aligned}$$

where, with abuse of notation, $X\left(v_n^{-i}, t\right) = \frac{\zeta_t}{n} + \frac{n-1}{n}\sum_{k=1}^{m} \zeta_k v_{n(k)}^{-i}$.

Notice that, if:

$$\begin{cases} \theta_{i,1} \leq \min\{X_1(s_{-i}, c), X_1(s_{-i}, lb)\} \\ \theta_{i,2} \leq \min\{X_2(s_{-i}, c), X_2(s_{-i}, lb)\} \end{cases}$$

then

$$\begin{aligned} & u(X(s_{-i}, c), \theta_i) - u(X(s_{-i}, lb), \theta_i) \\ = & \frac{1}{n}\left[\alpha(\theta_i)\zeta_{lb,1} + \zeta_{lb,2} - \alpha(\theta_i)\zeta_{c,1} - \zeta_{c,2}\right] = \frac{1}{n}D^c(\theta_i) \end{aligned}$$

Moreover:

$$\max_{s_{-i},c,t} \left| u\left(X\left(s_{-i},c\right),\theta_i\right) - u\left(X\left(s_{-i},lb\right),\theta_i\right)\right| = \frac{1}{n}\left[\alpha\left(\theta_i\right)+1\right]$$

Obviously, $X_1\left(s_{-i},\vec{0}\right) \leq \min\left\{X_1\left(s_{-i},c\right),X_1\left(s_{-i},lb\right)\right\}$, as well as $X_2\left(s_{-i},\vec{0}\right) \leq \min\left\{X_2\left(s_{-i},c\right),X_2\left(s_{-i},lb\right)\right\}$.[9] From Lemma 3 we know, for $n > \frac{m}{4\phi^{*2}\delta^*}+1$, that $\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*\right) > 1-\delta^*$. Because, for $n > \frac{m}{4\phi^{*2}\delta^*}+1$, $X\left(\bar{\mu}^{\sigma_{-i}} - \phi^*,\vec{0}\right) > X\left(\bar{\mu}^{\sigma}\right) - \vec{\varepsilon} \geq \theta_i$[10], we can deduce:

$$\begin{aligned}
& u\left(X\left(\sigma_{-i},c\right),\theta_i\right) - u\left(X\left(\sigma_{-i},lb\right),\theta_i\right) \\
= & \sum_{v_n^{-i}\in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i})\left[u\left(X\left(v_n^{-i},c\right),\theta_i\right) - u\left(X\left(v_n^{-i},lb\right),\theta_i\right)\right] < \\
< & \frac{1}{n}\left[\left(1-\delta^*\right)D + \delta^*(1+\alpha^*)\right] = \\
= & \frac{1}{n}\left[(1-\frac{-D}{1+\alpha^*-D})D + \frac{-D}{1+\alpha^*-D}(1+\alpha^*)\right] = 0
\end{aligned}$$

and, hence, $c \notin LB\left(\theta_i\right)$ cannot be a best reply for player i..

Analogously, the following Lemmas can be proved:

Lemma 11 *$\forall\varepsilon > 0$, $\exists n_0^{RT}$ such that $\forall n \geq n_0^{RT}$, if the game has n voters, if $\theta_i \geq X\left(\bar{\mu}^{\sigma}\right) + \vec{\varepsilon}$, and if $c \notin RT\left(\theta_i\right)$ then $\sigma_i^c = 0$.*

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[9] With abuse of notation $X\left(s_{-i},\vec{0}\right)$ denotes the outcome that would have been resulted if a party in $(0,0)$ existed and player i voted for it.

