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Value, complement reduction and interval TU games

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Abstract

In the framework of interval transferable-utility (TU) games, we introduce a generalization of the equal allocation of nonseparable cost (EANSC). Further, we extend the reduced game introduced by Moulin (1985) to interval TU games. By applying this extended reduction, two axiomatizations of this extended EANSC are proposed.

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1 Introduction

In a standard TU game, the utility produced by a coalition or the payoff assigned to a player could be computed as a single real value. However, the utilities or the payoffs may not be computed precisely in real situations, but it is sensible to verify rational intervals to which it belong. Methods of interval arithmetic and analysis (cf. Moore, 1979) have played a key role for new models of games based on interval uncertainty. A *interval TU game*, originally introduced by Branzei et al. (2003,2004) in the context of bankruptcy situations, is a generalization of a standard TU game in which all the utilities or all the payoffs could be rather expressed in the form of closed intervals. Similar to standard TU games, solutions on interval TU games could be applied in many fields such as economics, political sciences, accounting and even management. Related results may be found in Alparslan Gök et al. (2008,2009,2010), Branzei et al. (2003,2004,2010) and so on.

Consistency, originally introduced by Harsanyi (1959) under the name of bilateral equilibrium, is a crucial property of solutions. Consistency allows us to deduce, from the desirability of an outcome for some problem, the desirability of its restriction to each subgroup for the associated reduced game the subgroup faces. If a solution is not consistent, then a subgroup of agents might not respect the original compromise but revise the payoff distribution within the subgroup. The fundamental property of solutions has always been investigated in various classes of problems by applying *reduced games*. Various definitions of a reduced game have been proposed, depending upon exactly how the agents outside of the subgroup should be paid off.

Here we focus on the solution concept of *the equal allocation of non-separable costs (EANSC)*. The EANSC is a well-known solution concept in cooperative game theory. In the framework of standard TU games, Moulin (1985) introduced the complement reduced game to axiomatize the EANSC. Inspired by Hart and Mas-Colell (1989), Hwang (2009) characterized the EANSC by means of two-person standardness and related consistency. These mentioned above raise one question in the framework of interval TU games:

- whether these remarkable results of the EANSC could be described in the framework of interval TU games.

The note is aimed at answering the question. In this note, we firstly extend the EANSC and the complement reduction to interval TU games. Inspired by Hwang (2009) and Moulin (1985), we further characterize this extended EANSC by means of related consistency.

2 Preliminaries

Here we follow the notation and terminology of the published papers (2008,2009,2010). Let U be the universe of players and $N \subseteq U$ be a set of players. A **interval TU game** is a pair (N, w) where N is a non-empty and finite set of players and $w : 2^N \to I(\mathbb{R})$ is a characteristic function such that $w(\emptyset) = [0, 0]$. For each $S \in 2^N$, the worth interval $w(S)$ of the coalition S in the interval game (N, w) is of the form $[\overline{w}(S), \underline{w}(S)]$, where $\underline{w}(S)$ is the minimal reward which coalition S could receive on its own and $\overline{w}(S)$ is the maximal reward which coalition S could get. Denote the class of all interval TU games with player set N by IG^N. We also denote by $I(\mathbb{R})^N$ the set of all such interval payoff vectors.

Let $I, J \in I(\mathbb{R})$ with $I = [\underline{I}, \overline{I}]$, $J = [\underline{J}, \overline{J}]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \geq 0$. Then,

$$\begin{aligned} I + J &= [\underline{I} + \underline{J}, \overline{I} + \overline{J}], \\ \alpha I &= [\alpha \underline{I}, \alpha \overline{I}]. \end{aligned} \tag{1}$$

By equation (1) we see that $I(\mathbb{R})$ has a cone structure. We define $I - J$, only if $|I| \geq |J|$, by $I - J = [\overline{I} - \overline{J}, \underline{I} - \underline{J}]$. Note that $\overline{I} - \overline{J} \geq \underline{I} - \underline{J}$.

