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Weighted allocation rules for standard fixed tree games*

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Abstract. In this paper we consider standard fixed tree games, for which each vertex unequal to the root is inhabited by exactly one player. We present two weighted allocation rules, the weighted down-home allocation and the weighted neighbour-home allocation, both inspired by the *painting story* in Maschler et al. (1995) . We show, in a constructive way, that the core equals both the set of weighted down-home allocations and the set of weighted neighbour allocations. Since every weighted down-home allocation specifies a weighted Shapley value (Kalai and Samet (1988)) in a natural way, and vice versa, our results provide an alternative proof of the fact that the core of a standard fixed tree game equals the set of weighted Shapley values. The class of weighted neighbour allocations is a generalization of the nucleolus, in the sense that the latter is in this class as the special member where players have all equal weights.

Key words: Cooperative games, Tree games, Core, Weighted Shapley value, Nucleolus

JEL Classification: C71

1 Introduction

We consider cost sharing problems arising from *standard fixed tree enterprises*. There is a fixed and finite set of agents connected to a source through a fixed tree network. We seek to allocate the cost of this tree for cases where maintaining the connections within the network is costly. Many real-life situations can be modelled to fit in this general setting. For instance, consider the problem of allocating the maintenance cost of an irrigation network or a

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cablevision network, setting airport taxes for planes or setting dredging fees for ships. In a natural way each standard fixed tree problem gives rise to a *standard fixed tree game*, which relates each coalition of agents/players to the minimal expenses for maintaining the connections of all its members to the source. This makes it possible to investigate this type of problems with techniques from cooperative game theory. The same problem is studied in Megiddo (1978) , Koster et al. (2001) whereas Granot et al. (1996) and Maschler et al. (1995) study a generalization, where more than one player is allowed to occupy each vertex. A special case, where the underlying structure of the game is a chain, is also known as the airport problem and considered by Littlechild (1974) , Littlechild and Owen (1977), Littlechild and Thompson (1977) , Dubey (1982) , Potters and Sudhölter (1999), and Aadland and Kolpin (1998).

We are concerned with the core of the standard fixed tree game, and in section 3 we give some results on the core, essentially the same as in Koster *et al.* (2001) . Inspired by the *painting story* presented by Maschler et al. (1995) we introduce, in section 4, the *weighted down-home allocation*, where each player is allocated a share, according to his relative weight, of the cost of each arc along the path from the (local) source to his home. We show, by explicitly characterizing the corresponding weight system, that each core element can be obtained as a weighted down-home allocation. Especially, the core element as determined by the Shapley value corresponds to the weighted down-home allocation with equal weights to all players. Moreover, each weighted down-home allocation is equal to a weighted Shapley value, and therefore our results provide an alternative and constructive proof of the result in Monderer et al. (1992) , where it is shown that the core of a concave game, and so also the core of fixed tree games, equals the set of weighted Shapley values. In section 5 we introduce the *weighted neighbour-home allocation*, a generalization of the scheme in Maschler et al. (1995) for computing the nucleolus, and show that the set of weighted neighbour-home allocations equals the core. The weighted neighbour-home allocation is equal to the nucleolus in the special case where all players are given equal weight. But first, in section 2, we formally define the standard fixed tree problem and its corresponding game, and introduce necessary notation.

2 The fixed tree maintenance problem: the model and its game

In this paper we consider a *maintenance problem* $\mathcal{G} := (G, c, N)$. Here $G = (V, E)$ is a tree, i.e. a directed connected graph without cycles, with vertex set V and arc set E. The set V contains a distinguished vertex. We denote this vertex by r and refer to it as the *source*. The function $c : E \to \mathbb{R}_+$, called *cost function*, associates with each arc e a cost $c(e)$. It can be interpreted as the cost to maintain e. $N = \{1, 2, \ldots, n\}$ is a fixed and finite set of players. The players are located at the vertices $V \backslash \{r\}$, and it is assumed that at each vertex there is exactly one player[1]. The players find themselves connected to the source

[1] Basically, our approach differs from *standardness* as in Maschler *et al.* (1995), by assuming precisely one player per vertex outside the source. However, this assumption is not essential for any of our results. See Koster et al. (2001) for details.

through the costly arcs in E. The problem under consideration is to divide the cost of the complete network $\sum_{e\in E} c(e)$ among the players in N. A *vector of cost shares* is by definition a vector $x \in \mathbb{R}^N$ such that $\sum_{i\in N} x_i = \sum_{e\in E} c(e)$. Here x_i represents the amount that player i has to pay according to x. Below we will use the notation $x(S)$ to express the aggregate payments of a coalition of players $S \subseteq N$, i.e. $\sum_{i\in S} x_i$. $\Delta(S)$ stands for the unit simplex in $\mathbb{R}^S_+$, i.e. the set of all vectors $y \in \mathbb{R}^S_+$ such that $y(S) = 1$. In the sequel we identify vertices with players ($V = N \cup \{r\}$). For any subgraph G' of G, we will let $E(G')$ and $V(G')$ denote the corresponding arc set and vertex set, respectively. Sometimes we will also denote the player set corresponding to G' by $N(G') \subseteq V(G')$. For each vertex $i \in N$ there is a unique path P_i from the source to vertex i. If $V(P_i)$ consists of the vertices $j_0 = r, j_1, \dots, j_q = i$, then j_{q-1} is called the *predecessor* $\pi(i)$ of vertex i. In this fashion by $\pi^{-1}(i)$ we denote the set $\{j \in V \mid \pi(j) = i\}$. We put $N(P_i) := V(P_i) \setminus \{r\}$. We denote by e_i the arc $(\pi(i), i)$, and we will sometimes write $c_i := c(e_i)$. The *precedence relation* $(V, \preceq)$ on the set of vertices and/or players is defined by $i \preceq j$ if and only if $i \in V(P_j)$. Analogously we define the precedence relation $(E, \preceq)$ on the arcs. In this way, the arcs are considered to be directed away from the source. A *trunk* of $G = (V, E)$ is a set of vertices $T \subseteq N$, which is closed under the precedence relation defined above, i.e. if $i \in T$ and $j \preceq i$, then $j \in T$. Let the *followers* of a vertex i be denoted by $F(i) := \{j \in N : i \preceq j\}$. A vertex i is called a *leaf* if $F(i) = \{i\}$. If $e = (i, j)$, then B_e is the *branch* at i in the direction of j, as in Maschler et al. (1995), so $V(B_e) := \{i\} \cup F(j)$, $N(B_e) := F(j)$ and $E(B_e) := \{(k, \ell) \in E : k, \ell \in V(B_e)\}$. With each problem $\mathcal{G} = (G, c, N)$ can be associated a *cost game* $(N, c_{\mathcal{G}})$, where the cost $c_{\mathcal{G}}(S)$ of each coalition S is defined as the minimal cost needed to maintain all connections of the members of S to the source via a connected subgraph of (V, E), i.e.

$$c_{\mathcal{G}}(S) = \sum_{i\in T_S} c(e_i) \quad \text{for all } \emptyset \neq S \subseteq N \tag{1}$$

where $T_S = \{i \in N \mid \text{there is } j \in S \text{ with } i \preceq j\}$, and $c_{\mathcal{G}}(\emptyset) = 0$. So think of T_S as the smallest trunk containing S. In order to prove some of our results, we will need to represent our cost game using the basis $\{(N, u^*_S)\}_{S\subseteq N}$ of dual unanimity games. The game (N, u^*_S) is defined by

$$u^*_S(T) = \begin{cases} 1 & \text{if } S \cap T \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

When there is no confusion about the set of players then in the sequel we will slightly abuse notation by abbreviating a game (N, c) by c.

It is known (see Koster et al. (2001)), that if $\mathcal{G} = (G, c, N)$ is a maintenance problem, then the associated cost game $(N, c_{\mathcal{G}})$, or just $c_{\mathcal{G}}$, can be represented as a linear combination of dual unanimity games as follows:

$$c_{\mathcal{G}} = \sum_{e\in E} c(e) u^*_{N(B_e)}. \tag{2}$$

3 The core of a maintenance game

We seek to allocate the the total cost $c_{\mathcal{G}}(N)$ of maintaining the network corresponding to $\mathcal{G}$. A *vector of cost shares* or *cost vector* for the game $c_{\mathcal{G}}$ is a vector $x \in \mathbb{R}^{\mathbb{N}}$ such that $\sum_{i \in N} x_i = c_{\mathcal{G}}(N)$. The following notation will be useful. In the sequel, for $y \in \mathbb{R}^N$, $y^S \in \mathbb{R}^{\mathbb{S}}$ is the restriction of y to $S \subseteq N$, and $y(S) := \sum_{i \in S} y_i$. A solution concept for a class of cost games is no more than a correspondence that assigns to each of the cost games under consideration a subset of vectors of cost shares. In particular, the set of all cost shares for a game defines a solution concept. Nevertheless, it will be not very appealing since it is fairly large in general. In this paper we restrict ourselves to those vectors of cost shares that satisfy *collective stability*. This means that a specific vector of cost shares x will remain an eligible candidate for a solution concept only if it is stable with respect to coalitional deviation. In the present setting of maintenance games this means that for each coalition $S \subset N$ the aggregate payment of coalition S, i.e. $x(S)$, is smaller than the cost of maintaining the network that coalition S needs, i.e. $c_{\mathcal{G}}(S)$. The set of all such vectors of cost shares is called the *core* of the game $(N, c_{\mathcal{G}})$. So formally the definition reads as follows:

Definition 3.1. The *core* of a cost game (N, g) is the set

$$C(g) := \{x \in \mathbb{R}^N \colon x(S) \leq g(S) \text{ for all } S \subset N, x(N) = g(N)\}.$$

It is a well-known fact that each standard fixed tree games is *concave*[2] (Granot *et al.* (1996)) and that for this reason (Shapley (1971)) the corresponding core is nonempty. So in particular this is true for each member in the subclass of all maintenance games. Characterizations of the core of the game $(N, c_{\mathcal{G}})$ are found in Koster et al. (2001). The next proposition summarizes these results and adds a characterization of the core in terms of *overflows*.

