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This is a postprint of:

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Cahiers du Centre d'Études de Recherche Opérationnelle, 1981, vol. 23, pp. 143-152

Report 8031

EQUILIBRIUM POINTS OF SEMI-INFINITE BIMATRIX GAMES

M.J.M. Jansen and S.H. Tijs

ABSTRACT: In this paper, we consider semi-infinite bimatrix games with a non-empty equilibrium point set and investigate the structure of this latter set. Furthermore, we study the relationship of this set with the equilibria of finite subgames.

1. INTRODUCTION

In this paper, we combine the approaches in [2] and [6] in the study of the set of equilibrium points of the mixed extension of a bounded semi-infinite bimatrix game.

To this end we consider, in section 3, Nash subsets, i.e. sets of equilibrium points that have the interchangeability property. The equilibrium point set is the union of the maximal Nash subsets. We show that a maximal Nash subset is the Cartesian product of a compact convex set and a closed, convex set with a finite or countably infinite number of extreme points. In section 4, these extreme points are characterized in terms of square submatrices of the payoff matrix of the first player.

Finally, in section 5, the relationship between the set of equilibrium points of a bounded semi-infinite bimatrix game and the equilibria of the finite subgames is considered.

NOTATION: For $m \in \mathbb{N}$, let $\mathbb{N}_m := \{1,2,\ldots,m\}$ and let e_m be a vector or sequence with 1 on the m-th place and 0 on the other places. For a finite set S, $|S|$ is the number of elements of S. The convex hull of a set $S \subset \mathbb{R}^m$ is denoted by conv(S). If $C \subset \mathbb{R}^m$ is a convex set, then we write ext(C) and relint(C) for the set of extreme points of C and the relative interior of C, respectively. The closure of a set A is denoted by cl(A).

2. SEMI-INFINITE BIMATRIX GAMES AND EQUILIBRIUM POINTS

Let $A = [a_{ij}]_{i=1,j=1}^{m\ \ \infty}$ and $B = [b_{ij}]_{i=1,j=1}^{m\ \ \infty}$ be two bounded $m\times\infty$-matrices $(m \in \mathbb{N})$ with real coefficients.

Let $S^m := \{p \in \mathbb{R}^m;\ p \geq 0,\ \sum_{i=1}^m p_i = 1\}$

$S^\infty := \{q \in \ell^1;\ q \geq 0,\ \sum_{j=1}^\infty q_j = 1\}$,

where ℓ^1 is the normed linear space, consisting of those infinite sequences $x = (x_1, x_2, \ldots)$ of real numbers for which $\|x\|_1 := \sum_{i=1}^\infty |x_i| < \infty$,

$S^c := \{q \in S^\infty;$ there exists an $n \in \mathbb{N}$ such that $q_j = 0$ for all $j > n\}$.

We call the game $\langle S^m, S^\infty, E_A, E_B \rangle$ ($\langle S^m, S^c, E_A, E_B \rangle$), in which

$E_A(p,q) := \sum_{i=1}^m \sum_{j=1}^\infty p_i a_{ij} q_j = pAq^t$, and

$E_B(p,q) := pBq^t$, for all $p \in S^m$ and $q \in S^\infty$ (S^c),

the *mixed* (c-*mixed*) *extension of the* $m\times\infty$-*bimatrix game* (A,B). These games are often denoted by (A,B).

Let $\varepsilon \geq 0$. A pair $(\bar{p}, \bar{q}) \in S^m \times S^\infty$, is called an ε-*equilibrium point* if

$$\bar{p}A\bar{q}^t = \max_{p \in S^m} pA\bar{q}^t \tag{2.1}$$

$$\bar{p}B\bar{q}^t \geq \sup_{q \in S^\infty} \bar{p}Bq^t - \varepsilon \tag{2.2}$$

If $\varepsilon = 0$, then we call an ε-equilibrium point always an equilibrium point. The set of all equilibrium points (ε-equilibrium points) is denoted by $E(A,B)$ ($E^\varepsilon(A,B)$).

