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TERMINAL CONDITIONS AS EFFICIENT INSTRUMENTS FOR NUMERICAL DETECTION OF THE SADDLEPOINT PATHS: A LINEAR ALGEBRA NON-ROBUSTNESS ARGUMENT

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Abstract

In this paper, we address a criticism against the usual prescriptions on the introduction of terminal conditions as the principal numerical instruments for detecting the saddlepoint solutions of consistent expectations models. The argumentation is purely theoretical and it is conducted on a canonical linear infinite-time horizon model, approximated by the means of an elementary fixed-value terminal condition. Considering two equivalent algebraic representations of the model, we show that the asymptotic behavior of a backward solution method, associated to the fixed-value terminal condition, depends crucially on the selected algebraic formulation of the model.

Key Words

Consistent expectations, Numerical solutions, Terminal conditions, Saddlepoint paths.

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TERMINAL CONDITIONS AS EFFICIENT INSTRUMENTS FOR NUMERICAL DETECTION OF THE SADDLEPOINT PATHS : A LINEAR ALGEBRA NON-ROBUSTNESS ARGUMENT[1]

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SUMMARY

In this paper, we address a criticism against the usual prescriptions on the introduction of terminal conditions as the principal numerical instruments for detecting the saddlepoint solutions of consistent expectations models. The argumentation is purely theoretical and it is conducted on a canonical linear infinite-time horizon model, approximated by the means of an elementary fixed-value terminal condition. Considering two equivalent algebraic representations of the model, we show that the asymptotic behaviour of a backward solution method, associated to the fixed-value terminal condition, depends crucially on the selected algebraic formulation of the model.

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1. Introduction

The topic of terminal conditions is relatively recent in the litterature of economic dynamics, not only because it is extremely numerical, but more especially because it had not been clearly related to the solution techniques of economic models, until the last decade. Actually, this topic appears to be important once the use of the rational expectation hypothesis in economics was legitimated. Indeed, a major problem related to rational expectations models is their ability to generate an infinity of stable solutions. This crucial property holds, actually, for every model based on forward expectations, including consistent and perfect expectations. Given this indeterminacy problem, many authors (see for example Gouriéroux, Laffont and Monfort (1983)) argued against the introduction of rational expectations in economic modelling as the latter problem implies that forcasting exercises could be highly misleading.

Beginning with Begg (1982), a view emerged among the practitioners as solving forward-looking models makes sense as long as it can be ensured that they admit a unique stable solution. The latter statement founds the current preponderant doctrine concerning the resolution of forward-looking systems. Relatively to the simulation framework associated to the usual backward-looking forecasting models, an additional condition is required : to validate their predictions, the practitioners have to ensure that their models generate unique stable solutions, namely saddlepoint paths. The theoretical detection being somehow unreachable, especially when dealing with large scale models, a numerical approach remained unavoidable.

Taking advantage of the forward-looking configuration of the analyzed models, some practitioners proposed the introduction of ad-hoc terminal constraints for the purpose of selecting solutions and ensuring their uniqueness. This strategy received an important development by the ESRC Macroeconomic Modelling Bureau, directed by Kenneth Wallis (see for example, Wallis *et alii* (1985, 1986)). To understand more concretely this approach, let us present it more formally. Consider, for example, the following model, denoted (M) :

$$x_{t+1} = A\, x_t + B\, x_{t-1} \quad , x_0 \text{ given}$$

where x_t is the (nx1) vector of endogenous variables and t the time index. Assume that x_0 is nonzero ; in this case, the solution set of (M), for a given nonzero initial condition x_0, is indexed by the value of x_1 . As x_1 is unknown in economic practice, we need another boundary-value to solve numerically the model. A natural way to do it consists in imposing a finite distance constraint, a terminal condition at a certain period T, according to a given economic or mathematical reason. In practice, terminal conditions could be either exogenous or endogenous.

Exogenous terminal conditions are of a fixed-value form : $x_{T+1} = x^0$, and in this case, a natural value of x^0 is the stationary equilibrium if available.

Two endogenous terminal conditions are intensively used : the constant level formulation, $x_{T+1} = x_T$, and the constant growth form , $\dfrac{x_{T+1}}{x_T} = \dfrac{x_T}{x_{T-1}}$.

If we formalize terminal conditions by the equation $F(x_{T+1}, x_T, x_{T-1}) = 0$ where F(.) is a vector function of dimension n, the induced finite-time approximation system to solve is of the form :

$$\begin{cases} x_0 \text{ given} \\ x_{t+1} = A\, x_t + B\, x_{t-1} \text{ pour } 1 \leq t \leq T \\ F(x_{T+1}, x_T, x_{T-1}) = 0 \end{cases} .$$

Using first-order iterative schemes for solving such systems, the practitioners of the ESRC Macroeconomic Modelling Bureau provided a number of methodological prescriptions concerning the ability of each terminal condition formulation to detect the saddlepoint solutions. Although their conclusions are purely heuristical and related to small sample systems (namely small T values), they found the unique existing unified methodology until now. A major outcome of this methodology is that fixed-value terminal conditions allow the solution method to converge even in the presence of an infinity of stable solutions. That is why Fisher (1990), for example, prescribed the use of endogenous terminal conditions as they exhibit better performances from the point of view of saddlepoint paths selection. LE Van and Boucekkine (1993) provided a theoretical proof to the finding of the ESRC Bureau concerning the behaviour of the fixed-value terminal conditions, in a more general framework including all the solution methods of the relaxation type. Moreover, they showed that the ESRC Bureau's result still holds asymptotically, namely for large sample systems.