Given a maintenance problem $\mathcal{G} = (G, c, N)$ and a cost vector x, we define the *overflow* over the arc $e \in E(G)$ as the amount that the members of $N(B_e)$, i.e. the inhabitants of the branch B_e, pay in excess of the cost of the arcs of B_e. Formally, the definition of the overflow over e is given by

$$O_e(x) := \sum_{i \in N(B_e)} x_i - \sum_{f \in E(B_e)} c(f) = \sum_{i \in N(B_e)} (x_i - c_i).$$

If $e = (i, j)$, we will sometimes write $O_j(x)$ instead of $O_e(x)$, and it is easily seen that

$$O_j(x) = \sum_{\ell \in F(j)} (x_\ell - c_\ell) = (x_j - c_j) + \sum_{\ell \in \pi^{-1}(j)} O_\ell(x). \quad (3)$$

Proposition 3.2. *Let* $x \in \mathbb{R}^N$. *Then the following statements are equivalent:*

(*i*) $x \in C(c_{\mathcal{G}})$,
(*ii*) $x(N) = c_{\mathcal{G}}(N), x \geq 0$, *and* $x(T) \leq c_{\mathcal{G}}(T)$ *for every trunk* T,
(*iii*) $x(N) = c_{\mathcal{G}}(N)$, $x \geq 0$, *and* $O_e(x) \geq 0$ *for all* $e \in E$,
(*iv*) *There exist* $y^e \in \Delta(N(B_e))$ *for all* $e \in E$, *such that* $x_i = \sum_{e \in E(P_i)} y_i^e c(e)$ for all $i \in N$.

[2] A cost game (N, c) is *concave* if for all $i \in N$, $S \subset T \subset N \backslash \{i\}$ it holds that $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$. So the marginal costs of adding a player i to a coalition decrease with its cardinality.

Proof. These results essentially appear as Propositions 3.1 ($(i) \Leftrightarrow (ii)$), 3.2 ($(ii) \Leftrightarrow (iii)$), and 3.3 ($(i) \Leftrightarrow (iv)$) in Koster et al. (2001). □

Definition 3.3. A *pseudo subtree* of a tree $G = (V, E)$ is a connected subgraph $G' = (V', E')$ such that there exists an $r' \in V$ such that

(a) r' is the minimal element in V' with respect to $\preceq$,
(b) there is exactly one vertex in V' that has r' as predecessor.

Definition 3.4. A pseudo subtree $G' = (V', E')$ of G rooted at r' yields a *restricted maintenance problem* $\Gamma' = (G', c', N')$ where c' is the restriction of c to E' and $N' = V' \backslash \{r'\}$.Let $\mathcal{T} = (G^1, \ldots, G^p)$ be an ordered collection of pseudo subtrees of G. Then $\mathcal{T}$ is said to be a *partition of G into pseudo subtrees* if and only if

(a) for all $k = 1, \ldots, p$, there exists $r_k \in V(G^k)$ such that G^k is the pseudo subtree of G rooted at r_k,
(b) $(N(G^1), \ldots, N(G^p))$ is a partition of N.

Given an allocation vector x, let $E(x) := \{e \in E : O_e(x) > 0\}$. The graph $(V, E(x))$ contains p connected subgraphs, where $1 \leq p \leq n$. For each of these subgraphs, $1 \leq k \leq p$, we construct a pseudo subtree G^k with player set $N(G^k)$. Let $r_k \in V \setminus N(G^k)$ be such that $r_k \in V(P_i)$ for every $i \in N(G^k)$, and $r_k = \pi(i)$ for exactly one $i \in N(G^k)$. Let $V(G^k) := N(G^k) \cup \{r_k\}$ and $E(G^k) := \{e = (i, j) : i, j \in V(G^k)\}$. Then $G^k := (V(G^k), E(G^k))$ is a pseudo subtree rooted at r_k, and $\mathcal{T}(x) := (G^1, \ldots, G^p)$ is a partition of G into pseudo subtrees. We will refer to $\mathcal{T}(x)$ as the partition of G *induced by* x.

Example 3.5. Consider the maintenance problem $\mathcal{G} = (G, c, N)$ described by Fig. 1, where the arc weights are given by $c(e) := 10$ for all $e \in E$.

The allocation $x = (4, 5, 15, 16)$ is a core element, and the corresponding overflows are indicated next to the arcs in the figure. By removing all the arcs with zero overflows, we obtain the partition of G into the pseudo subtrees G^1 and G^2, where $N(G^1) = \{1, 4\}$, $N(G^2) = \{2, 3\}$, $r_1 = r$, and $r_2 = 1$.

For any $i \in N$, let $1 \leq k(i) \leq p$ be such that $i \in N(G^{k(i)})$. Let, for any $i \in N$, $\tilde{F}(i) := F(i) \cap V(G^{k(i)})$. For $1 \leq k \leq p$ and $e \in E(G^k)$, let $\tilde{B}_e$ be defined such that $V(\tilde{B}_e) := V(B_e) \cap V(G^k)$, $N(\tilde{B}_e) := N(B_e) \cap N(G^k)$, and $E(\tilde{B}_e) := E(B_e) \cap E(G^k)$. In an analogous manner, for $1 \leq k \leq p$ and $i \in V(G^k)$, define $\tilde{P}_i$. We will write $\tilde{O}_e(x) := \sum_{i \in N(\tilde{B}_e)} (x_i - c_i)$.

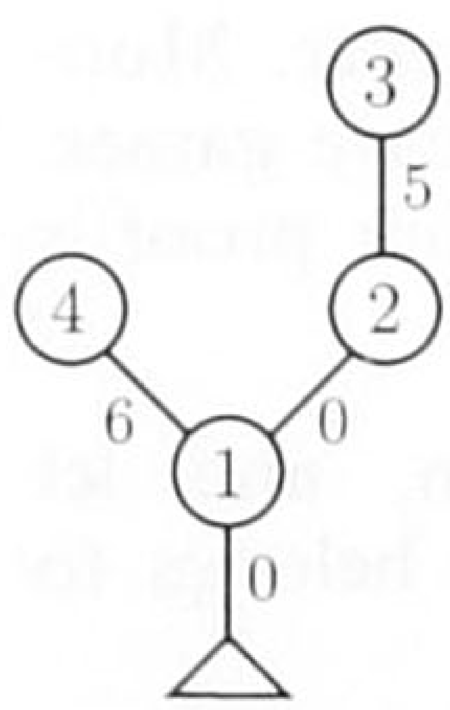

Fig. 1. The maintenance problem for Example 3.5

Proposition 3.6. *(i) Let $\mathcal{T}$ be a partition of G into pseudo subtrees. Then $\prod_{k=1}^{p} C(c_{\mathcal{G}^k}) \subseteq C(c_{\mathcal{G}})$, where $(N(G^k), c_{\mathcal{G}^k})$ is the cost game corresponding to the restricted maintenance problem $\mathcal{G}^k$.*

(ii) Let $x \in C(c_{\mathcal{G}})$, and $\mathcal{T} = (G^1, \ldots, G^p) := \mathcal{T}(x)$. Then $x \in \prod_{k=1}^{p} C(c_{\mathcal{G}^k})$.

Proof. (*i*) This result appears as Proposition 3.8(*i*) in Koster et al. (2001). (*ii*) A similar result appears as Proposition 3.8(*ii*) in Koster et al. (2001). In order to prove it, we use the core characterization in Proposition 3(*iii*). Let $1 \le k \le p$. Because $x \in C(c_{\mathcal{G}})$ is efficient with respect to the game $c_{\mathcal{G}}$, and since $\mathcal{T}$ has been constructed by removing only arcs with zero overflows, it is clear that $x^{N(G^k)}$ is efficient with respect to the game $c_{\mathcal{G}^k}$. Also, $x^{N(G^k)} \ge 0$ follows from $x \in C(c_{\mathcal{G}})$ and Proposition 3. We will complete the proof by showing that $\tilde{O}_i(x) = O_i(x) \ge 0$ for all $i \in N(G^k)$, where the inequality follows from $x \in C(c_{\mathcal{G}})$ and Proposition 3(*iii*). Note that, by (3) and the construction of $\mathcal{T}$, $\tilde{O}_i(x) = x_i - c_i = O_i(x)$ for any $i \in N(G^k)$ such that i is a leaf in G^k, since i must either be a leaf in G, or we must have $O_j(x) = 0$ for every $j \in \pi^{-1}(i)$. Then, for every $i \in N(G^k)$ such that i is not a leaf in G^k, $\tilde{O}_i(x) = (x_i - c_i) + \sum_{j \in \pi^{-1}(i) \cap \tilde{F}(i)} \tilde{O}_j(x) = (x_i - c_i) + \sum_{j \in \pi^{-1}(i)} O_j(x) = O_i(x)$. □

4 The core and the set of weighted down-home allocations

Proposition (3) shows that the core of a maintenance game is in general fairly large and multi-valued. In this section the focus is on a particular single-valued solution concept that assigns a single vector of cost shares to each game under consideration: the Shapley value (Shapley (1953)). It is one of the most celebrated single-valued solution concepts for transferable utility games. For concave cost games it has the geometric interpretation of the barycenter of the core. Moreover, for specific classes of games the Shapley value is a very intuitive concept and allows for an elegant expression. Two such classes are airport games and maintenance games (cf. Littlechild and Thompson (1977), Dubey (1982), Koster et al. (2001).

For both mentioned classes of games the Shapley value is calculated by an equal split of the costs of each arc in the network among its users. Here we will slightly change this procedure by the introduction of a dynamical process of uniformly distributing the costs of the network. This serves our goal: to treat the class of weighted Shapley values (Kalai and Samet (1988)). Starting with an arbitrary *weight system* the dynamic procedure results in a *weighted down-home allocation*. From the procedure itself we conclude that this weighted down-home allocation is a core element. Then we conclude that it represents a specific weighted Shapley value. Hence, we immediately obtain the well-known result that the set of all weighted Shapley values is a subset of the core. Moreover, with this dynamic approach we are able to show the converse, that the set of weighted Shapley values is exactly the core. Monderer et al. (1992) show this result for the large class of all concave games. Nevertheless, where their proof needed a fixed point argument, our proof is constructive.

Definition 4.1. Let $\mathcal{G} = (G, c, N)$ be a maintenance problem, and let $\mathcal{B}(\mathcal{G})$ denote the set of *weight systems* for $\mathcal{G}$. Then $\beta := (\mathcal{T}, w)$ belongs to $\mathcal{B}(\mathcal{G})$ if

(a) $\mathcal{T} = (G^1, \ldots, G^p)$ is a partition of G into pseudo subtrees,
(b) $w \in \mathbb{R}^N$, $w_i \geq 0$ for all $i \in N$, and
(c) $w(\bar{F}(i)) > 0$ for all $i \in N$ such that $c_i > 0$.

