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Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624 9875

REAL INCOME GROWTH AND REVEALED PREFERENCE INCONSISTENCY *

David Jerison[1] and Michael Jerison[2]

Abstract

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Keywords: weak axiom of revealed preference, real income growth, Ville cycle, consumer price index.

[1] Departament of Mathematics, MIT, Cambridge, MA 02139; E.mail: jerison@math.mit.edu
[2] Departament of Economics, SUNY, Albany, NY 12222; E.mail: m.jerison@albany.edu

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Real Income Growth and Revealed Preference Inconsistency*

David Jerison
Department of Mathematics,
MIT, Cambridge, MA 02139
jerison@math.mit.edu
and
Michael Jerison
Department of Economics
SUNY, Albany, NY 12222
m.jerison@albany.edu

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1. Introduction

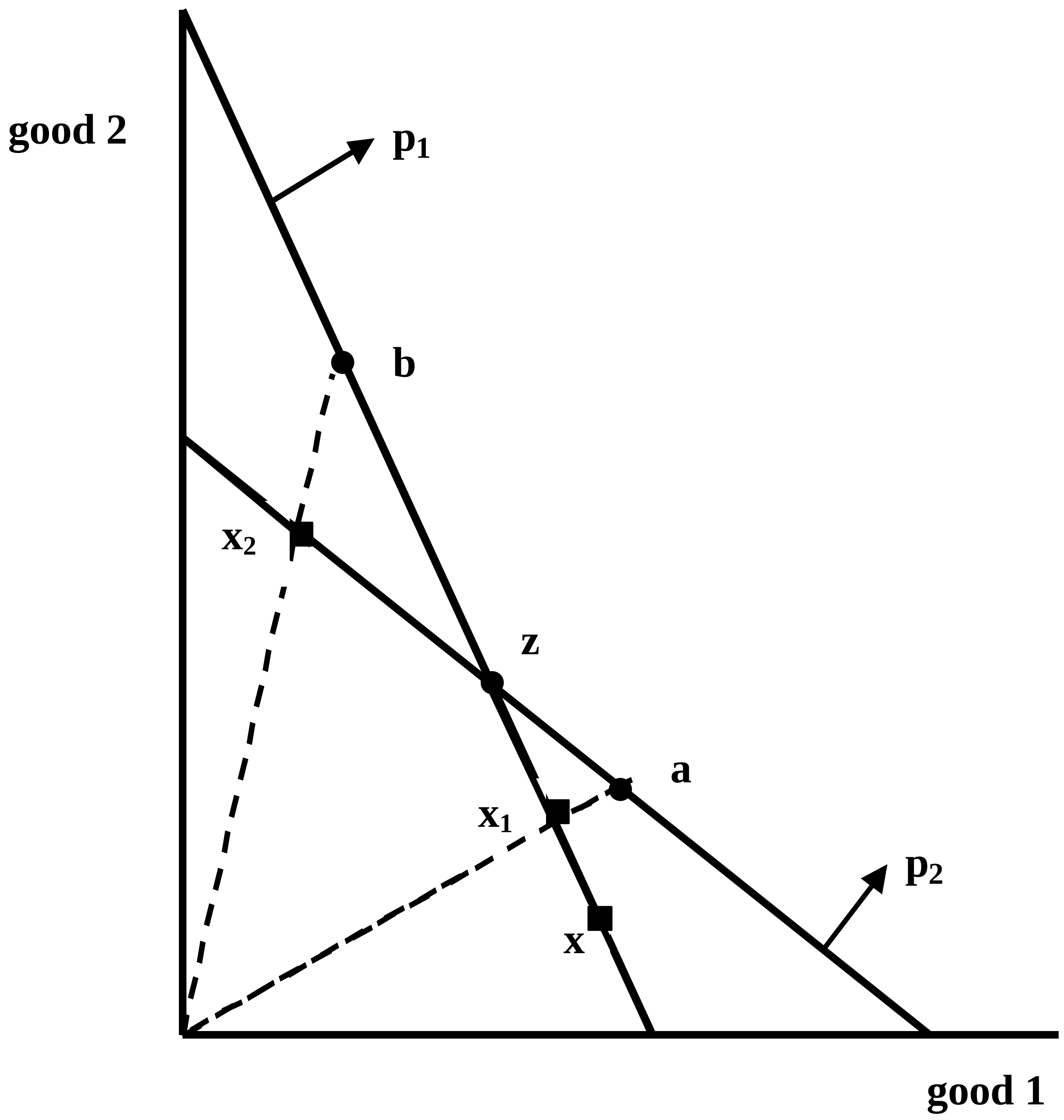
The neoclassical model of a competitive consumer plays a central role in economic analysis, yet it often conflicts with consumption data. Observed consumer behavior is often inconsistent in the sense that it violates the revealed preference axioms and therefore cannot be derived from utility maximization (Battalio, et. al. (1973), Koo and Hasenkamp (1972), Sippel (1997), Mattei (2000)). Such inconsistency can appear for many reasons (Sen (1997)). A consumer's preferences might depend on the budget situation, e.g., when prices provide information about quality. Preferences might change over time and the consumer might be subject to constraints that are not modeled. The analyst might not observe all the demands or might observe some of them with errors. (But none of these reasons can account for the inconsistencies found in consumption experiments by Sippel (1997) and Mattei (2000).) Alternatively, the consumer might have incomplete or random preferences or might make mistakes.

Whatever the reasons for the inconsistencies, they call into question inferences about economic welfare that are drawn from consumption data. The inconsistencies can also be important for positive analysis since various authors claim that small deviations from optimization can have large effects on the allocation of resources (e.g., Thaler (1992)). It is natural to ask to what extent the simple neoclassical model conflicts with the data. Are consumers' inconsistencies small enough to be ignored? To answer this question one needs a sensible measure of demand inconsistency.

Inconsistencies can be a problem when the neoclassical consumer model is applied to aggregate demand. When a representative consumer model is used to analyze changes in policy or technology, one implicitly assumes that aggregate demand is generated by utility maximization. This assumption does not conflict with most aggregate time series data: aggregate demands for many countries satisfy the strong axiom of revealed preference over long periods (e.g., Landsburg (1983)). However the aggregate time series might be consistent only because the historical path of prices and aggregate income is very special. When real income is continually rising with little relative price variation, it can be impossible to detect violations of the revealed preference axioms. It does not follow that aggregate demands would continue to be consistent after hypothetical policy or technology changes of interest.

The standard way to study demand responses to hypothetical policy or technology changes is to work with estimated parametric demand functions. Often these estimated demand functions are not derivable from utility maximization (e.g., Deaton and Muellbauer (1980)). But this should come as no surprise. Even if the individual consumers are utility maximizers, aggregate demand generically violates the strong axiom of revealed preference, and therefore behaves differently from the demand of a single competitive consumer (Gorman (1953), Jerison (1994)). How differently presumably depends on the sizes of the violations of the strong axiom. But how are these violations to be measured?

Figure 1 about here.


good 2
p1
b
x2
z
a
x1
p2
x
good 1

The measurement problem is illustrated in Figure 1. Suppose that a consumer demands the vector x_j when the price vector is p_j $(j = 1, 2)$. When x_j is chosen, x_k $(k \neq j)$ could have been bought with money left over. Therefore each x_j is revealed preferred to x_k $(k \neq j)$, and the choices violate the weak axiom. The question is by how much.

The best-known measures of the violations are due to Afriat (1973) and Varian (1985). Afriat's "cost-inefficiency" measure is based on the idea that an inconsistent consumer is wasting money since there is a consumption bundle both cheaper and revealed preferred to the bundle that is chosen. In Figure 1, when the price vector is p_2, the consumer could save the fraction $|a - x_1|/|a|$ of the available budget by switching from x_2 to x_1, which is revealed preferred. ($|\cdot|$ denotes the Euclidean norm.) Similarly, when the price vector is p_1, the consumer could save the fraction $|b - x_2|/|b|$ of the budget by switching from x_1 to x_2. The smaller of these fractional savings is Afriat's (1973) "cost-inefficiency" measure. If income is fixed at 1, the cost-inefficiency is the smallest income wastage that is consistent with the given demand data. (For more detailed discussion, see section 2, below.)

Varian (1985) proposes an alternative measure: the minimum distance from the given data to data that satisfy the generalized axiom of revealed preference (GARP), a slightly weakened version of the strong axiom. In Figure 1, that measure is the distance between x_1 and z. The inconsistency is larger according to either measure if the demand at price vector p_1 is x instead of x_1. Both measures can be defined for any number of observations.

Afriat's cost-inefficiency is the most commonly used measure of violations of the revealed preference axioms, and Varian (1990) recommends it as being an *economic*, as opposed to statistical, measure. However Gross (1991) and Jerison and Jerison (1993) point out that it has a drawback. It is very sensitive to the amount of price variation in the data. If in Figure 1, there were little price variation then the budget lines would be nearly parallel and the cost-inefficiency would be small no matter what choices the consumer made. But the amount of price variation is a property of the environment, not of the consumer's behavior, so the cost-inefficiency is not just a measure of behavioral inconsistency. There is no easy solution to this problem. With finite data, any reasonable measure of inconsistency will depend on the environment. Still, it is useful to know in what way an intuitive inconsistency measure like the cost-inefficiency depends on the amount of price variation.

The present paper shows how a slight modification of the cost-inefficiency depends locally on the amount of price variation and on behavioral inconsistency summarized by the Slutsky matrix. A smooth demand function is generated by utility maximization if and only if it has a symmetric, negative semidefinite Slutsky matrix. We relate the sizes of revealed preference inconsistencies to violations of the two Slutsky conditions. We use this result to justify a new measure of local inconsistency based on the Slutsky matrix. The new measure is behavioral since it is determined by the demand function and does not depend on the amount of price variation. The inconsistency measure can be applied to aggregate demand if the aggregate demand is a smooth function of aggregate income and prices (as happens when individual incomes are determined by aggregate income and prices through a smooth distribution or sharing rule). The inconsistency measure then indicates the extent to which the aggregate demand deviates from the demand of a single optimizing consumer. We characterize the new inconsistency measure by exploiting the close connection between the cost-inefficiency and real income growth along revealed preference cycles, which we describe next.

We will show that as one moves along a revealed preference cycle, real income grows from each observation to the next, yet one returns to where one started. The subtlety is that this proposition requires real income to be defined using a Laspeyres price index with a consumption base that is updated at each step along the cycle. (For details, see section 2.) Defined in this way, the real income growth along a revealed preference cycle is very different from growth in a business cycle. In a business cycle, real income rises then falls. In a revealed preference cycle real income is *always rising* yet the prices and consumer demands return to their initial levels. The real growth is illusory, like the climbers' ascent of Escher's staircase in "Ascending and Descending" (Escher (1982)). Demand inconsistency is equivalent to existence of such a cycle. When income is fixed at 1, Afriat's cost-inefficiency is the minimum real income growth rate along the worst revealed preference cycle. (For details, see section 2.)