The model of interval TU games is an extension of the model of classical TU games. We recall that a classical TU game[1] $< N, v >$ is defined by $v : 2^N \to \mathbb{R}$ and $v(\emptyset) = 0$. A classical TU game $< N, v >$ is **monotonic** if $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subseteq T$. We call an interval TU game (N, w) **size monotonic** if its length TU game $< N, |w| >$ is monotonic, where $|w|(S) = \overline{w}(S) - \underline{w}(S)$ for all $S \subseteq N$. We denote by $SMIG^N$ the class of size monotonic interval games with player set N. Let $(N, w) \in SMIG^N$, we call (N, w) **full monotonic** if for all $S \subseteq N$, $|w|(S) \geq \sum_{i \in S} \big[|w|(S) - |w|(S \setminus \{i\})\big]$. We denote by $FMIG^N$ the class of full monotonic interval games with player set N. In the sequel, we focus on the set of games, $FMIG$, where $FMIG = \cup_{N \subseteq U} FMIG^N$.

A **solution** on $FMIG$ is a map ϕ assigning to each interval TU game $(N, w) \in FMIG$ an element $\phi(N, w) \in I(R)^N$.

Definition 1 *The* **interval equal allocation of nonseparable costs (IEANSC)**, $\overline{\eta}$, *is the function on $FMIG$ which associates with each $(N, w) \in FMIG$ and each $i \in N$ the value*

$$\overline{\eta}_i(N, w) = \eta_i(N, w) + \frac{1}{|N|} \cdot \big[w(N) - \sum_{k \in N} \eta_k(N, w)\big],$$

[1] We use $< N, v >$ and (N, v) to denote a classical TU game and an interval TU game respectively.

where $\eta_i(N,w) = \big[w(N) - w(N\setminus\{i\})\big]$. *The value* $\eta_i(N,w)$ *is the* **interval marginal contributions** *of player* i.

3 Main results

In this section, we show that there exists reduced game that can be used to characterize the IEANSC.

Let ϕ be a solution on $FMIG$. ϕ satisfies **efficiency (EFF)** if for all $(N,w) \in FMIG$, $\sum_{i\in N} \phi_i(N,w) = w(N)$. ϕ satisfies **standard for two-person games (STPG)** if for all $(N,w) \in FMIG$ with $|N| \leq 2$, $\phi(N,w) = \overline{\eta}(N,w)$. ϕ satisfies **symmetry (SYM)** if for all $(N,w) \in FMIG$ with $w(S) - w(S\setminus\{i\}) = w(S) - w(S\setminus\{k\})$ for some $i,k \in N$ and for all $S \subseteq N$, $\phi_i(N,w) = \phi_k(N,w)$. ϕ satisfies **zero-independence (ZI)** if for all $(N,v),(N,w) \in \Gamma$ with $v(S) = w(S) + \sum_{i\in S} b_i$ for some $b \in \mathbb{R}^N$ and for all $S \subseteq N$, $\phi(N,v) = \phi(N,w) + b$. By Definition 1, it is easy to see that the IEANSC satisfies EFF, STPG, SYM and ZI.

Next, consider the complement reduction. Given a payoff vector chosen by a solution for some game, and given a subgroup of players, Moulin (1985) defined the reduced game as that in which each coalition in the subgroup could attain payoffs to its members only if they are compatible with the initial payoffs to "all" the members outside of the subgroup. A natural extended complement reduction on interval TU games is defined as follows.

Given $(N,w) \in FMIG$, $S \subseteq N\setminus\{\emptyset\}$, and a solution ϕ, the **reduced game** (S, w_S^{ϕ}) **with respect to** S **and** ϕ is defined by for all $T \subseteq S$,

$$w_S^{\phi}(T) = \begin{cases} [0,0] & \text{, if } T = \emptyset, \\ w(T\cup(N\setminus S)) - \sum\limits_{i\in N\setminus S} \phi_i(N,w) & \text{, otherwise.} \end{cases}$$

Consistency may be described informally as follows: Let ϕ be a solution that associates a payoff to every player in every game. For any two-person group of players in a game, one defines a " reduced game" among them by considering the amounts remaining after the rest of the players are given the payoffs prescribed by ϕ. Formally, a solution ϕ satisfies **consistency (CON)** if for all $(N,w) \in FMIG$ with $|N| \geq 2$, for all $S \subseteq N$ with $|S| = 2$ and for all $i \in S$, $\phi_i(N,w) = \phi_i(S, w_S^{\phi})$.

