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STRATEGIC EQUILIBRIUM IN ECONOMIES WITH A CONTINUUM OF AGENTS

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Abstract

In this note, a pure exchange economy with a continuum of agents who behave strategically in endowments and preferences is considered. A notion of equilibrium, namely, strategic equilibrium is defined. It is shown that price-taking and strategic behavior leads to identical results.

Key Words:
Strategic equilibrium, pure exchange economy, continuum economy.

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1 Introduction

Strategic behaviour does not yield, in general, the same result that is obtained under price-taking behaviour. However, recent work, such as Codognato and Gabszewicz (1993), suggests that in continuum economies some other notions of equilibrium, than the competitive one, may lead to identical results .

In the present note, we consider a pure exchange economy with a continuum of agents who behave strategically. Following Gabszewicz and Vial (1972), a notion af equilibrium, called strategic equilibrium, is defined. Our main objective is an attempt at a comparative study with respect to perfect competition models.

For this purpose, we analyse several situations according to the definition of a strategies set. First, the strategies are endowments and preferences. Second, the strategies are only endowments, keeping preferences invariant; this may be interpreted as a situation where goods are burnt to increase prices, such as the case of coffee in Brasil. Third, we consider an intermediate situation, where strategies are basically recourses. This last case covers the model developped by Codognato and Gabszewicz (1993); the same equivalence result is obtained, without imposing any restriction in the structure of initial holdings and without assuming uniqueness of equilibrium while generalizing the set of strategies. It is proved that the set of strategic equilibrium allocations coincides with the set of competitive allocations; in particular, true characteristics is a strategic equilibrium profile. Therefore, this note can be interpreted as a justification (slightly different from usual) for the assumption of competitive behaviour in continuum economies, which have been termed as perfectly competitive economies.

In section 2 we present the model, state the notations and introduce the strategic equilibrium concept. Section 3 is concerned with the main result. Some applications are provided in section 4. Section 5 is the conclusion.

2 The Model

Consider a pure exchange economy $\mathcal{E}$, with a continuum of agents represented by the real interval $I = [0, 1]$. Every consumption set is $\Re^{\ell}_{+}$, where ℓ represents the number of different commodities traded in the market. Each agent $t \in I$ is characterized by his initial endowment ω_t and his preference relation on his consumption set $X_t = \Re^{\ell}_{+}$, represented by a continous utility function $U_t : X_t \to \Re$. Thus, the economy $\mathcal{E}$ is specified by endowments and utilities (ω_t, U_t), for all agent $t \in I$.

An allocation is a μ-integrable function $x : I \to \Re^{\ell}_{+}$. An allocation x is said to be feasible if $\int_I x(t)d\mu \leq \int_I \omega_t d\mu$, where μ denotes the Lebesgue measure on the Borel subsets of I. A competitive (or walrasian) equilibrium for the economy $\mathcal{E}$ is a pair (p^*, x^*), consisting of a non zero price system $p^* \in \Re^{\ell}_{+}$ and a feasible allocation x^*, such that for almost all $t \in I$, $x^*(t) \in B_t(p^*) = \{x \in \Re^{\ell}_{+} : p^*x \leq p^*\omega_t\}$ and $B_t(p^*) \bigcap \{ x \in \Re^{\ell}_{+} : U_t(x) > U_t(x^*(t)) \} = \emptyset$.

Let us suppose that individuals behave strategically. Given the set of agents I of an exchange economy, characterized by their initial holdings and preferences; we can say that the strategies are misrepresentations of these characteristics and, therefore, misrepresentations of their supplies. Thus, given the exchange economy $\mathcal{E}$,let the set of strategies for an individual $t \in I$ be $\Theta_t = \{(\theta_t, U_{\theta_t})$, such that $0 \leq \theta_t \leq \omega_t$ and, $U_{\theta_t} : X_t \to \Re$ is continous$\}$. So, an agent

$t \in I$ of characteristics (ω_t, U_t) can claim to be of any other characteristic in Θ_t, i.e., he can send to the market not his whole holdings but a fraction, and he can also announce another utility function, which differs from his real one. However, the consumption set is invariant.

