A MODIFIED, IMPLICIT, DIRECTLY ADDITIVE DEMAND SYSTEM

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Abstract

A recently developed demand system, nicknamed AIDADS, offers a more general approach to capturing consumption preferences. AIDADS generalizes the LES by assuming marginal budget shares vary indirectly with expenditure. AIDADS is limited by the fact that the subsistence parameters are constant across expenditure. We modify AIDADS by replacing the constant subsistence parameters with a function which varies with utility, and hence expenditure. The modified AIDADS (MAIDADS) allows subsistence levels to vary with expenditure. This model is applied to the 1996 International Consumption Project data. As these data span a wide range of expenditure levels, MAIDADS offers a viable alternative when estimating "global demand systems". Results suggest subsistence values for livestock and other food products vary with expenditure, while those for grain are constant across expenditure.

Introduction

The choice of functional form has long vexed economists undertaking empirical work related to producer or consumer behaviour. Over time, however, attention has focused on functional forms which embody more general behavioural properties with respect to parameters outside the economic agents' control (i.e., prices, often income or fixed output levels). While these general functional forms prove useful, they are often difficult to estimate. Recognize, however, that subtle modifications to existing forms often appear in the literature as a means to relax an overly restrictive or untenable assumption of an existing functional form. The purpose of this paper is to motivate, develop and illustrate one such generalization in the context of consumer demand systems. Specifically, the model developed here is a generalization of Rimmer and Powell's (1992, 1996) AIDADS model. The value of such generalization becomes clear when one recognizes the usefulness of developing models which capture flexible or general price and expenditure (or income) effects. This issue is underscored by the genesis of existing demand systems.

Beginning as early as Houthakker (1957, 1965), demand analysts have strived to develop increasingly flexible representations of consumer preferences and resultant demand systems. For example, John Muellbauer and Angus Deaton worked to develop formulations that embody convenient aggregation properties. These efforts began with price independent (PI) preference structures, followed by PIGL and PIGLOG preference structures, the latter of which has its roots in Working's (1943) specification of demand as a function of expenditure. The development of PIGLOG structures was a watershed for the empirist as it led to Deaton and Muellbauer's (1980) Almost Ideal Demand System (AIDS). Generalizations of the AIDS model have followed, such as Cooper and McLaren's (1992) modified AIDS model

(MAIDS), Banks *et al.* (1997) quadratic AIDS (QUAIDS) model and Lewbel's (2003) rational rank four AIDS model (RAIDS). Beyond its convenient aggregation properties, the advantage of using AIDS as the starting point to these generalizations relates to the fact that one can then test whether restrictions consistent with nested models (contained within the general model) can be rejected. While useful from a statistical perspective (i.e., can we restrict a model to increase the degrees of freedom?), such tests often lead to useful economic information, such as the rank of a demand system and a better understanding of consumer preferences.

Another useful example has its roots in the Cobb-Douglas (CD) preference structure. Stone (1954) took the CD as a starting point in his development of the linear expenditure system. In turn, the LES was extended by Howe *et al.* (1979) to include a term that is quadratic in discretionary expenditure (i.e. the QES).[1] Other generalizations of the LES have focused on introducing marginal budget shares which vary with expenditure (e.g., Gamaletos, 1973; Lluch *et al.*, 1977). More recently, Rimmer and Powell (1992, 1996) introduced a variant of the LES that allows the marginal budget shares to vary with expenditure. This system, which is nicknamed AIDADS, is based on the assumption of direct, implicit additivity (see Hanoch, 1975). AIDADS is limited by the fact that the subsistence parameters (in the terminology of the LES) are constant across expenditure. In this paper AIDADS is modified by replacing the constant subsistence parameters with a function which varies with utility, and hence expenditure. The result is a modified AIDADS (MAIDADS) model that allows subsistence levels to vary across expenditure levels.

[1] Ryan and Wales (1999) developed further generalizations and extensions of the QES. These extensions merge the QES with other demand systems, such as the Translog, Normalized Quadratic and Generalized Leontief models. These resulting models offer Engel curves that are quadratic in expenditure, but also carry with them more flexible properties than the QES.

The next section of the paper provides a brief review of the AIDADS model – the general characteristics, as well as conditions needed for it to satisfy effective global regularity. Following this, the modified AIDADS model is developed and discussed. The econometric methods and data used to estimate MAIDADS are included in the subsequent sections, followed by empirical results and then conclusions.

An Implicitly, Directly Additive Demand System (AIDADS)

Rimmer and Powell (1992, 1996) introduced an implicitly, directly additive demand system that they nicknamed AIDADS. They view AIDADS as a generalization of the LES that overcomes one drawback of the LES – namely, constancy of the marginal budget shares. They cite a modest number of parameters (3*n* + 1, where *n* is the number of commodities), flexibility of marginal budget shares, and effective global regularity (given sufficient expenditure to satisfy subsistence) as the strengths of AIDADS.

AIDADS was investigated extensively by Cranfield *et al.* (2000). Here we draw on that work to summarize the properties of AIDADS. AIDADS is an implicit functional form. Thus, the relationship between utility and consumption levels is expressed in terms of an identity (see Hanoch, 1975 for further details). The relevant identity is:

$$\sum_{i=1}^{n} \frac{\alpha_i + \beta_i e^u}{1+e^u} \ln\left(\frac{x_i - \gamma_i}{Ae^u}\right) = 1 \qquad (1)$$

where α_i, β_i, γ_i and *A* are constant parameters, e^x is the exponential operator, *ln*(.) is the natural logarithm operator, *u* denotes utility arising from the consumption bundle and $x_i (> \gamma_i)$ is the level of consumption of the *i*-th good. Rimmer and Powell (1996) are more general in their specification of this function, but (1) is the form that has been used for most empirical work. Note that if $\alpha_i = \beta_i$ then equation (1) becomes equivalent to the LES. Like the LES, the γ_i s in AIDADS can be viewed as subsistence parameters. (The further simplification of setting $\gamma_i = 0$ yields demands equivalent to Cobb-Douglas.) It is for this reason that AIDADS is viewed as a generalization of the LES.

The implicit model of consumer behavior is that of maximizing utility, *u*, subject to (1) and a budget constraint. The resulting first-order optimality conditions for the x_i can then be re-arranged to solve for the AIDADS model stated in budget share form:

$$s_i = \frac{\gamma_i p_i}{c} + \frac{\alpha_i + \beta_i e^u}{1+e^u}\left(1 - \sum_{i=1}^{n} \frac{p_i}{c}\gamma_i\right), \qquad (2)$$

where p_i are prices of the goods in the consumption set and *c* it total expenditure on all final goods and services.

In keeping with restrictions on the parameters of the LES, the AIDADS parameters are restricted such that $\alpha_i \geq 0$ and $\beta_i \geq 0$ for all *i*=1,…,*n*, and $\sum_i \alpha_i = \sum_i \beta_i = 1$. Regularity conditions are only relevant for the region in primal space where $x_i > \gamma_i$ for all *i*=1,…,*n*. The utility relationship should be interpreted as undefined outside this region. Ideally, AIDADS would be regular (i.e., strictly increasing in each argument x_i and quasi-concave) over the region where the utility relationship is defined.