Consider a maintenance problem $\mathcal{G} = (G, c, N)$ and some weight system $\beta \in \mathcal{B}(\mathcal{G})$. For each pseudo subtree G^k, interpret the vertices in $N(G^k)$ as the homes of the different players and the arcs in $E(G^k)$ as the roads to the community center (r_k). The cost of a road is expressed as the number of days it takes (for one person) to paint the stripes on the road. The work is done by the players themselves according to the following rules[3]

(*i*) Every worker keeps painting as long as the road from the community center to his home has not been completed.
(*ii*) Every worker does his job on an unfinished segment between the community center and his home.
(*iii*) Every worker starts painting at the same moment.
(*iv*) Every worker $i \in N$ paints with velocity w_i.
(*v*) Each worker paints as close to the community center as the rules (*i*)–(*iv*) permit him to.

We call the resulting allocation the *weighted down-home allocation*, and denote it $\delta^\beta(\mathcal{G})$. It is given by, for any player $i \in N$,

$$\delta_i^\beta(\mathcal{G}) = \sum_{e \in E(\tilde{P}_i)} \frac{w_i}{w(N(\tilde{B}_e))} c(e).^4 \qquad (4)$$

So according to the weighted down-home allocation the users of an arc e share the costs $c(e)$ proportionally to their individual weights.[4]

Example 4.2. Consider the maintenance problem $\mathcal{G}$ that is graphically depicted in Fig. 1, where $c(e) := 10$ for every $e \in E$. Let $\mathcal{T} := (G^1, G^2)$ be the partition into pseudo subtrees of $\mathcal{G}$, where $N(G^1) := \{1, 2, 3\}$ and $N(G^2) := \{4\}$, and let $w := (1, 1, 3, 1)$. For $\beta := (\mathcal{T}, w)$ we have $\delta^\beta(\mathcal{G}) = \left(2, 4\frac{1}{2}, 23\frac{1}{2}, 10\right)$. Player 1 only contributes to the cost of arc $(r, 1)$, so his total contribution is $10 \cdot \frac{1}{5} = 2$. Player 2 contributes to the cost of arc $(r, 1)$ and $(1, 2)$, with relative weights of $\frac{1}{5}$ and $\frac{1}{4}$, respectively, so his total contribution is $10 \cdot \frac{9}{20} = 4\frac{1}{2}$. Player 3 contributes at arc $(r, 1)$, $(1, 2)$, and $(2, 3)$, with relative weights of $\frac{3}{5}$, $\frac{3}{4}$, and 1, respectively, hence his total contribution is $10 \cdot \frac{47}{20} = 23\frac{1}{2}$. Player 4 is the only player in his pseudo subtree, and contributes the entire cost of the arc that he uses, i.e. 10.

From Proposition 3.2(*iv*) it follows that each down-home allocation specifies a core-element. But as we are about to show, the converse also holds. For each core element x there is a weight system β such that the corre-

[3] These rules are inspired by the *painting story* presented in Maschler et al. (1995).

[4] Koster et al. (2001) treat the weighted *home-down* allocation, which results by replacing "the community center" in (v) by "his home". The resulting allocation is related to a weighted version of the *constrained egalitarian solution* of Dutta and Ray (1989) (see Koster (2002)).

sponding down-home allocation $\delta^{\beta}(\mathcal{G})$ equals x. We will show how such a weight system β is easily calculated for a given $x \in C(c_{\mathcal{G}})$.

First of all, the partition of the player set is derived from the partition of $\mathcal{G}$ into pseudo subtrees induced by x; this can be done by considering the overflows in the tree. Next the weights for the players are calculated for each such separate subproblem. Without loss of generality, we will assume that the partition into pseudo subtrees of $\mathcal{G}$ with respect to x is trivial, or, equivalently, all the overflows are positive except at the arc that leaves the source. We do this because the following procedures will be the same for each pseudo subtree.

It is assumed that player 1 is the player directly connected to the source. The cost of the corresponding arc e_1 is covered by the collective of players N. Suppose that x is a down-home allocation. Then our objective is, if at all possible, to find a suitable vector of weights w such that for $\beta = (\{N\}, w) \in \mathcal{B}(\mathcal{G})$ we have $\delta^{\beta}(\mathcal{G}) = x$. First of all, with the interpretation of the weights as painting speeds, the arc e_1 is painted in

$$\frac{c(e_1)}{w(N)} = c(e_1) \text{ units of time,}$$

if we assume that w is normalized such that $w(N) = 1$. Moreover, each of the painting players is finished with e_1 at the same time. In particular, if player 1 is painting at all (in case $x_1 > 0$) then he is also painting for $c(e_1)$ units of time. On the other hand he must complete x_1 by himself, at speed w_1, so we have the condition $\frac{x_1}{w_1} = c(e_1)$, and thus $w_1 = \frac{x_1}{c(e_1)}$. Note that $c(e_1) > 0$ by the fact that $x_1 > 0$ and x is a core-element.

After having calculated this first weight, we proceed by consecutively assigning weights to each of the players in the sets $\pi^{-1}(1), \pi^{-1}(\pi^{-1}(1)), \ldots,$ until all the players have a weight. Basically we repeat the above type of reasoning using an induction argument. Consider a player $i \notin \pi^{-1}(1)$. Then, according to x, his followers $F(i)$ contribute $O_i(x) > 0$ to the maintenance cost of the path from the root to his predecessor, player $\pi(i)$. Recall again the painting story. The speed at which the collective of players $F(i)$ operates on the path from r to $\pi(i)$ is given by the aggregate of the weights $w(F(i))$. Then the time that the group of players $F(i)$ needs to complete $O_i(x)$ is given by

$$\frac{O_i(x)}{w(F(i))}.$$

Similarly, it holds that the followers of $\pi(i)$ contribute $O_{F(\pi(i))}(x)$ to the path from the source to $\pi(\pi(i))$ *plus* the full cost of maintaining the arc $(\pi(\pi(i)), \pi(i))$. The collective of players $F(\pi(i))$ paints at speed $w(F(\pi(i)))$, which means that the time that it needs to complete their part of the path from the root to $\pi(i)$ equals

$$\frac{O_{\pi(i)}(x) + c(e_{\pi(i)})}{w(F(\pi(i)))}.$$

Then this indicates the time that each of the individuals in $F(\pi(i))$ is working on the path from r to $\pi(i)$, and especially each of the players in $F(i)$. But then we must have the equality

$$\frac{O_i(x)}{w(F(i))} = \frac{O_{\pi(i)}(x) + c(e_{\pi(i)})}{w(F(\pi(i)))}.$$

This determines an iterative procedure for calculating all the weights $w(F(i))$ for each $i \in F(1) \setminus \{1\}$, since

$$w(F(i)) = w(F(\pi(i))) \frac{O_i(x)}{O_{\pi(i)}(x) + c(e_{\pi(i)})}$$

for all $i \in N$, and

$$w_i = w(F(i)) - \sum_{j \in \pi^{-1}(i)} w(F(j)).$$

Example 4.3. Consider the network $\mathcal{G}$ depicted in Fig. 2.

As in earlier examples the maintenance costs of the different arcs are all 10. Check that $x = (5, 13, 12)$ is a core element for $c_{\mathcal{G}}$. The numbers at the arcs in Figure 2 denote the overflows corresponding to x. Firstly, observe that the partition $\mathcal{T}$ of $\mathcal{G}$ into pseudo subtrees induced by x is trivial. Assume that x is a down-home allocation: there is a vector of weights w with $w_i > 0$ for all $i \in \{1, 2, 3\}$ such that $\delta^{\beta}(\mathcal{G}) = x$ for $\beta = (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$. Recall the painting story for the weighted down-home allocation. The players 1,2, and 3 respectively paint at velocities w_1, w_2, and w_3 respectively at e_1 as long as $c(e_1) = 10$ is not completed. Furthermore, the contribution of player 1 together with the overflows $O_2(x)$ and $O_3(x)$ respectively determine the parts of $c(e_1)$ that are individually covered by the players 1,2 and 3 respectively. Given the velocities we can compute the time that the players need to finish these parts in three ways, as

$$\frac{x_1}{w_1}, \quad \frac{O_2(x)}{w_2}, \quad \text{and} \quad \frac{O_3(x)}{w_3}.$$

These numbers are equal by the fact that all the players will continue painting on e_1 until it is finished, which implies that the finishing time of the collective of players equals the individual finishing times.

Since we are completely informed about the individual contribution of player 1 and the overflows corresponding to each branch emanating from the node of player 1, we must therefore have

$$\frac{5}{w_1} = \frac{3}{w_2} = \frac{2}{w_3},$$

and thus $w = (w_1, \frac{3}{5}w_1, \frac{2}{5}w_1)$. Since w is a vector in the unit simplex, we get $w_1 = \frac{1}{2}$, $w = (\frac{1}{2}, \frac{3}{10}, \frac{2}{10})$. The reader may verify that indeed $\delta^{\beta}(\mathcal{G}) = x$ for $\beta = (\{G\}, w)$.

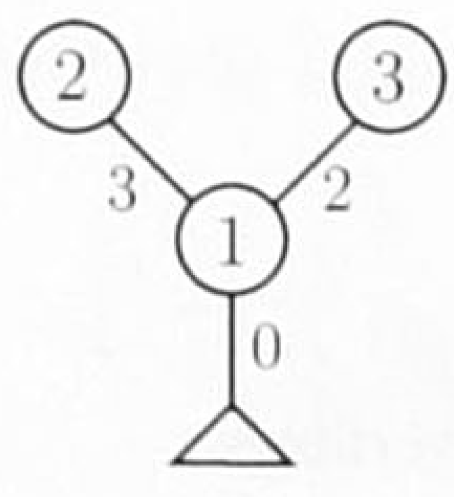

Fig. 2. The tree network corresponding to Example 4.3 with overflows

Example 4.4. Consider the network as in Figure 3. All arcs have equal maintenance cost 10.