However, there is a strong presumption that all the previous outcomes are insufficiently robust to the experimental framework, chosen by the practitioner for a reason or another. Principally and by construction, the performances of terminal conditions depend on the selected solution method. When using the Newton-Raphson algorithm, found out by Laffargue (1990), Boucekkine (1993) theoretically showed that exogenous and endogenous terminal conditions do not significantly matter in the convergence conditions of this specific algorithm.

This paper provides an additional non-robustness proof of the usual methodological prescriptions concerning terminal conditions, and especially the ones of a fixed-value form. The main outcome of the paper is that terminal conditions performances depend not only on the solution method but also on the algebraic forms of the models, which significantly widens the scope of the non-robustness argument.

To this end, we use exclusively the elementary model (M) given above. On the basis of a specific diagonalizability assumption, we construct two equivalent algebraic representations of the model , denoted (MF_1) and (MF_2). Contrary to the first representation, the algebraic form (MF_2) exhibits a formal separation between the dynamics induced by the n largest eigenvalues and by the n lowest ones. For each representation, we build up the corresponding finite-time approximation system by the means of a basic fixed-value terminal condition. Then, each system is solved by a natural backward solution method, taking advantage of the finite-difference configuration of the original model (M). Whereas the differences between the two representations are purely formal, the asymptotic behaviour of the backward solution method on the finite-time systems is shown to depend on the chosen algebraic representation, in the presence of an infinity of stable solutions. More precisely, while the representation (MF_1) does not allow the solution method to locate a unique solution in the case of non-saddlepoint stable models, the other does. The outcome is somehow "dramatical" as our setting is chosen deliberately elementary. It is especially dramatical as many of the usual solution methods (for exampel, the first-order iterative schemes) are based on the possibility of writing the models into some specific forms. It is very likely that such non-robustness problems are much more important in the presence of nonlinearities, and especially in this case, it does not seem to us reasonable to follow blindly any prescription concerning any terminal condition.

The paper is organized as follows. Section 2 is devoted to present the two algebraic representations of the model (M) and some regularity assumptions necessary to make the finite-time systems backward solvable. In this section, some useful preliminary mathematical results are proved. Section 3 reports the analysis of the behaviour of the solution method on the representation (MF_1) : we prove that a unique stable solution is asymptotically located if and only if the saddlepoint conditions are fulfilled. Section 4 provides the same type of analysis on the representation (MF_2) : in particular, it is shown why the solution method selects a unique stable solution even in the case of non-saddlepoint stable models. We conclude by some methodological remarks on the use of terminal conditions and other experimental instruments when simulating forward-looking models.

2. The algebraic representations of the model and the regularity assumptions

As announced in the introductive section, we will use a terminal condition of the fixed value type (ie $x_{T+1} = x^0$, x^0 arbitrarily chosen) as a second boundary-value. If the long run equilibrium of the model is independent of the initial conditions, the most rigorous choice of x^0 is exactly this equilibrium. To ensure that, we assume that the model (M) does not include a unit root. In this case, the stationary equilibrium is the null vector and the corresponding fixed-value terminal condition is $x_{T+1} = 0$.

Consequently , the general form of the finite-time system, denoted (TM), approximating the model (M) is :

$$(TM) \quad \begin{cases} x_0 \text{ given} \\ x_{t+1} = A\, x_t + B\, x_{t-1} \text{ pour } 1 \le t \le T \\ x_{T+1} = 0 \end{cases} .$$

The solution path $\{x_t(TM), 1 \le t \le T\}$ of such a system depends of course on the value of T, whereas the original problem is of an infinite-time support. It is then mathematically necessary to find out the conditions which ensure that the solutions of the finite-time systems (TM) equal, at least approximately, the ones of the initial model for certains values of T. In our setting, the adequate concept is the convergence of the T-dependent path $\{x_t(TM), 1 \le t \le T\}$ to the solution path of the original model (M), and the corresponding consistency conditions could be stated as follows :

Proposition

A T-dependent solution path $\{x_t(TM), 1 \le t \le T\}$ is admissible if and only if :

C1) the limit of x_1 when T goes to infinity is finite.

C2) the limit of x_T when T goes to infinity is zero.

The consistency conditions C1) and C2) are quite immediate. Given the special structure of the model (M), they allow to guarantee that the limit of $x_t(TM)$, for every t such as $1 \le t \le T$, is finite when T goes to infinity, and that the limit path reaches the same equilibrium value as the original model (M). Consequently and by construction of the finite time approximations, the two conditions ensure that the limit paths equal the stable solutions of the initial model. Given this admissibility setting, we can now describe the adopted backward method on the announced two algebraic representations of the model (M). As the algebraic representations are based on a diagonalizability assumption, we begin by precisely stating the latter assumption.