For the game $\langle S^m, S^c, E_A, E_B \rangle$ (ε-) equilibrium points are analogously defined. The set of equilibrium points (ε-equilibrium points) of this game is denoted by $E_c(A,B)$ ($E_c^\varepsilon(A,B)$).

EXAMPLE 2.1. Let $A = [0\ 0\ \ldots]$, $B = [0\ \frac{1}{2}\ \frac{2}{3}\ \frac{3}{4}\ \ldots]$. Then $E(A,B) = \emptyset$, $E_c(A,B) = \emptyset$; but, for each $\varepsilon > 0$, $E^\varepsilon(A,B) \neq \emptyset$ and $E_c^\varepsilon(A,B) \neq \emptyset$.

In Tijs [5], [7], the following theorem was proved.

THEOREM 2.2. *Let* (A,B) *be a bounded* $m\times\infty$-*bimatrix game. Then* $E_c^\varepsilon(A,B) \neq \emptyset$, *for all* $\varepsilon > 0$.

In this paper, we shall make frequent use of the following equivalence, which can be proved in a straighforward manner.

LEMMA 2.3. *Let* (A,B) *be a bounded semi-infinite* $m\times\infty$*-bimatrix game and let* $(p,q) \in S^m \times S^\infty$. *Then* $(p,q) \in E(A,B)$ *if and only if* $C(p) \subset M(A;q)$ *and* $C(q) \subset M(p;B)$, *where*

$$C(p) := \{i \in \mathbb{N}_m;\ p_i > 0\},\quad C(q) := \{j \in \mathbb{N};\ q_j > 0\},$$

$$M(A;q) := \{i \in \mathbb{N}_m;\ e_i A q^t = \max_k e_k A q^t\} \text{ and}$$

$$M(p;B) := \{j \in \mathbb{N};\ pBe_j^t \geq pBe_k^t, \text{ for all } k \in \mathbb{N}\}.$$

For the description of the relationship between E(A,B) and $E_c(A,B)$, we need the following result of Blackwell and Girshick [1], p.48.

LEMMA 2.4. *Let* $v_1, v_2, \ldots$ *be an infinite sequence of elements in* $\mathbb{R}^m$. *Let* $q \in S^\infty$ *be such that* $\sum_{j=1}^\infty q_j v_j \in \mathbb{R}^m$. *Then there is a* $\hat{q} \in S^c$ *such that*

$$\sum_{j=1}^\infty q_j v_j = \sum_{j=1}^\infty \hat{q}_j v_i \in \operatorname{conv}\{v_1, v_2, \ldots\}.$$

THEOREM 2.5. *Let* (A,B) *be a bounded* $m\times\infty$*-bimatrix game. Then* $E(A,B) = \operatorname{cl}(E_c(A,B))$. *Consequently,* $E(A,B) \neq \emptyset$ *iff* $E_c(A,B) \neq \emptyset$.

PROOF. (a) Let $(p,q) \in E(A,B)$. We show that $E(A,B) \subset \operatorname{cl}(E_c(A,B))$ by proving that for each $\varepsilon > 0$ there is a $q_\varepsilon \in S^c$ with $\|q-q_\varepsilon\|_1 < \varepsilon$ and $(p,q_\varepsilon) \in E_c(A,B)$. Let $\varepsilon > 0$. As in the proof of theorem 3 in [6], one can find, with the aid of lemma 2.4, a $q_\varepsilon \in S^c$ with $\|q-q_\varepsilon\|_1 < \varepsilon$, $C(q_\varepsilon) \subset C(q)$ and $Aq_\varepsilon^t = Aq^t$. Then $M(A;q_\varepsilon) = M(A;q)$. Consequently, by lemma 2.3, $(p,q_\varepsilon) \in E_c(A,B)$.