For a given level of utility, the choice of level of consumption of goods in response to prices and expenditure is straightforward. As such, the nature of regularity[2] can be examined by considering the number of solutions to the defining equation of utility, where the consumption levels (i.e. the x_i) have been replaced by the AIDADS model stated in consumption levels form:

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[2] Note that AIDADS is in the family of functional forms which satisfy the conditions for effective global regularity (see Cooper and McLaren 1992).

$$\sum_{i=1}^{n}\frac{\alpha_i+\beta_i e^u}{1+e^u}\ln\left[\frac{1}{p_i}\,\frac{\alpha_i+\beta_i e^u}{1+e^u}\left(c-\sum_{i=1}^{n}p_i\gamma_i\right)\right]-\ln(A)-u=1\,. \qquad (3)$$

From the consumer's perspective, the only variable in this problem is *u*. Solutions to the consumer's problem with AIDADS will be unique if and only if the solution to (3) is unique.

Based on examination of this relationship, Powell *et al.* (2002) find restrictions involving relative prices and differences in the parameters are needed to define the region over which AIDADS is regular. One extremely useful relationship they develop is the following:

$$\sum_{i=1}^{n}(\beta_i-\alpha_i)\ln(p_i)\geq-\frac{(1-e^u)^2}{e^u}+\sum_{i=1}^{n}\ln\left(\frac{\alpha_i+\beta_i e^u}{1+e^u}\right)(\beta_i-\alpha_i). \qquad (4)$$

Some important insights can be obtained from this inequality. First, violations of the relationship cannot occur in the LES case (i.e. when $\alpha_i=\beta_i$). Second, violations are driven by differences in the relative prices or in the discretionary budget shares, both of which depend upon inputs to the consumer choice problem that are not *parameters* of the AIDADS relationship. Third, the subsistence quantities, γ_i, are not involved. (Note, however, that extreme relative prices suggest that the quantity level for the good with the high price may be near its subsistence value.) Regularity during empirical exercises where relative price changes are typically modest is most likely to be maintained. However to guard against unexpected cases, it may be circumspect to test (4) during post-processing of numerical solutions to verify that the solution has not strayed into a dangerous region.

A Modification of AIDADS

The goal of generalizing AIDADS is to increase the flexibility of the price and expenditure effects as we move across the expenditure spectrum. One approach to the generalization of

AIDADS is to allow the subsistence quantities, γ_i, to change as a function of the utility level. A simple approach to this is to choose γ_i as follows:

$$\gamma_i(u) = \frac{\delta_i + \tau_i e^{\omega u}}{1 + e^{\omega u}} \tag{5}$$

where δ_i, τ_i, and ω are positive constants. The magnitude of δ_i and τ_i prove useful when characterizing the subsistence term's pattern of adjustment. If $\delta_i > \tau_i$ ($\delta_i < \tau_i$), and as expenditure grows from the subsistence level without bound, then δ_i represents the upper (lower) bound of the subsistence term and τ_i the lower (upper) bound. For our purposes we assume that $\delta_i \le \tau_i$. This means that as utility increases, the subsistence quantities increase, following a logistic pattern with a lower asymptote of δ_i and an upper asymptote of τ_i.

The use of subsistence quantities that vary with the level of utility is a bit at variance with the usual interpretation of γ_i as the consumption levels essential for human survival. The motivation for the alternative treatment is that as a country develops, the bundle of goods that is viewed as necessary also seems to increase. For example, in most developed countries, a telephone is considered a virtual necessity despite the fact that it is not essential to survival. When $\delta_i \le \tau_i$, the δ_i may still be interpreted as the levels of consumption needed for human survival, while the levels of τ_i may be interpreted as the consumption levels deemed necessary by a very wealthy consumer. The purpose of ω is to allow the transition from the low to high subsistence bundle to occur at a different rate than the transition from the low to high discretionary budget share.

The defining equation for the MAIDADS model appears as:

$$\sum_{i=1}^{n}\frac{\alpha_i+\beta_i e^u}{1+e^u}\ln\left(x_i-\frac{\delta_i+\tau_i e^{\omega u}}{1+e^{\omega u}}\right)-\ln(A)-u=1. \qquad (6)$$

Maximizing utility, subject to this defining equation of utility and a budget constraint results in the following demand system:

$$s_i=\frac{p_i}{c}\frac{\delta_i+\tau_i e^{\omega u}}{1+e^{\omega u}}+\frac{\alpha_i+\beta_i e^u}{(1+e^u)}\left(1-\sum_{j=1}^{n}\frac{p_j}{c}\frac{\delta_j+\tau_j e^{\omega u}}{1+e^{\omega u}}\right). \qquad (7)$$

Clearly, if $\delta_i=\tau_i$ for all i, then the AIDADS model results.

Following Hanoch (1975) and Rimmer and Powell (1992), the partial elasticities of substitution remain as they do for AIDADS:

$$\sigma_{ij}=\frac{(x_i-\gamma_i(u))(x_j-\gamma_j(u))}{x_i x_j}\times\frac{c}{c-\Sigma_{i=1}^{n}p_i\gamma_i(u)}. \qquad (8)$$

This is unchanged from the case where the γ_i s were not functions of *u*, which is appropriate since these partial elasticities are evaluated with constant utility. While the form of (8) is unchanged from AIDADS, note that the values are nonetheless functions of the γ_i s. Thus, these elasticities change not only with x_i, but also with *u*. Depending on how the parameters of the γ_i functions are realized (particularly ω), the convergence of these substitution elasticities to one (i.e. Cobb-Douglas preferences) may be delayed over the range of the data.

Now let us consider the MAIDADS Engel elasticities, which are expressed as:

$$\eta_i=\frac{c}{p_i x_i}\left\{\frac{\alpha_i+\beta_i e^u}{1+e^u}+\left(c-\sum_{j=1}^{n}p_j\frac{\delta_j+\tau_j e^{\omega u}}{\left(1+e^{\omega u}\right)^2}\right)\frac{(\beta_i-\alpha_i)e^u}{(1+e^u)^2}\lambda\right.$$
$$\left.+\frac{(\tau_i-\delta_i)\omega e^{\omega u}}{\left(1+e^{\omega u}\right)^2}p_i\lambda-\frac{\alpha_i+\beta_i e^u}{1+e^u}\sum_{j=1}^{n}\frac{(\tau_j-\delta_j)\omega e^{\omega u}}{\left(1+e^{\omega u}\right)^2}p_j\lambda\right\}. \qquad (9)$$

where λ is defined as:

$$\lambda = -\left[\sum_{i=1}^{n} \frac{(\beta_i - \alpha_i)e^u}{(1+e^u)^2} \ln\left(x_i - \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right) - \frac{\alpha_i + \beta_i e^u}{1+e^u} \left(x_i - \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right)^{-1} \frac{(\tau_i - \delta_i)\omega e^{\omega u}}{(1+e^{\omega u})^2} - 1 \right]^{-1} \times \left(c - \sum_i p_i \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right)^{-1}$$

(see Appendix A for derivation of the Engel elasticity). In equation (9), the term multiplying the expression in brackets is one over the expenditure share on the *i*-th good. Inside the brackets, the first term is the discretionary budget share of the *i*-th good. The second term in the brackets is total discretionary expenditure times the derivative of the discretionary budget share of the *i*-th good with respect to total expenditure. To this point, the elasticity is identical to the unmodified AIDADS except that in the second term, the subsistence level of consumption is a function of utility. The next term in the brackets is the derivative of the *i*-th subsistence share with respect to total expenditure, and the fourth and last terms are the derivative of total discretionary expenditure with respect to total expenditure. With the restriction that $\delta_i \le \tau_i$, it appears that the sign of the sum of the last two terms is ambiguous. Thus, the effect of utility on the Engel elasticity can be determined on empirical grounds.