To do that, we rewrite the model (M) as follows :

$$(M') \qquad \begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix} = \begin{bmatrix} A & B \\ I(n) & 0(n) \end{bmatrix} \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} \quad \text{for } 1 \le t \le T, \text{ and } x_0 \text{ nonzero given}$$

where I(n) (Resp. 0(n)) is the identity-matrix (Resp. null-matrix) of dimension n.

Set $F = \begin{bmatrix} A & B \\ I(n) & 0(n) \end{bmatrix}$. We assume that :

H1) matrix F is diagonalizable.

Observe that assumption H1) implies that the eigenvalue problem $(\lambda_i^2 I(n) - \lambda_i A - B)x = 0$, admits 2n real solutions, which induces a clear restriction on A and B. Denoting by P the eigenvectors matrix and by Λ the eigenvalues matrix, assumption H1) is rewritten as :

$$H1) \quad F = P \Lambda P^{-1}.$$

We can always assume that the elements of Λ are in an increasing order. P, Λ and P^{-1} are partitioned as follows :

. $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ where P_{11} is the submatrix of P (Resp. P_{22}) corresponding to the n first (Resp. last) rows and to the n first (Resp. last) columns, and P_{12} (Resp. P_{21}) is the submatrix of P corresponding to the n first (Resp. last) rows and to the n last (Resp. first) columns.

. $P^{-1} = \begin{bmatrix} p^{11} & p^{12} \\ p^{21} & p^{22} \end{bmatrix}$ with the same conventions as before.

. $\Lambda = \begin{bmatrix} \Lambda_1 & 0(n) \\ 0(n) & \Lambda_2 \end{bmatrix}$ where Λ_1 (Resp. Λ_2) contains the n lowest (Resp. greatest) eigenvalues of F.

For convenience, we assume that there is no common element to Λ_1 and Λ_2, which can be formalized as follows :

$$\text{H2)} \quad \text{Max}(\Lambda_1) < \text{Min}(\Lambda_2)$$

where Max(.) (Resp. Min(.)) is the operator returning the maximal (Resp. minimal) element in modulus of the matrix between parenthesis.

Let us present now the backward solution method on the announced two algebraic representations of (M).

2.1. The backward solution method on the representation (MF_1)

The representation (MF_1) is designed to be the formulation (M') given above :

$$(MF_1) \qquad \begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix} = F \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} \quad \text{for } 1 \leq t \leq T, \text{ and } x_0 \text{ nonzero given.}$$

The corresponding truncated system is :

$$\begin{cases} x_0 \text{ nonzero given} \\ \begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix} = F \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} \quad \text{for} \quad 1 \leq t \leq T \\ x_{T+1} = 0 \end{cases} .$$

Observe that the terminal condition $x_{T+1}=0$ is equivalent to the following equation :

$$[A \quad B] \begin{bmatrix} x_T \\ x_{T-1} \end{bmatrix} = 0 .$$

Conducting backward computations on the representation (MF_1) from T-1 to 1 yields the equation :

$$(E_{1,1}) \qquad \begin{bmatrix} x_T \\ x_{T-1} \end{bmatrix} = F^{T-1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} .$$

Using simultaneously the two previous equalities, it follows that :

$$(E_{1,2}) \qquad [A \quad B] \; F^{T-1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = 0 .$$

The natural corresponding solution scheme is consequently :

i) First, we use equation ($E_{1,2}$) to study the existence and the uniqueness of x_1 when T goes to infinity.

ii) Then, we use equation ($E_{1,1}$) and the outcomes of step i) to determine the limit of x_T when T goes to infinity.

Observe that steps i) and ii) are precisely devoted to check respectively the consistency conditions C1) and C2), presented in the admissibility setting above. Such a procedure is then perfectly adapted to analyze the status of the solutions attained by the backward solution method on the considered truncated system.

2.1. The backward solution method on the representation (MF_2)

To obtain the representation (MF_2), we use the partitioned forms of Λ and P^{-1}. As the approximated form of the model (M) could be written as :

$$\begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix} = F \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} \text{ for } 1 \le t \le T, \text{ and } x_0 \text{ nonzero given,}$$

and given the diagonalizability assumption H1), it follows that :

$$P^{-1} \begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix} = \Lambda \; P^{-1} \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} \text{ for } 1 \le t \le T, \text{ and } x_0 \text{ nonzero given.}$$