A strategy profile is a mapping $\theta : I \rightarrow \bigcup_{t \in I} \Theta_t$, which associates to each agent $t \in I$ a strategy $\theta(t) = (\theta_t, U_{\theta_t}) \in \Theta_t$. In this sense, agents can affect economies that differ from the initial one $\mathcal{E}$. Let $\mathcal{E}_\theta$ be the virtual economy which effects if individual declare a strategy profile θ, that is, $\mathcal{E}_\theta \equiv \left(X_t = \Re^\ell_+, \ (\theta_t, U_{\theta_t}), t \in I \right)$.

An allocation mechanism is a mapping that associate to each possible economy a feasible allocation. A mechanism f is said to be incentive compatible individually if given any initial economy $\mathcal{E}$, it's satisfied that for almost all $t \in I$, $U_t(f_t(\mathcal{E})) \geq U_t(f_t(\mathcal{E}_{\theta(t)}))$, for every $\theta(t) \in \Theta_t$, where $\mathcal{E}_{\theta(t)}$ denotes the economy which coincides with $\mathcal{E}$ except for one agent $t \in I$, who should be characterized by (θ_t, U_{θ_t}), instead of (ω_t, U_t), and $f_t(\mathcal{E})$ denotes the allocation received by the individual t in the economy $\mathcal{E}$. Let f be an allocation mechanism such that, given a virtual economy $\mathcal{E}_\theta$, $f(\mathcal{E}_\theta)$ is a Walrasian allocation for $\mathcal{E}_\theta$.

As it is shownn in the next section, we deal with economies for which there exists a competitive equilibrium. Given the possibilities of multiple equilibrium, some selection can be prescribed in order to define the mechanism f correctly. Following Roberts (1980), and for convenience of analysis and notation, f is defined in a way such that prices change by the smallest amount necessary to restore equilibrium. That is, f is required to satisfy the following property: if (p, x) is competitive equilibrium of the basis economy $\mathcal{E}$ and we define $x = f(\mathcal{E})$, then, given any other economy $\mathcal{E}'$, $x' = f(\mathcal{E}')$ if there exists p' such that (p', x') is Walras equilibrium of $\mathcal{E}'$ and $\|p' - p\| \leq \|p'' - p\|$ for all p'' Walras equilibrium prices of $\mathcal{E}'$; in other case the mechanism selects any competitive allocation for $\mathcal{E}'$. This selection is justified in the final remarks. However, as it may be noticed in the proof of theorem 1, the price that minimize the distance to a previous one not only exists but is unique.

With this approach, we denine a strategic equilibrium for the economy $\mathcal{E}$ as a pair consisting of a strategy profile θ^* and a feasible allocation x^*, such that $x^*(t) = \tilde{x}(t) + \omega_t - \theta^*_t$, with $\tilde{x}$ walrasian allocation for the apparent economy $\mathcal{E}_{\theta^*}$, where θ^* verifies that no set of agents of positive measure can benefit from deviating unilaterally. Note, that for all $\mathcal{E}_\theta$ we have $x(t) = f_t(\mathcal{E}_\theta) + \omega_t - \theta_t$ as a feasible allocation for the basis economy $\mathcal{E}$. Towards defining the notion of strategic equilibrium formally, let us denote by $\mathcal{E}_{\theta \setminus \theta'(t)}$ the economy which coincides with $\mathcal{E}_\theta$, except for one agent $t \in I$, who declares characterists $(\theta'_t, U_{\theta'_t})$.

Definition 1 . *A* **strategic equilibrium** *for the economy $\mathcal{E}$ is a pair (θ^*, x^*), where θ^* is a strategy profile and x^* is a feasible allocation, such that*

a) x^* can be written as $x^*(t) = f_t(\mathcal{E}_{\theta^*}) + \omega_t - \theta^*_t$, and

b) $U_t(x^*(t)) \geq U_t(f_t(\mathcal{E}_{\theta^* \setminus \theta(t)}) + \omega_t - \theta_t)$, for all $\theta(t) \in \Theta_t$, for almost all $t \in I$.