(b) Now we show that $\operatorname{cl}(E_c(A,B)) \subset E(A,B)$. Let $(p^1,q^1), (p^2,q^2), \ldots$ be a sequence in $E_c(A,B)$, converging to some $(p,q) \in S^m \times S^\infty$. Then, for all $k \in \mathbb{N}$,

$$e_i A(q^k)^t \le p^k A(q^k)^t, \qquad \text{for all } i \in \mathbb{N}_m \tag{2.3}$$
$$p^k B e_j^t \le p^k B(q^k)^t, \qquad \text{for all } j \in \mathbb{N}$$

In view of the fact that, for a bounded m×∞-matrix M, $x \mapsto e_i M x^t$ $(x \in S^\infty)$ is a continuous function on S^∞, (2.3) implies that

$$e_i A q^t \le pAq^t, \qquad \text{for all } i \in \mathbb{N}_m$$
$$pBe_j^t \le pBq^t, \qquad \text{for all } j \in \mathbb{N}.$$

So $(p,q) \in E(A,B)$. □

3. THE STRUCTURE OF MAXIMAL NASH SUBSETS

In this section, we only consider bounded semi-infinite bimatrix games (A,B) with $E(A,B) \neq \emptyset$.

Let (A,B) be such a bimatrix game of size m×∞. For $(\tilde{p},\tilde{q}) \in S^m \times S^\infty$, put

$$K(\tilde{q}) := \{p \in S^m;\ (p,\tilde{q}) \in E(A,B)\}$$
$$L(\tilde{p}) := \{q \in S^\infty;\ (\tilde{p},q) \in E(A,B)\}.$$

Then $K(\tilde{q})$ is a convex, compact subset of S^m and $L(\tilde{p})$ is a closed, convex subset of S^∞, as one easily verifies.

DEFINITION 3.1. Let (A,B) be a bounded semi-infinite bimatrix game. A subset $S \subset E(A,B)$ is called a *Nash subset* for the game (A,B) if for all $(p,q),(p',q') \in S$ also $(p,q'),(p',q) \in S$. A Nash subset S is called a *maximal Nash subset* for the game (A,B) if there exists no Nash subset for the game (A,B) properly containing S.

REMARK 3.2. If S is a maximal Nash subset for a bimatrix game (A,B), then $S = \pi_1(S) \times \pi_2(S)$, where

$$\pi_1(S) := \{p \in S^m;\ \text{there exists a } q \in S^\infty \text{ with } (p,q) \in S\}$$
$$\pi_2(S) := \{q \in S^\infty;\ \text{there exists a } p \in S^m \text{ with } (p,q) \in S\}$$

[redacted]

Combining the fact that $(\overset{\circ}{p},q) \in E(A,B)$ with (3.3) and (3.4), we obtain

$$C(p) \subset C(\overset{\circ}{p}) \subset M(A;q)$$

$$C(q) \subset M(\overset{\circ}{p};B) \subset M(p;B).$$

Consequently, $(p,q) \in E(A,B)$ and (3.1) is proved.

(c) Now suppose that $L(\overset{\circ}{p}) \neq \pi_2(S)$. Then, in view of (a) and (b), $\pi_1(S) \times L(\overset{\circ}{p})$ is a Nash subset for (A,B) properly containing S. This contradicts with the fact that S is a maximal Nash subset. So $L(\overset{\circ}{p}) = \pi_2(S)$. □

4. EXTREME EQUILIBRIUM POINTS

We start with a generalization of a result of H. Mills [4].

LEMMA 4.1. *Let* (A,B) *be a bounded* $m\times\infty$*-bimatrix game and let* $(p,q) \in S^m \times S^\infty$. *Then* $(p,q) \in E(A,B)$ *if and only if there exist scalars* α *and* β *such that*

$$\begin{aligned} &e_iAq^t \le \alpha, \quad \text{for all } i \in \mathbb{N}_m, \\ &pBe_j^t \le \beta, \quad \text{for all } j \in \mathbb{N} \end{aligned} \tag{4.1}$$

and $pAq^t + pBq^t = \alpha+\beta$.

PROOF. If $(p,q) \in E(A,B)$, then (4.1) is satisfied if one takes $\alpha = pAq^t$ and $\beta = pBq^t$.