Using the partitions of Λ and P^{-1}, it yields :

$$\text{(M1)} \qquad P^{11} x_{t+1} + P^{12} x_t = \Lambda_1 (P^{11} x_t + P^{12} x_{t-1})$$

and

$$\text{(M2)} \qquad P^{21} x_{t+1} + P^{22} x_t = \Lambda_2 (P^{21} x_t + P^{22} x_{t-1})$$

for $1 \le t \le T$, and x_0 nonzero given. We denote by (MF_2) the representations (M1) and (M2). As announced, the representation (MF_2) allows for a formal separation between the dynamics generated by Λ_1 and Λ_2 respectively. This separation is, of course, purely formal as the models (M), (M1) and (M2) are necessarily equivalent. Intuitively, it is easy to understand that the formulation (M1) (Resp. (M2)) is obtained by rewriting (M) through a factorization involving mainly Λ_1 (Resp. Λ_2). To give a concrete idea of that, let us consider a real sequence : (M) $x_{t+1} = 2.5\, x_t - x_{t-1}$. The eigenvalues of this sequence are 0.5 and 2, and the corresponding (M1) and (M2) formulations are :

$$\text{(M1)} \qquad \frac{-2}{3} x_{t+1} + \frac{4}{3} x_t = \frac{1}{2} \left(\frac{-2}{3} x_t + \frac{4}{3} x_{t-1}\right)$$

and

$$\text{(M2)} \qquad \frac{2}{3} x_{t+1} + \frac{-1}{3} x_t = 2 \left(\frac{2}{3} x_t + \frac{-1}{3} x_{t-1}\right).$$

Observe, as expected, that the three forms (M), (M1) and (M2) are equivalent.

It remains to present how to apply a backward method in this case. First, consider the model (M1) as t=T ; using the boudary-value $x_{T+1}=0$, we obtain :

$$P^{12} \ x_T = \Lambda_1 (P^{11} x_T + P^{12} \ x_{T-1}).$$

Conducting backward computations on the second side of the previous equality using equations (M1) from T-1 to 1, we get an equation in terms of x_T and x_1:

$$(E_{2,1}) \qquad P^{12} \ x_T = \Lambda_1^T (P^{11} x_1 + P^{12} \ x_0).$$

Following a similar approach on equations (M2) truncated by the fixed-value terminal condition, we obtain an analogous equation :

$$(E_{2,2}) \qquad P^{22} \ x_T = \Lambda_2^T \ (P^{21} x_1 + P^{22} \ x_0).$$

Now, it is worth pointing out that whereas equations (M1) and (M2) are equivalent, the introduction of a terminal condition and the iteration effects of the backward solution method make equations $(E_{2,1})$ and $(E_{2,2})$ independent. Moreover, the system $(E_{2,1})$-$(E_{2,2})$, if solvable, provides the values of x_T and x_1, such as x_T is the one corresponding to x_1 according to the model (M). The corresponding solution strategy is then to solve simultaneously for x_T and x_1 using the system $(E_{2,1})$- $(E_{2,2})$. It will be then possible to study the limits of both the solutions when T goes to infinity, in order to check the consistency conditions C1) and C2).

The developments given below are exactly devoted to identify the cases where the solution paths corresponding to each approach are asymptotically unique, especially if the original model (M) is stable but non-saddlepoint. Of course, if this model admits a unique stable solution, it is worthwhile to check that the two approaches lead to the same stable solution.

To do that, we need some preliminary linear algebra results.

2.3. Regularity conditions

To ensure the solvability of the two proposed problems, we need to a single regularity condition. We assume that :

H3) P_{11} and P_{12} are invertible.

Assumption H3) allows to check both the non-verticality conditions and the short run solvability requirements, as it will appear in the following sections. To make clear this implication, we need some prior results :

Proposition

Under H1) and H3), we have :

R1) The submatrices P_{ij} and P^{ij} are invertible for all i and j.

R2) The matrices C and D defined by : $[C \quad D] = [A \quad B]\, P$, are invertible.

R3) Let $Q = P^{12} - P^{11} (P^{21})^{-1} P^{22}$, and Ξ the R^n set defined by :

$$\Xi = \{x_0 \in R^n \,/\, \exists\ i\ ,\ 1 \leq i \leq n,\ [\, Q\, x_0]_i = 0\}$$

where $[\ .\ ]_i$ designates the ith component of the vector between brackets.

Ξ is of a null measure.

Proof : To establish property R1), we use the very particular form of the matrix $F = \begin{bmatrix} A & B \\ I(n) & 0(n) \end{bmatrix}$. For such a matrix, observe that if λ_i is an eigenvalue, then the associated eigenvector is of the form : $E_i = \begin{bmatrix} \lambda_i e_i \\ e_i \end{bmatrix}$ where e_i belongs to the kernel of the n-dimensioned matrix λ_i^2 I(n)- λ_i A - B, which is non-empty by assumption H1).

Consequently, the submatrices matrices P_{ij} of P satisfy the equalities :

$$P_{11}= P_{21}\,\Lambda_1 \text{ and } P_{12}=P_{22}\,\Lambda_2\,.$$

It follows that if P_{11} and P_{12} are invertible, P_{21} and P_{22} are invertible too. Hence, all the submatrices of P are invertible. Observe also that the invertibility of P_{11} implies the invertibility of Λ_1, which signifies that we implicitely assume that there is no null eigenvalue. This implicit assumption is obviously equivalent to the invertibility of the matrix B.