On Escher's staircase you cannot tell whether you are going up or down. The same can happen to real income under proposed reforms of the U.S. consumer price index. Revealed preference inconsistency makes real income growth incoherent when it is defined using a frequently updated Laspeyres price index. The incoherence does not arise if the base for the price index remains fixed. But frequent updating of the consumption base is one of the reforms recommended by the recent CPI Commission (Boskin, et. al. 1998). The Commission also recommended replacing the Laspeyres index by a "superlative" index (Diewert (1976)) based on both current and past consumption. In practice, however, information on current consumption is unavailable since collecting and processing the data takes time. So, for the foreseeable future, the U.S. consumer price index is likely to remain a Laspeyres index, possibly with more frequent updating of the consumption base. Our paper provides a framework for measuring the incoherence that such frequent updating can introduce. For a given small amount of price variation we find the highest constant real income growth rate compatible with prices and demands returning to their initial values.

Our main theorem (Theorem 1) shows how this highest constant real income growth rate depends on the amount of price variation and on an inconsistency index computed from the Slutsky matrix. When each price can vary over the interval $[1-r, 1+r]$ the real income growth rate along the worst revealed preference cycle is approximately r^2 times the Slutsky inconsistency index. If the Slutsky index is 1 and if prices can vary up or down by up to 10% then there is a path along which real income grows at a rate of approximately 1% from each observation to the next, yet nominal income and prices return to their starting values. If prices vary up or down by only 1% then the real income growth rate along the worst cycle is reduced to approximately 0.01%.

The Slutsky index is the only inconsistency measure derived from the Slutsky matrix that satisfies a set of reasonable axioms (Theorem 2). In particular, it does not vary when the commodity units change or when goods with fixed relative prices are aggregated. If the Slutsky matrix is known, the index and a nearly worst cycle can be computed using an efficient quadratic programming algorithm due to Coleman and Li (1996). Even if the Slutsky matrix is not known, Theorem 1 provides information about the way that real growth along the worst cycle depends on the variation in prices.

It is not at all obvious how violations of the Slutsky conditions should be measured or interpreted. The problem of interpretation arises, for example, when Browning and Chiappori (1998) use the Slutsky matrix to test for consumption efficiency in the "collective" model of household demand. For a two-member household with efficient consumption, the Slutsky matrix can be asymmetric, but if it is, then it must equal a symmetric matrix plus a matrix of rank one. The amount of Slutsky asymmetry in an empirical analysis has a statistical interpretation. It indicates the strength of rejection of a "unitary" model in which the household acts like a single optimizer. Our main theorem provides an *economic* interpretation for sizes of Slutsky violations by relating them to real income growth along cycles.

The Slutsky conditions have been linked separately to revealed preference inconsistencies in two classic papers. Kihlstrom, et. al. (1976) show that Slutsky negative semidefiniteness is equivalent to a weak version of the weak axiom. Hurwicz and Richter (1979) show that Slutsky symmetry is equivalent to Ville's (1946) axiom, a differential version of the strong axiom. Jerison and Jerison (1992, 1993) provide quantitative theorems showing how violations of Slutsky symmetry or negative semidefiniteness control respectively the sizes of violations of Ville's axiom or the weak axiom. These results are discussed below in section 6.

Testing for Ville's axiom requires a continuum of demand data. The present paper extends the previous literature by analyzing discrete cycles and by relating them to the two Slutsky conditions together. The relationship allows us to solve a long-standing open problem in demand theory. Samuelson (1938) asked if a finite upper bound could be placed on the number of observations needed to reject the hypothesis that a demand function is generated by utility maximization. Shafer (1977) gives a partial answer. He first shows that there cannot be an a priori bound. For each K he exhibits a smooth demand function that violates Slutsky symmetry and has no revealed preference cycle with fewer than K observations. It follows that any finite bound must depend on information about the specific demand function. Shafer then derives a bound that depends on the Slutsky matrix. The more asymmetric the matrix, the smaller the number of observations required to form a revealed preference cycle. But Shafer's bound applies only to demand functions that are linear in income. In section 5, below, we obtain a bound that applies to every C^1 demand function and is tighter than Shafer's bound in the case of demands that are linear in income.

In the next section we present notation and preliminary results. We show that real income grows continually as one moves along a revealed preference cycle if the consumption base of the Laspeyres price index is updated at each step. We also show how Afriat's cost-inefficiency is related to the real income growth along cycles. In Section 3 we introduce a class of inconsistency measures derived from the Slutsky matrix. Then we state the main theorem relating the real growth along cycles to the Slutsky measures and the amount of price variation. In Section 4, we pick one Slutsky measure, the "Slutsky index" of local demand inconsistency, and characterize it by means of reasonable axioms. In Section 5 we derive an upper bound on the number of observations required to uncover revealed preference inconsistency. In Section 6 we discuss related literature and remaining open problems. The longer proofs are in Section 7.

2. Preliminaries

We consider a C^1 *demand function*, i.e., a function $h : \mathrm{IR}_{++} \times R^n_{++} \longrightarrow \mathrm{IR}^n_+$ that satisfies the *budget identity* $ph(y,p) = y$ for all $(y,p) \gg 0$. (Note that we do not require the demand function to be homogeneous of degree zero.) We write dot products of vectors, omitting the dot. The column vector $h(y,p) = (h^1(y,p), \ldots, h^n(y,p))^T$ represents the vector of demands at income $y > 0$ and price vector $p \gg 0$, with jth component p^j, the price of good j. (A vector x is treated as a column vector with jth component x^j. Superscript T denotes the transpose.) A vector (y,p) of income and prices is called a *budget situation.* Whenever we use the term "demand function," we mean for the function to be C^1. The *Slutsky matrix of* h *at* (y,p) is

$$S(y,p) \equiv h_p(y,p) + h_y(y,p)h(y,p)^T,$$

with ij component $(\partial h^i/\partial p^j) + (\partial h^i/\partial y)h^j$ evaluated at (y,p).

The demand function h is said to be *generated by a utility function* $u : \mathrm{IR}^n_+ \longrightarrow \mathrm{IR}$ if for every $(y,p) \gg 0$, $h(y,p)$ is the unique maximizer of $u(x)$ over the budget set $\{x \in \mathrm{IR}^n_+ : px \leq y\}$. The C^1 demand function h is generated by some utility function u if and only if the Slutsky matrix of h at each $(y,p) \gg 0$ is symmetric and negative semidefinite (Theorem 1 of Jerison and Jerison, 1993). (An $n \times n$ matrix M (not necessarily symmetric) is *negative semidefinite* [respectively, *positive semidefinite*] if $x^T M x \leq [\geq] 0$ for each n-vector x.)

We will focus on demand behavior near a fixed budget situation (y^*, p^*). Let $\bar{S}$ and A denote respectively the symmetric and antisymmetric parts of the Slutsky matrix of h at (y^*, p^*). This means that

$$\bar{S} \equiv (1/2)[S(y^*,p^*) + S(y^*,p^*)^T] \quad \text{and} \quad A \equiv (1/2)[S(y^*,p^*) - S(y^*,p^*)^T]. \tag{1}$$

Demand functions are typically not generated by utility maximization if there are more than two goods. In that case asymmetry of the Slutsky matrix is a generic property. Perturbations of any smooth demand function typically violate Slutsky symmetry.

If a demand function is not generated by a utility function then it exhibits inconsistencies called revealed preference cycles. One budget situation (y,p) is *strictly revealed preferred* (for h) to another, (z,q), if the demand vector chosen in the second situation could have been afforded in the first with money left over, i.e., if $y > ph(z,q)$. A sequence $\{a_k\}_{k=0}^K$ of elements of a Euclidean space is called a *cycle of length* K or a K*-cycle* if $a_K = a_0$. A *strict revealed preference cycle* (for h) is a cycle of budget situations $\{(y_k,p_k)\}_{k=0}^K$ such that (y_k,p_k) is strictly revealed preferred to (y_{k-1},p_{k-1}) for $k = 1, \ldots, K$.

The demand function h satisfies the *weak weak axiom of revealed preference* if it has no strict revealed preference cycles of length two. If h is generated by the utility function u and situation

(y,p) is strictly revealed preferred to (z,q), then it is easy to see that $u(h(y,p)) > u(h(z,q))$. It follows that if h is generated by utility maximization it cannot have any strict revealed preference cycles. The converse is also true, so utility maximization is equivalent to the absence of *strict* revealed preference cycles. (See Corollary 3, below.)

Strict revealed preference can be quantified through its connection with real income growth. The connection comes from the fact that one situation is strictly revealed preferred to another whenever a move from the latter to the former raises real income, as defined below. Therefore, as one moves along a revealed preference cycle, real income is always growing yet one returns to the starting point. The rate of real income growth can be used to measure the demand inconsistency. The relation between real growth and inconsistency is relevant for the debate over price index reform.

We now formulate these ideas more precisely. Real income growth is defined by adjusting the actual (nominal) income change to take account of price changes. Real income growth is connected with demand inconsistency when it is defined using a Laspeyres price index. Consider a cycle of budget situations $\{(y_k, p_k)\}_{k=0}^{K}$ and let $x_k \equiv h(y_k, p_k)$ be the demand vector in situation k. The *Laspeyres price index* for situation k with base situation m is $L(p_k, p_m; x_m) \equiv (p_k x_m)/(p_m x_m)$. There is real income growth from observation $k-1$ to situation k (using the Laspeyres price index with base $k-1$) if the nominal income in situation k is larger than the income in situation $k-1$ multiplied by the price index for k, i.e., if $y_k > L(p_k, p_{k-1}; x_{k-1})y_{k-1} = p_k x_{k-1}$. We call $y_k - p_k x_{k-1}$ the *real income growth rate from situation* $k-1$ *to* k. It is the income in situation k minus the Laspeyres "cost of living" (the cost of buying at current prices what was bought in the previous situation). This real income growth rate is positive whenever situation k is strictly revealed preferred to situation $(k-1)$. So with these definitions, as one moves along a strict revealed preference cycle, real income is always growing, yet one returns to the initial situation. The real growth is illusory, like the ascent of Escher's staircase.

What is special about the definition of real income above is that the base of the Laspeyres price index is updated at each step along the cycle. Such a price index is said to be *chained.* The updating in the chained Laspeyres index makes the concept of real income growth incoherent when the demand has revealed preference inconsistencies. In that case continual real growth can bring one back to where one started. On the other hand, if the growth is defined using a Laspeyres index with a fixed base situation then it is impossible for real income to grow at each step along a cycle. To see this, let the base be fixed at observation 0. Then real income at observation k is greater than at $k-1$ if and only if $y_k/y_{k-1} > L(p_k, p_0; x_0)/L(p_{k-1}, p_0; x_0) = (p_k x_0)/(p_{k-1} x_0)$. This inequality cannot hold for every $k = 1, \ldots, K$. If it did then one would obtain the contradiction: $1 = \Pi_{k=1}^{K}(y_k/y_{k-1}) > \Pi_{k=1}^{K}[(p_k x_0)/(p_{k-1} x_0)] = 1$. Thus, with a fixed consumption base, real income growth is coherent even if the demand is inconsistent. Working with a chained index can introduce incoherence, but the incoherence can be used to measure the demand inconsistency.