[10] We only prove the inequality for the first coordinate because the proof for the second coordinate is completely analogous.

$$X_1(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*,\vec{0}) =$$

$$X_1(\bar{\mu}^{\sigma}) - \frac{1}{n}\sum_k \sigma_i^k\zeta_{k,1} - \frac{n-1}{n}\sum_k \phi^*\zeta_{k,1} >$$

$$X_1(\bar{\mu}^{\sigma}) - \frac{1}{n} - m\phi^*$$

Hence this step of the proof is concluded by noticing that δ_ε^* is by definition less than $\frac{1}{2}$, hence

$$X_1(\bar{\mu}^{\sigma}) - \frac{1}{n} - m\phi^* > X_1(\bar{\mu}^{\sigma}) - m\phi^* - \frac{2\phi^{*2}}{m} =$$

$$X_1(\bar{\mu}^{\sigma}) - (-2+2\sqrt{2})\varepsilon - \frac{24-16\sqrt{2}}{m^3}\varepsilon^2 > X_1(\bar{\mu}^{\sigma}) - \varepsilon$$

Lemma 12 *$\forall \varepsilon > 0$, $\exists n_0^{LT}$ such that $\forall n \geq n_0^{LT}$, if the game has n voters, if $\theta_{i,1} \leq X_1(\bar{\mu}^\sigma) - \varepsilon$, if $\theta_{i,2} \geq X_2(\bar{\mu}^\sigma) + \varepsilon$, and if $c \notin LT(\theta_i)$ then $\sigma_i^c = 0$.*

Lemma 13 *$\forall \varepsilon > 0$, $\exists n_0^{RB}$ such that $\forall n \geq n_0^{RB}$, if the game has n voters, if $\theta_{i,1} \geq X_1(\bar{\mu}^\sigma) + \varepsilon$, if $\theta_{i,2} \leq X_2(\bar{\mu}^\sigma) - \varepsilon$, and if $c \notin RB(\theta_i)$ then $\sigma_i^c = 0$.*

Setting $n_0 = \max\{n_0^{LB}, n_0^{RT}, n_0^{LT}, n_0^{RB}\}$ completes the proof. ■

6 Conclusion

In this short paper we have studied strategic voting in a proportional representation model, when the policy space is multidimensional. This model is the extension of the unidimensional model presented in De Sinopoli and Iannantuoni (2000), in which single-peakdness of voters' preferences is the only assumption needed to prove that voters, in any pure strategy equilibrium, essentially vote only for the two extremist parties.

We first show, via an example, that the assumption of strict-quasi-concavity of the utility function is not sufficient to obtain the result that only the *extreme* parties get votes: stronger assumptions on the utility function are needed.

We then prove that, if four parties located at the corners of the policy space exist (i.e. they are the *extreme* parties), if voters' preferences are single-peaked in each dimension with the peak independent from the other dimension, and if the utility function is continuously differentiable with respect to the policy, voters essentially vote in any equilibrium only for the *extreme* parties. Because the assumption that there are four parties located at the corners is very strong, we drop it, at the cost to deal with a more specific utility function, i.e. linear and separable in the dimensions. In this case, we show that, in any equilibrium, only parties located on the boundary of the convex hull of parties' positions take a relevant amount of votes.

References

[1] De Sinopoli F., and G. Iannantuoni (2000), A Spatial Voting Model where Proportional Representation Leads to Two-Party Equilibria, CORE DP 2000/37.

[2] Schofied N. and I. Sened (2002), Local Nash Equilibrium in Multiparty Politics, Annals of Operational Research, 109: 193-210.

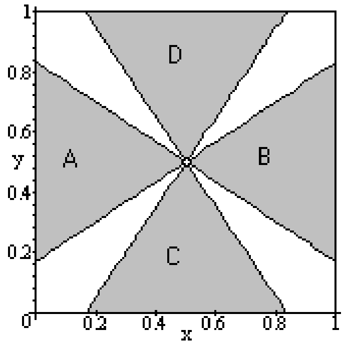

Figure 1: $x = \theta_{i,1}$, $y = \theta_{i,2}$.

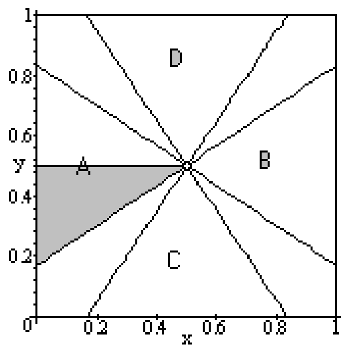

Figure 2: $x = \theta_{i,1}$, $y = \theta_{i,2}$.