Now, given that all the submatrices of P are invertible, it follows that all the corresponding submatrices of P^{-1} are invertible, which establishes property R1).

We prove now property R2). The invertibility arguments for C and D being very similar, we focuse on the second matrix. Using the partitioned form of P and the equality defining D, we obtain :

$$D= A\,P_{12} + B\,P_{22}\,.$$

On the other hand, we know that the submatrices P_{12} and P_{22} are exactly of the form :

$$P_{12} = [\lambda_{n+1}\,e_{n+1}\ \ \lambda_{n+2}\,e_{n+2}\ldots\ \lambda_{2n}\,e_{2n}] \text{ and } P_{22} = [e_{n+1}\ \ e_{n+2}\ldots\ e_{2n}]\,.$$

It follows that :

$$D= [\lambda_{n+1}\,A\,e_{n+1}+B\,e_{n+1}\quad \lambda_{n+2}\,A\,e_{n+2} +B\,e_{n+2}\ldots\ \lambda_{2n}\,A\,e_{2n}+B\,e_{2n}\,]\,.$$

As the vectors (e_k), $n+1 \leq k \leq 2n$, are in the kernel of the matrix λ_i^2 I(n)- λ_i A - B, matrix D could be rewritten as :

$$D = [\lambda_{n+1}^2 e_{n+1} \; \lambda_{n+2}^2 e_{n+2} \ldots \lambda_{2n}^2 e_{2n}].$$

Given that the submatrix $P_{22} = [e_{n+1} \; e_{n+2} \ldots e_{2n}]$ is invertible, $(e_{n+1}, e_{n+2}, \ldots, e_{2n})$ is a base of R^n. Consequently, D must be invertible as no zero eigenvalue is allowed, which ensures property R2).

Property R3) is analogous to a regularity property considered in LE Van and Boucekkine (1993). However, as our framework is simpler, the corresponding proof is more immediate than in the previous paper.

Let us first establish that the matrix $Q = P^{12} - P^{11}(P^{21})^{-1}P^{22}$ is invertible. If Q is singular, it exists a nonzero vector x such as $Q\,x = (P^{12} - P^{11}(P^{21})^{-1}P^{22})x = 0$. Introduce a vector y by the relation : $y = -(P^{21})^{-1}P^{22}x$. It remains that : $P^{12}x + P^{11}y = 0$.

Now, observe that, given the definition of y, the equation : $P^{21}y + P^{22}x = 0$, is also satisfied. Writing the two previous equations in a stacked form yields : $P^{-1}\begin{bmatrix} y \\ x \end{bmatrix} = 0$, where $\begin{bmatrix} y \\ x \end{bmatrix}$ is nonzero, which contradicts the invertibility of P^{-1}. Thus, Q is invertible.

The null measure characteristic of the set $\Xi = \{x_0 \in R^n / \exists\, i,\ 1 \le i \le n,\ [Q\,x_0]_i = 0\}$ follows quite automatically from the latter result. Indeed, observe that Ξ could be written as :

$$\Xi = \cup\, \Xi_i, \text{ for } 1 \le i \le n, \text{ where } \Xi_i = \{x_0 \in R^n / [Q\,x_0]_i = 0\}.$$

As Q is invertible, we can conclude that every set Ξ_i, for $1 \le i \le n$, is an hyperplane of R^n, and so, it is of a null measure. As Ξ is a finite union of null measure sets, it is also of a null measure. QED.

Given property R3), we can formulate the weak assumption :

H4) x_0 does not belong to Ξ .

We are now able to study the behaviour of the two solution approaches proposed above.

3. The asymptotic behaviour of the backward method under the representation (MF_1)

As announced, we will prove in this section that the backward method applied on the representation (MF_1), as described in subsection 2.1, locates a unique stable limit path if and only if the original model (M) is saddlepoint. In our case, the saddlepoint conditions are checked if only if the matrix F admits n eigenvalues greater than unity (in modulus) and n less than unity.

Let us state the corresponding proposition :

Proposition

Under assumptions H1), H2), H3) and H4), the backward method described in subsection 2.1, locates a unique stable limit path satisfying the consistency conditions C1) and C2), if and only if the model (M) fulfills the saddlepoint conditions.

Proof : Let us write again the system corresponding to the indicated backward method :

$$(E_{1,1}) \qquad \begin{bmatrix} x_T \\ x_{T-1} \end{bmatrix} = F^{T-1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$$

$$(E_{1,2}) \qquad [A \quad B] \; F^{T-1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = 0$$

i) We prove first that if the saddlepoint conditions hold, it asymptotically exists a unique solution path satisfying the consistency conditions C1) and C2). Using the diagonalizability assumption H1), equation ($E_{1,2}$) could be rewritten as :

$$[A \quad B]\ P \begin{bmatrix} \Lambda_1^{T-1} & 0(n) \\ 0(n) & \Lambda_2^{T-1} \end{bmatrix} P^{-1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = 0.$$