Consider the core element $x=(4,12,12,12)$ of the corresponding 4-player maintenance game. The overflows corresponding to x are the numbers to the arcs in Figure 3. The partition into pseudo subtrees by x is (again) trivial. Assume that x is a down-home allocation, i.e. there is a vector $w \in \mathbb{R}^4$ with all positive coordinates such that for $\beta = (\{G\}, w)$ we have $\delta^{\beta}(\mathcal{G}) = x$. We will see that similar reasoning as in the above example 4.3 leads to conditions that determine w. Basically, the only difference with the situation in example 4.3 is that it is not directly clear what are the individual contributions of the players 3 and 4 at e_1. We are only able to monitor their aggregate efforts by means of $O_3(x)$. The same considerations as in the above example lead to the conclusion that players 1,2, and the collective of players 3 and 4 finish in $\frac{x_1}{w_1}, \frac{O_2(x)}{w_2}$ and $\frac{O_3(x)}{w_3+w_4}$ time units, respectively. Since these numbers are all equal we have

$$\frac{4}{w_1} = \frac{2}{w_2} = \frac{4}{w_3 + w_4}.$$

Therefore, at this stage we are able to express w_2 and $w_3 + w_4$ in terms of w_1, i.e. $w_2 = \frac{1}{2}w_1, w_3 + w_4 = w_1$. But now we can calculate w_1 by the equality $w(N) = 1$, i.e. $w_1 = \frac{2}{5}$. This means that we only have to consider w_3 and w_4 since $w_2 = \frac{1}{2} w_1 = \frac{1}{5}$. Consider the path from the root to vertex 3. The players 3 and 4 reach vertex 3 at the same time. The time they need to complete the entire path equals the time for completing e_1 plus the time necessary for completing e_3, i.e.

$$\frac{O_3(x)}{w_3 + w_4} + \frac{c(e_3)}{w_3 + w_4} = \frac{O_3(x) + c(e_3)}{w_3 + w_4}.$$

At this precise moment player 4 has completed exactly $O_4(x)$. Using the velocity of player 4, w_4, therefore the time that player 4 must spend equals $\frac{O_4(x)}{w_4}$ and thus

$$\frac{O_4(x)}{w_4} = \frac{O_3(x) + c(e_3)}{w_3 + w_4}, \text{ so that } \frac{2}{w_4} = \frac{14}{w_3 + w_4} = 35,$$

from which we see that $w_4 = \frac{2}{35}$ and $w_3 = w_1 - w_4 = \frac{2}{5} - \frac{2}{35} = \frac{12}{35}$. Thus $w = \left(\frac{2}{5}, \frac{1}{5}, \frac{12}{35}, \frac{2}{35}\right)$.

Now we will formalize the above ideas. For any core allocation x, we define a weight system $\beta \in \mathcal{B}(\mathcal{G})$ such that $x = \delta^{\beta}(\mathcal{G})$. First, find the partition

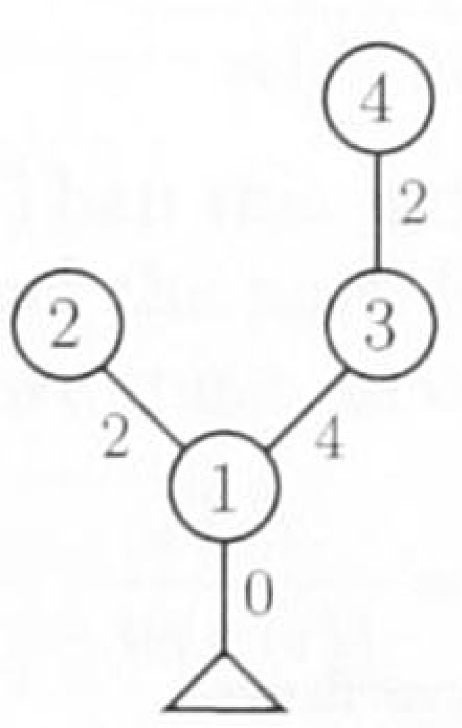

Fig. 3. The tree network corresponding to Example 4.4 with overflows

$\mathcal{T} = (G^1, \ldots, G^p)$ of G into pseudo subtrees induced by x. Then a weight vector w can be found by first, for all $i \in N$, calculating the sums

$$w(\tilde{F}(i)) = \begin{cases} 1 & \text{if } \pi(i) = r_{k(i)}, \\ \frac{\tilde{O}_i(x)}{\tilde{O}_{\pi(i)}(x) + c_{\pi(i)}} w(\tilde{F}(\pi(i))) & \text{else,} \end{cases} \tag{5}$$

in a recursive manner, and then the individual weights are given by

$$w_i = w(\tilde{F}(i)) - \sum_{j \in \tilde{F}(i)} w(\tilde{F}(j)) \quad \text{for all } i \in N. \tag{6}$$

Proposition 4.5. *Let $x \in C(c_{\mathcal{G}})$. There exists $\beta := (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$ such that $x = \delta^\beta(\mathcal{G})$, where $\mathcal{T} = \mathcal{T}(x)$, and w satisfies* (5) *and* (6).

Proof. First we show the existence part. Observe that $\mathcal{T}(x)$ exists and that if, for some $i \in N$, we have $|N(G^{k(i)})| = 1$, then (5) and (6) imply $w_i = 1$. To prove existence for w, it is therefore sufficient to show that

$$\tilde{O}_i(x) + c_i > 0 \text{ for all } i \in N \text{ such that } |N(G^{k(i)})| > 1. \tag{7}$$

Since $c_i \geq 0$ for all $i \in N$, and since $\tilde{O}_i(x) > 0$ for all $i \in N$ such that $\pi(i) \neq r_{k(i)}$, the only possible problem arises if $c_i = 0$ for a player i such that $\pi(i) = r_{k(i)}$. Suppose that this is the case. Then, since, by the construction of $\mathcal{T}$, $x^{N(G^{k(i)})}$ is a vector of cost shares with respect to the game $c_{\mathcal{G}^{k(i)}}$, we must have $\tilde{O}_j(x) = 0$ for all $j \in \pi^{-1}(i) \cap \tilde{F}(i)$, contradicting the fact that $\mathcal{T}$ is induced by x. It holds that $\beta \in \mathcal{B}(\mathcal{G})$: clearly, $\mathcal{T} = (G^1, \ldots, G^p)$ is a partition of G into pseudo subtrees. From (5), (7), and because $\tilde{O}_i(x) > 0$ if $\pi(i) \neq r_{k(i)}$, it follows that

$$w(\tilde{F}(i)) > 0 \quad \text{for all } i \in N. \tag{8}$$

Also, for any $i \in N$, we have from (5) and (6) that

$$\begin{aligned} w_i &= w(\tilde{F}(i)) \left\{ 1 - \sum_{j \in \pi^{-1}(i) \cap \tilde{F}(i)} \frac{\tilde{O}_j(x)}{\tilde{O}_i(x) + c_i} \right\} \\ &= w(\tilde{F}(i)) \frac{\sum_{j \in \tilde{F}(i)} (x_j - c_j) + c_i - \sum_{j \in \tilde{F}(i) \setminus \{i\}} (x_j - c_j)}{\tilde{O}_i(x) + c_i} \\ &= w(\tilde{F}(i)) \frac{x_i}{\tilde{O}_i(x) + c_i} \geq 0. \end{aligned} \tag{9}$$

Here the last inequality follows from (7) and (8), and because Proposition 3.2 and $x \in C(c_{\mathcal{G}})$ imply $x \geq 0$.

Finally we show that $x = \delta^\beta(\mathcal{G})$. For any $i \in N$ it follows from (9) that

$$x_i = w_i \frac{\tilde{O}_i(x) + c_i}{w(\tilde{F}(i))}. \tag{10}$$

For any k such that $1 \leq k \leq p$, and $i \in N(G^k)$, define the number

$$t_j^k := \begin{cases} 0 & \text{if } i = r_k, \\ \frac{\tilde{O}_i(x) + c_i}{w(\tilde{F}(i))} & \text{else.} \end{cases} \tag{11}$$

From this definition follows, for any $i \in N(G^k)$, that $t_i^k = \sum_{j \in N(\bar{P}_i)} (t_j^k - t_{\pi(j)}^k)$. Also, by (10) we have $x_i = w_i t_i^k$ for all $i \in N(G^k)$. We will complete the proof by showing that $t_j^k - t_{\pi(j)}^k = \frac{c_j}{w(\tilde{F}(j))}$ for all $j \in N(G^k)$, and by referring to the definition given in (4). If $\pi(j) = r_{k(j)}$, then $\tilde{O}_j(x) = 0$, so the result follows from (11). Else

$$t_j^k - t_{\pi(j)}^k = \frac{\tilde{O}_j(x) + c_j}{w(\tilde{F}(j))} - \frac{\tilde{O}_{\pi(j)}(x) + c_{\pi(j)}}{w(\tilde{F}(\pi(j)))} = \frac{\tilde{O}_j(x) + c_j - \tilde{O}_j(x)}{w(\tilde{F}(j))} = \frac{c_j}{w(\tilde{F}(j))},$$

where the second equality follows from (5). □

Theorem 4.6. *The core of the maintenance game $(N, c_{\mathcal{G}})$ equals the set of down-home allocations, i.e.* $\{\delta^\beta(\mathcal{G}) \mid \beta \in \mathcal{B}(\mathcal{G})\} = C(c_{\mathcal{G}})$.

Proof. That the weighted down-home allocations form a superset of $C(c_{\mathcal{G}})$ follows from Proposition 4.5. To show the inclusion, suppose $\beta \in \mathcal{B}(\mathcal{G})$. The proof is complete by first noting that $\delta^\beta(\mathcal{G}^k) \in C(c_{\mathcal{G}^k})$ for every $k = 1, \ldots, p$ by (iv) in Proposition 3.2, and from Proposition 3.6(i). □

In Monderer et al. (1992), it is shown by a non-constructive proof that the set of weighted Shapley values equals the core for convex games. Here we show this result for maintenance games in a constructive way. But first we will develop the notion of the weighted Shapley values.

Definition 4.7. Call a weight system for the game (N, g) a pair $\mu := (\mathcal{S}, \lambda)$, where $\mathcal{S} = (S_1, \ldots, S_q)$ is an ordered partition of the player set N, and $\lambda^{S_\ell} \in \mathbb{R}_{++}^{S_\ell} \cap \Delta(S_\ell)$ for all $\ell = 1, \ldots, q$. Let $\mathcal{M}(g)$ be the set of all such weight systems for the game (N, g).