Lemma 1 *The solution* $\overline{\eta}$ *satisfies CON.*

Proof. Given $(N,w) \in FMIG$ with $|N| \geq 2$ and $S \subseteq N$ with $|S| = 2$.

By definitions of η and $w^{\overline{\eta}}_S$, for all $i \in S$,

$$\begin{aligned}
\eta_i(S, w^{\overline{\eta}}_S) &= \Big[w^{\overline{\eta}}_S(S) - w^{\overline{\eta}}_S(S \setminus \{i\})\Big] \\
&= \Big[w(N) - \sum_{k \in N \setminus S} \overline{\eta}_k(N, w) - w(N \setminus \{i\}) + \sum_{k \in N \setminus S} \overline{\eta}_k(N, w)\Big] \\
&= \Big[w(N) - w(N \setminus \{i\})\Big] \\
&= \eta_i(N, w).
\end{aligned} \tag{2}$$

Hence, for all $i \in S$,

$$\begin{aligned}
\overline{\eta}_i(S, w^{\overline{\eta}}_S) &= \eta_i(S, w^{\overline{\eta}}_S) + \tfrac{1}{|S|} \cdot \Big[w^{\overline{\eta}}_S(S) - \sum_{k \in S} \eta_k(S, w^{\overline{\eta}}_S)\Big] \\
&= \eta_i(N, w) + \tfrac{1}{|S|} \cdot \Big[w^{\overline{\eta}}_S(S) - \sum_{k \in S} \eta_k(N, w)\Big] \text{ (by equation (1))} \\
&= \eta_i(N, w) + \tfrac{1}{|S|} \cdot \Big[w(N) - \sum_{k \in N \setminus S} \overline{\eta}_k(N, w) - \sum_{k \in S} \eta_k(N, w)\Big] \\
&= \eta_i(N, w) + \tfrac{1}{|S|} \cdot \Big[\sum_{k \in S} \overline{\eta}_k(N, w) - \sum_{k \in S} \eta_k(N, w)\Big] \text{ (by EFF of } \overline{\eta}) \\
&= \eta_i(N, w) + \tfrac{1}{|S|} \cdot \Big[\sum_{k \in S} \tfrac{1}{|N|} \cdot \big[w(N) - \sum_{p \in N} \eta_p(N, w)\big]\Big] \\
&= \eta_i(N, w) + \tfrac{1}{|S|} \cdot \Big[\tfrac{|S|}{|N|} \cdot \big[w(N) - \sum_{p \in N} \eta_p(N, w)\big]\Big] \\
&= \eta_i(N, w) + \tfrac{1}{|N|} \cdot \Big[w(N) - \sum_{k \in N} \eta_k(N, w)\Big] \\
&= \overline{\eta}_i(N, w).
\end{aligned}$$

■

Next, we characterize the IEANSC by means of related properties of two-person standardness and consistency.

Theorem 1 *A solution ϕ on FMIG satisfies STPG and CON if and only if $\phi = \overline{\eta}$.*

Proof. By Lemma 1, $\overline{\eta}$ satisfies CON. Clearly, $\overline{\eta}$ satisfies STPG.

To prove the uniqueness, suppose ϕ satisfies STPG and CON. By STPG and CON of ϕ, it is easy to derive that ϕ also satisfies EFF; hence, we omit it. Let $(N, w) \in FMIG$. If $|N| \leq 2$, then by STPG of ϕ, $\phi(N, w) = \overline{\eta}(N, w)$. The case $|N| > 2$: For all $i, k \in L^N$ with $i \neq k$, let