Let us suppose that the equilibrium profile θ^* is given by the true characteristics and define the mechanism g by $g_t(\mathcal{E}_\theta) = f_t(\mathcal{E}_\theta) + \omega_t - \theta_t$. It is worth noting that, in this case, condition b) is equivalent to the individual incentive compatibility of g.

3 The Main Result

We are interested in showing that in the stated situation the set of strategic equilibrium allocations coincides with the set of competitive allocations. Let us suppose that the initial economy $\mathcal{E}$ verifies the assumptions stated by Aumann (1964) for existence of competitive equilibrium in continuum economies

(H.1) $\int_I \omega_t d\mu >> 0$, i.e., the total endowment is strictly positive in every component,

(H.2) the utility functions are strictly monotone, and

(H.3) the functions $U_t(x)$ are measurable in x and t with respect to the compact-open topology.

In order to guarantee the existence of competitive equilibrium for every apparent economy, denote by $A(\mathcal{E})$ the set of economies $\mathcal{E}_\theta$ that agents can create with θ admissible strategy profile. A strategy profile θ is said to be admissible if the utility functions U_{θ_t} verify (H.2) and (H.3), and $\int_I \theta_t d\mu >> 0$. So, if $\mathcal{E}_\theta \in A(\mathcal{E})$ we can assert that $\mathcal{E}_\theta$ satisfy the hypothesis above and, therefore, the mechanism f is well defined on $A(\mathcal{E})$. Under this assumptions we can state the main result. If (p^*, x^*) is a competitive equilibrium for the economy $\mathcal{E}$, there exists a strategy profile θ^*, such that (θ^*, x^*) is a strategy equilibrium; conversely, if (θ^*, x^*) is a strategic equilibrium for the economy $\mathcal{E}$, there exists p^* such that (p^*, x^*) is a Walras equilibrium.

Theorem 1 . *x^* is a competitive equilibrium allocation for the economy $\mathcal{E}$ if and only if x^* is a strategic equilibrium allocation for the economy $\mathcal{E}$.*

Proof. Let (p^*, x^*) be a competitive equilibrium for the economy $\mathcal{E}$. Denote by θ^* the strategy profile which associates to each agents his true characteristics. i.e., $\theta^*(t) = (\omega_t, U_t)$ and $\mathcal{E}_{\theta^*} = \mathcal{E}$. Defining the auxiliary mechanism g as $g_t(\mathcal{E}_\theta) = f_t(\mathcal{E}_\theta) + \omega_t - \theta_t$ one obtains $x^*(t) = f_t(\mathcal{E}) = g_t(\mathcal{E})$. Let us first show that g is incentive compatible individually, which in this situation is equivalent to condition (b) in definition 1 and, consequently we could conclude that (θ^*, x^*) is strategic equilibrium for $\mathcal{E}$. To this end, associated to each economy $\mathcal{E}_\theta \in A(\mathcal{E})$ and to each agent $t \in I$, let us define the set $B_{\theta_t}(p(\theta)) = \{x \in \Re^\ell_+ \text{ such that } p(\theta)x \leq p(\theta)\theta_t\}$, where $p(\theta)$ is a competitive equilibrium price system for the economy $\mathcal{E}_\theta$. Consider $p(\theta^*) = p^*$ and $f(\mathcal{E}_{\theta^*}) = x^*$. Because μ is atomless, we have that p^* is also walrasian price system for the economy $\mathcal{E}_{\theta^* \setminus \theta(t)} = \mathcal{E}_{\theta(t)}$, whatever caharacteristic $\theta(t) \in \Theta_t$ declared by the agent t may be. So, we can state $(p^*, f(\mathcal{E}_{\theta(t)}))$ walrasian equilibrium for $\mathcal{E}_{\theta(t)}$. Clearly, we also have $B_{\theta_t}(p^*) \subseteq B_t(p^*)$, for all $\theta_t \leq \omega_t$. By definition of f and the sets $B_{\theta_t}(p(\theta))$, one obtains that the allocation $f_t(\mathcal{E}_{\theta(t)}) + \omega_t - \theta_t \in B_t(p^*)$, for all $\theta(t) \in \Theta_t$, for almost all $t \in I$. Therefore, since $f_t(\mathcal{E})$ maximizes U_t in $B_t(p^*)$ for almost all $t \in I$, we can conclude that $U_t(f_t(\mathcal{E})) \geq U_t(f_t(\mathcal{E}_{\theta(t)}) + \omega_t - \theta_t) = U_t(g_t(\mathcal{E}_{\theta(t)}))$ for all $\theta(t) \in \Theta_t$, for almost all $t \in I$, what this means is that g es incentive compatible individually. Consequently, the pair consisting of the strategy profile define by the true characteristics an the allocation x^* is a strategic equilibrium.