Definition 4.8. Take a weight system $\mu = (\mathcal{S}, \lambda)$ for all games in Γ^N, i.e. all games with player set N. The corresponding weighted Shapley value is the linear function $\Phi^\mu : \Gamma \to \mathbb{R}^N$ that is defined for each unanimity game u_S as follows. Let $m(S) := \min\{j : S_j \cap S \neq \emptyset\}$, and let $\bar{S} := S \cap S_{m(S)}$. Then

$$(\Phi^\mu)_i(u_S) = \begin{cases} \frac{\lambda_i}{\lambda(\bar{S})} & \text{if } i \in \bar{S}, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Note that with the unanimity games as a basis for Γ^N, this completes the definition of Φ^μ.

In the unanimity game u_S, the importance of the players depend on how they are "ranked", i.e. where they are located in the ordered collection $\mathcal{S}$. In a cost game the most important players, i.e. those in $\bar{S}$, carry the entire cost. In the case of our cost game $c_{\mathcal{G}}$, because of (2), we only need to consider the (dual) unanimity games corresponding to users of arcs, i.e. the games $u^*_{N(B_e)}$ for all $e \in E$. If, for some $e \in E$ and $i \in N$, we have $i \in N(B_e)$, we say that i *is present* at e. For some $e \in E$, let

$$S(e) := N(B_e) \cap S_{\min\{j : N(B_e) \cap S_j \neq \emptyset\}},$$

and if $i \in S(e)$, we say that i is a *senior player at e*. If i is present, but not a senior player, at e, there must exist some $j \neq i$ such that $j \in S(e)$, and we say that i *is dominated by* j *at* e. The weighted Shapley value for a maintenance game is given by the value of the dual unanimity game for each arc,

$$(\Phi^{\mu})_i(u^*_{N(B_e)}) = \begin{cases} \frac{\lambda_i}{\lambda(S(e))} & \text{if } i \in S(e), \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Example 4.9. Consider the example illustrated in Figure 4, where $c(e) := 10$ for all $e \in E$.Let $\mathcal{S} := (\{2,3\},\{1,4,5\})$ and $\lambda := (\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{2})$, hence $\mu = (\mathcal{S},\lambda) \in \mathcal{M}(c_{\mathcal{G}})$. The corresponding weighted Shapley value is $\Phi^{\mu}(c_{\mathcal{G}}) = (0,15,15,10,10)$. Player 1 pays nothing, since he is not among the senior players at any arc. Players 2 and 3 dominate all other players at arc $(r,1)$, and since they both have the same weight, they both pay 5 here. Only player 2 is present at $(1,2)$, so he pays for this arc alone. Since he is not present at any other arc except $(r,1)$, his total contribution is $5+10=15$. Player 3 dominates all other players at $(1,3)$, and since he is not present at any other arc except $(r,1)$, his total contribution is $5+10=15$. Players 4 and 5 are dominated by other players at all arcs where they are present, except at the arcs e_4 and e_5, respectively, where they make up the entire set of senior players, and therefore they contribute 10 each.

Theorem 4.10. (*i*) *For any* $\beta \in \mathcal{B}(\mathcal{G})$, *there exists* $\mu \in \mathcal{M}(c_{\mathcal{G}})$ *such that* $\Phi^{\mu}(c_{\mathcal{G}}) = \delta^{\beta}(\mathcal{G})$.

(*ii*) For any $\mu \in \mathcal{M}(c_{\mathcal{G}})$, *there exists* $\beta \in \mathcal{B}(\mathcal{G})$ such that $\Phi^{\mu}(c_{\mathcal{G}}) = \delta^{\beta}(\mathcal{G})$.

Proof. (*i*) Let $\beta = (\mathcal{T},w) \in \mathcal{B}(\mathcal{G})$ for some maintenance problem $\mathcal{G}$. Note that the elements of $\mathcal{T} = (G^1,\ldots,G^p)$ can be ordered arbitrarily without affecting $\delta^{\beta}(\mathcal{G})$, and we choose an ordering such that $k(i) < k(j) \Rightarrow j \notin N(P_i)$ for any pair $i,j \in N$. Let, for every $k = 1,\ldots,p,$, $S_k := \{i \in N(G^k) \mid w_i > 0\}$ and $S_{p+k} := \{i \in N(G^k) \mid w_i = 0\}$. The ordered collection $(S_1,\ldots,S_p,S_{p+1},\ldots,S_{2p})$ contains q nonempty elements, where $p \le q \le 2p$, and let $\mathcal{S} := (S_1,\ldots,S_q)$ be the ordered collection obtained by deleting the empty elements. Also, for every $i \in N$, let

$$\lambda_i := \begin{cases} \frac{w_i}{w(S_{\ell(i)})} & \text{if } w_i > 0, \\ \frac{1}{|S_{\ell(i)}|} & \text{otherwise,} \end{cases} \tag{14}$$

where $\ell(i) = \ell$ if and only if $i \in S_\ell$. It is easily seen that $\mu := (\mathcal{S},\lambda) \in \mathcal{M}(\mathcal{G})$. For any $i \in N$, we have

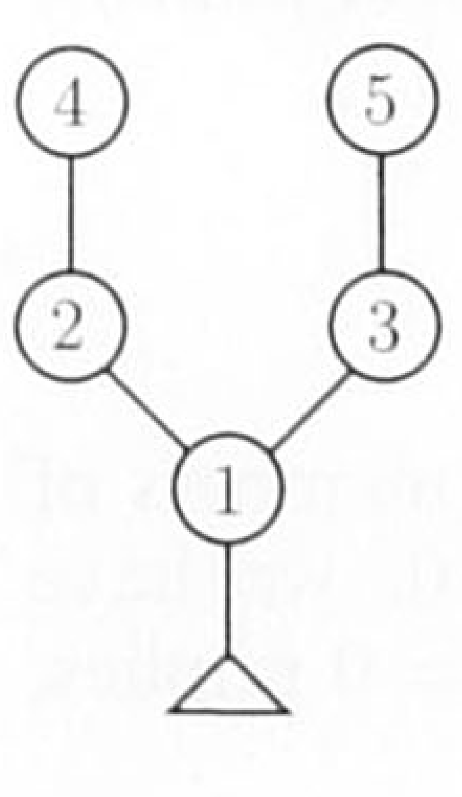

Fig. 4. The maintenance problem as in Example 4.9

$$\Phi_i^{\mu}(c_{\mathcal{G}}) = \sum_{\substack{e \in E \\ S(e) \ni i}} \frac{\lambda_i}{\lambda(S(e))} c(e) = \sum_{\substack{e \in E(\tilde{P}_i) \\ S(e) \ni i}} \frac{\lambda_i}{\lambda(S(e))} c(e)$$

$$= \sum_{\substack{e \in E(\tilde{P}_i) \\ S(e) \ni i}} \frac{w_i}{w(S(e))} c(e) = \sum_{e \in E(\tilde{P}_i)} \frac{w_i}{w(N(\tilde{B}_e))} c(e) = \delta_i^{\beta}(\mathcal{G}).$$

The first equality follows from (2), the additivity of the weighted Shapley value, and (13). The second equality follows from the fact that we can have $i \in S(e)$ only if $e \in E(\tilde{P}_i)$. Suppose, on the contrary, that $i \in S(e)$ for some $e \in E \setminus E(\tilde{P}_i)$. Since we can have $i \in S(e)$ only if i is present at e, we must have $e \in E(P_i)$. Then, by the construction of $\mathcal{S}$, we must have $N(B_e) \cap S_j \neq \emptyset$ for some $j < \ell(i)$, implying $i \notin S(e)$, a contradiction. In order to prove the third equality, it is sufficient to show that if $i \in S(e)$ for some $i \in N$ and $e \in E(\tilde{P}_i)$ such that $c(e) > 0$, then $\lambda_i = \frac{w_i}{w(S_{\ell(i)})}$, and hence $\lambda(S(e)) = \frac{w(S(e))}{w(S_{\ell(i)})}$. Suppose that this is not true. Then $w_i = 0$ by (14), and $\beta \in \mathcal{B}(\mathcal{G})$ implies that there exists some $j \in N(\tilde{B}_e)$ such that $w_j > 0$. Then, by the construction of $\mathcal{S}$, $i \notin S(e)$, a contradiction. The fourth equality follows because, for any $e \in E$ and $i \in N$, $i \in N(\tilde{B}_e) \setminus S(e)$ implies $w_i = 0$ (by the construction of $\mathcal{S}$), so $w(N(\tilde{B}_e)) = w(S(e))$. The last equality follows from (4). (*ii*) Let $\mu = (\mathcal{S}, \lambda) \in \mathcal{M}(c_{\mathcal{G}})$ for some maintenance problem $\mathcal{G}$. We construct $\mathcal{T}$ by applying algorithm 4.11.

Algorithm 4.11.

Initialization
Let $S'_m := S_m$ for every $m = 1, \dots, q$, $w := \lambda$, and $\ell := 1$.