$S = \{i, k\}$, we derive that

$$
\begin{aligned}
& \phi_i(N,w) - \phi_k(N,w) \\
= \; & \phi_i(S, w_S^{\phi}) - \phi_k(S, w_S^{\phi}) \quad \text{(by CON of } \phi) \\
= \; & \overline{\eta}_i(S, w_S^{\phi}) - \overline{\eta}_k(S, w_S^{\phi}) \quad \text{(by STPG of } \phi) \\
= \; & \eta_i(S, w_S^{\phi}) - \eta_k(S, w_S^{\phi}) \quad \text{(by Definition 1)} \\
= \; & \Big[w_S^{\phi}(S) - w_S^{\phi}(\{k\})\Big] - \Big[w_S^{\phi}(S) - w_S^{\phi}(\{i\})\Big] \\
= \; & \Big[w_S^{\phi}(\{i\}) - w_S^{\phi}(\{k\})\Big] \\
= \; & \Big[w(N \setminus \{k\}) - \sum_{t \in N \setminus S} \phi_t(N,w) - w(N \setminus \{i\}) + \sum_{t \in N \setminus S} \phi_t(N,w)\Big] \\
= \; & \Big[w(N \setminus \{k\}) - w(N \setminus \{i\})\Big] \\
= \; & \Big[w(N) - w(N \setminus \{i\})\Big] - \Big[w(N) - w(N \setminus \{k\})\Big]
\end{aligned}
\tag{3}
$$

Similarly, $\overline{\eta}$ instead of ϕ in equation (3), we can derive that

$$
\begin{aligned}
& \overline{\eta}_i(N,w) - \overline{\eta}_k(N,w) \\
= \; & \Big[w(N) - w(N \setminus \{i\})\Big] - \Big[w(N) - w(N \setminus \{k\})\Big]
\end{aligned}
\tag{4}
$$

Hence, by equations (3) and (4),

$$
\phi_i(N,w) - \phi_k(N,w) = \overline{\eta}_i(N,w) - \overline{\eta}_k(N,w). \tag{5}
$$

This implies that $\phi_i(N,w) - \overline{\eta}_i(N,w) = d$ for all $(i \in N$ and for some $d \in \mathbb{R}$. It remains to show that $d = [0,0]$. By EFF of ϕ and $\overline{\eta}$ and equation (5),

$$
[0,0] = w(N) - w(N) = \sum_{i \in N} \Big[\phi_i(N,w) - \overline{\eta}_i(N,w)\Big] = |N| \cdot d.
$$

Hence, $d = 0$. ■

Finally, we characterize the IEANSC by means of related properties of efficiency, symmetry, zero-independence and consistency.

Lemma 2 *If a solution ϕ on FMIG satisfies EFF, SYM and ZI, then ϕ satisfies STPG.*

Proof. Assume that a solution ϕ satisfies EFF, SYM and ZI. Given $(N, v) \in \Gamma$ with $N = \{i, k\}$ for some $i \neq k$. We define a game (N, w) to be that for all $S \subseteq N$,

$$
w(S) = v(S) - \sum_{i \in S} \eta_i(N, v).
$$

By the definition of w,

$$\begin{aligned} w(\{i,k\}) - w(\{k\}) &= v(\{i,k\}) - \eta_i(N,v) - \eta_k(N,v) - v(\{k\}) + \eta_k(N,v)\} \\ &= v(\{i,k\}) - \eta_i(N,v) - v(\{k\})\} \\ &= v(\{i,k\}) - v(\{k\}) - \eta_i(N,v) \\ &= \eta_i(N,f,v) - \eta_i(N,f,v) \\ &= [0,0]. \end{aligned}$$

Similarly, $w(\{i,k\}) - w(\{i\}) = [0,0]$. Since $w(\{i,k\}) - w(\{k\}) = w(\{i,k\}) - w(\{i\}) = 0$, by SYM of ϕ, $\phi_i(N,w) = \phi_k(N,w)$. By EFF of ϕ,

$$w(N) = \phi_i(N,w) + \phi_k(N,w) = 2 \cdot \phi_i(N,w).$$

Therefore,

$$\phi_i(N,w) = \frac{w(N)}{2} = \frac{1}{2} \cdot \big[v(N) - \eta_i(N,v) - \eta_k(N,v)\big].$$

By ZI of ϕ,

$$\phi_i(N,v) = \eta_i(N,v) + \frac{1}{2} \cdot \big[v(N) - \eta_i(N,v) - \eta_k(N,v)\big] = \overline{\eta_i}(N,v).$$

Similarly, $\phi_k(N,v) = \overline{\eta_k}(N,v)$. Hence, ϕ satisfies STPG. ■

Theorem 2 *On FMIG, the IEANSC is the only solution satisfying EFF, SYM, ZI and CON.*

Proof. By Definition 1, $\overline{\eta}$ satisfies EFF, SYM and ZI. The remaining proofs follow from Theorem 1 and Lemmas 1, 2. ■

The following examples are to show that each of the axioms used in Theorems 1 and 2 is logically independent of the remaining axioms.