Reciprocally, let (θ^*, x^*) be a strategic equilibrium for the economy $\mathcal{E}$. By definition one has $x^*(t) = f_t(\mathcal{E}_{\theta^*}) + \omega_t - \theta_t^*$, and $U_t\left(f_t(\mathcal{E}_{\theta^*}) + \omega_t - \theta_t^*\right) \geq U_t\left(f_t(\mathcal{E}_{\theta^* \setminus \theta(t)}) + \omega_t - \theta_t\right)$, for almost all $t \in I$ and whatever $\theta(t) \in \Theta_t$ may be. Let $p(\theta^*)$ be the equilibrium price system associated with $f(\mathcal{E}_{\theta^*})$. Consider $p^* = p(\theta^*)$; and let us show that (p^*, x^*) is a competitive equilibrium for $\mathcal{E}$. Since $f_t(\mathcal{E}_{\theta^*}) \in B_{\theta_t^*}(p^*)$ for almost all $t \in I$, we have $x^*(t) = f_t(\mathcal{E}_{\theta^*}) + \omega_t - \theta_t^* \in B_t(p^*)$ for almost all $t \in I$. It remains to be shown that $x^*(t)$ is a maximal element in the budget set $B_t(p^*)$ for almost all $t \in I$. Suppose, it is not so. Then there exists $S \subseteq I$, with $\mu(S) > 0$ and there exist consumption vectors $x(t) \in B_t(p^*)$, such that $U_t(x(t)) > U_t(x^*(t))$ for all agent t in S. The strategy profile θ^* must verify *(i)* $\theta^*(t) = (\omega_t, U_t)$, for all $t \in S' \subset S$, with $\mu(S') > 0$, or *(ii)* $\theta^*(t) \neq (\omega_t, U_t)$, for almost all $t \in S$. If *(i)* occurs, one obtains $U_t(x(t)) > U_t(f_t(\mathcal{E}_{\theta^*}))$ for almost all $t \in S$, but then $(p^*, f(\mathcal{E}_{\theta^*}))$ would not be walrasian equilibrium for the economy $\mathcal{E}_{\theta^*}$. Which is not in accordance with the definition of f and is contrary to condition (a) in the definition of strategic equilibrium. If *(ii)* occurs, let us consider the strategy $\theta(t) = (\omega_t, U_t)$ for almost all $t \in S$. As was mentioned earlier p^* is also a competitive equilibrium price for the economy $\mathcal{E}_{\theta^*\setminus\theta(t)}$. Thus, we can state $f(\mathcal{E}_{\theta^*\setminus\theta(t)})$ as a competitive allocation with prices p^*. Then, since $x(t) \in B_t(p^*)$ for almost all $t \in S$, we have $U_t(f_t(\mathcal{E}_{\theta^*\setminus\theta(t)})) \geq U_t(x(t)) > U_t(f_t(\mathcal{E}_{\theta^*}) + \omega_t - \theta_t^*)$ for almost all $t \in S$. Which is contrary to condition b) in the strategic equilibrium definition. Q.E.D.