If the quadruplet (p,q,α,β) satisfies (4.1), then for $i_0 \in C(p)$, $e_{i_0}Aq^t = \alpha$, as one can prove as follows.

If $e_{i_0}Aq^t < \alpha$, then

$$\begin{aligned} \alpha+\beta = pAq^t+pBq^t &= p_{i_0}e_{i_0}Aq^t + \sum_{i\neq i_0} p_ie_iAq^t + \sum_j pBe_j^tq_j \\ &< p_{i_0}\alpha + \sum_{i\neq i_0} p_i\alpha + \sum_j q_j\beta = \alpha+\beta, \end{aligned}$$

which is impossible. Hence, $\alpha = pAq^t$ and $pAq^t \geq e_iAq^t$, for all $i \in \mathbb{N}_m$. (4.2)
Similarly, if $j_0 \in C(q)$, then $pBe^t_{j_0} = \beta$. So $\beta = pBq^t$ and $pBq^t \geq pBe^t_j$, for all $j \in \mathbb{N}$. Together with (4.2) this implies that $(p,q) \in E(A,B)$.

□

Inspired by this lemma, we introduce the convex and closed set

$$Q_A := \{(q,\alpha) \in S^\infty \times \mathbb{R} ;\ e_iAq^t \leq \alpha, \text{ for all } i \in \mathbb{N}_m\}.$$

In [3], Lindenstrauss proved that a closed, bounded and convex subset of ℓ^1 has extreme points and that such a set is the closed convex hull of its extreme points. In view of this result and remark 3.2 it is meaningful to investigate the extreme points of a maximal Nash subset.

DEFINITION 4.2. An equilibrium point of a bounded $m\times\infty$-bimatrix game (A,B) is called an *extreme equilibrium point* if it is an extreme point of some maximal Nash subset for the game (A,B). If (p,q) is an extreme equilibrium point of (A,B), then we call p (q) an *extreme strategy* for player I (II).

LEMMA 4.3. *Let* (A,B) *be a bounded* $m\times\infty$*-bimatrix game. If* (p,q) *is an extreme equilibrium point of* (A,B)*, then*

(1) $(q,pAq^t) \in \text{ext}(Q_A)$,

(2) $C(q)$ *is a finite set; even* $|C(q)| \leq m$,

(3) the rank of the matrix $\tilde{M} = \begin{bmatrix} M & -1 \\ 1 & 0 \end{bmatrix}$, *where*

$M := [a_{ij}]_{i\in M(A;q);j\in C(q)}$ *equals* $|C(q)|+1$.

PROOF. Suppose that $(p,q) \in \text{ext}(S)$ and $\mathring{p} \in \text{relint}\ \pi_1(S)$, where S is a maximal Nash subset for (A,B). Then, by theorem 3.3,

$(p,q) \in \text{ext}(\pi_1(S) \times L(\mathring{p})) = \text{ext}\ \pi_1(S) \times \text{ext}\ L(\mathring{p})$. So $q \in \text{ext}\ L(\mathring{p})$.

(a) Now suppose that

$$(q,pAq^t) = \tfrac{1}{2}(q',\alpha') + \tfrac{1}{2}(q'',\alpha''),$$

where $(q',\alpha'),(q'',\alpha'') \in Q_A$ and $(q',\alpha') \neq (q'',\alpha'')$. Note that this last inequality implies that $q' \neq q''$.

Since $e_iAq^t = \tfrac{1}{2}e_iA(q')^t + \tfrac{1}{2}e_iA(q'')^t \leq \tfrac{1}{2}\alpha' + \tfrac{1}{2}\alpha'' = pAq^t$, for all $i \in \mathbb{N}_m$, we have, for $i \in M(A;q)$,

$$e_iA(q')^t = \alpha' \text{ and } e_iA(q'')^t = \alpha'',$$

which implies that $M(A;q) \subset M(A;q')$ and $M(A;q) \subset M(A;q'')$.