The modification of AIDADS does not affect the first-order conditions with respect to the consumption variables. However, as with AIDADS, regularity is dependent on the defining equation of utility. To see this, use MAIDADS, stated in consumption levels form, to substitute out for the consumption levels in (6):

$$\sum_{i=1}^{n} \frac{\alpha_i + \beta_i e^u}{1+e^u} \ln\left[\frac{1}{p_i} \frac{\alpha_i + \beta_i e^u}{1+e^u} \left(c - \sum_{i=1}^{n} p_i \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right) \right] - \ln(A) - u = 1. \tag{10}$$

There remain concerns about uniqueness of solutions to (10). However, from an intuitive perspective, it appears that the modification of AIDADS does not worsen regularity concerns. Consider that for extremely negative values of *u*, the left-hand side of (10) is positive and for extremely positive values of *u*, the same expression is negative. With the modified form, a term that was constant (i.e. γ_i) is now increasing with *u* (because we have restricted $\delta_i \leq \tau_i$). However, that term is subtracted from expenditure, multiplied by a non-negative function of *u*, transformed by the (strictly increasing) logarithmic function, and then multiplied by another non-negative function of *u*. Thus, the modification would seem to increase the negative slope of the left-hand side of (10). It is intuitively appealing that the regular region may increase as a result of the modification.

Re-deriving the equivalent of (4), we find that the new region of regularity has become:

$$\left(c - \sum_{i=1}^{n} p_i \gamma_i\right)^{-1} \sum_{i=1}^{n} \frac{\alpha_i + \beta_i e^u}{1+e^u} \sum_{j=1}^{n} \frac{(\tau_j - \delta_j)\omega e^{\omega u}}{\left(1+e^{\omega u}\right)^2} \geq \sum_{i=1}^{n} (\beta_i - \alpha_i)\left[\ln\left(\frac{\alpha_i + \beta_i e^u}{1+e^u}\right) - \ln(p_i)\right] - \frac{(1+e^u)^2}{e^u} \quad (11)$$

which is a less stringent condition than (4). To see this, note that (4) is equivalent to requiring that the right-hand side of (11) to be non-positive. However given that $\delta_i \leq \tau_i$, the left-hand side of (11) is strictly positive. Hence, the modified form of AIDADS is regular for a greater range of prices than the original.

Estimation Framework

MAIDADS is difficult to estimate as the unobservable level of utility is an argument in the demand function. As this unobservable level of utility does not have an analytical solution, one cannot solve for *u* and substitute it out in the demand system. Rather, one must appeal to numerical methods to aid in the estimation of AIDADS. In this regard, we estimate MAIDADS using maximum likelihood techniques in the context of a mathematical programming model.

To estimate MAIDADS, error terms, denoted by v_{it} where *t* indexes observations, are appended to each equation in the demand system. By the adding up property of demands, $\sum_{i=1}^{n} v_{it} = 0$ for all *t*, so the resulting covariance matrix is singular. Dropping the last equation from each observation allows one to define Σ, a (*n*-1)x(*n*-1) covariance matrix, in terms of the *n*-1 vector $\boldsymbol{v}_t$. Upon concentration, and ignoring terms that are independent of the unknown parameters, the log-likelihood function can be written as $-0.5\ln\left|\hat{\Sigma}\right|$, where $\hat{\Sigma}$ is the estimate of Σ with typical element $\hat{\Sigma}_{ij} = T^{-1}\sum_{t=1}^{T}\hat{v}_{it}\hat{v}_{jt}$ for all $i \neq n, j \neq n$. Evaluation of the objective function is simplified by noting that $\hat{\Sigma} = \boldsymbol{R}^t \boldsymbol{R}$, where $\boldsymbol{R}$ is an upper triangular matrix of conformable dimension. Assuming $\hat{\Sigma}$ has full row rank, then $-0.5\ln\left|\hat{\Sigma}\right|$ can be expressed as $-0.5\ln\prod_{i=1}^{n-1} r_{ii}^2$, where r_{ii} are the diagonal elements of $\boldsymbol{R}$. This is the objective function of the optimization problem used here (since a concentrated log-likelihood function is used, the optimization operand changes from a maximum to a minimum). The choice variables in the optimization problem are: α_i, β_i, $\kappa(=1+\ln(A))$, u_t, $\hat{s}_{it}$, v_{it}, r_{ij} for all $i < j$, δ_i, τ_i and ω.

The constraints in this minimization problem include the fitted MAIDADS model itself, as well as the defining equation of MAIDADS. Note that including the latter assists in the estimation of the utility levels. In addition, the relationship between the residuals and elements of $\boldsymbol{R}$ are defined via:

$$T^{-1}\sum_{t=1}^{T} v_{it}v_{jt} = \sum_{k=1}^{n-1} r_{ki}r_{kj} \;\; \forall\, i \neq n, j \neq n\,, \tag{12}$$

while upper trianguarity of $\boldsymbol{R}$ is imposed with the following restriction: $r_{kl} = 0$ for all $k > l$. The residuals are defined according to the following constraint:

$$v_{it} = s_{it} - \hat{s}_{it} \quad \forall i, t, \tag{13}$$

where $\hat{s}_{it}$ is the fitted budget share. In addition, parametric restrictions on α_i and β_i (*i.e.*, $0 \le \alpha_i, \beta_i \le 1$ for all *i*, and $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 1$) are used to ensure the predicted budget shares satisfy the adding up property of demand and regularity of the predicted budget shares. As well, δ_i and τ_i are restricted to be non-negative.

To prevent the estimation procedure from attempting mathematically impossible operations, and to ensure the properties of demand are satisfied, the logarithm term in the defining equation of utility must be positive. As $\left(\alpha_i + \beta_i \exp(u_t)\right)/\left(1 + \exp(u_t)\right)$ is bounded between zero and one, and $p_i \in \Re_{++}$, then discretionary expenditure must be positive. Consequently, the following constraint is also included:

$$0.99\, y_t \ge \mathbf{p}_t' \gamma(u) \quad \forall t. \tag{14}$$

While the scaling factor (i.e. the 0.99 on the left hand side of (14)) on expenditure is somewhat arbitrary, experiments with this value suggest results are robust to this value.

Since AIDADS is a non-linear model, and the constraint set has non-linear equality and linear inequality constraints, using starting values that are at least feasible, and preferably close to optimal, helps reduce the computational burden of finding an optimal solution. In addition, appropriate choice of upper and lower bounds on the parameters, fitted budget shares, utility levels, and error terms helps to reduce the space over which the solution algorithm searches for an optimal solution. In this regard, we follow the solution strategy outlined in Cranfield *et al.* (2002). The exception to this relates to starting values and bounds on the subsistence function parameters. Lower bounds on δ_i and τ_i are set at zero, while upper bounds are set so as not to be active in the optimal solution. No bounds are placed on

ω. Estimates of γ_i (i.e., a constant subsistence parameter) are used as the starting values for δ_i and τ_i, while the starting value for ω is arbitrarily set at 0.01. This mathematical programming problem is implemented in the General Algebraic Modeling System (GAMS) and solved using the MINOS solver.

Data

The 1996 International Comparisons Project (ICP) data are used for this analysis. These data are useful in analyzing international demand patterns since they are provided in identical units (*i.e.*, international dollars). The raw data are composed of real and nominal expenditure on 26 final goods and services in 114 countries which range in expenditure levels from Malawi to the USA. For estimation, the data are aggregated into six goods: grains, livestock, other food, other non-durables, durables, and services. Expenditure on each aggregate good is computed as the sum of nominal expenditure on each good in the aggregate group. Total per capita expenditure equals total nominal expenditure divided by population. Unit prices for each good equals nominal expenditure divided by real expenditure. Nominal expenditure is defined in exchange rate converted US dollars, while real expenditure is defined in purchasing power parity converted international dollars. Finally, budget shares are computed as the ratio of nominal expenditure on the good to total nominal expenditure. Table 1 provides summary statistics for the prices, shares and expenditure.