It is worth noting that this result depends critically on the continuum assumption. As in continuum economies an individual has no ability to influence price formation and has no gains from non-competitive behaviour. Furthermore, in atomless economies the influence of each agent (or of a set of measure zero) is null because the integral does not change if the behaviour of such set of agents is altered. However, this does not happen in finite economies and the equilvalence result does not hold if a finite set of agents is considered. In fact, an example similar to which appears in Codognato and Gabszewicz (1993) can be stated to show that in finite economies the strategic equilibrium may differ from the competitive outcome.

4 Some Applications

In this note we have formalized a model of a pure exhange economy, with a continuum of agents who behave strategically in their endowments and preferences. However other strategic behaviour can be considered. For example, that strategies are only endowments and preferences are invariant. Even an intermediate situation can be analized, with strategies consisting of endowments and preferences. The latter determined by the former and by true preferences. Nevertheless, both examples lead to the same result. As we show below, they are applications of the main result obtained in the previous section.

First, let us suppose that the set of strategies for each agent $t \in I$ is reduced to $\Theta_t = \{\theta_t \in X_t, \text{such that} \theta_t \leq \omega_t\}$, keeping utility functions U_t fixed. Now the strategies are not pairs consisting of endowments and preferences but only recourses. Thus, we may write $\theta(t) = \theta_t$. Let us also assume that the initial economy $\mathcal{E}$ satisfies assumptions (H.1), (H.2) and (H.3), which assert the existence of competitive equilibrium. In this case a strategy profile is said to be admissible if it verifies (H.1). In this way, we can state (as a particular case of theorem 1) that x^* is a competitive allocation for the economy $\mathcal{E}$ if and only if x^* is a strategic equilibrium allocation for the economy $\mathcal{E}$. For this, it is sufficient to realize that (in this case) the equilibrium profile θ^* verifies either $\theta_t^* = \omega_t$, for almost all $t \in S$ or $\theta_t^* \neq \omega_t$ and for almost all $t \in S$.

Given that agents declare their true preferences, one can interpret that they misrepresent their holdings not to consume but only to impact on prices. This suggests that condition b) in definition 1 may be replaced by condition (b') $U_t(f_t(\mathcal{E}_{\theta^*})) \geq U_t(f_t(\mathcal{E}_{\theta^*\setminus\theta(t)}))$, for all $\theta(t) \in \Theta_t$, for almost all $t \in I$. If so, the equivalence result does not alter because the first part of the proof would be reduced to show the incentive compatibility of competitive mechanism f, and the second one follows immediately, taking into account that $f_t(\mathcal{E}_\theta) \in B_t(p(\theta))$, whatever strategy profile θ may be. It is interesting to note that this first application provides explicit economic interpretations. It represents situations where commodities, as was in our earlier example, are burnt. It is known that there are countries and regions where the excess of productivity leads to the owners to throw away part of the output in order to increase prices. That is how it happens, for example, with coffee in Brasil and with cherries in Jerte Valley (Spain).

Let us now consider the intermediate situation, where the set of strategies for each agent $t \in I$ is defined by $\Theta_t = \{(\theta_t, U_{\theta_t}),$ such that $0 \leq \theta_t \leq \omega_t$ and $U_{\theta_t}(x) = U_t(x + \omega_t - \theta_t),$ with $x \in X_t = \Re^\ell_+\}$. This can be interpreted as follows; if an individual declares endowments θ_t, then he will say that his utility function is the true one evaluated on what he receives plus what he hides. As above, the initial economy $\mathcal{E}$ is required to satisfy (H.1), (H.2) and (H.3). Note, that the utility functions U_{θ_t} satisfy (H.2) and (H.3) because the functions U_t verify both assumptions. Therfore, in order to guarantee the existence of competitive equilibrium in all apparent economies it is sufficient to say that a strategy profile is admissible if it verifies (H.1). In this intemediate situation we once more obtain that price-taking and strategic behaviour lead to identical results. The proof is just like the one stated for theorem 1.