Hence, $C(\mathring{p}) \subset M(A;q')$ and $C(\mathring{p}) \subset M(A;q'')$.

Since $q = \tfrac{1}{2}q' + \tfrac{1}{2}q''$, we can conclude that

$$C(q') \subset C(q) \subset M(\mathring{p};B) \quad\text{and}\quad C(q'') \subset C(q) \subset M(\mathring{p};B).$$

So, in view of lemma 2.3, $q',q'' \in L(\mathring{p})$ while $q' \neq q''$. This contradicts the fact that $q \in \text{ext } L(\mathring{p})$. So $(q,pAq^t) \in \text{ext } Q_A$.

(b) Suppose that $q \notin S^c$. In view of lemma 2.4, there exists a $\hat{q} \in S^c$ such that $Aq^t = A\hat{q}^t$ and $C(\hat{q}) \subset C(q)$. This implies that $M(A;q) = M(A;\hat{q})$; so $(\mathring{p},\hat{q}) \in E(A,B)$.

Now consider, for $\varepsilon > 0$, the vector

$$q(\varepsilon) := (1+\varepsilon)q - \varepsilon\hat{q}.$$

Then, for $\varepsilon > 0$ sufficiently small, $q(\varepsilon) \in S^\infty$.

Moreover, $C(q(\varepsilon)) \subset C(q) \cup C(\hat{q}) = C(q)$ and $Aq(\varepsilon)^t = (1+\varepsilon)Aq^t - \varepsilon A\hat{q}^t = Aq^t$.

Consequently, $(\mathring{p},q(\varepsilon)) \in E(A,B)$, for ε sufficiently small.

However, $q = (1+\varepsilon)^{-1}q(\varepsilon) + \varepsilon(1+\varepsilon)^{-1}\hat{q}$ and $q(\varepsilon),\hat{q} \in L(\mathring{p})$. Since $\hat{q} \neq q(\varepsilon)$, this contradicts the fact that $q \in \text{ext } L(\mathring{p})$. So $q \in S^c$ i.e. $C(q)$ is a finite set.

(c) Let $\gamma := |C(q)| \in \mathbb{N}$. Without loss of generality, we may suppose that $C(q) = \{1,2,\ldots,\gamma\}$. Suppose that rank $\tilde{M} < \gamma+1$. Then there exists a vector $y = (y_1,\dots,y_\gamma,0,\dots) \in \mathbb{R}^{\mathbb{N}}$ and a constant $s \in \mathbb{R}$ such that

(i) $(y_1,\dots,y_\gamma,s) \neq 0$

(ii) $\sum_{j=1}^{\infty} y_j = 0$

(iii) $\sum_{j=1}^{\gamma} a_{ij}y_j - s = 0$, for all $i \in M(A;q)$.

Consider, for each $\varepsilon > 0$, the vectors

$$u(\varepsilon) := q+\varepsilon y \text{ and } v(\varepsilon) := q-\varepsilon y.$$

Then there exists an $\hat{\varepsilon} > 0$ such that, for all $\varepsilon \in (0,\hat{\varepsilon})$,

$$u(\varepsilon),v(\varepsilon) \in S^{\infty}$$

and $e_iAq^t < \mathring{p}Aq^t + \varepsilon(s - \max_{k\notin M(A;q)} e_kAq^t)$, for each $i \notin M(A;q)$. This implies that, for all $\varepsilon \in (0,\hat{\varepsilon})$,

$$e_iAu(\varepsilon)^t = e_iAq^t + \varepsilon e_iAy^t = \mathring{p}Aq^t + \varepsilon s \quad \text{if} \quad i \in M(A;q)$$

and

$$e_iAu(\varepsilon)^t = e_iAq^t + \varepsilon e_iAy^t \le e_iAq^t + \varepsilon \max_{k\notin M(A;q)} e_kAy^t$$
$$< \mathring{p}Aq^t + \varepsilon s \qquad \text{if} \quad i \notin M(A;q).$$

Consequently, $(u(\varepsilon),\mathring{p}Aq^t+\varepsilon s) \in Q_A$, for all $\varepsilon \in (0,\hat{\varepsilon})$.