Given the partitioned form of P^{-1} and the definition : $[C \quad D] = [A \quad B]\ P$, it follows that :

$$C\,\Lambda_1^{T-1}(P^{11}\,x_1+P^{12}\,x_0) + D\,\Lambda_2^{T-1}\,(P^{21}\,x_1+P^{22}\,x_0) = 0$$

and consequently :

$$\{C\,\Lambda_1^{T-1}P^{11} + D\,\Lambda_2^{T-1}\,P^{21}\}\,x_1 = -\{\,C\,\Lambda_1^{T-1}P^{12} + D\,\Lambda_2^{T-1}\,P^{22}\}\,x_0\,.$$

In the saddlepoint case, all the elements of Λ_1 (Resp. Λ_2) are less (Resp. greater) than unity. Hence, when increasing T sufficiently, the terms involving Λ_1 vanish, and the previous equation becomes approximately : $D\,\Lambda_2^{T-1}\,P^{21}\,x_1 \approx - D\,\Lambda_2^{T-1}\,P^{22}\,x_0$. As D is invertible given property R2), it remains : $\Lambda_2^{T-1}\,P^{21}\,x_1 \approx - \Lambda_2^{T-1}\,P^{22}\,x_0$. Consequently, the quantity $\Lambda_2^{T-1}\,P^{21}\,x_1 + \Lambda_2^{T-1}\,P^{22}\,x_0 = \Lambda_2^{T-1}\,(P^{21}\,x_1 + P^{22}\,x_0)$ goes to zero when T goes to infinity, which can not hold if the limit of $P^{21}\,x_1 + P^{22}\,x_0$ is not zero. We can conclude that, if the saddlepoint conditions are checked, x_1 admits a unique finite limite value, given by the following formula : $x_1 = -(P^{21})^{-1}\,P^{22}\,x_0$.

It remains to prove that the corresponding limit of x_T is zero. To this end, we use equation ($E_{1,1}$) :

$$(E_{1,1}) \qquad \begin{bmatrix} x_T \\ x_{T-1} \end{bmatrix} = F^{T-1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}.$$

Multplying it by P^{-1} and using the partitioned forms of Λ and P^{-1}, yields :

$$P^{-1}\begin{bmatrix} x_T \\ x_{T-1} \end{bmatrix} = \begin{bmatrix} \Lambda_1^{T-1} & 0(n) \\ 0(n) & \Lambda_2^{T-1} \end{bmatrix} P^{-1}\begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} \Lambda_1^{T-1}(P^{11}\, x_1+P^{12}\, x_0) \\ \Lambda_2^{T-1}\,(P^{21}\, x_1+P^{22}\, x_0) \end{bmatrix}.$$

It is straightforward that the limit of the vector $\begin{bmatrix} \Lambda_1^{T-1}(P^{11}\, x_1+P^{12}\, x_0) \\ \Lambda_2^{T-1}\,(P^{21}\, x_1+P^{22}\, x_0) \end{bmatrix}$ is zero. The second component goes to zero given the reasoning just before; the first one goes to zero because of the magnitude of the elements of Λ_1 and given the finite limit value of x_1 found before. Hence, $P^{-1}\begin{bmatrix} x_T \\ x_{T-1} \end{bmatrix}$ goes to zero and x_T too.

We can conclude now that in the saddlepoint case, the backward method asymptotically locates a unique stable solution path, satisfying the consistency conditions C1) and C2).

ii) Let us study now the case of non-saddlepoint models. We begin by analyzing explosive models ; let us assume that the matrix F contains more than n unstable eigenvalues : p eigenvalues are less than unity, $p < n$, and $n+p$ are greater than unity. In other terms, the $n-p$ last elements of Λ_1 are greater than unity.

Suppose that the backward method succeeds at locating a limit path satisfying the consistency condition C2), namely with x_T of a null limit. It follows that the vector $F\begin{bmatrix} x_T \\ x_{T-1} \end{bmatrix}$, and consequently the vector $F^{T-1}\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$, go also to zero. The latter result allows to deduce two properties :

PR1) the limit of $\Lambda_1^{T-1}\,(P^{11}\, x_1+P^{12}\, x_0)$ is zero.

PR2) the limit of $\Lambda_2^{T-1}\,(P^{21}\, x_1+P^{22}\, x_0)$ is zero.

See that because all the elements of Λ_2 are also greater than unity, property PR2), exactly as in the saddlepoint case analyzed before, allows to compute a unique finite limit value for x_1 : $x_1 = -(P^{21})^{-1} P^{22} x_0$. Using this limit value in the statement of property PR1), we get :

$$\text{PR3)} \quad \text{the limit of } \Lambda_1^{T-1} (P^{12} - P^{11} (P^{21})^{-1} P^{22}) x_0 \text{ is zero.}$$

We prove now that property PR3) is impossible.