Observe, that this situation covers the model described by Codognato and Gabszewicz (1993). In such a model there are a continuum of agents who behave strategically and a continuum of agents who behave as price-takers. The former are called oligopolists and all of them manipulate the market for the same single good. The structure of initial endowments prevents them from acting strategically on the others, since they do not share the ownership of other goods. As it was mentioned in the introduction, under uniqueness of equilibrium and defining endowments as strategies, they show that the set of Cournot-Walras equilibrium coincides with the set of Walras equilibrium. The same result is obtained here. Defining the notion of strategic equilibrium without restricting the structure of initial endowments and generalizing the set of strategies and without assuming uniqueness of competitive equilibrium. Moreover, we could have considered that some agents behave strategically while the others remain as price-takers. (As introducing agents who adopt a price-taking behaviour would not alter the main result.)

5 Final Remarks

We prove that in continuum economies the set of strategic equilibrium allocations coincides with the set of competitive allocations. Consequently, there exists strategic equilibrium if and only if there exists competitive equilibrium. This equivalence is stated for three different cases, on the basis of the stategies considered. First, the set of strategies is defined by endowments and preferences. Second, the strategies are only endowments. Third, the strategies are basically endowments. Regardless, we show that the price-taking behaviour leads to an identical result to that of the strategic one.

We have worked in the commodity space $\Re^\ell_+$, but this requirement is not essential. The same result can be extended to the infinite dimensional case by adding the hypothesis that guarantees the existence of competitive equilibrium in continuum economies defined on an infinite dimensional commodity space. (See Khan and Yannelis (1991), ch.4). The competitive equilibrium is not required to be unique. The possibility of multiple equilibria justified the selection effected in section 2. Such a selection is made for convenience of analysis, in order to obtain a certain continuity, for clarity of notation, and to facilitate the proof of the main result.

The result obtained by Codognato and Gabszewicz (1993) is a special case of the intermediate situation considered in this note. The same equivalence result is obtained here but within a more general approach. We assume no particular structure of initial endowments, nor uniqueness of competitive equilibrium and the set of strategies is generalized; moreover, agents not only behave strategically on all goods but can trade with all of them, even with those that are misrepresented. In the intermediate situation it is worth emphasizing that the consumption sets in the economies $\mathcal{E}_\theta$ are defined by $\Re^\ell_+$. If the consumption set for an agent $t \in I$ in the economy $\mathcal{E}_\theta$ would be defined as $X_t = \Re^\ell_+ - \omega_t + \theta_t$, then it is easy to show that (p, x) is Walras equilibrium for $\mathcal{E}$ if and only if $(p, \bar{x})$ is Walras equilibrium for $\mathcal{E}_\theta$, with $\bar{x}(t) = x(t) - \omega_t + \theta_t$. Thus, the equilibrium price system for $\mathcal{E}$ and $\mathcal{E}_\theta$ would be the same, therefore, the proof of theorem 1 would be reduced to show the incentive compatibility of the auxiliar mechanism g. Let us also point out that the definition of strategic equilibrium has been stated following the idea of the Cournot-Walras equilibrium concept introduced by Codognato and Gabszewicz (1993), however (as noted) both notions differ somewhat. If the initial situation is considered as in Codognato and Gabszewicz (1993), one obtains a single Cournot-Walras equilibrium profile and two strategic equilibrium profiles. One, to declare the true characteristics and the other to declare the competitive equilibrium supply. Both profiles resulting in the same equilibrium allocation.

As is pointed out our main result does not hold in economies with a finite set of agents. In finite economies an agent may be able to manipulate prices to his benefit and he would then have an incentive to adopt non-competitive behaviour. Despite this, one would expect an approximation theorem, showing that as the number of agents increases the strategic equilibrium tends to the competitive equilibrium. In fact, economists typically assume that consumers in large economies will adopt a price-taking behaviour. This note should provide a basis for a more systematic study of an asymptotic version of the main result.

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