Main step
Repeat
    For $i \in S'_\ell$ **do**
        For $j \in N(P_i)$ **do**
            If $\ell(j) > \ell(i)$ **then**
                $S'_{\ell(i)} := S'_{\ell(i)} \cup \{j\}$
                $S'_{\ell(j)} := S'_{\ell(j)} \setminus \{j\}$
                $w_j := 0$
    $\ell := \ell + 1$
until $\ell > q$

The algorithm will give as output the ordered set of coalitions $S'_1, \dots, S'_q$. Suppose that this ordered set has q' nonempty members. Delete the empty members, and for every $1 \leq \ell \leq q'$, let $G_1^\ell, \dots, G_{i_\ell}^\ell$ be the collection of pseudo subtrees corresponding to maximal connected, with respect to G, components of S'_ℓ. Clearly, the ordered set

$$G_1^1, \dots, G_{i_1}^1, G_1^2, \dots, G_{i_2}^2, \dots, G_1^{q'}, \dots, G_{i_{q'}}^{q'}$$

is a partition of G into pseudo subtrees. Let p be the number of members of this partition, re-index, and set $\mathcal{T} := (G^1, \dots, G^p)$. Since $\lambda \geq 0$, we have $w \geq 0$. $\beta := (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$ then follows, since, for any $i \in N$, $w_i = 0$ implies, by algorithm 4.11, that there exists some $j \in \tilde{F}(i) \setminus \{i\}$ such that $w_j > 0$. Now, for every $i \in N$,

$$\delta_i^{\beta}(\mathcal{G}) = \sum_{e \in E(\tilde{P}_i)} c(e) \frac{w_i}{w(N(\tilde{B}_e))} = \sum_{\substack{e \in E(\tilde{P}_i) \\ S(e) \ni i}} c(e) \frac{\lambda_i}{\lambda(S(e))}$$

$$= \sum_{\substack{e \in E(P_i) \\ S(e) \ni i}} c(e) \frac{\lambda_i}{\lambda(S(e))} = \sum_{\substack{e \in E \\ S(e) \ni i}} c(e) \frac{\lambda_i}{\lambda(S(e))} = \phi_i^{\mu}(c_{\mathcal{G}}).$$

The first equality follows from (4), and the second equality from the fact that $w_i = 0$ if $i \notin S(e)$ for some $e \in E(\tilde{P}_i)$, and since $w(N(\tilde{B}_e)) = \lambda(S(e))$ for every $e \in E$. To see that the latter equality is correct, consider some $e \in E$. After applying algorithm 4.11, the vertices in $N(P_j) \cap N(B_e)$ will be included in $S'_{\ell(j)}$ for every $j \in S(e)$. Hence the vertex set $\cup_{j \in S(e)}(N(P_j) \cap N(B_e))$ will be connected, with respect to G, and we must therefore have $S(e) \subseteq N(\tilde{B}_e)$. Also, $j \in N(\tilde{B}_e) \setminus S(e)$ implies $w_j = 0$, and $j \in S(e)$ implies $w_j = \lambda_j$, hence the desired result. The third equality follows because $e \in E(P_i) \setminus E(\tilde{P}_i)$ implies, from algorithm 4.11, that $N(B_e) \cap S_j \neq \emptyset$ for some $j < \ell(i)$, i.e. i is dominated by the members of S_j ($i \notin S(e)$). The fourth equality follows because $e \in E \setminus E(P_i)$ implies that i is not present at e, hence $i \notin S(e)$, and the last equality follows from (2), the additivity of the weighted Shapley value, and (13). □

Example 4.12. Consider the maintenance problem in Figure 5, and the weight system $\beta = (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$, where $\mathcal{T} = (G^1, G^2)$ and $w = (1, 1, 3, 1)$. Here, the corresponding $\mu = (\mathcal{S}, \lambda) \in \mathcal{M}(c_{\mathcal{G}})$ is uniquely given by $\mathcal{S} := (\{1, 2, 3\}, \{4\})$ and $\lambda = \left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1\right)$.

Example 4.13. Consider the maintenance game in example 4.9 (see the below Figure 6) and the weight system $\mu = (\mathcal{S}, \lambda) \in \mathcal{M}(c_{\mathcal{G}})$, where $\mathcal{S} = (\{2, 3\}, \{1, 4, 5\})$ and $\lambda = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}\right)$. By applying algorithm 4.11, we obtain the partition $\mathcal{S}' = (\{1, 2, 3\}, \{4, 5\})$ of the player set, and the weight vector $w = \left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}\right)$. Note that player 1 has been absorbed by the partition member containing 2 and 3, since 1 is dominated by these two players, and that, accordingly, his weight is now zero. By taking maximal connected subsets of each partition member, we obtain a partition of G into pseudo subtrees, equal to $\mathcal{T} = (G^1, G^2, G^3)$, where $N(G^1) = \{1, 2, 3\}$, $N(G^2) = \{4\}$, and $N(G^3) = \{5\}$.

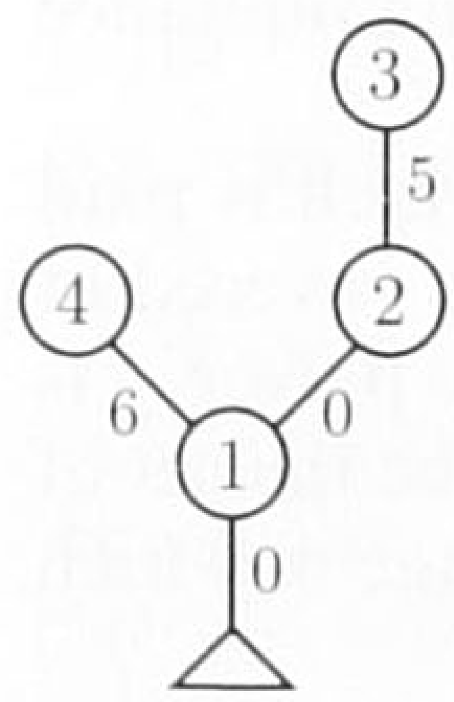

Fig. 5. The maintenance problem for Example 4.12

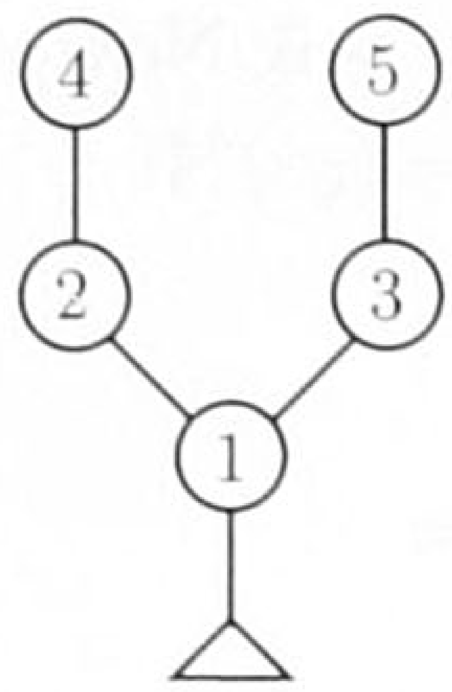

Fig. 6. The maintenance problem as in Example 4.13

Theorems 4.6 and 4.10 together imply:

Corollary 4.14. The core of the maintenance game $(N, c_{\mathcal{G}})$ equals the set of weighted Shapley values, i.e. $\{\Phi^{\mu}(c_{\mathcal{G}}) : \mu \in \mathcal{M}(c_{\mathcal{G}})\} = C(c_{\mathcal{G}})$.

Monderer et al. (1992) show a more general result, that the set of all weighted Shapley values equals the core of any concave cost game. However, in proving this they needed a fixed point theorem.

5 The core and the set of weighted neighbour-home allocations

In the case of the weighted down-home allocation, the players have an obligation to help their neighbours (predecessors), since they are required to start working from the community center towards their own home. A less extreme social obligation results by applying rules (i)-(iv) in section 4, as well as (v) and (vi) below. The resulting allocation will be called the *neighbour-home* allocation.

(v) If, for any worker $i \in N$, the road between $r_{k(i)}$ and $\pi(i)$ has not been finished yet, then i is working outside his own arc e_i.

(vi) Each worker paints as close to his home as the rules $(i) - (iii), (iv), (v)$ permit.

The algorithm in Maschler et al. (1995) produces a special case of the weighted neighbour-home allocation, the nucleolus, where $\mathcal{T} = \{G\}$ and $w_i = \frac{1}{|N|}$ for all $i \in N$. We will show, analogous to the treatment in Section 4 for the weighted down-home allocation, that the set of weighted neighbour-home allocations equals the core, when the weight systems vary over the set $\mathcal{B}(\mathcal{G})$. In order to do this, we need to present the scheme implied by rules $(i) - (v)$ in a more formal manner, and this is done in Algorithm 5.1. Let $\beta = (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$, where $\mathcal{T} = (G^1, \ldots, G^p)$. The neighbour-home allocation, denoted $\eta^{\beta}(\mathcal{G})$, is obtained by, for each of the restricted maintenance problems $\mathcal{G}^k$, $1 \leq k \leq p$, applying 5 to the restricted maintenance problem $\mathcal{G}^k$.

Let $x(e, q) \in [0, c(e)]$ be the part of the cost of arc $e \in E(G^k)$ which is paid before stage q. Let $E_q \subseteq E(G^k)$ be the subset of arcs whose cost is covered at stage q, and let $E(q) := \cup_{j<q} E_j$. Let $e(i, q)$ be the arc to which player i is contributing in stage q, and let $S(e, q) := \{i \in N(G^k) \mid e(i, q) = e\}$ be the set of players contributing to arc e in stage q. Let $Q(i)$ denote the first stage in which i stops contributing.

Algorithm 5.1.

Step 0

$q := 1, x(e,1) := 0$ for all $e \in E(G^k), E(1) := \emptyset$,

$$e(i,1) := \begin{cases} e_{\pi(i)} & \text{if } \pi(i) \neq r_k, \\ e_i & \text{otherwise.} \end{cases}$$

Step 1

For any $e \in E(G^k) \setminus E(q)$ such that $S(e,q) \neq \emptyset$, it would take $t(e,q) := \frac{c(e)-x(e,q)}{w(S(e,q))}$ units of time to finish for arc e. Thus, the first arc will be finished after $t(q) := \min\{t(e,q) \mid e \in E(G^k) \setminus E(q) \text{ and } S(e,q) \neq \emptyset\}$ units of time. Then $w(S(e,q))t(q)$ is the fraction of an arc $e \in E(G^k) \setminus E(q)$ which is constructed at stage q, and therefore $x(e,q+1) := x(e,q) + w(S(e,q))t(q)$. Let $E_q := \{e \in E(G^k) \setminus E(q) \mid t(e,q) = t(q)\}$ be the subset of arcs finished at stage q, and let $E(q+1) := E(q) \cup E_q$ be the subset of arcs finished at or before stage $q+1$. Consider every $i \in S(e,q)$ and $e \in E_q$. If there exists an unfinished arc between $e = e(i,q)$ and the source, i.e. $f \preceq e$ such that $f \in E(G^k) \setminus E(q+1)$, then choose such an f as close to e as possible, and set $e(i,q+1) := f$. If such an arc does not exist, and if i's own arc is not finished, i.e. $e_i \in E(G^k) \setminus E(q+1)$, set $e(i,q+1) := e_i$. Otherwise, set $Q(i) := q$ and $\eta_i^\beta(\mathcal{G}) := \sum_{q=1}^{Q(i)} t(q)w_i$.