Results

Note that in what is reported below, equation (14), which was included to ensure discretionary expenditure was positive, was not active – the strong inequality always prevailed, so discretionary expenditure was positive in the solution to the mathematical programming problem used for estimation. For comparative purposes, Table 2 shows the estimate parameters for AIDADS – recall these assume the subsistence parameters, γ_i, do not vary with expenditure. These estimates are in-keeping with those previously reported (see, for instance, Rimmer and Powell 1992, 1996; Cranfield *et al.* 2000, 2002). For all food goods, the values of α_i and β_i indicate that marginal budget shares for the respective goods fall as one progresses through higher levels of expenditure. Such result is in-keeping with Engel's Law. Moreover, the relative share of each additional dollar of expenditure devoted to non-food goods rises in expenditure. For three goods (grain, other food and other non-durables), the subsistence parameters are positive, while the γ_i s for all other goods are zero.

To investigate whether the restriction that $\delta_i \leq \tau_i$ plays a role in shaping estimates in MAIDADS, two versions of the MAIDADS models have been estimated. The first MAIDADS model does not include the restriction that $\delta_i \leq \tau_i$ (and is called the unrestricted model), while the second model includes the restriction $\delta_i \leq \tau_i$ (and is called the restricted model). Table 3 shows the parameter estimates for the unrestricted MAIDADS model. As with the AIDADS model, the value of the estimated α_i and β_i in the unrestricted MAIDADS model suggests the marginal budget shares for the food products fall as expenditure rises, while the marginal budget shares for the other goods rise with expenditure. For non-food goods, the unrestricted MAIDADS estimates of α_i and β_i have similar magnitudes to those from the model. However, the magnitude of α_i for grain and β_i for livestock and other foods are a bit different, suggesting that the nature of the subsistence parameter cannot be overlooked.

Unrestricted estimates of δ_i and τ_i are both zero for other non-durables and durables. For grain and services, $\hat{\delta}_i \geq \hat{\tau}_i$, indicating that the subsistence level for these two goods fall as per capita expenditure increases. In contrast, unrestricted estimates of δ_i and τ_i indicate that subsistence values for livestock and other food increase in per capita expenditure. To better illustrate the pattern of subsistence level adjustment, Figure 1 plots the calculated values of $\gamma_i(\hat{u})$ for those goods with non-zero estimates of δ_i and τ_i, this figure also shows the value of $\gamma_i(\hat{u})$ when prices are held fixed at their means. The subsistence parameters evaluated at the means of prices have been included to remove price-related variation from the plots and to focus on the role of expenditure in shaping the gamma function's value. The plots of the gamma function for livestock and other food show the subsistence values to rise as expenditure grows, while those for grains and services show that subsistence levels for these goods fall. Moreover, the value of the gamma function for these goods appears to reach its upper asymptotic value at a per capita expenditure level (in natural logarithms) of about 8.5, and this is true regardless of whether prices are fixed at the means or allowed to vary. This would suggest that as a country progresses through development spectrum, the subsistence levels initially adjust, then reach a stabilized value.

Table 4 shows the parameter estimates for the MAIDADS model estimated with the $\delta_i \leq \tau_i$ restriction. Note that the estimated values of α_i and β_i are not terribly different from the corresponding values in the unrestricted MAIDADS model. However, the estimates of the parameters in the subsistence function do differ for some goods. Specifically, δ_i and τ_i are all zero for other non-durables, durables and services, thus indicating these goods have a constant, zero value of subsistence. For grains, $\delta_i = \tau_i$, thus indicating the subsistence levels for grain are independent of utility (and hence expenditure). For livestock and other food, the strong inequality between δ_i and τ_i holds (so $\delta_i < \tau_i$), which tells us that the subsistence levels of these two food goods vary with utility (and hence expenditure), and that subsistence rises as expenditure grows. These points are illustrated in Figure 2, which shows the gamma function values for the three food goods as expenditure rises. As before, the gamma values at each data point are plotted, as are the gamma function values evaluated at the means of the prices but allowing expenditure to vary. As mentioned above, the subsistence level for grain is constant across expenditure levels, while those for livestock and other food increase in expenditure and reach their asymptotic limits when the natural log of per capita expenditure reaches about 8.5.

But which model is preferred? Is the AIDADS model preferred over the restricted MAIDADS model, or is the former preferred over the unrestricted MAIDADS model? In the context of nested models, such questions can typically be answered by calculating a likelihood ratio test statistic based on the value of the log of the likelihood functions for the restricted and unrestricted models. In the current context, such an approach is a bit at odds with the fact that the parameters have been estimated with bounds. Moreover, the key to comparing AIDADS to both versions of the MAIDADS model relates to equality of a set of parameters that in one case, the restricted MAIDADS model, involve weak inequality constraints which are active in some cases. Nevertheless, the trade-off is that one gains at least some insight as to which model ought to be preferred over the other.

In this regard, note that the log of the likelihood function value for the AIDADS, restricted MAIDADS and unrestricted MAIDADS models are 464.74, 476.88 and 477.6, respectively. The likelihood ratio test statistic with respect to imposition of the equation (14), which is a weak inequality restriction, is 1.44. The question is, how many degrees of freedom does one assume? There are six restrictions arising from equation (14), but also note that the parameters of the gamma function are bounded below, and in some instances these bounds are active for both the restricted and unrestricted MAIDADS models. Given this, it is not clear exactly how many degrees of freedom are relevant. However, in the unrestricted model, both δ_i and τ_i are at their lower bounds for two goods (and hence are equal to each other), but are not both at their lower bounds for four goods. In this sense, it seems appropriate the use four degrees of freedom. Given this, one fails to reject the null hypothesis of the restrictions in equation (14) at the ten percent level. The likelihood test statistic comparing the AIDADS and restricted MAIDADS model is 24.28. Since $\delta_i = \tau_i$ for four of the goods, the relevant degrees of freedom, following the logic laid out above, is two. In this case, one fails to accept the AIDADS restrictions. Given these results, one can conclude the restricted MAIDADS model is preferred over its unrestricted version and the AIDADS model. Again, however, it must be emphasized that these comparisons using the LRT are clouded by the fact that the relevant restriction involves a weak inequality which is active in some instances.

The question now becomes whether there is an economic difference between the three models. Tables 5, 6 and 7 report the marginal budget shares, fitted budget shares and Engel elasticities, calculated at the means of the data, using the estimated AIDADS, unrestricted MAIDADS and restricted MAIDADS models, respectively. Focusing on the Engel elasticities, it would appear that for the food goods, the economic behaviour characterized via Engel elasticities differs across the three models. Specifically, Engel elasticities for other non-durables, durables and services are relatively similar across the models.[3] Moreover, the

[3] The most noticeable difference is with respect services from the restricted MAIDADS and AIDADS model, where the Engel elasticity based on the latter is about 20 percent larger than the former.

qualitative nature of the Engel elasticities for the non-food goods, at the means of the data, is identical across models – these are goods are luxuries.[4]

Drastic differences are noted between the Engel elasticities for the food goods. Engel elasticities for grain in the both unrestricted and restricted MAIDADS models are about four times as great as those for AIDADS. The same is true of livestock and other food, where Engel elasticities from the MAIDADS models are about three and two times as large as those from the AIDADS model, respectively. What is not clear is how the Engel elasticities adjust as expenditure increases. This issue is addressed in Figures 3, 4 and 5, which show the Engel elasticities for the food goods from the AIDADS, unrestricted MAIDADS and restricted MAIDADS models, respectively. To better capture adjustments, these figures show not only the Engel elasticity at each observations respective price and income levels, but also smoothed patterns of adjustment fitted to the Engel elasticities using high order polynomials.[5] The Engel elasticities for the food goods estimated in the AIDADS model generally exhibit a downward trend. This is certainly true for livestock and other food, whose Engel elasticities fall from a level consistent with these goods being luxuries, to near around 0.2 at the highest level of per capita expenditure. The Engel elasticity shows an initial rising trend, reaches a maximum and the falls to around 0.1 at the largest value of per capita expenditure. As AIDADS has asymptotic Cobb-Douglas behaviour, it would appear that expenditure levels are not high enough in this sample to bring the Engel elasticities for the food goods close to unity, as would be expected as expenditure grows without bound.