To measure the inconsistency we consider the minimum real income growth rate along the worst revealed preference cycle in a region. The *minimum real growth rate* of a cycle of budget situations $\{(y_k, p_k)\}_{k=0}^{K}$ is

$$G(\{(y_k, p_k)\}_{k=0}^{K}) \equiv min\{y_k - p_k h(y_{k-1}, p_{k-1}) : k = 1, \ldots, K\}.$$

The minimum real growth rate depends on the commodity units. In our definition of the inconsistency index in Section 4, we will choose units so that the base budget situation (y^*, p^*) becomes a vector of ones. The *worst cycle* in a region is the cycle with the highest minimum real growth rate. In the regions considered below, the range of allowed variation in nominal income is an interval. In such a region, for any revealed preference cycle with nonconstant real income growth rate, it is always possible to raise the minimum real growth rate by adjusting the nominal income levels. Therefore the worst cycle in such a region has a constant real growth rate. We sometimes refer to the minimum real growth rate of a cycle simply as the *real growth rate* of the cycle. A strict revealed preference cycle is a cycle with a strictly positive minimum real growth rate. It would be possible to define inconsistency measures based on the average or the maximum real growth rate along cycles, but for such measures it is not clear how to prove limit theorems like the ones presented below.

To state the theorems we need notation for the real growth rate of the worst cycle in a region. Given a set X in Euclidean space, let $C_K(X)$ be the set of K-cycles with elements in X and let $C(X) \equiv \cup_{K\geq 1} C_K(X)$ be the set of *all* cycles with elements in X. Define

$$\begin{aligned} \mathcal{G}_K(X) &\equiv sup\{G(c) : c \in C_K(X)\} \text{ and} \\ \mathcal{G}(X) &\equiv sup\{G(c) : c \in C(X)\}, \end{aligned} \tag{2}$$

the suprema of the real growth rates of all K-cycles in X and of *all* cycles in X, respectively. Since G is continuous and $C_K(X)$ is homeomorphic to the Cartesian product X^K, the supremum in (2) is attained at a *worst* K-cycle if X is compact.

We conclude this section by defining Afriat's cost-inefficiency and showing that for a set X of budget data with income fixed at 1, the cost-inefficiency is $\mathcal{G}(X)$, the minimum real growth rate of the worst cycle in the data. Afriat (1973) treats the consumer as having preferences but not necessarily choosing optimally. The consumer is choosing at "efficiency level" ϵ or higher if no vector costing less than the fraction ϵ of the budget is ever preferred to what is actually chosen. This means that the consumer is never wasting more than the fraction $1-\epsilon$ of the budget. The problem for the analyst is that the consumer's preferences are not known. Afriat's solution is to look for the highest efficiency level compatible with *some* preference relation.

To express this formally, call a relation R on $\mathbb{R}_+^n$ a *weak preference* if it is reflexive (xRx) and transitive (xRy and yRz imply xRz). Let x_m be the demand vector chosen when the price vector is p_m ($m \in M$). A scalar ϵ is called an *efficiency level compatible with the relation* R if, for all m and k in M, conditions (a) and (b) are satisfied: (a) $p_m x_k \leq \epsilon p_m x_m$ implies $x_m R x_k$; (b) $x_k R x_m$ implies $p_m x_k \geq \epsilon p_m x_m$. In that case, the consumer's choice is always preferred (according to R) to vectors that cost less than ϵ times what is actually spent. The *cost-efficiency* is the supremum of the efficiency levels compatible with some weak preference. The *cost-inefficiency* of the data is 1 minus the cost-efficiency.

Remark 1. *The cost-inefficiency of the price and demand data* $\{p_m, x_m\}_{m\in M}$ *is the supremum of* $min\{(p_kx_k - p_kx_{k-1})/(p_kx_k) : k = 1,\ldots,K\}$*, with the supremum taken over all* K *and all* K*-cycles* $\{p_k, x_k\}_{k=0}^K$ *in the data. If the income and total expenditure are always equal to 1, then the cost-inefficiency is* $\mathcal{G}(\{(p_m\cdot x_m, x_m)\}_{m\in M})$*, the supremum of the minimum real growth rates of cycles in the data.*

Proof. Afriat (1973) shows that ϵ is a compatible efficiency level for some weak preference if and only if for every cycle $\{p_k, x_k\}_{k=0}^K$ in the data, $p_kx_{k-1} \leq \epsilon p_kx_k$, $\forall k$, implies $p_kx_{k-1} = \epsilon p_kx_k$, $\forall k$. Let w^* be the cost-inefficiency of the data. If for some w there is a cycle $\{p_k, x_k\}_{k=0}^K$ in the data with $(p_kx_k - p_kx_{k-1})/(p_kx_k) > w$ for all $k \geq 1$, then $(1-w)p_kx_k > p_kx_{k-1}$ for $k \geq 1$, so $1-w$ is larger than the cost-efficiency $1-w^*$ and $w < w^*$. This shows that no cycle has $min_k\{(p_kx_k - p_kx_{k-1})/(p_kx_k)\}$ higher than w^*. On the other hand, for each w less than w^*, $1-w$ is larger than the cost-efficiency, so there is a cycle with $(1-w)p_kx_k \geq p_kx_{k-1}$, $\forall k \geq 1$, and hence $min_k\{(p_kx_k - p_kx_{k-1})/(p_kx_k)\} \geq w$. Therefore w^* is the supremum of the terms $min_k\{(p_kx_k - p_kx_{k-1})/(p_kx_k)\}$ for cycles in the data. If the income is always $p_kx_k = 1$ then w^* is the supremum of the minimum real growth rates.

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3. The Slutsky Matrix and Real Income Growth Along Cycles

In this section we consider revealed preference cycles in neighborhoods of a fixed budget situation (y^*, p^*). In each neighborhood, the real growth rate of the worst cycle depends on both the extent of the price variation and on the degree of inconsistency of the demand function. We show how the real growth rate of the worst cycle decreases as the neighborhood shrinks. We also relate the real growth rate to violations of the Slutsky conditions. The main result is stated in Theorem 1.

We begin with the Slutsky matrix S at (y^*, p^*). Given a K-cycle $\{q_k\}_{k=0}^K \in C_K(\mathrm{I\!R}^n)$, define

$$I(S,\{q_k\}_k) \equiv (1/K)\sum_{k=1}^{K} q_k^T S(q_k - q_{k-1}) \tag{3}$$

$$= (1/K)\sum_{k=1}^{K}[(1/2)(q_k - q_{k-1})^T\bar{S}(q_k - q_{k-1}) + q_{k-1}^T A q_k]. \tag{4}$$

From (4), we see that $I(S,\{q_k\}_k)$ is an average of quadratic forms of the Slutsky matrix S and terms involving its antisymmetric part A defined in (1). We will define measures of Slutsky violations by taking suprema of $I(S,\cdot)$ over K-cycles or over all cycles in a nonempty "base" set $Q \subset \mathrm{I\!R}^n$. Given Q, define

$$\mathcal{I}_K(S,Q) \equiv sup\{I(S,\{q_k\}_k) : \{q_k\}_k \in C_K(Q)\}$$
$$\mathcal{I}(S,Q) \equiv \sup_{K\geq 1} \mathcal{I}_K(S,Q).$$

We will refer to I, $\mathcal{I}_K$ and $\mathcal{I}$ as *Slutsky measures.* Their definitions were chosen to make the limit equations in Theorem 1 correct. But the following remark justifies our interpreting $\mathcal{I}(S,Q)$ as a measure of violations of the Slutsky conditions if Q is a neighborhood of the origin.

Remark 2. $\mathcal{I}(S,Q)$ is nonnegative, and is zero if S is symmetric and negative semidefinite. Suppose now that Q is a compact neighborhood of the origin. Then $\mathcal{I}(S,Q)=0$ if and only if S satisfies the Slutsky conditions. If $\hat{S}-S$ is symmetric and positive semidefinite for another Slutsky matrix $\hat{S}$, then $\mathcal{I}(\hat{S},Q)\geq\mathcal{I}(S,Q)$ and $\mathcal{I}_K(\hat{S},Q)\geq\mathcal{I}_K(S,Q)$. The latter inequality is strict if the rank of $\hat{S}-S$ is $n-1$ (the highest possible) and $\mathcal{I}_K(S,Q)>0$. Also, $\mathcal{I}_K(S+tA,Q)$ is nondecreasing in $t\geq 0$ and is strictly increasing if the antisymmetric part A affects the Slutsky measure of S, i.e., if $\mathcal{I}_K(S,Q)\neq\mathcal{I}_K(\bar{S},Q)$, where $\bar{S}$ is the symmetric part of S, defined in (1). Finally, if $\bar{S}=0$, so that $S=A$, then $\mathcal{I}(S,Q)$ is a norm of the antisymmetric part A.

The proof is in section 7. Remark 2 says essentially that if the base set Q is a neighborhood of the origin, $\mathcal{I}(S,Q)$ detects whether either Slutsky condition is violated, and increases (weakly) when the violation worsens.

As a further illustration, we compute $\mathcal{I}(S,Q)$ for a case in which Q consists of just four points.

Example. Let Q consist of vectors of the form $(x,0)$ where x is one of the following: $(1,1)$, $(1,-1)$, $(-1,-1)$ or $(-1,1)$. Then only the prices of the first two goods are allowed to change and $\mathcal{I}(S,Q)$ depends only on the 2×2 leading principal minor matrix of S. Let this matrix be

$$\hat{S}\equiv\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Every cycle in Q is a combination of K-cycles with $K\leq 4$, so $\mathcal{I}(S,Q)=\mathcal{I}_K(\hat{S},Q)$ for some $K\leq 4$. By considering all possible 2, 3 and 4-cycles, we find that $\mathcal{I}(S,Q)$ is the maximum of the terms

$$2a, 2d, 2(a+b+c+d), 2(a+d-b-c), a+d+|b-c|, 0.$$

The first four of these terms are values of the quadratic form of $\hat{S}$. They are nonpositive if the Slutsky matrix is negative semidefinite. If the maximum of these four terms is positive, it measures the worst violation of Slutsky negative semidefiniteness in directions determined by Q.

Asymmetry of $\hat{S}$ means that $b\neq c$. The asymmetry can be measured by $|b-c|$. It affects the Slutsky measure $\mathcal{I}(S,Q)$ only when $\mathcal{I}(S,Q)=a+d+|b-c|$, which occurs only if the asymmetry $|b-c|$ is sufficiently large. In that case, the index combines the asymmetry $|b-c|$ with $a+d$, the trace of $\hat{S}$. If $\hat{S}$ is negative semidefinite then its trace is nonpositive, and the Slutsky measure $\mathcal{I}(S,Q)$ is positive only when the asymmetry $|b-c|$ dominates the trace. Otherwise the Slutsky measure is 0 even though $\hat{S}$ is asymmetric: the negative semidefiniteness removes the effect of the asymmetry. In that case, detecting Slutsky asymmetry requires more than the four observations in the example. By Remark 2, an asymmetric Slutsky matrix S (no matter how small the asymmetry) has a positive Slutsky measure $\mathcal{I}(S,Q)$ when the base set of price variations is a neighborhood of the origin.

The discussion above links $\mathcal{I}(S,Q)$ to violations of the two Slutsky conditions. Next, we link it to local demand inconsistency as measured by real income growth along cycles.