Similarly, an $\check{\varepsilon}$ can be found such that $(v(\varepsilon),\mathring{p}Aq^t-\varepsilon s) \in Q_A$, for all $\varepsilon \in (0,\check{\varepsilon})$. So, for $0 < \varepsilon < \min\{\hat{\varepsilon}\ \check{\varepsilon}\}$,

$$(q,\mathring{p}Aq^t) = \tfrac{1}{2}((u(\varepsilon),\mathring{p}Aq^t+\varepsilon s)+(v(\varepsilon),\mathring{p}Aq^t-\varepsilon s)),$$

which contradicts (1). So rank $\tilde{M} = \gamma+1$.

(d) That $|C(q)| \le m$ follows now from (3). □

In the following theorem we associate the extreme strategies of player II with certain square submatrices of A. Compare this result with the Vorobev-Kuhn theorem (cf. [2], theorem 3.5).

THEOREM 4.4. *Let* (A,B) *be a bounded* $m\times\infty$*-bimatrix game. If* $(p,q) \in E(A,B)$

and $pAq^t \neq 0$, *then the following two assertions are equivalent*

(1) q *is an extreme strategy for player* II,

(2) there exists a nonsingular square submatrix K *of* A *such that*

$$q_K^t = pAq^t K^{-1} 1_K^t \quad \textit{and} \quad pAq^t = (1_K K^{-1} 1_K^t)^{-1}.$$

[Here q_K is the vector obtained from q by removing the coordinates corresponding to the columns of A which play no role in K.]

PROOF. (a) We show that (2) implies (1). Let S be a maximal Nash subset for (A,B) containing (p,q) and let $\mathring{p} \in \text{relint } \pi_1(S)$. Suppose that there are $\hat{q}, \check{q} \in L(\mathring{p})$ with $q = \frac{1}{2}\hat{q} + \frac{1}{2}\check{q}$. Then $C(\hat{q}) \subset C(q)$ and $C(\check{q}) \subset C(q)$. Since $(\mathring{p},q), (\mathring{p},\hat{q}), (\mathring{p},\check{q}) \in E(A,B)$, for $i \in M(A;q)$, we have

$$e_iAq^t = \tfrac{1}{2}e_iA\hat{q}^t + \tfrac{1}{2}e_iA\check{q}^t \leq \tfrac{1}{2}\mathring{p}A\hat{q}^t + \tfrac{1}{2}\mathring{p}A\check{q}^t = \mathring{p}Aq^t = e_iAq^t$$

and therefore $e_iA\hat{q}^t = \mathring{p}A\hat{q}^t$ and $e_iA\check{q}^t = \mathring{p}A\check{q}^t$; so $M(A;q) \subset M(A;\hat{q})$ and $M(A;q) \subset M(A;\check{q})$. Consequently,

$$e_iK\hat{q}_K^t = pA\hat{q}^t \quad \text{and} \quad e_iK\check{q}_K^t = pA\check{q}^t,$$

for all suitable values of i. Because K is nonsingular and $\hat{q}_K$ and $\check{q}_K$ are probability vectors, this implies that $\hat{q}_K = \check{q}_K$. So $\hat{q} = \check{q}$. Therefore, $q \in \text{ext } L(\mathring{p})$.

(b) Suppose that (1) holds. Let $\tilde{M}$ be the matrix as defined in lemma 4.3(3) and let $\gamma := |C(q)|$. Since rank $\tilde{M} = \gamma+1$ by that lemma, we can find $\gamma+1$ rows of $\tilde{M}$ (including the last row) which are linearly independent. These rows form a nonsingular $(\gamma+1)\times(\gamma+1)$-submatrix

$$\tilde{K} := \begin{bmatrix} K & -1 \\ 1 & 0 \end{bmatrix}$$

of $\tilde{M}$.