Figures 4 and 5 show the plots of the Engel elasticities for the food goods based on the unrestricted and restricted MAIDADS models. Note that in these figures, the smoothed plots

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[4] Although those for the MAIDADS models are close to the threshold between normal and luxury goods.
[5] These polynomial trends were fitted typically using a fourth order or higher polynomial of the log of per capita expenditure.

of the Engel elasticities for grain and livestock were generated using a Box-Tidwell regression model[6], while those for other food were generated using a high order polynomial. For both versions of the MAIDADS model, the Engel elasticities for all food goods seem to be converging on Cobb-Douglas like behaviour – the Engel elasticities appear to be approaching unity as expenditure grows without bound. Those for livestock and grain fall and rise, respectively, as per capita expenditure grows, while those for other food initially fall and then begin to rise after the about the mid-point in the sample. Compared to the AIDADS model, the Engel elasticities for other food appear rather different. It is difficult to attribute these differences to the generalization of the gamma terms alone.[7] Nevertheless, comparing Figures 1 and 2 to Figure 4 and 5 illustrates that as the gamma for other food, for example, begins to increase dramatically, the Engel elasticity for other food begins to change its direction of movement (from a downward trend to an upward trend). In more general terms, one might conclude that the modified AIDADS model offers a preferred approach to modeling global demands.

Conclusion

In this paper we develop a modified version of Rimmer and Powell's (1992, 1996) AIDADS model. Following their lead, AIDADS is modified by replacing the constant subsistence parameters with a function which varies with utility, and hence expenditure. The result is a modified AIDADS (MAIDADS) model that allows subsistence levels to vary across expenditure levels. This model is applied to the 1996 International Consumption Project database. These data span a wide range of expenditure levels, and countries at various stages

[6] The Engel elasticities were regressed in the log of per capita expenditure, where the latter was transformed using the Box-Cox transformation.

[7] This difficulty arises because all of the parameters of the AIDADS model change value when the gamma function is incorporated.

of development. As preferences are likely to vary across these countries, the MAIDADS model offers a viable alternative when estimating “global demand systems”. Casual and statistical comparison (via likelihood ratio tests) suggests the MAIDADS model is preferred to the AIDADS model. Results suggest subsistence values for livestock and other food products vary across expenditure levels, while those for grain are constant across different expenditure levels.

Table 1. Summary statistics of the 1996 ICP dataset.

	Grain	Livestock	Other Food	Other non-durables	Durables	Service
			Prices			
Mean	0.696	0.615	0.672	0.601	0.785	0.519
Std. dev.	0.373	0.287	0.279	0.476	0.462	0.445
			Shares			
Mean	0.080	0.107	0.153	0.223	0.085	0.353
Std. dev.	0.075	0.052	0.074	0.059	0.036	0.127
			Per capita expenditure			
Mean			5436.75			
Std. dev.			7293.07			

Table 2. Estimated Parameters of AIDADS

	Grain	Livestock	Other Food	Other non-durables	Durables	Service
α_i	0.090	0.188	0.259	0.199	0.068	0.197
β_i	0.000	0.009	0.005	0.266	0.117	0.604
γ_i	0.533	0.000	0.128	0.102	0.000	0.000
$\kappa = 1 + \ln(A)$	2.589					

Table 3. Estimated parameters of the unrestricted MAIDADS

	Grain	Livestock	Other Food	Other non-durables	Durables	Service
α_i	0.202	0.204	0.170	0.224	0.063	0.138
β_i	0.000	0.037	0.036	0.265	0.115	0.547
δ_i	0.302	0.000	0.353	0.000	0.000	0.090
τ_i	0.000	1.163	3.432	0.000	0.000	0.000
$\kappa = 1 + \ln(A)$	1.498					
ω	2.617					

Table 4. Estimated parameters of the restricted MAIDADS

	Grain	Livestock	Other Food	Other non-durables	Durables	Service
α_i	0.192	0.209	0.163	0.229	0.064	0.144
β_i	0.000	0.038	0.039	0.265	0.115	0.543
δ_i	0.321	0.000	0.379	0.000	0.000	0.000
τ_i	0.321	1.125	3.390	0.000	0.000	0.000
$\kappa = 1 + \ln(A)$	1.435					
ω	2.804					