Let us denote by Λ_{12} the n-p last elements of Λ_1 which are greater than unity by assumption. P^{12}_{n-p} (Resp. P^{11}_{n-p}) is the submatrix of P^{12} (Resp. of P^{11}) corresponding to the n-p last rows of this matrix. Property PR3) leads to :

$$\text{PR4) the limit of } \Lambda_{12}^{T-1} (P^{12}_{n-p} - P^{11}_{n-p} (P^{21})^{-1} P^{22}) x_0 \text{ is zero.}$$

Observe that $P^{12}_{n-p} - P^{11}_{n-p} (P^{21})^{-1} P^{22}$ is the submatrix corresponding to the n-p last rows of the matrix $Q = P^{12} - P^{11} (P^{21})^{-1} P^{22}$. Given assumption H4) induced by property R3), it follows that all the components of the vector ($P^{12}_{n-p} - P^{11}_{n-p} (P^{21})^{-1} P^{22}) x_0$ are nonzero. Consequently, property PR4) could not hold as the term Λ_{12}^{T-1} is explosive.

It is then impossible to locate a limit path path satisfying the consistency condition C2) in the presence of more than n unstable eigenvalues.

iii) It remains to analyze the case where it exists more than n eigenvalues less than unity. To do that, we assume that the p first elements of the submatrix Λ_2 are less than one. Let us assume, as before, that the backward method locates a limit path checking the consistency condition C2), ie x_T goes to zero when T grows indefinitely. As before, we get the properties PR1) and PR2) :

$$\text{PR1)} \quad \text{the limit of } \Lambda_1^{T-1} (P^{11} x_1 + P^{12} x_0) \text{ is zero.}$$

$$\text{PR2) the limit of } \Lambda_2^{T-1} (P^{21} x_1 + P^{22} x_0) \text{ is zero.}$$

Property PR1) does not provide any information concerning x_1 as all the elements of Λ_1 are less than unity. On the other hand and contrary to the case ii), property PR2) does not allow to locate a limit value for x_1, as the p first elements of Λ_2 are less than unity.

Property PR2) implies that :

$$\text{PR5) the limit of } \Lambda_{21}^{T-1} (P_p^{21} x_1 + P_p^{22} x_0) \text{ is zero.}$$

with Λ_{21} the submatrix of Λ_2 corresponding to its n-p last elements, which are all greater than unity. P_p^{21} (Resp. P_p^{22}) is the submatrix corresponding to n-p last rows of P^{21} (Resp. P^{22}).

PR5) asymptotically implies : $P_p^{21} x_1 + P_p^{22} x_0 = 0$. As the matrix P^{21} is invertible, given property R1), the submatrix P_p^{21} is of rank n-p. Consequently, it could not exist a unique solution for the n-dimensioned vector x_1.

It is then impossible to locate a unique limit path path satisfying the consistency condition C1) and C2) in the presence of more than n stable eigenvalues.

This ends the proof of the whole proposition. QED.

The result contradicts the outcome of the ESRC Bureau concerning the performances of the terminal conditions of the fixed-value type. This contradiction is only apparent as the latter outcome is related to small sample solution paths and to first-order iterative solution schemes. However, this gives a quite dramatical idea on the non-robutness of the numerical prescriptions concerning such control problems.

Our result contradicts also LE Van and Boucekkine's result (1993) as the authors argued that fixed-value terminal conditions should allow the relaxation algorithms to asymptotically locate unique stable solutions in the stable non-saddlepoint case, if the relaxations are initialized with the long run values of the underlying models. To demonstrate such a result, the authors used a theoretical framework somehow analogous to the one developed in subsection 2.2, involving the representation (MF_2) of the model (M). The following section is devoted to confirm LE Van and Boucekkine's result in the latter framework.

4. The asymptotic behaviour of the backward method under the representation (MF_2)

In this section, we prove that the backward method applied on the representation (MF_2), as described in subsection 2.2, locates a unique stable limit path if and only if the original model (M) is stable. The difference with the previous setting is that the saddlepoint requirement is not necessary to reach a unique limit solution path ; only the stability condition matters. Let us state the corresponding proposition :

Proposition

Under assumptions H1), H2), H3) and H4), the backward method described in subsection 2.2, locates a unique stable limit path satisfying the consistency conditions C1) and C2), if and only if the model (M) is stable.

Proof : Let us write again the system corresponding to the indicated backward method :

$$(E_{2,1}) \qquad P^{12}\, x_T = \Lambda_1^T (P^{11}\, x_1 + P^{12}\, x_0)$$

$$(E_{2,2}) \qquad P^{22}\, x_T = \Lambda_2^T (P^{21}\, x_1 + P^{22}\, x_0).$$

Given property R1), the system ($E_{2,1}$) - ($E_{2,2}$) allows to compute x_1 and x_T as follows :

S1) $\left(P^{22}\,(P^{12})^{-1}\,\Lambda_1^T\,P^{11} - \Lambda_2^T\,P^{21}\right) x_1 = \left(\Lambda_2^T\,P^{22} - P^{22}\,(P^{12})^{-1}\,\Lambda_1^T\,P^{12}\right) x_0$

and

S2) $\left(P^{12} - \Lambda_1^T\,P^{11}\,(P^{21})^{-1}\,\Lambda_2^{-T}\,P^{22}\right) x_T = \left(\Lambda_1^T\,P^{12} - \Lambda_1^T\,P^{11}\,(P^{21})^{-1}\,P^{22}\right) x_0$.