Example 1 *Define a solution ϕ by for all $(N,w) \in FMIG$ and for all $i \in N$,*

$$\phi_i(N,w) = [0,0].$$

Clearly, ϕ satisfies CON, but it violates STPG.

Example 2 *Define a solution ϕ by for all $(N,w) \in FMIG$ and for all $i \in N$,*

$$\phi_i(N,w) = \begin{cases} \overline{\eta}_i(N,w) & , \text{ if } |N| \le 2, \\ \overline{\eta}_i(N,w) - [\varepsilon,\varepsilon] & , \text{ otherwise.} \end{cases}$$

where $\varepsilon \in \mathbb{R} \setminus \{0\}$. Clearly, ϕ satisfies STPG, but it violates CON.

Example 3 *Define a solution ϕ by for all $(N,w) \in FMIG$ and for all $i \in N$,*

$$\phi_i(N,w) = \sigma_i(N,w) + \frac{1}{|N|} \cdot \Big[w(N) - \sum_{k \in N} \sigma_k(N,w) \Big],$$

where $\sigma_i(N,w) = \frac{1}{2^{|N|-1}} \sum_{S \subseteq N \setminus \{i\}} [w(S \cup \{i\}) - w(S)]$. Clearly, ϕ satisfies EFF, SYM, ZI, but it violates CON.

Example 4 *Define a solution ϕ by for all $(N,w) \in FMIG$ and for all $i \in N$,*

$$\phi_i(N,w) = \frac{w(N)}{|N|}.$$

Clearly, ϕ satisfies EFF, SYM, CON, but it violates ZI.

Example 5 *Define a solution ϕ by for all $(N,w) \in FMIG$ and for all $i \in N$,*

$$\phi_i(N,w) = w(N) - w(N \setminus \{i\}).$$

Clearly, ϕ satisfies SYM, ZI, CON, but it violates EFF.

Example 6 *Define a solution ϕ by for all $(N,w) \in FMIG$ and for all $i \in N$,*

$$\phi_i(N,w) = \eta_i(N,w) + \frac{r_i}{\sum_{k \in N} r_k} \cdot \Big[w(N) - \sum_{k \in N} \eta_k(N,w) \Big],$$

where $R = \{r_t \mid t \in U\}$ be a collection of positive real numbers. Clearly, ϕ satisfies EFF, ZI, CON, but it violates SYM.

References

Alparslan Gök S.Z., Branzei R., Tijs S. (2008) Cores and stable sets for interval-valued games. Preprint No 90, Institute of Applied Mathematics, METU, 2008 and Tilburg University, Center for Economic Research, The Netherlands, Discussion Paper, 17-2008.

Alparslan Gök S.Z., Miquel S., Tijs S. (2009) Cooperation under interval uncertainty. Mathematical Methods of Operations Research 69:99-109

Alparslan Gök S.Z., Branzei R., Tijs S. (2010) The interval Shapley value: an axiomatization. Central European Journal of Operations Research 18:131-140

Branzei R., Branzei O., Alparslan Gök S.Z., Tijs S. (2010) Cooperative interval games: a survey. Central European Journal of Operations Research 18:397-411

Branzei R., Dimitrov D., Tijs S. (2003) Shapley-like values for interval bankruptcy games. Economics Bulletin 3:1-8

Branzei R., Dimitrov D., Tijs S. (2004) How to cope with division problems under interval uncertainty of claims. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 12:191-200

Harsanyi J.C. (1959) A bargaining model for the cooperative n-person game. In Contributions to the Theory of Games IV (Annals of Mathematics Studies 40), e.d. by A.w. Tucker and R.D. Luce Princeton: Princeton University Press, 325-355

Hart S., Mas-Colell A. (1989) Potential, value and consistency. Econometrica 57:589-614

Hwang Y.A. (2009) An NTU value under complement reduced game. International Journal of Game Theory 38:305-324

Moore R. (1979) Methods and applications of interval analysis. SIAM Philadelphia

Moulin H. (1985) The separability axiom and equal-sharing methods. Journal of Economic Theory 36:120-148