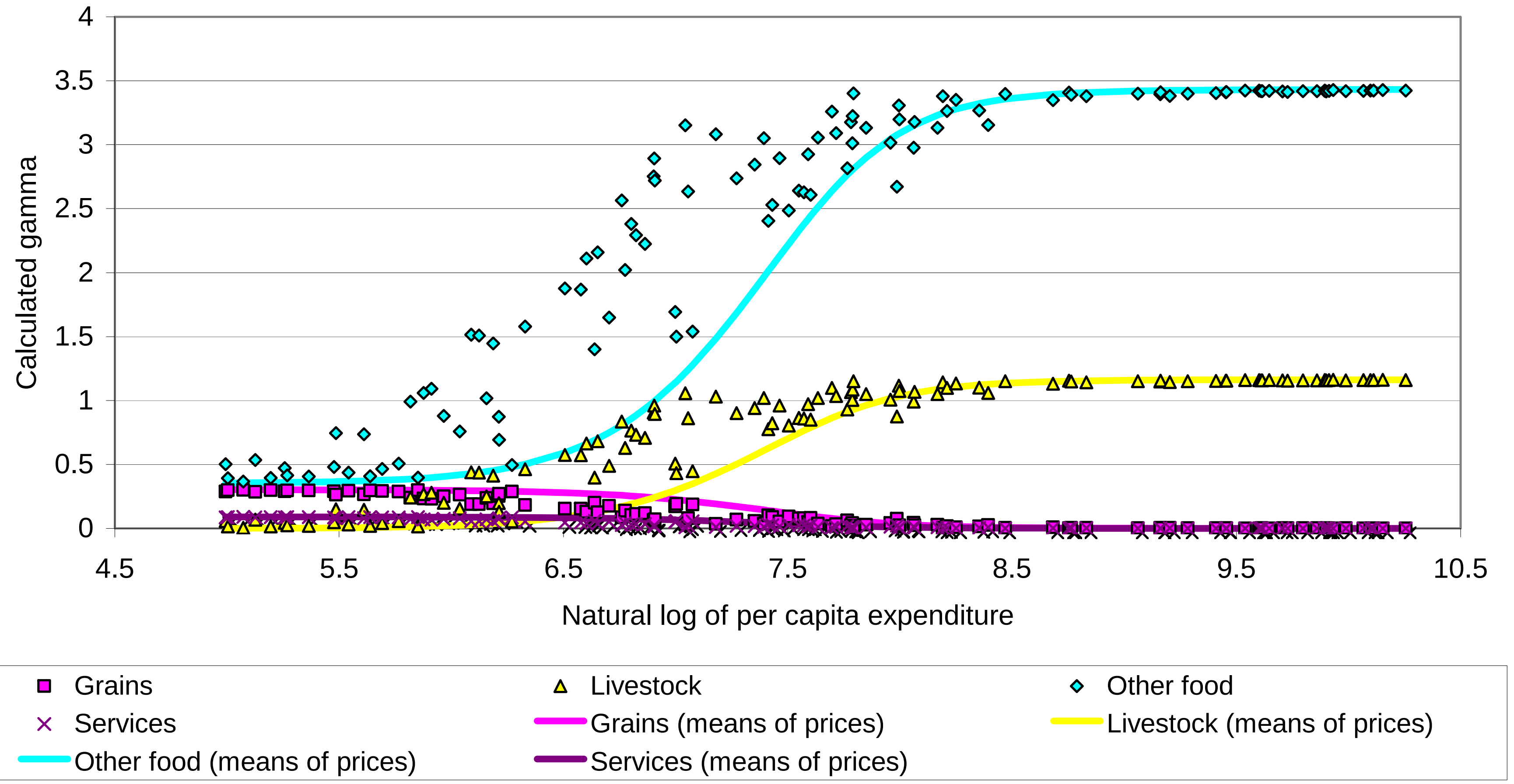

Figure 1. Estimated subsistence shares from the unrestricted MAIDADS evaluated at each data point, and with prices fixed at the sample means.

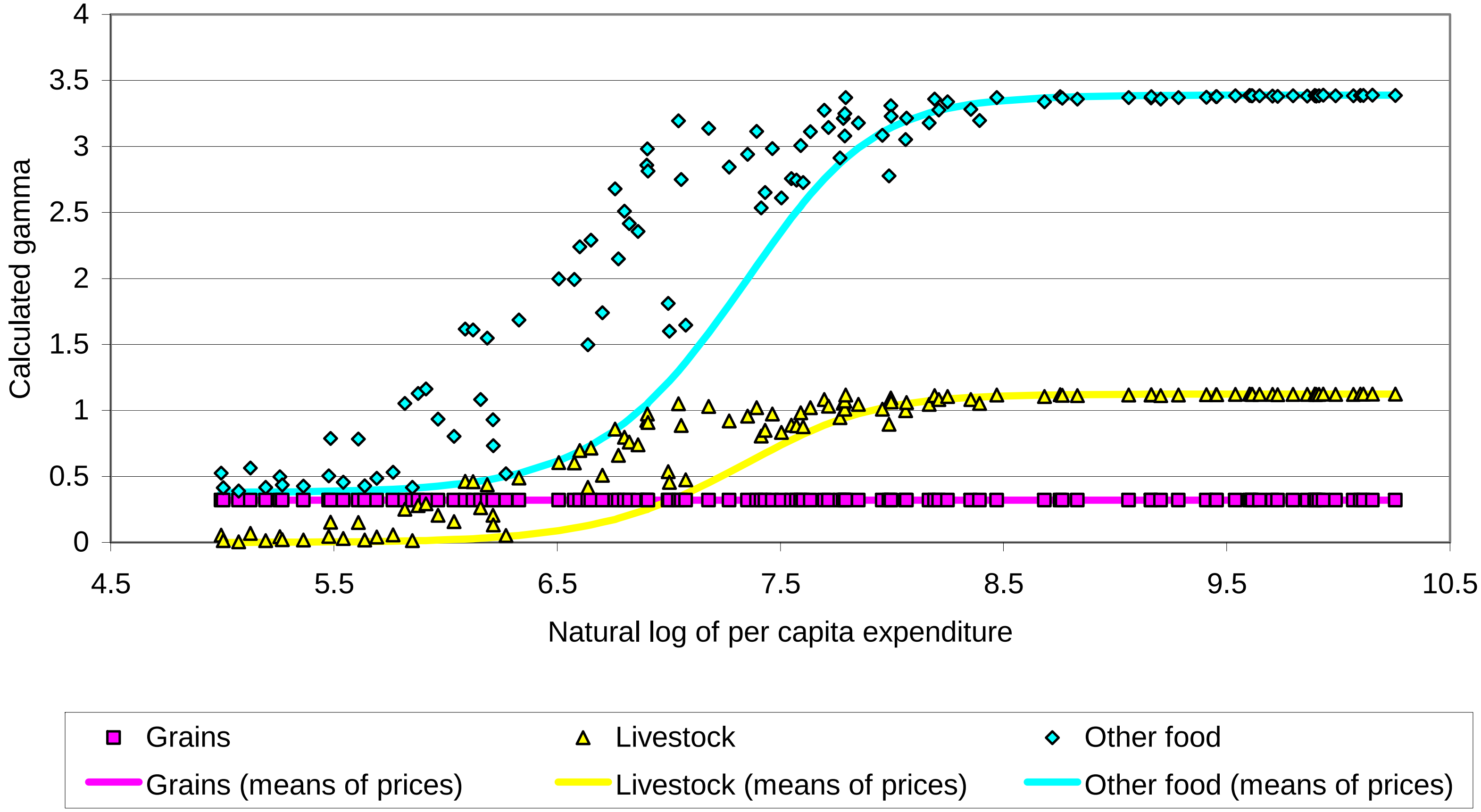

Figure 2. Estimated subsistence shares from the restricted MAIDADS evaluated at each data point, and with prices fixed at the sample means.

Table 5. Estimated Marginal Budget Shares, Fitted Budget shares and Engel Elasticities for AIDADS, evaluated at the sample means of the data.

	Grain	Livestock	Other Food	Other non-durables	Durables	Service
MBS	0.008	0.024	0.027	0.259	0.113	0.567
$\hat{s}_i$	0.041	0.078	0.104	.239	0.097	0.440
Engel	0.193	0.318	0.262	1.089	1.163	1.288

Table 6. Estimated Marginal Budget Shares, Fitted Budget shares and Engel Elasticities for the unrestricted MAIDADS, evaluated at the sample means of the data.

	Grain	Livestock	Other Food	Other non-durables	Durables	Service
MBS	0.029	0.062	0.056	0.259	0.107	0.486
$\hat{s}_i$	0.033	0.075	0.098	0.244	0.100	0.450
Engel	0.905	0.822	0.576	1.062	1.070	1.079

Table 7. Estimated Marginal Budget Shares, Fitted Budget shares and Engel Elasticities for the restricted MAIDADS, evaluated at the sample means of the data.

	Grain	Livestock	Other Food	Other non-durables	Durables	Service
MBS	0.027	0.062	0.056	0.260	0.107	0.487
$\hat{s}_i$	0.033	0.074	0.097	0.244	0.100	0.450
Engel	0.797	0.831	0.580	1.065	1.073	1.082