Theorem 1. *Consider a compact set $Q \subset I\!R^n$, a cycle $\{q_k\}_k$ in $C_K(Q)$ and a scalar $\gamma > max\{|q^T h(y^*, p^*)| : q \in Q\}$. Let $N \equiv [-\gamma, \gamma] \times Q$. Then*

$$\lim_{r\to 0} r^{-2}\, sup\{G(\{(y^* + rz_k, p^* + rq_k)\}_{k=0}^K) : |z_k| \leq \gamma, \forall k\} = I(S, \{q_k\}) \tag{5}$$

$$\lim_{r\to 0} r^{-2}\mathcal{G}_K((y^*, p^*) + rN) = \mathcal{I}_K(S, Q) \quad and$$

$$\lim_{r\to 0} r^{-2}\mathcal{G}((y^*, p^*) + rN) = \mathcal{I}(S, Q).$$

The expression $sup\{G(\{(y^* + rz_k, p^* + rq_k)\}_{k=0}^K) : |z_k| \leq \gamma, \forall k\}$ on the left side of (5) is the highest minimum real growth rate for cycles in which the kth price vector is $p^* + rq_k$ and the kth income differs from y^* by no more than $r\gamma$. According to the theorem, when the right side of (5) is positive, this highest minimum real growth rate shrinks approximately in proportion to r^2 as r approaches 0. The theorem provides the same information when the q_k are allowed to vary over an arbitrary base set Q. The constant of proportionality in each case is the corresponding Slutsky measure.

In the proof of Theorem 1 we construct a nearly worst revealed preference K-cycle in $(y^*, p^*) + rN$ for small r. The price vectors in this cycle can be computed by maximizing the right side of (3). Thus, the problem of finding a nearly worst K-cycle in a small region is reduced to a quadratic programming problem. As we see from the definition of N, nominal income y is restricted to an exogenously given interval: $|y - y^*| \leq r\gamma$. This restriction is not likely to matter since γ can be chosen to be as large as one wishes. We conjecture that the restriction on the income variation can be removed altogether. In any case, for sufficiently small r, the exogenous bound on the income variation is not binding. The income levels in the nearly worst cycle constructed in the proof of Theorem 1 differ from y^* by strictly less than $r\gamma$.

We conclude this section by applying Theorem 1 to prove part of the classic characterization of the weak weak axiom by Kihlstrom, et. al. (1976).

Corollary 1. *If a C^1 demand function satisfies the weak weak axiom of revealed preference, then at each point of its domain, its Slutsky matrix is negative semidefinite.*

Proof. Suppose that the Slutsky matrix fails to be negative semidefinite. Then there exists p with $p^T \bar{S} p > 0$. Let $q_0 = q_2 = p^*$ and let $q_1 = p^* + p$. Then $(q_1 - q_k)^T \bar{S}(q_1 - q_k) > 0$ for $k = 0, 2$. By (3) and (4), $I(S, \{q_k\}_{k=0}^2) > 0$, since $A^T = -A$. By Theorem 1, there is a strict revealed preference 2-cycle.

□

4. An Index of Local Inconsistency

We have not yet defined an inconsistency measure that depends on demand behavior alone. The Slutsky measures defined above depend on commodity units and on the base set Q. In this section we propose a local inconsistency index that removes this dependence, namely,

$\mathcal{I}(S^*, [-1,1]^n)$, where the normalized Slutsky matrix S^* is defined in (6), below. The base set of price variations is a box. We show in Theorem 2 that this is the only choice of base set (up to scalar multiple) that satisfies a set of reasonable axioms. The most important of the axioms is that the measure does not change when goods with fixed relative prices are aggregated.

A simple way to deal with the dependence on commodity units is to specify the units in such a way that income and all prices in the base situation are equal to 1. Then absolute deviations from the base values of income and prices are also fractional deviations. Suppose we start with arbitrary commodity units and an arbitrary base situation (y^*, p^*). We define new units in the following way. One new unit of good j is equal to y^*/p^{*j} old units, and one new unit of money is y^* old units. Then when the old price vector is p^*, the cost of one new unit of good j is y^* old units of money or one new unit of money. Therefore the old income and price vector (y^*, p^*) is replaced by a vector of 1's in the new units. In the new units, the demand function h becomes $\tilde{h}$ with $\tilde{h}^j(y,p) \equiv (p^{*j}/y^*)h^j(yy^*, p^1p^{*1}, \ldots, p^np^{*n})$. The Slutsky matrix of this modified demand function evaluated at the $(n+1)$-vector of ones is

$$S^* \equiv (1/y^*)P^*S(y^*, p^*)P^*, \tag{6}$$

where P^* is the diagonal matrix with p^{*j} as its jj component. Note that S^* is unaffected by changes in commodity or money units. Also, S^* is symmetric [resp. negative semidefinite] if and only if $S(y^*, p^*)$ is.

Next we consider the choice of Q, the base set of price variations. While this choice is somewhat arbitrary, the Slutsky measures are easier to describe and interpret if Q has a simple shape. Letting Q be a ball simplifies the computation, but creates problems when one compares demand functions with different dimensional commodity spaces. For such comparisons it is better to let Q be a box. To justify this claim we will need some additional terminology. In what follows, we assume that units have been chosen so that the base budget situation is a vector of 1's. Let 1_m denote the m-vector of 1's. Let π_j be the projection of a vector onto the jth coordinate:

$$\pi_j(q^1, \ldots, q^m) \equiv q^j, \text{ for } q = (q^1, \ldots, q^m) \in \mathrm{I\!R}^m$$

and let $\psi_j(q)$ be the vector obtained from q by removing its jth coordinate:

$$\psi_j(q^1, \ldots, q^m) \equiv (q^1, q^2, \ldots, q^{j-1}, q^{j+1}, \ldots, q^m).$$

To be able to compare demand functions in different dimension commodity spaces we must specify a base set of price variations $Q_n \subset \mathrm{I\!R}^n$ for each dimension $n = 1, \ldots, \infty$. Consider the n-good demand function h and its worst K-cycle with price vectors in $1_n + rQ_n$. The real growth rate along this cycle (divided by r^2) is a measure of the inconsistency of h over this set of budgets. But suppose that in h the demand for some good j is always zero. Then the inconsistency of h, the demand function for n goods, should be the same as the inconsistency of $\hat{h}$ formed from h by dropping the jth good. The worst K-cycle for $\hat{h}$ must have the same real growth rate as the worst cycle for h. But the only way to ensure this for all demand functions is to require that in the original cycle, when the jth price is removed from each price vector we obtain a price vector in $1_{n-1} + rQ_{n-1}$. This justifies imposing the following restriction on the base sets Q_n:

(A1) For each integer $n \geq 2$, and each positive integer $j \leq n$, if $q \in Q_n$ then $\psi_j(q) \in Q_{n-1}$.

A second restriction on the base sets is suggested by the effects of Hicks-Leontief aggregation of commodities. Let $\hat{h}$ be a demand function for $n+1$ goods, and suppose that along the cycle $\{(y_k, p_k)\}_k$ the price of good n equals the price of good $n+1$, i.e., $p_k^n = p_k^{n+1}$ for each observation k. For these observations, goods n and $n+1$ can be aggregated, with the price of good n treated as the price of the new aggregated good. This yields the aggregated demand function h defined by $h^j(y, p^1, \ldots, p^{n-1}, p^n) \equiv \hat{h}^j(y, p^1, \ldots, p^{n-1}, p^n, p^n)$ for $j = 1, \ldots, n-1$, and

$$h^n(y, p^1, \ldots, p^{n-1}, p^n) \equiv \hat{h}^n(y, p^1, \ldots, p^{n-1}, p^n, p^n) + \hat{h}^{n+1}(y, p^1, \ldots, p^{n-1}, p^n, p^n).$$

It is easy to verify that this h satisfies the budget identity.

The real income growth from one observation to the next is unaffected when goods with equal prices are aggregated. To see this, note that the real income growth from observation k to $k+1$ for $\hat{h}$ is $y_{k+1} - p_{k+1} \cdot \hat{h}(y_k, p_k) = y_{k+1} - \psi_{n+1}(p_{k+1}) \cdot h(y_k, \psi_{n+1}(p_k))$. Thus aggregation or disaggregation of goods with equal prices should have no effect on measured demand inconsistency. This is assured by requiring

(A2) For integers n and j, $1 \leq j \leq n$, if $q = (q^1, \ldots, q^n) \in Q_n$ then $(q, \pi_j(q)) \in Q_{n+1}$.

According to (A2), if we start with an n-good price vector in the base set Q_n and we create an $n+1$-good price vector by repeating one of the original prices, then the new price vector is in the base set Q_{n+1} in the $n+1$-good model.

It is natural to assume that the range of allowed variation for each price is unbroken:

(A3) If $x, w \in Q_n$ and $q \in \mathbb{R}^n$ satisfy $\psi_j(x) = \psi_j(w) = \psi_j(q)$ and $w^j \leq q^j \leq x^j$, then $q \in Q_n$.

Axiom (A3) states that if the vectors x, w and q differ only in their jth components and if q^j is between w^j and x^j and w and x are in Q_n then q is in Q_n too.

Finally, we make a simplifying assumption: that each Q_n is symmetric about the origin in each coordinate.

(A4) For each integer $n > 0$, if $x \in Q_n$, $z \in \mathbb{R}^n$, $\psi_j(x) = \psi_j(z)$ and $x^j = -z^j$ for some j, then $z \in Q_n$.

Axioms (A1) through (A4) imply that the base sets Q_n are boxes.

Theorem 2. *For each positive integer n let Q_n be a nonempty compact subset of $\mathbb{R}^n$. If the sets Q_n satisfy axioms (A1) through (A4), then there is a scalar $\alpha > 0$ such that $Q_n = [-\alpha, \alpha]^n$ for each n.*

Proof. Combining (A3) and (A4) we see that $Q_1 = [-\alpha, \alpha]$ for some $\alpha > 0$. By (A1), $Q_n \subset [-\alpha, \alpha]^n$ for every n. We claim that $Q_n = [-\alpha, \alpha]^n$ for each n. Suppose that the claim is true for some n. Then the n-vector with all components equal to α is in Q_n and, by (A2), the $(n+1)$-vector with all components equal to α is in Q_{n+1}. By (A4), the $(n+1)$-vector with first component $-\alpha$ and all other components equal to α is in Q_{n+1}. Suppose that $x \in [-\alpha, \alpha]^{n+1}$. Axiom (A3) implies that the $(n+1)$-vector $(x^1, \alpha, \ldots, \alpha)$ is in Q_{n+1}. Repeating this argument we find that $(x^1, x^2, \alpha, \ldots, \alpha)$ and x are in Q_{n+1}. Therefore, $[-\alpha, \alpha]^{n+1} \subset Q_{n+1}$, and the claim holds for all n by induction.

$\square$

Since α in Theorem 2 is arbitrary, we can let it equal 1. We therefore propose as a measure of inconsistency for h at (y^*, p^*) the

Slutsky index: $\mathcal{I}(S^*, [-1,1]^n)$,
where S^* is the transformed Slutsky matrix defined in (6), above.

By Theorem 1, for the n-good demand function h and for small r, the real growth rate along the worst cycle with prices in $[1-r, 1+r]$ is approximately $r^2\mathcal{I}(S^*, [-1,1]^n)$. As the range of price variation shrinks the real growth rate shrinks approximately in proportion to the square of the price variation. The proportionality constant is the Slutsky index.

The Slutsky index can be computed using the very efficient box-constrained quadratic programming algorithm of Coleman and Li (1996). For example, computing the index and the worst 10-cycles in an 8 good demand model (requiring search over a space of dimension 80) took less than one second on a 500 megahertz PC.