Now $(q_K, \mathring{p}Aq^t)$ is the unique solution of the system $\tilde{K}x^t = e_{\gamma+1}^t$. The theorem is proved if we apply Cramer's rule. □

With the aid of theorem 4.4, we can prove the following two results.

THEOREM 4.5. *Let* (A,B) *be a bounded* $m\times\infty$*-bimatrix game. Let* S *be a maximal Nash subset for* (A,B) *with* $\overset{\circ}{p} \in \text{relint}\ \pi_1(S)$. *Then*

$$L(\overset{\circ}{p}) = \text{cl}(\text{conv}(\text{ext}(L(\overset{\circ}{p}))))$$

and $\text{ext}\ L(\overset{\circ}{p})$ *is a finite or a countably infinite set.*

PROOF. The first assertion about $L(\overset{\circ}{p})$ is a consequence of remark 3.2 and the result of Lindenstrauss mentioned earlier.Further, theorem 4.4 associates with an extreme point of $L(\overset{\circ}{p})$ a certain nonsingular submatrix of A.
It is easy to see that with a nonsingular square submatrix of A there corresponds at most one such an extreme point. Because A has only a countably infinite number of square submatrices, we can conclude that $\text{ext}\ L(\overset{\circ}{p})$ is a finite or countably infinite set. □

REMARK 4.6. Also $\pi_1(S)$ is the convex hull of its extreme points. This is a consequence of the theorem of Krein-Milman. Note that the number of extreme points of $\pi_1(S)$ may be uncountable. An example of this phenomena can be found in the proof of theorem 2 of [6]. That the number of extreme points of $L(\overset{\circ}{p})$ may be countably infinite is illustrated by the example where A and B are two $1\times\infty$-matrices with all coefficients equal to zero. Then $E(A,B) = S^1 \times S^\infty$ and $\text{ext}\ L(e_1) = \{e_1, e_2, \ldots\}$.

REMARK 4.7. It follows from the proof of theorem 4.5 that the number of extreme strategies of player II is at most countably infinite.

Let (A,B) be a bounded semi-infinite bimatrix game and let $(p,q) \in E(A,B)$. Since $\{(p,q)\}$ is a Nash subset, we can find, with the help of the lemma of Zorn, a maximal Nash subset containing (p,q). Consequently, every equilibrium point of (A,B) is contained in a maximal Nash subset and $E(A,B)$ is the union of such subsets.

THEOREM 4.8. *The set of equilibrium points of a bounded semi-infinite bimatrix game is a union of maximal Nash subsets.*

Now we give an example of a bounded semi-infinite bimatrix game with a countably infinite number of maximal Nash subsets.

EXAMPLE 4.9. Let $A = \begin{bmatrix} 0 & 0 & \dots \\ 0 & 0 & \dots \end{bmatrix}$ and let B be the $2\times\infty$-matrix with, for all $n \in \mathbb{N}$,

$$b_{1n} = \cos(2^{-n}\pi) \quad \text{and} \quad b_{2n} = \sin(2^{-n}\pi).$$

Let $p^n := (\cos(2^{-n}\pi) + \sin(2^{-n}\pi))^{-1}(b_{1n}, b_{2n})$.

Then, for all $n \in \mathbb{N}$, $(p^n, e_n) \in E(A,B)$. Furthermore, the equilibrium points $(p^1,e_1),(p^2,e_2),\dots$ are contained in different maximal Nash subsets.

5. APPROXIMATION BY FINITE SUBGAMES

In this section, we study the relationship between the set of equilibrium points of a bounded semi-infinite bimatrix game (A,B) with the equilibria of the games $(A_1,B_1),(A_2,B_2),\dots$, where, for $n \in \mathbb{N}$, $A_n := [a_{ij}]_{i=1,j=1}^{m\ \ n}$ and $B_n := [b_{ij}]_{i=1,j=1}^{m\ \ n}$.