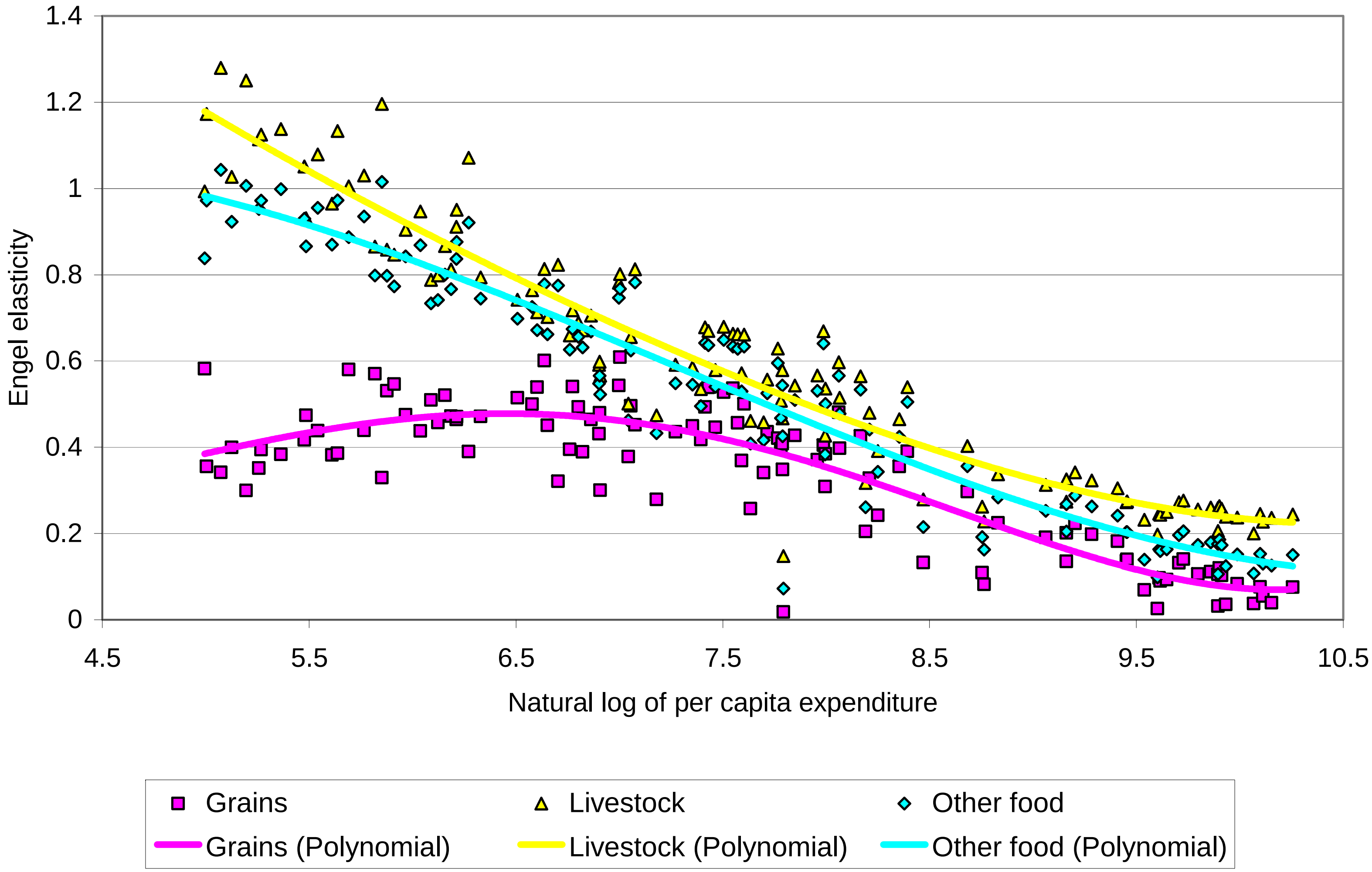

Figure 3: Estimated Engel elasticities from the AIDADS model evaluated as expenditure grows, and at the price levels (points) and smoothed (lines)

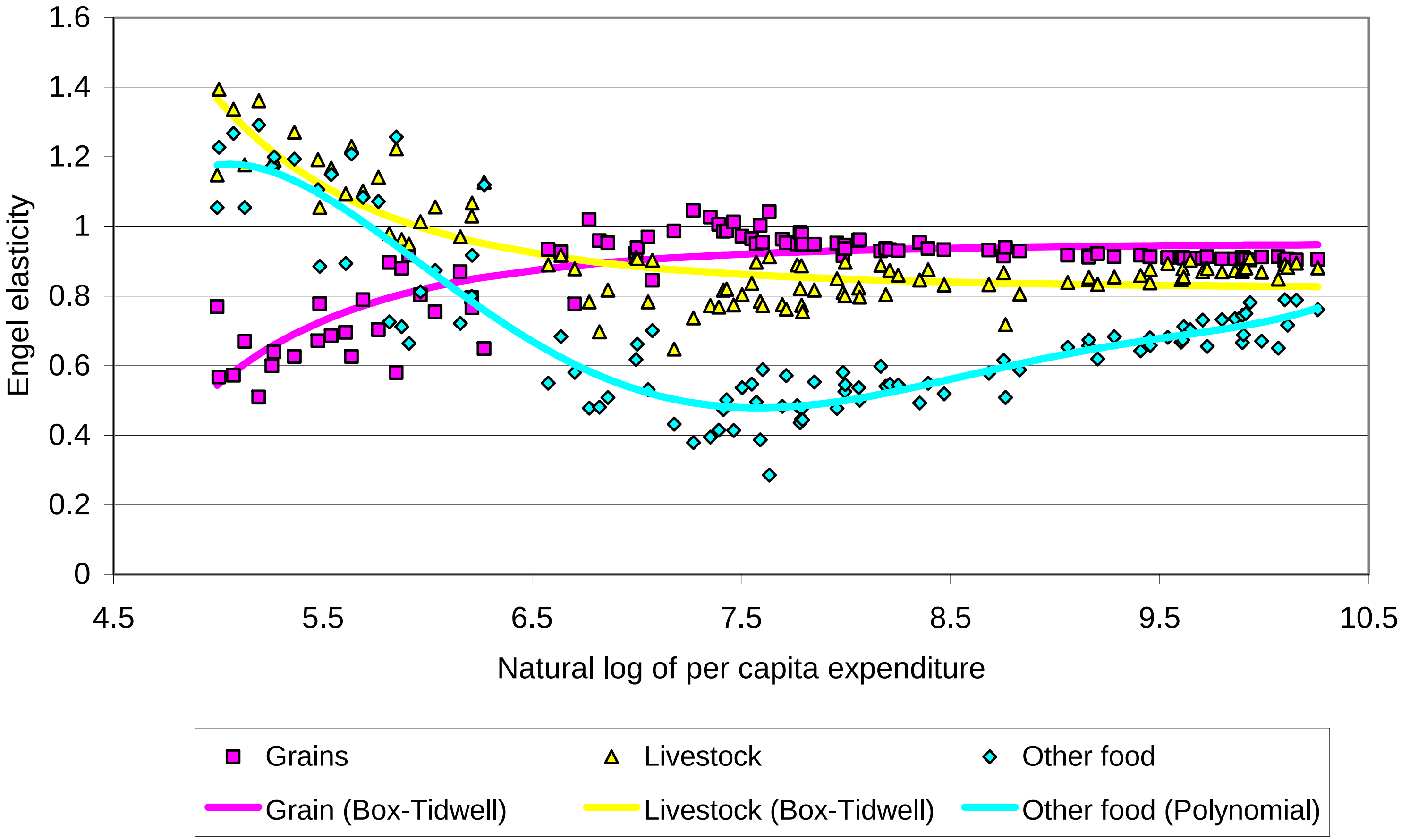

Figure 4: Estimated Engel elasticities from the unrestricted MAIDADS model evaluated as expenditure grows, and at the price levels (points) and smoothed (lines)