Step 2

If $E(q+1) = E(G^k)$, terminate. Otherwise, set $q := q+1$, and repeat step 1.

We will first illustrate the algorithm by an example.

Example 5.2. For example 4.2 we have $\eta^\beta(\mathcal{G}) = (4, 4, 22, 10)$. Player 4 is alone in his pseudo subtree G^2, so he will contribute the entire cost of the arc $(1,4)$, i.e. 10. For pseudo subtree G^1 we apply algorithm 5.1. Initially, $e(1,1) = e(2,1) = (r,1)$ and $e(3,1) = (1,2)$. The first arc is finished after $t(1) = \min\{\frac{10}{2}, \frac{10}{3}\} = \frac{10}{3} = t((1,2),1)$ units of time, and the set of arcs finished in the first stage is $E_1 = \{(1,2)\}$. Now $e(3,2) = (r,1)$ and $S((r,1),2) = \{1,2,3\}$, i.e. all three players will be contributing to arc $(r,1)$ in the second stage. Then $t(2) = t((r,1),2) = \frac{10-\frac{10}{3}\cdot(1+1)}{5} = \frac{2}{3}$, and $E_2 = \{(r,1)\}$. Players 1 and 2 stop contributing after the second stage, i.e. $Q(1) = Q(2) = 2$, and they each contribute, in total, $1 \cdot \left(\frac{10}{3} + \frac{2}{3}\right) = 4$. Player 3 now starts contributing to his own arc, i.e. $e(3,3) = (2,3)$. He will finish this arc in $t(3) = \frac{10}{3}$ units of time, and then stop contributing ($Q(3) = 3$). His total contribution is $3 \cdot \left(\frac{10}{3} + \frac{2}{3} + \frac{10}{3}\right) = 22$. Since all the arcs have been finished after stage 3, the algorithm terminates.

Now we turn to the following question: Given a core element y, can we find a weight system $\beta = (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$ such that $y = \eta^\beta(\mathcal{G})$? It turns out that the answer is yes, and Proposition 3.6(ii) suggests that we choose $\mathcal{T} := \mathcal{T}(y)$. We will now illustrate, using two examples, how the weight vector w can be found.

Example 5.3. Consider Example 4.4 again, and the core element $y = (4, 12, 12, 12)$, for wich the partition $\mathcal{T}(y)$ is trivial. First, note that arc e_1 will be finished after $\frac{4}{w_1}$ units of time. Moreover, players 2 and 3 will be contributing at this arc until it is finished, and will return home (to their own arcs) exactly when this is the case. In order to calculate how long 2 and 3 will be contributing at arc e_1, we need to find their *far-away contributions*, i.e. how much they contribute at arcs other than their own. We will do this by first finding their *home contributions*, i.e. how much they contribute at their own arcs. Player 2's home contribution is obviously given by the cost of his own arc, i.e. 10, since he has no followers other than himself. Thus his far-away contribution, i.e. the amount that he will contribute at arc e_1, is 12 - 10 = 2. For player 3 the picture is more complicated, since he has a follower, player 4. Rule (*vi*) implies that if 4 contributes anything above the cost of his own arc, this contribution will first be used to arc e_3. This is indeed the case, since player 4 contributes $12 - 10 = 2$ in excess of the cost of his own arc. The home contribution of player 3 will thus be only $10 - 2 = 8$. Since he contributes 12 in total, his far-away contribution will be $12 - 8 = 4$. To sum up, we know now that players 1, 2, and 3 contributes 4, 2, and 4, respectively, at arc e_1. This implies $\frac{4}{w_1} = \frac{2}{w_2} = \frac{4}{w_3}$. Player 4 will be contributing at e_3 until this arc is finished. His contribution at this arc is 2, as we stated above. Player 3 will stop contributing at all exactly when his own arc is finished, at which point he will have contributed 12. This implies $\frac{12}{w_3} = \frac{2}{w_4}$. A weight vector that satisfies both equalities above is $w = (4, 2, 4, \frac{4}{6})$.

To formalize the notions of home and far-away contributions, let $y = \eta^\beta(\mathcal{G})$ for some weight system $\beta \in \mathcal{B}(\mathcal{G})$, and let $i \in N(G^k)$, $1 \leq k \leq p$. Note that the contribution of the players in $\bar{F}(i) \setminus \{i\}$ at or below e_i will be given by what they contribute in excess of the cost of their own arcs, i.e. by $\sum_{j \in \bar{F}(i) \setminus \{i\}} (y_j - c_j)$. Because of rule (*vi*), this excess contribution will first be used at arc e_i. Player i will cover the remaining part $c_i - \sum_{j \in \bar{F}(i) \setminus \{i\}} (y_j - c_j)$ of the cost of his own arc, if this expression is positive. Hence, player i's home contribution is given by

$$h_i(y) := \left(c_i - \sum_{j \in \bar{F}(i) \setminus \{i\}} (y_j - c_j) \right)_+ = \left(c_i - \sum_{j \in \pi^{-1}(i) \cap \bar{F}(i)} \tilde{O}_j(y) \right)_+ ,$$

and his far-away contribution is $f_i(y) := y_i - h_i(y)$. Next we will consider an example where some players contribute nothing, which makes finding the weight vector slightly more complicated.

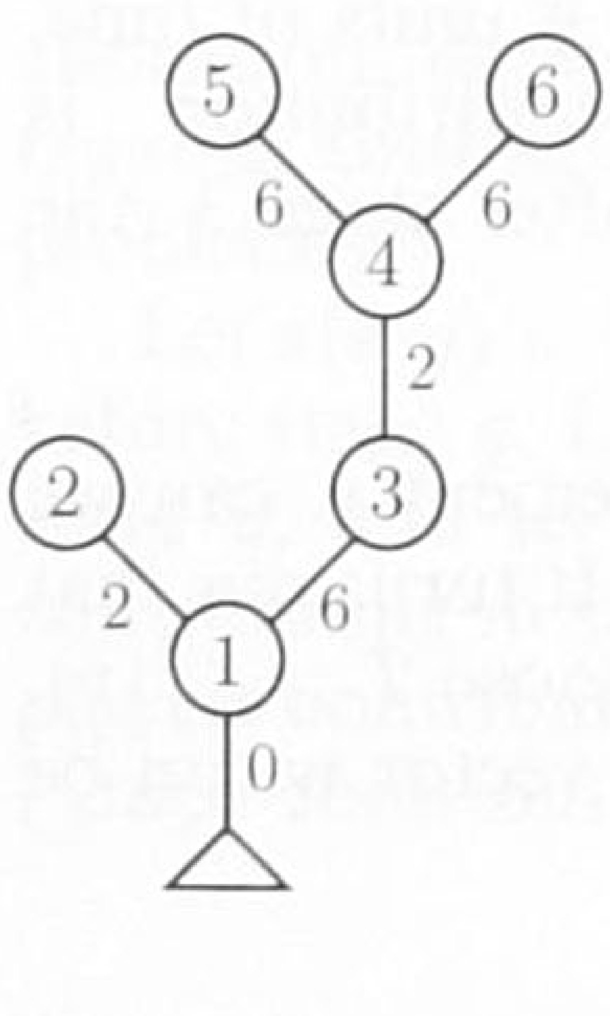

Fig. 7. The maintenance problem as in Example 5.4

Example 5.4. Consider the maintenance problem depicted illustrated in Fig. 7, and the corresponding core element $y = (0, 12, 16, 0, 16, 16)$. We set the weights of players which do not make any contribution, to zero, i.e. $w_1 := w_4 := 0$. Player 2 has no followers other than himself, he will have to contribute to the entire cost of his own arc e_2, i.e. $h_2(y) = 10$. The remaining 12 - 10 = 2 $(= f_2(y))$ that he contributes, will be used to pay for the cost of arc e_1. Player 3 will also contribute at this arc, but how much? The answer can be found by noting that the followers of 3 (except himself), i.e. players 4, 5 and 6, contribute $0 + 16 + 16 = 32$, while the total cost of their own arcs is only 30. Hence we have $h_3(y) = 10 - 2 = 8$ and $f_3(y) = 16 - 8 = 8$. Players 2 and 3 will return home at exactly the same time, i.e. when arc e_1 is finished. This will happen after

$$\frac{f_2(y)}{w_2} = \frac{2}{w_2} \quad \text{or} \quad \frac{f_3(y)}{w_3} = \frac{8}{w_3} \tag{15}$$

units of time. Note that the weights of players 2 and 3 are not related to the weight of the player in front of them, as was the case in the Example 5.3 $h_5(y) = h_6(y) = 10$, since neither player 5 nor player 6 have followers other than themselves, and therefore $f_5(y) = f_6(y) = 16 - 10 = 6$. Because of rule (v), they cannot return home until the players in front of them have all finished. The last such player will be the closest one that makes a positive contribution, i.e. player 3, who finishes after $\frac{16}{w_3}$ units of time. Our weight vector must therefore satisfy

$$\frac{6}{w_5} = \frac{6}{w_6} = \frac{16}{w_3}. \tag{16}$$

A weight vector that satisfies (15) and (16), such that weight zero is assigned to players that do not contribute anything, is given by $w = (0, 4, 16, 0, 6, 6)$.

For any $i \in N(G^k)$, let $\pi^+(i)$ be the first predecessor of i in G^k such that $y_i > 0$. If no such predecessor exists, let $\pi^+(i) := r_k$. Also, let $N^+(G^k) := \{i \in N(G^k) \mid y_i > 0\}$. Note that if $i \in N(G^k)$ is such that $\pi(i) \neq r_k$ and $\tilde{O}_i(y) > 0$, then he will contribute a nonzero amount to the cost of the arcs in $E(\tilde{P}_{\pi^+(i)})$, and will return home exactly when all the arcs in $E(\tilde{P}_i) \setminus \{e_i\}$ have been finished. Since $\frac{f_i(y)}{w_i}$ is the total time that player i spends contributing to arcs other than his own, we have

$$\pi^+(i) = \pi^+(j) \neq r_k \Rightarrow \frac{f_i(y)}{w_i} = \frac{f_j(y)}{w_j} \quad \text{for all } i, j \in N^+(G^k). \tag{17}$$

Also, if a player contributes to the cost of the arcs of his predecessors, he will return home exactly when the last one of his predecessors stops contributing, i.e.

$$\frac{f_i(y)}{w_i} = \frac{y_{\pi^+(i)}}{w_{\pi^+(i)}} \quad \text{for all } i \in N^+(G^k) \text{ such that } \pi^+(i) \neq r_k. \tag{18}$$

Let, for $k = 1, \ldots, p$, $B^k(x) := \{i \in N(G^k) : \pi^+(i) = r_k\}$.