i) In a first step, we prove that the consistency condition C1) holds in all the cases, given assumption H2). To this end, consider equation S1) and multiply it by Λ_2^{-T}. As any element of Λ_2 is strictly greater than any element of Λ_1, the term $\Lambda_2^{-T}\,P^{22}\,(P^{12})^{-1}\,\Lambda_1^T\,P^{11}$ goes to zero when T goes to infinity. The term $\Lambda_2^{-T}\,P^{22}\,(P^{12})^{-1}\,\Lambda_1^T$ also vanishes for the same reason. Thus, asymptotically, equation S1) remains : $-\,P^{21}\,x_1 = P^{22}\,x_0$. It is then worth pointing out that the second approach leads to the same limit value for x_1, although under a clearly weaker condition.

ii) In a last step, we show that the consistency property C2) holds if and only if the model is stable. To ensure that, we use equation S2).

First, observe that the coefficient of x_T, namely $P^{12} - \Lambda_1^T\,P^{11}\,(P^{21})^{-1}\,\Lambda_2^{-T}\,P^{22}$, converges always to the matrix P^{12}, given assumption H2). The asymptotic behaviour of the right side of the equality, $\Lambda_1^T\,(P^{12} - P^{11}\,(P^{21})^{-1}\,P^{22})\,x_0 = \Lambda_1^T\,Q\,x_0$, depends on the three terms Λ_1, Q and x_0. Using assumption H4), we know that all the components of the vector $Q\,x_0$ are nonzero. It follows that the limit of x_T is zero if and only if all the elements of Λ_1 are less than unity. Otherwise, x_T is explosive. This ends the proof of the proposition. QED.

5. Concluding remarks

The paper provides a specific theoretical argument against the usual prescriptions concerning the use of terminal conditions for the purpose of saddlepoint paths identification. On a canonical linear model, we showed that almost all the known outcomes concerning the fixed-value terminal conditions could be clearly challenged. The main outcome of the paper is that, even when using this type of terminal constraints, the possibility of differentiating between saddlepoint models and stable non-saddlepoint ones depends not only on the solution method, but also on the algebraic representations considered for the models. Iterating on a stacked form or on disagregated one is likely to produce different outcomes with respect to the saddlepoint extraction problem. As one of the main principles of numerical resolution of simultaneous systems is to use alternative algebraic forms until locating the most efficient one in terms of convergence domain, the outcome points at a relevant issue.

This issue is clearly far from being completely investigated, although some major results had emerged since the last decade. One important point, also clearly apparent in this paper, is that increasing the solution time horizon is not always sufficient to make the difference between saddlepoint models and stable non-saddlepoint ones. The so-called Fair's criterium, which is the basis of Fair and Taylor's algorithm (1983), seems definitely non relevant. As the numerical investigation based on the terminal conditions analytical forms is dramatically non robust, the practitioners are facing a very sophisticated puzzle. It is at least sure that no single instrument is sufficient to solve rigorously forward-looking models. Neither terminal conditions nor the relaxation initialization, and surely not the simulation time horizon. Especially when taking into account the problems related to nonlinearities or to the computability of long run equilibria, the problem seems itself non-robust. Consequently, the practitioners dealing with forward-looking models should proceed with no *a priori* and multiply *a posteriori* sensitivity tests. Actually, that is the unique reasonable prescription in the story.

References

Begg, D (1982), The Rational Expectation Revolution in Macroeconomics, *Oxford University Press*.

Boucekkine, R (1993), " Some New Developments on The Analysis of the Numerical Solutions of Consistent Expectations Models", *Journal of Economic Dynamics and Control*, forthcoming.

Boucekkine, R and C. LE Van (1993), " How to Detect Saddlepoint Finite-Difference Linear Models Using Relaxation Algorithms ?" , Pre-accepted for publication in *Journal of Economic Dynamics and Control*, second version under examination.

Fair, R and J-B. Taylor (1983), " Solution and Maximum Likelihood Estimation of Dynamic Rational Expectations Models", *Econometrica*, 51 (4).

Gouriéroux, C, J-J. Laffont and A. Monfort (1983) , "Révision Adaptative des Anticipations et Convergence vers les Anticipations Rationnelles", *Economie Appliquée*, 36.

Laffargue, J-P (1990) , " Résolution d'un Modèle Macroéconomique à Anticipations Rationnelles", *Annales d'Economie et Statistique*, 17.

Wallis, K (ed.), M. Andrews, D. Bell, P. Fisher and J. Whitley (1985), Models of the UK Economy : a Second Review by the ESRC Macroeconomic Modelling Bureau, *Oxford University Press*.

Wallis, K (ed.), M. Andrews, D. Bell, P. Fisher and J. Whitley (1986), Models of the UK Economy : a Third Review by the ESRC Macroeconomic Modelling Bureau, *Oxford University Press*.