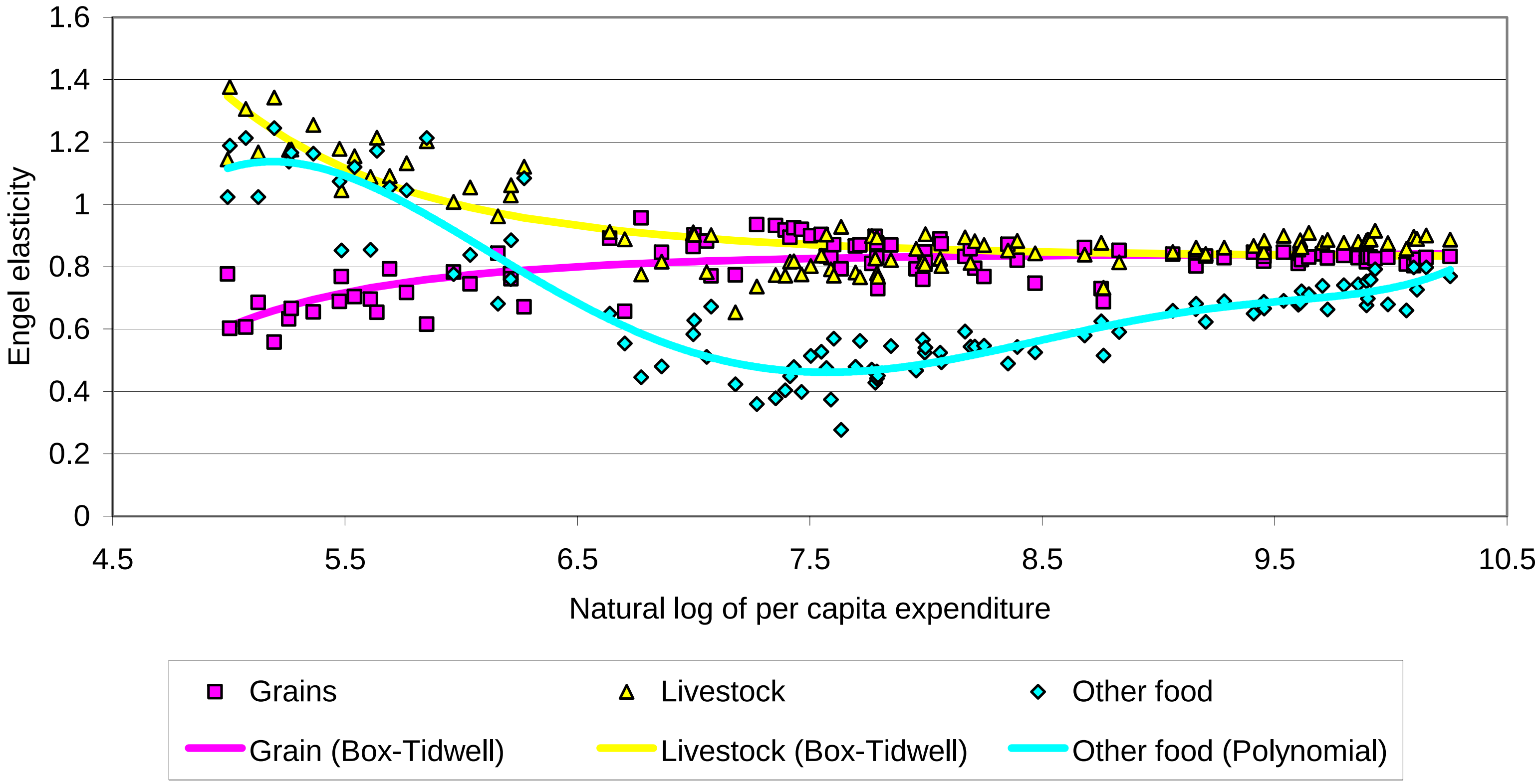

Figure 5: Estimated Engel elasticities from the restricted MAIDADS model evaluated as expenditure grows, and at the price levels (points) and smoothed (lines)

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Appendix A- Derivation of the Engel Elasticity for MAIDADS

Derivation of the Engel elasticities for MAIDADS is rather complicated. As such, begin with the consumer's utility maximization problem. Recognize that since the defining equation for MAIDADS is implicit in utility, it cannot be treated as the objective function in the consumer's problem. Rather, the objective function is defined as the value of utility, and is maximized subject to the defining equation of utility and a budget constraint:

$$\begin{aligned} &\underset{u,x}{\text{maximize}} \quad u \\ &\text{subject to}: \sum_{i=1}^{n} \frac{\alpha_i + \beta_i e^u}{1+e^u} \ln\left(x_i - \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right) - \ln(A) - u = 1 \\ &\qquad\qquad \sum_{i=1}^{n} p_i x_i = c. \end{aligned} \tag{A1}$$

Let μ and λ denote the Lagrange multipliers associated with the first and second constraints in (A1). The solution with respect to x_i is:

$$-\mu \frac{\alpha_i + \beta_i e^u}{1+e^u} \frac{1}{x_i - \dfrac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}}} - \lambda p_i = 0. \tag{A2}$$

A2 can be manipulated to show that

$$\lambda = -\mu \left(c - \sum_i p_i \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right)^{-1}. \tag{A3}$$

The first-order condition with respect to u is:

$$\begin{aligned} 1 - \mu \Bigg[&\sum_{i=1}^{n} \frac{(\beta_i - \alpha_i) e^u}{(1+e^u)^2} \ln\left(x_i - \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right) - \\ &\frac{\alpha_i + \beta_i e^u}{1+e^u} \left(x_i - \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right)^{-1} \frac{(\tau_i - \delta_i)\omega e^{\omega u}}{(1+e^{\omega u})^2} - 1 \Bigg] = 0. \end{aligned} \tag{A4}$$

This means that the Lagrange multiplier on the first constraint is:

$$\mu = \left[\sum_{i=1}^{n} \frac{(\beta_i - \alpha_i) e^{u}}{(1+e^{u})^2} \ln\left(x_i - \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right) - \right.$$
$$\left. \frac{\alpha_i + \beta_i e^{u}}{1+e^{u}} \left(x_i - \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right)^{-1} \frac{(\tau_i - \delta_i)\omega e^{\omega u}}{(1+e^{\omega u})^2} - 1 \right]^{-1}. \tag{A5}$$

As such, the constraint on the defining equation of utility can be expressed as:

$$\lambda = -\left[\sum_{i=1}^{n} \frac{(\beta_i - \alpha_i) e^{u}}{(1+e^{u})^2} \ln\left(x_i - \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right) - \right.$$
$$\left. \frac{\alpha_i + \beta_i e^{u}}{1+e^{u}} \left(x_i - \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right)^{-1} \frac{(\tau_i - \delta_i)\omega e^{\omega u}}{(1+e^{\omega u})^2} - 1 \right]^{-1} \times \left(c - \sum_i p_i \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} \right)^{-1} \tag{A6}$$

From (A2) and (A3), we can denote the demand for the *i*-th good as follows:

$$x_i = \frac{\delta_i + \tau_i e^{\omega u}}{1+e^{\omega u}} + \frac{\alpha_i + \beta_i e^{u}}{(1+e^{u}) p_i} \left(c - \sum_{j=1}^{n} p_j \frac{\delta_j + \tau_j e^{\omega u}}{1+e^{\omega u}} \right). \tag{A7}$$

Following Hanoch (1975) and Rimmer and Powell’s (1992) derivation strategies, the Engel elasticity is then expressed as:

$$\eta_i = \frac{c}{p_i x_i} \left\{ \frac{\alpha_i + \beta_i e^{u}}{1+e^{u}} + \left(c - \sum_{j=1}^{n} p_j \frac{\delta_j + \tau_j e^{\omega u}}{\left(1+e^{\omega u}\right)^2} \right) \frac{(\beta_i - \alpha_i) e^{u}}{(1+e^{u})^2} \lambda \right.$$
$$\left. + \frac{(\tau_i - \delta_i)\omega e^{\omega u}}{\left(1+e^{\omega u}\right)^2} p_i \lambda - \frac{\alpha_i + \beta_i e^{u}}{1+e^{u}} \sum_{j=1}^{n} \frac{(\tau_j - \delta_j)\omega e^{\omega u}}{\left(1+e^{\omega u}\right)^2} p_j \lambda \right\}. \tag{A8}$$