5. How Many Observations Are Required To Form a Cycle?

In this section we will use Theorem 1 to place a finite upper bound on the number of observations required to form a strict revealed preference cycle. The bound depends on the Slutsky matrix. Shafer (1977) derives such a bound for demand functions that are linear in income. His bound and its proof are both quite complicated. We obtain a simpler bound that applies to *all* smooth demand functions and is tighter than Shafer's in his case of demands that are linear in income.

To obtain a general bound, it is enough to restrict attention to cycles with price vectors evenly spaced around a circle. For these cycles, the Slutsky inconsistency measure is given in the following.

Theorem 3. *Fix u and v in $\mathbb{R}^n$ and $K \geq 3$, and let $q_k \equiv [cos(2\pi k/K)]u + [sin(2\pi k/K)]v$ for $k = 0, 1, \ldots, K$. Then*

$$I(S, \{q_k\}_k) = (1/2)[1 - cos(2\pi/K)](u^T\bar{S}u + v^T\bar{S}v) + [sin(2\pi/K)]u^T Av. \tag{7}$$

By combining Theorems 1 and 3 we obtain information about the number of observations required to form revealed preference cycles. We present several applications in the rest of this section

Corollary 2. *If the Slutsky matrix of h is skew-symmetric and nonzero, then there exists a strict revealed preference cycle consisting of three observations.*

Proof. If the Slutsky matrix S is skew-symmetric and nonzero, then $S = A \neq 0$. In that case, by Lemma 1 in Section 7, there exist orthogonal unit vectors u and v such that $u^T Av > 0$. With $K = 3$, the right side of (7) becomes $(\sqrt{3}/2)u^T Av > 0$. The price vectors in the corresponding cycle are vertices of an equilateral triangle.

□

A possibly discontinuous demand function is generated by utility maximization if it satisfies mild regularity conditions and has no revealed preference cycles (Richter, 1979). Theorem 3 implies that a smooth demand function is generated by utility maximization if it has no *strict* revealed preference cycles.

Corollary 3. *If h has no strict revealed preference cycles, then h is generated by a utility function, and at each budget situation the Slutsky matrix of h is symmetric and negative semidefinite.*

Proof. If there are no 2-cycles, then S is negative semidefinite by Corollary 1. If the Slutsky matrix is asymmetric then there are vectors u and v such that $u^T Av > 0$. Since $sin\,\theta/(1 - cos\,\theta)$ approaches infinity as θ goes to 0, the right side of (7) can be made positive by letting K be sufficiently large. In that case, by Theorem 1, h has a strict revealed preference K-cycle. So under the hypothesis, h satisfies the budget identity and the Slutsky conditions, hence is generated by utility maximization (Theorem 1 of Jerison and Jerison (1993)).

□

The next theorem provides a simple upper bound on the minimum number of observations needed to form a strict revealed preference cycle.

Theorem 4. *If for some $K \geq 3$ the C^1 demand function h has no K-cycles in some neighborhood of (y^*, p^*), then for all vectors u and v in $I\!R^n$, and every i, $j = 1, \ldots, n$,*

$$|u^T Av| \leq (1/2)[tan(\pi/K)]|u^T \bar{S}u + v^T \bar{S}v| \quad \textit{and} \tag{8}$$

$$|S_{ij} - S_{ji}| \leq 2[tan(\pi/K)]|S_{ii}|^{1/2}|S_{jj}|^{1/2}. \tag{9}$$

Theorem 4 provides the desired bound since there must be a strict revealed preference cycle of length K if $K \geq 3$ is large enough to violate (8) or (9). The theorem shows how the minimum number of observations required to form a revealed preference cycle is related to the sizes of violations of Slutsky symmetry and negative semidefiniteness. With greater Slutsky asymmetry in the plane spanned by u and v, the left side of (8) is larger, so the largest K satisfying the inequality is smaller. Under the hypothesis, the demand function satisfies the weak weak axiom and has a negative semidefinite Slutsky matrix. If we change S to make it "less negative semidefinite" (i.e. if we add to it a symmetric matrix that is positive definite on the span of u and v) then the term $|u^T \bar{S}u + v^T \bar{S}v|$ is reduced, and so is the largest K satisfying inequality (8). If for a large K there are no strict revealed preference cycles of length K in the plane spanned by u and v, then in that plane the antisymmetric part of the Slutsky matrix must be small relative to the symmetric part. Similar reasoning applies to inequality (9).

Remark 3. For the case of demands that are linear in income, Shafer (1977) showed that when there are no K-cycles, with $K \geq 3$,

$$|S_{ij} - S_{ji}| \leq \phi(K)|S_{ii} - (h^i)^2|^{1/2}|S_{jj} - (h^j)^2|^{1/2}, \tag{10}$$

where the Slutsky matrix and the demands h^i and h^j are evaluated at (y^*, p^*), and where

$$\phi(K) \equiv \begin{cases} 2K^{1/2}/(K-2) & \text{for } K \text{ even,} \\ \dfrac{2K^{1/2}}{\left[(K-2)^2 - \dfrac{K^2-9}{K^2-1}\right]^{1/2}} & \text{for } K \text{ odd.} \end{cases}$$

The inequality (9) in Theorem 4 applies to every C^1 demand function. Furthermore, for demands linear in income, (9) is strictly tighter than (10) unless one of the demands h^i or h^j is 0 (in which case, (9) and (10) are equivalent, both implying $S_{ij} = S_{ji}$).

If h^i and h^j are strictly positive at (y, p) then the bound (9) is tighter than (10) for two reasons. First, S_{ii} and S_{jj} are nonpositive since S is negative semidefinite, and therefore $|S_{ii} - (h^i)^2| > |S_{ii}|$ (and similarly with i replaced by j). The second reason is that $2\,tan(\pi/K) \leq \phi(K)$ for $K \geq 3$, and this inequality is strict for $K > 4$ (see Lemma 4, below). For large K, $2\,tan(\pi/K)$ is approximately $2\pi/K$ whereas $\phi(K)$ is approximately $2K^{-1/2}$. Table 1 contains some sample values of the bounding functions $2\,tan(\pi/K)$ and $\phi(K)$.

Table 1. Comparison of $2\,tan(\pi/K)$ and $\phi(K)$.

K	$2\,tan(\pi/K)$	$\phi(K)$
3	3.464	3.464
4	2	2
5	1.453	1.549
6	1.155	1.225
9	.728	.865
50	.126	.295
100	.063	.204
1000	.0063	.0633

The proof of Theorem 4 shows that if the Slutsky matrix at (y^*, p^*) is sufficiently asymmetric then every neighborhood of (y^*, p^*) contains a strict revealed preference cycle with price vectors evenly spaced around a circle. It might be possible to improve on our bound by considering cycles with other shapes.

One might also try to find a *lower* bound on the number of observations required to detect inconsistency. The results in this section provide some relevant information. Suppose that there are no K-cycles in a neighborhood of (y^*, p^*). By Theorem 1 there is a neighborhood Q of the origin such that $\mathcal{I}_K(S(y,p), Q) = 0$ for all (y,p) near (y^*, p^*). We conjecture that the converse is false—that is, that requiring $\mathcal{I}_K(S(y,p), Q)$ to be 0 on a neighborhood of (y^*, p^*) is not strong enough to rule out K-cycles in a neighborhood of (y^*, p^*). Information about higher order derivatives of the demand function might be needed to rule out cycles with few observations.

6. Conclusion

Smooth demand functions that are not generated by utility maximization have revealed preference cycles. As one moves along these cycles, real income (defined with a chained Laspeyres price index) grows at each step, yet prices and demands return to their starting values. The "real" growth is illusory. The rate of real income growth along the worst cycle in a region is a variant of Afriat's cost-inefficiency, the most commonly used measure of demand inconsistency. Our main theorem shows how this worst growth rate depends on the amount of price variation and on behavioral inconsistency, measured by an index of violations of the Slutsky conditions. When prices can vary between $1 - r$ and $1 + r$ the highest constant real income growth rate along a cycle is approximately r^2 times the Slutsky index.

Our results show how violations of the two Slutsky conditions combine in a single index to produce given levels of revealed preference inconsistency. The connection with revealed preference provides an economic interpretation for violations of the Slutsky conditions. The Slutsky index itself is easy to compute and depends only on behavior, not on the environment. Our main theorem also allows us to place a finite upper bound on the number of observations required to detect revealed preference inconsistency. This solves a classic problem in demand theory.

The existence of cycles with continually increasing real income depends on the way in which real income is defined. Our definition uses a chained Laspeyres price index. The consumption base for the index is updated at each step along the cycle. The recent CPI Commission (Boskin, et. al. (1998)) recommended such frequent updating of the consumption base, but they also recommended replacing the Laspeyres index with a "superlative" price index (Diewert (1976)). Using a superlative index seems to reduce the likelihood of real growth incoherence, but does not rule it out altogether. We expect that the methods introduced above will also be useful for studying cycles of real income growth defined with a superlative index.

We know of only two other papers that relate revealed preference cycles to the Slutsky matrix. The first, Shafer (1977), is discussed in the introduction and in section 5, above. The second, Jerison and Jerison (1993), provides a global upper bound on a variant of Afriat's cost-inefficiency when there are just two observations. With two observations, the only possible revealed preference inconsistencies are violations of the weak weak axiom. The bound on these inconsistencies depends on the price variation, the quadratic form of the Slutsky matrix and the marginal propensity to consume.

Jerison and Jerison (1992, 1993) show how real income growth along smooth (Ville) cycles is related to the degree of Slutsky asymmetry and the amount of price variation. The 1992 paper shows that when prices are confined to a ball of radius r, the instantaneous real growth rate along the worst smooth cycle is approximately r times the norm of the antisymmetric part of the Slutsky matrix. As the ball shrinks, the worst real growth rate shrinks in proportion to r instead of the r^2 found in the present paper. The reason for the difference is that the instantaneous real growth rate is defined assuming that the velocity of price change is 1. For discrete cycles, this is comparable to taking the real income growth from one observation to the next (as defined above) and dividing it by the length of the vector of price changes. The resulting ratio is of order r. Our 1993 paper derives a global upper bound on the instantaneous real income growth rate along smooth cycles. The bound is determined by the amount of price variation, the Slutsky asymmetry and the marginal propensity to consume. The problem with smooth cycles is that they involve a continuum of data.

Jerison and Jerison (1996) provide a test for Slutsky symmetry using discrete cycles, but the cycles are not necessarily revealed preference cycles. The test depends on an "antisymmetric" income growth rate that is different from the growth rate used above. The antisymmetric growth rate from observation $k-1$ to observation k is defined to be the real income growth rate from $k-1$ to k (as in section 2, above) minus the real income growth rate in the opposite direction (from k to $k-1$). Subtracting the growth rate in the opposite direction removes the effect of violations of the weak axiom. When the Slutsky matrix is symmetric, the antisymmetric growth rate along the worst cycle in a ball of radius r is of order $o(r^2)$. Thus it shrinks faster than r^2 as r goes to zero.

The results in the present paper and in the previous literature leave many open problems. Perhaps the most important is to develop reasonable inconsistency measures when demand is stochastic. The results presented above provide some insight even in this case. They can be used to characterize the inconsistency of a smooth mean demand function when a consumer's demand deviates stochastically from the mean. Of course it is still important to consider stochastic demands explicitly.