Proposition 5.5. Let $x \in C(c_{\mathcal{G}})$. There exists $\beta := (\mathcal{T}, w) \in \mathcal{B}(\mathcal{G})$ such that $x = \eta^\beta(\mathcal{G})$, where $\mathcal{T} = \mathcal{T}(x)$, and w satisfies, for every $k = 1, \ldots, p$,

$$\frac{f_i(x)}{w_i} = \frac{f_j(x)}{w_j} \quad \text{for all } i, j \in N^+(G^k) \cap B^k(x), \tag{19}$$

$$\frac{f_i(x)}{w_i} = \frac{x_{\pi^+(i)}}{w_{\pi^+(i)}} \quad \text{for all } i \in N^+(G^k) \backslash B^k, \tag{20}$$

$$w_i = 0 \quad \text{for all } i \in N(G^k) \backslash N^+(G^k). \tag{21}$$

Proof. Claim 1: existence. Clearly, $\mathcal{T}(x)$ exists. Let $1 \leq k \leq p$. In order to show that

(19)–(21) have a solution, note that, for every $i \in N(G^k)$,

$$\begin{aligned} f_i(x) &= x_i - h_i(x) = x_i - \left(c_i - \sum_{j \in \tilde{F}(i) \backslash \{i\}} (x_j - c_j) \right)_+ \\ &= x_i - \left(c_i + x_i - c_i - \sum_{j \in \tilde{F}(i)} (x_j - c_j) \right)_+ \\ &= x_i - \left(x_i - \tilde{O}_i(x) \right)_+ . \end{aligned}$$

The construction of $\mathcal{T}$ implies, for every $i \in N(G^k)$, that $\tilde{O}_i(x) > 0$ if $\pi(i) \neq r_k$, hence

$$\pi(i) \neq r_k \quad \text{and} \quad x_i > 0 \Rightarrow f_i(x) > 0. \tag{22}$$

A solution can be found by arbitrarily fixing $w_i^* > 0$ for some $i^* \in N^+(G^k) \cap B^k(x)$. Note that, since the subtree G^k has exactly one node adjacent to the source, $\pi(i^*) = r_k$ implies $|N^+(G^k) \cap B^k(x)| = 1$, in which case (19) places no further restrictions on the weight vector. If $|N^+(G^k) \cap B^k(x)| > 1$, and hence $\pi(i^*) \neq r_k$, which means that (22) applies, we can use (23) to determine w_j for every $j \in N^+(G^k) \cap B^k(x) \setminus \{i^*\}$. For the players in $N^+(G^k) \setminus B^k(x)$, we determine the weights from (20) in a recursive manner. In the rest of the proof, let $y := \eta^\beta(\mathcal{G})$. Then, for $i \in N(G^k)$ and $1 \leq k \leq p$,

$$x_i > 0 \Leftrightarrow w_i > 0 \Leftrightarrow y_i > 0 \quad \text{for all } i \in N, \tag{23}$$

where the first equivalence follows from (22) and the construction of w described above. Since y is a result of Algorithm 5.1, where only players with positive weights have to pay anything, we have $y_i > 0 \Rightarrow w_i > 0$. Finally, $w_i > 0$ implies $x_i > 0$, and from $x \in C(c_{\mathcal{G}})$ and Proposition 3 (iv), there must exist some arc $e \in E(\bar{P}_i)$ such that $c(e) > 0$. Then, since y has been constructed using Algorithm 5.1, we have $w_i > 0 \Rightarrow y_i > 0$.

Claim 2: $\beta \in \mathcal{B}(\mathcal{G})$. Clearly, $\mathcal{T} = \mathcal{T}(x)$ is a partition of G into pseudo subtrees. Also, $x \geq 0$, together with (23), imply $w \geq 0$. Let $c_i > 0$ for some $i \in N(G^k)$, $1 \leq k \leq p$. Since $x^{N(G^k)} \in C(c_{\mathcal{G}^k})$ by Proposition 3.6 (ii), we must have $\tilde{O}_i(x) = \sum_{j \in \tilde{F}(i)} (x_j - c_j) \geq 0$ by Proposition 3.1 (iii). Since $x_j \geq 0$ and $c_j \geq 0$ for all $j \in \tilde{F}(i)$, there must exist some $\ell \in \tilde{F}(i)$ such that $x_\ell > 0$, and $w_\ell > 0$ then follows from (23).
Claim 3: $x = \eta^\beta(\mathcal{G})$. We have

$$y(N(G^k)) = c_{\mathcal{G}^k}(N(G^k)) = x(N(G^k)), \tag{24}$$

for $1 \leq k \leq p$, where the first equality follows from Algorithm 5.1 and $\beta \in \mathcal{B}(\mathcal{G})$, and the second from $x \in C(c_{\mathcal{G}})$ and Proposition 3.6 (ii). If $N(G^k) = \{i\}$ for some $i \in N$, then $x_i = y_i$ follows directly, so we will assume in the following that $|N(G^k)| > 1$. Suppose, contrary to our claim, that $x^{N(G^k)} \neq y^{N(G^k)}$. By (24), there must exist $i, j \in N(G^k)$ such that $x_i < y_i$ and $x_j > y_j$. We will complete the proof by showing that this leads to a contradiction. Consider node i. We will first show that

$$x_\ell \leq y_\ell \quad \text{for all } \ell \in \tilde{F}(i) \tag{25}$$

To prove (25), first note that $x_\ell = 0 \Leftrightarrow y_\ell = 0$ follows from (23). Next, note that w satisfies (19)–(21) with respect to x, by definition, and with respect to y, since we derived (19)–(21) from Algorithm 5.1 Also, because of (23), we have $B^k(x) = B^k(y)$, and the definitions of $\pi^+(\bullet)$ and $N^+(G^k)$ are unambiguous. Then (20) implies that $f_m(x) < f_m(y)$ for every $m \in (\pi^+)^{-1}(i)$, so there must exist some $\ell \in \tilde{F}(i) \setminus \{i\}$ such that $x_\ell < y_\ell$. The argument can be repeated for $i := \ell$, and by continuing in this manner, we will eventually have shown that there is a leaf $\ell \in \tilde{F}(i)$ such that $x_\ell < y_\ell$. Now we will show that $x_\ell \leq y_\ell$ for *every* leaf $\ell \in \tilde{F}(i)$. Suppose, on the contrary, that this was not true, i.e., there exists a leaf $m \in \tilde{F}(i)$ such that $x_m > y_m$. Then it must be possible to find two branches $\tilde{B}_{pq}$ and $\tilde{B}_{ps}$, both rooted at $p \in \tilde{F}(i)$, such that $m \in N(\tilde{B}_{pq})$ and $x_t \geq y_t$ for all $t \in N(\tilde{B}_{pq})$, and such that $x_t \leq y_t$ for all $t \in N(\tilde{B}_{ps})$. This implies $h_t(x) \leq h_t(y)$ and $f_t(x) \geq f_t(y)$ for all $t \in N(\tilde{B}_{pq})$, and $x_m > y_m$ together with (20) give

$$f_q(x) > f_q(y) \Rightarrow x_p > y_p.$$

On the other hand, for every $t \in N(\tilde{B}_{ps})$ we have $h_t(x) \geq h_t(y)$ and $f_t(x) \leq f_t(y)$, which together with (5) gives

$$f_q(x) \leq f_q(y) \Rightarrow x_p \leq y_p,$$

hence we have a contradiction. Now, since every $\ell \in \tilde{F}(i)$ such that ℓ is a leaf of G^k satisfies $x_\ell \leq y_\ell$, we can use (20) in a recursive manner to prove (25). Then $x_i < y_i$ and (27) together imply $f_i(x) < f_i(y)$, which, by (5), implies $x_{\pi^+(i)} < y_{\pi^+(i)}$. Setting $i := \pi^+(i)$, we can successively repeat this argument until we have $i \in N^+(G^k) \cap B^k(x)$. We have thus shown that there exists $i \in N^+(G^k) \cap B^k(x)$ such that $x_i < y_i$. Using the same line of argument for node j, we can show that there exists $j \in N^+(G^k) \cap B^k(x)$ such that $x_j > y_j$. If $|N^+(G^k) \cap B^k(x)| = 1$, this is in itself a contradiction, otherwise the contradiction follows from (19). □

In the same way that Proposition 4.5 enabled us to prove Theorem 4.6, Proposition 5.5 enables us to prove that the set of neighbour-home allocations equals the core.

Theorem 5.6. *The core of the maintenance game* $(N, c_{\mathcal{G}})$ *equals the set of neighbour allocations, i.e.* $\{\eta^\beta(\mathcal{G}) \mid \beta \in \mathcal{B}(\mathcal{G})\} = C(c_{\mathcal{G}})$.

We know, from Maschler et al. (1995), that $\eta^{(\mathcal{T},w)}(\mathcal{G})$ is equal to the nucleolus of $c_{\mathcal{G}}$ if we set $\mathcal{T} = \{G\}$ and $w_i = \frac{1}{|N|}$ for all $i \in N$. In Yanovskaya (1992) , the *weighted nucleolus* is defined by replacing the ordinary excess function by a weighted excess function, and it is shown that every point in the relative interior of the core can be obtained as a weighted nucleolus. For the game (N, g), and some pre-imputation x, this weighted excess function is given by, for any $S \neq N, \emptyset$, $\epsilon^p(S,x) := p_S(g(S) - x(S))$, where $p_S > 0$. Let $\beta := ((G^1, \ldots, G^p), w) \in \mathcal{B}(\mathcal{G})$ for some maintenance problem $\mathcal{G}$, and let $k = 1, \ldots, p$. Suppose we set $p_S := f(w^{N(G^k)})$ for all $S \subset N(G^k)$ such that $S \neq \emptyset$, where $f : \mathbb{R}^{N(G^k)} \to \mathbb{R}$. An interesting open problem is whether we can pick the function f such that $\eta^\beta(\mathcal{G})$, when restricted to the members of $N(G^k)$, is the weighted nucleolus of the game $c_{\mathcal{G}^k}$.

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