We start with a

DEFINITION 5.1. Let $E_1,E_2,\dots$ be a sequence of closed subsets of $S^m \times S^\infty$

$(m \in \mathbb{N})$. Then we write $\limsup_{n\to\infty} E_n$ for the possibly empty set of points $x \in S^m \times S^\infty$, for which there exist a subsequence $n(1), n(2), \ldots$ of $1,2,\ldots$ and points $x_{n(1)}, x_{n(2)}, \ldots$ such that $x_{n(k)} \in E_{n(k)}$, for all $k \in \mathbb{N}$, and $\lim_{k\to\infty} x_{n(k)} = x$.

Note that $\limsup_{n\to\infty} E_n = \bigcap_{n\in\mathbb{N}} \text{cl}\,[\bigcup_{k\geq n} E_k]$.

THEOREM 5.2. *Let* (A,B) *be a bounded semi-infinite* m×∞*-bimatrix game. Then*

$$\limsup_{n\to\infty} E(A_n,B_n)^{\sim} = E(A,B),$$

where $E(A_n,B_n)^{\sim} := \{(p,\tilde{q});\ \tilde{q} := (q,0,0,\ldots) \in S^\infty,\ (p,q) \in E(A_n,B_n)\}$.

PROOF. (1) Let $(p,q) \in E(A,B)$. We show that $E(A,B) \subset \limsup_{n\to\infty} E(A_n,B_n)^{\sim}$ by proving that for each $\varepsilon > 0$, there is an $n(\varepsilon) \in \mathbb{N}$ and a $q_\varepsilon \in S^{n(\varepsilon)}$ with $\|q-(q_\varepsilon,0,0,\ldots)\|_1 < \varepsilon$ and $(p,q_\varepsilon) \in E(A_{n(\varepsilon)},B_{n(\varepsilon)})$. As in part (a) of the proof of theorem 2.5, one can find a $\tilde{q}_\varepsilon \in S^c$ with $\|q-\tilde{q}_\varepsilon\|_1 < \varepsilon$ and $(p,\tilde{q}_\varepsilon) \in E_c(A,B)$.

Now we can write $\tilde{q}_\varepsilon = (q_\varepsilon,0,0,\ldots)$, where $q_\varepsilon \in S^n$ for some $n \in \mathbb{N}$.

Obviously, $(p,q_\varepsilon) \in E(A_n,B_n)$, because

$$pA_nq_\varepsilon^t = pA\tilde{q}_\varepsilon^t \geq e_iA\tilde{q}_\varepsilon^t = e_iA_nq_\varepsilon^t, \text{ for all } i \in \mathbb{N}_m,$$

$$pB_nq_\varepsilon^t = pB\tilde{q}_\varepsilon^t \geq pBe_j^t = pB_ne_j^t, \quad \text{for all } j \in \mathbb{N}_n.$$

(2) Let $(p^{n(1)},\tilde{q}^{n(1)}),(p^{n(2)},\tilde{q}^{n(2)}),\ldots$ be a sequence in $S^m \times S^\infty$ converging to (p,q) such that

$$(p^{n(k)},\tilde{q}^{n(k)}) \in E(A_{n(k)},B_{n(k)})^{\sim}.$$

We can write $\tilde{q}^{n(k)} = (q^{n(k)},0,0,\ldots)$, where $q^{n(k)} \in S^{n(k)}$. Now

$$e_iA(\tilde{q}^{n(k)})^t = e_iA_{n(k)}(q^{n(k)})^t \leq p^{n(k)}A_{n(k)}(q^{n(k)})^t$$
$$= p^{n(k)}A(\tilde{q}^{n(k)})^t, \quad \text{for all } i \in \mathbb{N}_m.$$

So $e_iAq^t \leq pAq^t$, for all $i \in \mathbb{N}_m$.

$$p^{n(k)}B(\tilde{q}^{n(k)})^t = p^{n(k)}B_{n(k)}(q^{n(k)})^t \geq p^{n(k)}B_{n(k)}e_j^t$$
$$= p^{n(k)}Be_j^t.$$

Taking limits, we obtain $pBq^t \leq pBe_j^t$.

So $(p,q) \in E(A,B)$. □

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