The inconsistency measures in our analysis apply to demand functions that are known exactly. However, in most applications, the demand functions are estimated. Statistical tests of the Slutsky conditions in the empirical literature have the advantage that they take into account the imprecision of the demand estimation. But the statistical tests lack the economic interpretation that our results provide for the Slutsky index. It would be desirable to find an economically interpretable measure of Slutsky violations that allows for uncertainty about the demand function.

Violations of the Slutsky conditions are the basis for tests of efficient consumption in the collective demand model (Browning and Chiappori (1998)). In that model the aggregate demand of a group of consumers can violate the Slutsky conditions because of the way group members share their aggregate income. The connection between Slutsky violations and revealed preference inconsistency in the present paper might provide additional insight into the structure of the collective model. We expect that there are also connections between smooth integrability conditions and discrete cycles in other economic contexts. For example, the path-dependence of consumer's surplus (Chipman and Moore (1976)) can undoubtedly be linked to discrete cycles.

The results in the present paper are local. It would be worthwhile extending them to obtain global bounds relating the Slutsky matrix to real growth rates along cycles in arbitrary regions of income and price space. Our main theorem is a limit theorem. It makes a statement about an infinite set of discrete cycles. In order to extend this to a statement about finite data we need a global theorem.

To obtain a global extension of our main theorem it would be useful to have more specific information about the shapes of worst revealed preference cycles. We conjecture that when the price vectors are confined to a ball, the price vectors in the worst cycle lie in a three dimensional space. Equation (4) suggests that this space is generated by the eigenvector corresponding to the largest eigenvalue of the symmetric part of the Slutsky matrix and by the two vectors u_k and v_k corresponding to the largest λ_k in the decomposition of the antisymmetric part of the Slutsky matrix in equation (11), below. We conjecture that when the price vectors are confined to a box the worst cycles also lie in low dimensional spaces, but this remains to be shown.

We derived an upper bound on the number of observations needed to detect revealed preference inconsistency. A referee has asked whether a lower bound can be obtained from our results. The problem is to find a condition on the Slutsky matrix that rules out cycles of length less than K in some fixed region. We offered some ideas in section 5, above, but the problem remains unsolved.

The choice of a measure of demand inconsistency must depend on what the measure is to be used for. For smooth demand, the Slutsky index measures local behavioral inconsistency. It is approximately equal to the cost-inefficiency adjusted for the variation in prices. For welfare analysis it might be better to use the cost-inefficiency without the price adjustment. The cost-inefficiency is a lower bound on the fraction of income that an inconsistent consumer appears to be wasting. If there is very little price variation, the wastage can be small even if the Slutsky index is large. Then the behavioral inconsistency is large, but it need not be costing the consumer much.

On the other hand, for positive comparative static analysis, the question of interest might be what size errors in demand elasticities are introduced if inconsistent behavior is modeled as utility-maximizing, i.e., if one assumes that the Slutsky conditions are satisfied when in fact they are not. In that case, the Slutsky index is likely to be useful. The Slutsky index might help clarify the literature that claims that small deviations from optimization can have large effects on resource allocation (e.g., Thaler (1992)). The "small" deviations in that literature refer to welfare loss. We conjecture that in order for consumer deviations from optimization to have large allocation effects the inconsistencies must be large according to the behavioral measure introduced above.

7. Proofs

Let (y^*, p^*) be fixed in IR^{n+1}_{++} and let S, $\bar{S}$ and A be respectively the Slutsky matrix of h and its symmetric and antisymmetric parts evaluated at (y^*, p^*), as defined in (1) in Section 2. Let h, h_y and h_p denote respectively the function h and its derivatives with respect to y and p, evaluated at (y^*, p^*). Let δ_{kl} be the Kronecker delta, equal to 1 if $k = l$ and equal to 0 otherwise.

Lemma 1. $$A = \sum_{k=1}^{m} \lambda_k (u_k v_k^T - v_k u_k^T) \tag{11}$$
for scalars $\lambda_k > 0$ and real vectors u_k, v_k, satisfying $u_k \cdot u_l = \delta_{kl} = v_k \cdot v_l$ and $u_k \cdot v_l = 0$ for $k, l = 1, \ldots, m$.

Proof. Since A is skew–symmetric, iA is Hermitian and has m positive eigenvalues $\lambda_k > 0$ and m negative eigenvalues $-\lambda_k$ corresponding to the eigenvectors $u_k - iv_k$ and $u_k + iv_k$ respectively, where u_k and v_k are real vectors. The remaining eigenvalues of iA are zero. Thus $A(u_k + iv_k) = i\lambda_k(u_k + iv_k)$ and $A(u_k - iv_k) = -i\lambda_k(u_k - iv_k)$. This implies $Au_k = -\lambda_k v_k$ and $Av_k = \lambda_k u_k$, and $Aw = 0$ for w orthogonal to all u_k and v_k vectors. Let $\langle \cdot, \cdot \rangle$ be the standard (Hermitian) inner product. The eigenvectors of iA can be chosen so that

$$\begin{aligned} \langle u_k + iv_k, u_l + iv_l \rangle &= u_k \cdot u_l + v_k \cdot v_l + i(v_k \cdot u_l - u_k \cdot v_l) \\ &= 2\delta_{kl} \end{aligned} \tag{12}$$

where δ_{kl} is the Kronecker delta. Also for all k and l,

$$0 = \langle u_k + iv_k, u_l - iv_l \rangle = u_k \cdot u_l - v_k \cdot v_l + i(u_k \cdot v_l + v_k \cdot u_l). \tag{13}$$

By (13), $u_k \cdot u_l = v_k \cdot v_l$ and by (12) $u_k \cdot u_l = -v_k \cdot v_l$ for $k \neq l$. So for $k \neq l$, $u_k \cdot u_l = v_k \cdot v_l = 0$. Also by (13), $u_k \cdot v_l = -v_k \cdot u_l$, and by (12), $u_k \cdot v_l = v_k \cdot u_l$, so $u_k \cdot v_l = 0$ for all k, l. Finally $u_k \cdot u_k + v_k \cdot v_k = 2 \cdot u_k \cdot u_k = 2$, so u_k and v_k are unit vectors. Letting $u_{j+m} \equiv v_j$ for $1 \leq j \leq m$, there are vectors u_j for $j > m$ such that the set of u_j, $j = 1, \ldots, n$ forms an orthonormal basis for IR^n. Then $Au_j = \sum_{k=1}^{m} \lambda_k (u_k v_k^T - v_k u_k^T) u_j$ for $j = 1, \ldots, n$, which completes the proof.

□

Lemma 2. *Given scalars x_j, $j = 1, \ldots, m$, if $|\sum_{j=1}^m x_j| \leq 1$ and $|x_j| \leq 1$ for all j and if $0 < \rho < 1/2$, then*

$$(1 + \rho x_1)(1 + \rho x_2) \cdots (1 + \rho x_m) \geq e^{-\rho - \rho^2 m}.$$

Proof. Define $M_k \equiv \sum_{j=1}^m x_j^k$ for each integer $k \geq 1$. Under the hypotheses of the lemma, $|M_1| \leq 1$ and $|M_k| \leq m$ for each k. Also $\sum_{k=2}^\infty \rho^k = \rho^2/(1-\rho)$, and therefore $\sum_{k=2}^\infty \rho^k/k \leq \rho^2/[2(1-\rho)] \leq \rho^2$ since $\rho < 1/2$. Using the Taylor series $ln(1+a) = \sum_{k=1}^\infty (-1)^{k+1} k^{-1} a^k$, we have

$$\begin{aligned} \sum_{j=1}^m ln(1 + \rho x_j) &= \sum_{k=1}^\infty (-1)^{k+1} k^{-1} \rho^k M_k \\ &\geq \rho M_1 - \sum_{k=2}^\infty k^{-1} \rho^k m \geq -\rho - \rho^2 m \end{aligned}$$

□

Proof of Theorem 1. In this proof, we let h, h_y and h_p denote the function h and its matrices of partial derivatives, all evaluated at (y^*, p^*). Define $R^*(z,q) \equiv h(y^*+z, p^*+q) - h - zh_y - h_p q$. Consider an arbitrary K-cycle $\{(y^* + rz_k, p^* + rq_k)\}_{k=0}^K$ with each z_k in $\mathbb{R}$ and q_k in $\mathbb{R}^n$. We will compute the real growth rate of this cycle using the following notation.

Define $\Lambda \equiv \sum_k \lambda_k u_k v_k^T$, using the notation in (11). Let $\hat{q}$ be the cycle $\{q_k\}_{k=0}^K$. Define

$$\Gamma(\hat{q}) \equiv (1/K)\sum_{k=1}^{K}[(1/2)(q_k - q_{k-1})^T \bar{S}(q_k - q_{k-1}) + q_{k-1}^T A q_k]. \tag{14}$$

Let $\epsilon_0 \equiv 0$, and define ϵ_k recursively by

$$\epsilon_k \equiv \epsilon_{k-1} + (1/2)(q_k - q_{k-1})^T S(q_k - q_{k-1}) + q_{k-1}^T A q_k + q_k^T \Lambda q_k - q_{k-1}\Lambda q_{k-1} - \Gamma(\hat{q})$$

for $k = 1, \ldots, K$. Summing $(\epsilon_k - \epsilon_{k-1})$ over $k = 1, \ldots, K$ yields

$$\epsilon_K = \sum(\epsilon_k - \epsilon_{k-1}) = \sum[(1/2)(q_k - q_{k-1})^T S(q_k - q_{k-1}) + q_{k-1}^T A q_k - \Gamma(\{q_k\}_k)] = 0 = \epsilon_0.$$

Thus the sequence $\{\epsilon_k\}_{k=0}^K$ is a cycle.

The income level at observation k in the cycle is $y^* + rz_k$. It will be convenient to write z_k in the following form:

$$z_k = h^T q_k + (1/2) r q_k^T S q_k + r q_k^T \Lambda q_k - r\epsilon_k + \delta_k \tag{15}$$

for $k = 0, 1, \ldots, K$. (Use (15) to define δ_k.) The sequence $\{\delta_k\}$ is a K-cycle since $\{z_k\}_{k=0}^K$, $\{\epsilon_k\}_{k=0}^K$ and $\hat{q}$ are K-cycles.

We next find an expression for the growth rate of the cycle $\{(y^* + rz_k, p^* + rq_k)\}_{k=0}^K$. For a fixed integer $m \in [1, K]$, let $p \equiv rq_m$ and $q \equiv rq_{m-1}$. Then the real income growth from observation $m-1$ to observation m is

$$\begin{aligned}
&rz_m - rz_{m-1} - (p-q)[h + rz_{m-1}h_y + h_p q + R^*(rz_{m-1}, q)] \\
&= (p-q)^T h + (1/2)p^T S p - (1/2) q^T S q + p^T \Lambda p - q^T \Lambda q + r^2\epsilon_{m-1} - r^2\epsilon_m + r\delta_m - r\delta_{m-1} \\
&\qquad - (p-q)^T h - (p-q)^T h_y h^T q - r\delta_{m-1}(p-q)^T h_y - (p-q)^T h_p q + o(r^2) \\
&= (1/2)p^T S p - (1/2)q^T S q - (p-q)^T S q + p^T \Lambda p - q^T \Lambda q \\
&\qquad + r^2\epsilon_{m-1} - r^2\epsilon_m + r\delta_m - r\delta_{m-1} - r\delta_{m-1}(p-q)^T h_y + o(r^2) \\
&= (1/2)(p-q)^T S(p-q) + q^T A p + p^T \Lambda p - q^T \Lambda q \\
&\qquad + r^2\epsilon_{m-1} - r^2\epsilon_m + r\delta_m - r\delta_{m-1} - r\delta_{m-1}(p-q)^T h_y + o(r^2) \\
&= r^2\Gamma(\hat{q}) + r\delta_m - r\delta_{m-1} - r\delta_{m-1}(p-q)^T h_y + o(r^2).
\end{aligned} \tag{16}$$

Now suppose that $\delta_k = 0$ for $k = 0, 1, \ldots, K$. In that case, by (16) the cycle $\{(y^* + rz_k, p^* + rq_k)\}_{k=0}^K$ has minimum real income growth rate $r^2\Gamma(\hat{q}) + o(r^2)$, so $lim_{r\to 0}\, r^{-2} G(\{(y^* + rz_k, p^* +$

$rq_k)\}_{k=0}^{K}) = \Gamma(\tilde{q}) = \mathcal{I}_K(S, \hat{q})$. For this choice of z_k we have $|z_k| \leq \gamma$ when r is sufficiently small. Therefore

$$\underset{r\to 0}{lim\,inf}\, r^{-2}\, sup\{G(\{(y^* + rz'_k, p^* + rq_k)\}_{k=0}^{K}) : |z'_k| \leq \gamma, \forall k\} \geq I(S, \hat{q}).$$

The right side of this inequality attains a maximum with respect to $\hat{q}$ in $C_K(Q)$ since Q is compact and Γ is continuous. Therefore we can take the supremum of both sides with respect to $\hat{q} \in C_K(Q)$ and obtain $lim\,inf_{r\to 0}\, r^{-2}\mathcal{G}_K((y^*, p^*) + rN) \geq \mathcal{I}_K(S, Q)$.

Given any $\alpha > 0$ there exists K such that $\mathcal{I}_K(S, Q) > \mathcal{I}(S, Q) - (\alpha/2)$. The argument in the previous paragraph shows that for sufficiently small $r > 0$ there is a cycle $\hat{c}(r) \equiv \{(y^* + rz_k, p^* + rq_k)\}_{k=0}^{K}$ in $C((y^*, p^*) + rN)$ with $r^{-2}G(\hat{c}(r)) > \mathcal{I}_K(S, Q) - (\alpha/2) > \mathcal{I}(S, Q) - \alpha$. This shows that $lim\,inf_{r\to 0}\, r^{-2}\mathcal{G}((y^*, p^*) + rN) \geq \mathcal{I}(S, Q)$.

In order to complete the proof, we will show that $lim\,sup_{r\to 0}\, r^{-2}\mathcal{G}((y^*, p^*) + rN) \leq \mathcal{I}(S, Q)$. A special case of the same proof shows that this last inequality holds when the subscript K is added to $\mathcal{G}$ and $\mathcal{I}$. Another special case shows that

$$\underset{r\to 0}{lim\,sup}\, r^{-2}\, sup\{G(\{(y^* + rz'_k, p^* + rq_k)\}_{k=0}^{K}) : |z'_k| \leq \gamma, \forall k\} \leq I(S, \hat{q}),$$

which, combined with the argument above, implies (5).

To prove $lim\,sup_{r\to 0}\, r^{-2}\mathcal{G}((y^*, p^*) + rN) \leq \mathcal{I}(S, Q)$ we suppose instead that this inequality is false. Then there exists a scalar $\alpha > 0$ and a sequence $\{r_i\}_{i=1}^{\infty}$ converging to zero, and, for each i, a $K(i)$-cycle $\{(y^* + r_i z_k(i), p^* + r_i q_k(i))\}_{k=0}^{K(i)}$ in $C_{K(i)}((y^*, p^*) + r_i N)$ that has minimum real income growth rate greater than $r_i^2(\mathcal{I}(S, Q) + \alpha)$. In the derivation above, replace r by r_i and z_k, q_k and K by $z_k(i)$, $q_k(i)$ and $K(i)$. Then (15) determines $\delta_k(i)$ for $k = 0, 1, \ldots, K(i)$; and $\{\delta_k(i)\}_k$ is a $K(i)$-cycle for each i. Since each $q_k(i)$ is in Q, there is a fixed compact set that contains every $\epsilon_k(i)$. Since each $z_k(i)$ is in $[-\gamma, \gamma]$, equation (15) implies that $|\delta_k(i)| < 2\gamma$, $\forall k \leq K(i)$ for sufficiently large i. By (16), for each i,

$$\begin{aligned}\underset{m}{min}\{\Gamma(\{q_k(i)\}_k) + (1/r_i)(\delta_m(i) - \delta_{m-1}(i)) - \delta_{m-1}(i)(q_m(i) - q_{m-1}(i))^T h_y + r_i^{-2} o(r_i^2)\} \\ \geq \mathcal{I}(S, Q) + \alpha \geq \Gamma(\{q_k(i)\}_k) + \alpha.\end{aligned}$$

Therefore there is some $\nu > 0$ satisfying

$$(1/r_i)(\delta_m(i) - \delta_{m-1}(i)) - \delta_{m-1}(i)(q_m(i) - q_{m-1}(i))^T h_y \geq \nu$$

for sufficiently large i and for all $m = 1, \ldots, K(i)$. Let β be larger than γ and larger than $sup\{qh_y : q \in Q\}$. Define

$$x_m(i) \equiv (2\beta)^{-1}(q_m(i) - q_{m-1}(i))^T h_y,$$

and define $\rho_i \equiv 2\beta r_i$. Then for sufficiently large i and for all $m = 1, \ldots, K(i)$, we have $\rho_i < 1/2$ and

$$\delta_m(i) - \delta_{m-1}(i) - \rho_i \delta_{m-1}(i) x_m(i) \geq r_i \nu.$$

In what follows, we restrict attention to i large enough so that these inequalities hold. Since $|q_k(i)h_y| < \beta$, we have $|\sum_{m=a}^{b} x_m(i)| \leq 1$ for each a and b with $1 \leq a \leq b \leq K(i)$. Also, since $\{q_k(i)\}_{k=0}^{K(i)}$ is a cycle, $\sum_{m=1}^{K(i)} x_m(i) = 0$. Therefore, omitting i as an argument of the functions K, δ_m and x_m,

$$\begin{aligned}\delta_K &\geq r_i\nu + \delta_{K-1}(1+\rho_i x_K) \\ &\geq r_i\nu + (1+\rho_i x_K)r_i\nu + \delta_{K-2}(1+\rho_i x_K)(1+\rho_i x_{K-1}) \\ &\geq r_i\nu[1 + (1+\rho_i x_K) + (1+\rho_i x_K)(1+\rho_i x_{K-1}) + \\ &\qquad \ldots + (1+\rho_i x_K)(1+\rho_i x_{K-1})\cdots(1+\rho_i x_2)] \\ &\qquad + \delta_0(1+\rho_i x_K)(1+\rho_i x_{K-1})\cdots(1+\rho_i x_1).\end{aligned}$$

Since $\delta_K(i) = \delta_0(i)$, we have (again omitting i as an argument of the functions K and x_m for each m)

$$\begin{aligned}&\delta_0(i)[1 - (1+\rho_i x_K)(1+\rho_i x_{K-1})\cdots(1+\rho_i x_1)] \\ &\qquad \geq r_i\nu[1 + (1+\rho_i x_K) + (1+\rho_i x_K)(1+\rho_i x_{K-1}) + \\ &\qquad \ldots + (1+\rho_i x_K)(1+\rho_i x_{K-1})\cdots(1+\rho_i x_2)].\end{aligned} \tag{17}$$

Note that $(1+\rho_i x_K)(1+\rho_i x_{K-1})\cdots(1+\rho_i x_1) \leq 1$ since $\sum_{k=1}^{K(i)} x_k(i) = 0$. Since the right-hand side of (17) is strictly positive, it follows that $\delta_0(i) > 0$. Since $2\beta > \delta_0(i)$, Lemma 2 implies that the left-hand side of (17) is less than $2\beta(1 - e^{-K(i)\rho_i^2})$. Next, consider the right-hand side of (17). Since $|\sum_{k=1}^{K(i)} x_k| \leq 1$, Lemma 2 implies that

$$\begin{aligned}&1 + (1+\rho_i x_K) + (1+\rho_i x_K)(1+\rho_i x_{K-1}) + \ldots + (1+\rho_i x_K)(1+\rho_i x_{K-1})\cdots(1+\rho_i x_2) \\ &\quad \geq \sum_{m=0}^{K-1} exp(-\rho_i - m\rho_i^2) = e^{-\rho_i}(1 - e^{K(i)\rho_i^2})/(1 - e^{-\rho_i^2}).\end{aligned}$$

It follows that

$$2\beta(1 - e^{-K(i)\rho_i^2}) \geq \nu\rho_i(2\beta)^{-1}e^{-\rho_i}(1 - e^{K(i)\rho_i^2})/(1 - e^{-\rho_i^2})$$

for sufficiently large i. But this last inequality is violated for sufficiently large i since $\rho e^{-\rho}/(1 - e^{-\rho^2})$ approaches infinity as ρ goes to zero. This contradiction implies that the hypothesis

$$\underset{r\to 0}{lim\,sup}\; r^{-2}\mathcal{G}((y^*, p^*) + rN) > \mathcal{I}(S, Q)$$

is false. Therefore

$$\underset{r\to 0}{lim}\; r^{-2}\mathcal{G}((y^*, p^*) + rN) = \mathcal{I}(S, Q).$$

The same argument, with $K(i)$ taken to be fixed, proves that $lim_{r\to 0}\, r^{-2}\mathcal{G}_K((y^*, p^*) + rN) = \mathcal{I}_K(S, Q)$ for each K. Equation (5) follows from the same proof, letting $q_k(i)$ and $K(i)$ be q_k and K in the hypothesis of the theorem for every i.

$\square$

Lemma 3. *Let K be an integer greater than 2 and let $\theta \equiv 2\pi/K$. Then*

$$\frac{1}{K}\sum_{k=1}^{K} cos\, k\theta[cos\, k\theta - cos((k-1)\theta)] = \frac{1}{K}\sum_{k=1}^{K} sin\, k\theta[sin\, k\theta - sin((k-1)\theta)] \tag{18}$$

$$= (1 - cos\,\theta)/2, \qquad and \tag{19}$$

$$\sum_{k=1}^{K}[sin\, k\theta\, cos((k-1)\theta) + cos\, k\theta\, sin((k-1)\theta)] = \sum_{k=1}^{K} sin((2k-1)\theta) = 0. \tag{20}$$

Proof. For any τ,

$$\sum_{k=1}^{K} e^{(2k+\tau)i\theta} = e^{\tau i\theta}e^{2i\theta}\ \frac{e^{2i\theta K}-1}{e^{2i\theta}-1} = 0.$$

The real and imaginary parts of this geometric sum are respectively

$$\sum_{k=1}^{K} cos[(2k+\tau)\theta] = 0 \tag{21}$$

and

$$\sum_{k=1}^{K} sin[(2k+\tau)\theta] = 0. \tag{22}$$

To prove (18), we use angle addition identities and note that the lefthand side of (18) minus the righthand side is

$$(1/K)\sum_{k=1}^{K}\{cos^2\, k\theta - sin^2\, k\theta + sin\, k\theta\, sin[(k-1)\theta] - cos\, k\theta\, cos[(k-1)\theta]\}$$

$$= (1/K)\sum_{k=1}^{K} cos\, 2k\theta - (1/K)\sum_{k=1}^{K} cos[(2k-1)\theta] = 0,$$

where the last equation follows from (21). To prove (19), use equation (18) and note that the sum of the left and righthand sides of (18) is

$$(1/K)\sum_{k=1}^{K}\{cos^2\, k\theta + sin^2\, k\theta - [cos\, k\theta\, cos((k-1)\theta) + sin\, k\theta\, sin((k-1)\theta)]\}$$

$$= 1 - cos\,\theta.$$

Equation (20) follows from an angle addition identity and (22).

□

Proof of Theorem 3. The right side of (4) is

$$\Gamma \equiv \frac{1}{K}\sum_{k=1}^{K}[(1/2)(q_k - q_{k-1})^T S(q_k - q_{k-1}) + q_{k-1}^T A q_k]$$

$$= \frac{1}{K}\sum_{k=1}^{K}[q_k^T S q_k + q_{k-1}^T (A - \bar{S})q_k].$$

Let $\theta \equiv 2\pi/K$. Under the hypothesis of the theorem, $q_k = (cos\, k\theta)u + (sin\, k\theta)v$. Using these identities, Lemma 3 and the fact that $x^T Ax = 0$ for every n-vector x, we obtain

$$\begin{aligned}
\Gamma = \frac{1}{K}\sum_{k=1}^{K}\{&(cos^2\, k\theta)u^T\bar{S}u + (sin^2\, k\theta)v^T\bar{S}v + (sin\, 2k\theta)u^T\bar{S}v \\
&- [cos\, k\theta\, cos(k-1)\theta]u^T\bar{S}u - [sin\, k\theta\, sin(k-1)\theta]v^T\bar{S}v \\
&+ [sin\, k\theta\, cos(k-1)\theta]u^T(A-\bar{S})v + [cos\, k\theta\, sin(k-1)\theta]v^T(A-\bar{S})u\} \\
= \frac{1}{K}\sum_{k=1}^{K}\{&(cos\, k\theta)[cos\, k\theta - cos(k-1)\theta]u^TSu + (sin\, k\theta)[sin\, k\theta - sin(k-1)\theta]v^TSv \\
&- [sin\, k\theta\, cos(k-1)\theta + cos\, k\theta\, sin(k-1)\theta]u^T\bar{S}v \\
&+ [sin\, k\theta\, cos(k-1)\theta - cos\, k\theta\, sin(k-1)\theta]u^TAv\} \\
= (1/2)(1 - &cos\,\theta)(u^TSu + v^TSv) + (sin\,\theta)u^TAv,
\end{aligned}$$

which proves (7).

$\square$

Proof of Remark 2. $\mathcal{I}(S,Q)$ is nonnegative since it is at least as great as $I(S,\cdot)$ evaluated at a cycle with $q_k = q_{k-1}$ for all k. If S is symmetric and negative semidefinite, then $A = 0$ and the right side of (4) is nonpositive, so $\mathcal{I}(S,Q) = 0$. Suppose now that Q is a compact neighborhood of the origin. If S is not negative semidefinite, then there is some $q \in Q$ such that $q^TSq > 0$, and $\mathcal{I}(S,Q) \geq I(S,\{0,q,0\}) = q^TSq > 0$. If S is asymmetric, then by Theorem 3 in section 5, below, there is a cycle q in Q such that $0 < I(S,q) \leq \mathcal{I}(S,Q)$.

Suppose that $D \equiv \hat{S} - S$ is symmetric and positive semidefinite. Let $\{q_k\}$ be the cycle that maximizes (3) over cycles in Q and let $s_k \equiv q_k - q_{k-1}$. By (4), $I(S+D,\{q_k\}) \geq I(S,\{q_k\})$. If this holds with equality, then $\sum_{k=1}^{K} s_k^TDs_k = 0$ and $s_k^TDs_k = 0$ for all k. The budget identity implies $\hat{S}^Tp^* = S^Tp^* = 0$, hence $Dp^* = 0$. Suppose that D has rank $n-1$. Then each s_k is collinear with p^*, and there exist scalars γ_k such that $q_k = q_0 + \gamma_k p^*$ for each k. Since $\gamma_K = \gamma_0 = 0$ and $\sum_{k=1}^{K}\gamma_k = \sum_{k=1}^{K}\gamma_{k-1}$, we have $\mathcal{I}_K(S,Q) = I(S,\{q_k\}) = (1/K)\sum_{k=1}^{K}(q_0 + \gamma_k p^*)^TS(\gamma_k - \gamma_{k-1})p^* = 0$. This proves that if $\mathcal{I}_K(S,Q) > 0$ then $\mathcal{I}_K(\hat{S},Q) \geq I(\hat{S},\{q_k\}) > I(S,\{q_k\})$.

$\mathcal{I}_K(S+tA,Q)$ is nondecreasing in $t \geq 0$ because whenever a cycle $\{q_k\}$ maximizes the right side of (4), the term $\sum_{k=1}^{K} q_{k-1}^TAq_k$ is nonnegative. Otherwise, one obtains a larger value for (4) by following the cycle in the opposite direction, using the cycle $\{q'_k\}$ with $q'_k = q_{K-k}$. By the envelope theorem, the derivative of $\mathcal{I}_K(S+tA,Q)$ with respect to t evaluated at $t = 0$ is $\sum_{k=1}^{K} q_{k-1}^TAq_k$, which is 0 only if $\mathcal{I}_K(S,Q) = \mathcal{I}_K(\bar{S},Q)$.

Given Q, a compact neighborhood of the origin, $I(S,q)$ is linear in S for each cycle q in Q. Therefore $\mathcal{I}(S,Q)$ is linear in S and satisfies the triangle inequality $\mathcal{I}(S,Q) + \mathcal{I}(\hat{S},Q) \geq \mathcal{I}(S+\hat{S},Q)$. $\mathcal{I}(S,Q)$ is zero only if S satisfies the Slutsky conditions, so if $\mathcal{I}(A,Q) = 0$ then $A = 0$. This shows that $\mathcal{I}(\cdot,Q)$ is a norm on the space of skew-symmetric matrices.

$\square$

Proof of Theorem 4. Suppose that h has no K-cycles for some $K \geq 3$. Then h satisfies the weak weak axiom of revealed preference and has a negative semidefinite Slutsky matrix S at (y,p). By Theorem 3, the right side of (7) is nonpositive. Since $-u^T\bar{S}u - v^T\bar{S}v = |u^T\bar{S}u + v^T\bar{S}v|$,

$$u^T Av \le \frac{1-cos(2\pi/K)}{2\,sin(2\pi/K)}|u^T\bar{S}u + v^T\bar{S}v|.$$

This implies (8) since the double angle formulas $cos\,2\alpha = cos^2\alpha - sin^2\alpha$ and $sin\,2\alpha = 2\,sin\,\alpha\,cos\,\alpha$ imply $[1-cos(2\pi/K)]/\,sin(2\pi/K) = tan(\pi/K)$.

To prove (9), let e_i be the unit vector with ith component equal to 1. Substituting $u = \sqrt{S_{jj}}\,e_i$ and $v = \sqrt{S_{ii}}\,e_j$ into (8) yields

$$\begin{aligned}(1/2)\sqrt{S_{ii}S_{jj}}(S_{ij}-S_{ji}) &\le (1/2)[tan(\pi/K)]|2S_{ii}S_{jj}|, \text{ hence}\\ S_{ij}-S_{ji} &\le 2[tan(\pi/K)]|S_{ii}|^{1/2}|S_{jj}|^{1/2}.\end{aligned}$$

The same steps with the definitions of u and v interchanged show that (9) holds.

□

Lemma 4. *$tan\,\dfrac{\pi}{K} < \dfrac{\sqrt{K}}{K-2}$ for $K > 4$.*

Proof. Let $x = 1/K$. We want to show that $tan\,\pi x < \dfrac{x^{-1/2}}{(1/x)-2} = \dfrac{x^{1/2}}{1-2x}$ for $x \in (0,1/4)$. This is equivalent to showing that

$$F(x) \equiv \frac{x^{1/2}}{1-2x}\cdot\frac{cos\,\pi x}{sin\,\pi x} > 1 \quad \text{on } (0,1/4).$$

Note that $F > 0$ on $(0,1/4)$ and that $F(1/4) = 1$. Define $G(x) \equiv ln\,F(x)$ on $(0,1/4]$. Then $G(1/4) = 0$. We will show that $G'(x) < 0$ and hence $G(x) > 0$ and $F(x) > 1$ for $x \in (0,1/4)$. Since

$$\begin{aligned}G'(x) &= \frac{1}{2x} + \frac{2}{1-2x} - \frac{\pi\,sin\,\pi x}{cos\,\pi x} - \frac{\pi\,cos\,\pi x}{sin\,\pi x}\\ &= \frac{1-2x+4x}{2x(1-2x)} - \frac{2\pi}{2(sin\,\pi x)(cos\,\pi x)} = \frac{1-2x}{2x(1-2x)} - \frac{2x}{sin\,2\pi x},\end{aligned}$$

we have $G' < 0$ on $(0,1/4)$ if and only if $sin\,2\pi x < \dfrac{4\pi x(1-2x)}{1+2x}$ for all $x \in (0,1/4)$.

Note that $8x(1-2x) < \dfrac{4\pi x(1-2x)}{1+2x}$ for $x \in (0,1/4)$ since $8x(1+2x) < 4\pi x$ is equivalent to $2+4x < \pi$, which holds for $x < 1/4$. Therefore to prove that $G' < 0$ on $(0,1/4)$, it is sufficient to show that $sin\,2\pi x \le 8x(1-2x)$ for all $x \in (0,1/4)$. Define $f(x) \equiv 8x(1-2x) - sin\,2\pi x$. We will show that f is nonnegative on $[0,1/4]$. Note first that $f(0) = f(1/4) = 0$. Since $f'(x) = 8(1-2x) - 16x - 2\pi\,cos\,2\pi x$, we have $f'(0) = 8-2\pi > 0$ and $f'(1/4) = 0$. In addition, $f''(x) = -32 + 4\pi^2\,sin\,2\pi x$ is strictly increasing on $[0,1/4]$ with $f''(0) < 0 < f''(1/4)$. Finally, $f'(1/8) = 4 - \pi\sqrt{2} < 0$ and $f''(1/8) = -32 + 2\pi^2\sqrt{2} < 0$. This shows that f' is negative at $1/8$ and first decreases, then increases on the interval $[1/8,1/4]$, and equals 0 at $1/4$. Therefore

$f' < 0$ on $[1/8, 1/4)$. Since $f(1/4) = 0$, we have $f > 0$ on $[1/8, 1/4)$. Since $f''(1/8) < 0$ and f'' is increasing on $(0, 1/8)$, f is concave on $(0, 1/8)$, with $f(0) = 0$. This shows that $f > 0$ on $(0, 1/8)$, and f is nonnegative on $[0, 1/4]$.

$\square$

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