# On the Nash Bargaining Solution with Noise*

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**Abstract**

Suppose two parties have to share a surplus of random size. Each of the two can either commit to a demand prior to the realization of the surplus - as in the Nash demand game with noise - or remain silent and wait until the surplus was publicly observed. Adding the strategy to wait to the noisy Nash demand game results in two strict equilibria, in each of which one player takes almost the whole surplus, provided uncertainty is small. If commitments concern only who makes the first offer, the more balanced Nash bargaining solution is approximately restored. In all cases commitment occurs in equilibrium, even though this entails the risk of breakdown of negotiations.

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# 1 Introduction

Bargaining models in economics are a key tool in understanding distributional conflicts. By abstracting from the concrete matters that are at stake, they reduce the problem of distribution to its bare bones. Though it appears as if at such an abstract level not much can be said, economics has made considerable progress in this area. In particular, Nash ((1950) and (1953)) showed that, in addition to symmetry and the axioms underlying expected utility, only independence of irrelevant alternatives is needed to pin down the outcome that will be agreed upon by rational bargaining partners.

But Nash also already felt the need to complement his cooperative solution with a non-cooperative underpinning. Specifically, he addressed the question whether a stripped-down version of a bargaining process would lead to the same outcome as the cooperative solution. More generally, the endeavour to provide non-cooperative foundations for solution concepts from cooperative game theory has become known as the "Nash program". Since Nash's pioneering contribution many non-cooperative bargaining procedures have been studied and several natural bargaining processes, including models in which parties alternate in making offers (Ståhl (1972), Krelle (1976), and Rubinstein (1982), among many others) and models in which the proposer is randomly selected (Rubinstein and Wolinsky (1985), Binmore, Rubinstein, and Wolinsky (1986), among others) were shown to have (at least in the 2-person case) a unique subgame perfect equilibrium and to lend support to Nash's cooperative solution. In the bilateral case, when frictions, like discount rates, time costs, or break-down probabilities, are small enough and of uniform relative magnitude, the equilibrium outcome is near the outcome of the cooperative Nash bargaining solution, and it converges to it when the frictions vanish.

Most of the previous work assumes that the underlying bargaining situation is deterministic, but the frictions that are added to the models to obtain uniqueness of the non-cooperative solution are frequently of a stochastic nature. Yet, players are typically not given all the strategic options that would be natural in such a stochastic situation. Nash's (1953) own non-cooperative model is a clear instance of this. In his "demand game", two players state simultaneously demands. If the combined demands are feasible, they receive what they asked for, otherwise they obtain nothing. To select among the many equilibria of this game, Nash introduces small uncertainty about the size of the pie and, thereby, obtains a balanced outcome, consistent with cooperative bargaining.

In Nash's argument uncertainty is a technical device to select among equilibria. For the present purpose, however, we prefer to view the presence of some amount of uncertainty as a matter of realism. Hence, we view Nash's (1953) solution as (potentially) relevant when there is small uncertainty. But, in this situation, we can only be confident in the prediction if the non-cooperative game allows the players all the strategic options that they may want to use. In this respect Nash's model seems less satisfactory. When there is uncertainty, one natural option is to wait until the uncertainty has been resolved. But Nash's model does not allow for this: players are *forced* to move before they know the size of the pie.

In this paper we investigate what happens to Nash's demand game when players also have this natural option to wait. We show that, if players are allowed to postpone their claims until the size of the pie is revealed (and if uncertainty is small enough), precisely one of them will always choose to do so. In the resulting equilibria, the player, who chooses not to wait, always demands almost all the surplus, yielding a distribution far away from the Nash bargaining solution. These equilibrium outcomes thus resemble the subgame perfect equilibrium of ultimatum bargaining (Güth (1976)), rather than the Nash bargaining solution. Adding the opportunity to wait to the Nash demand game, therefore, has dramatic consequences.

Though we phrase this point in terms of an abstract bargaining problem, it also has implications for models from the industrial organization literature. (In fact, the endogenous timing models, that we will study here, have grown out of a literature on oligopolies; see below.) In industrial production, for instance, the option to postpone claims may correspond to a switch towards a more flexible production technology. Admittedly, this may be associated with a different cost function, and this effect is ignored when focussing on a pure bargaining problem. But the advantage of commitment has been an issue in oligopoly theory ever since von Stackelberg (1951), even when leader and follower have identical costs. Therefore, we view bilateral bargaining as an abstract - but sparse - paraphrase for a distributional conflict under uncertainty.

The observation that the option to postpone demands leads to an asymmetric solution also begs a question. How will it be decided, who makes the first offer and who waits? A small, but growing literature (Hamilton and Slutsky (1990), Spencer and Brander (1992), Mailath (1993), Sadanand and Sadanand (1996), Amir and Grilo (1999), and van Damme and Hurkens

(1999)) has addressed this issue of endogenous timing in the context of duopoly models. These papers aim at an endogenous determination of Stackelberg versus Cournot behavior. They provide two different game forms that both allow firms to decide upon whether to become a Stackelberg-leader or -follower or play Cournot.

In the first game form, similar to the bargaining game with noise described above, firms can either choose an output at stage 1 or wait for stage 2 and then choose. Hamilton and Slutsky (1990) refer to such a game as one of "action commitment". Such a game form applies when commitment is inevitably tied to a quantitative demand. Retail firms for durable consumption goods, like automobiles, for instance, may compete by either posting prices or making it known that prices are negotiable. The first represents a commitment, that entails a quantitative claim, the latter corresponds to waiting and retaining flexibility.

But in other institutional settings, a choice of business strategy may be involved. For instance, retail firms may choose mail-order distribution systems, that require them to decide on the assortment of goods and prices at the stage of printing the catalogue. Competing shopkeepers and chainstores will know that mail-order firms are committed, but as long as the catalogue has not yet been published, mail-order prices remain unknown. Similarly, contracts with wholesalers may commit a retailer without revealing to competitors or consumers to what the retailer is committed. Likewise, in a bargaining context, a different model is required, when negotiations are carried out by delegates with rigid missions. If it is known that the other side in the negotiations will send a delegate without the authority to adjust a yet unknown claim, this is captured by the second game form which Hamilton and Slutsky (1990) call a "game with observable delay".

In this game, parties simultaneously choose *when* they intend to make demands. Only after these intentions are revealed, quantitative demands are actually stated. In the context of duopoly this means that initially firms choose whether to strive to become Stackelberg-leader or -follower (two leaders or two followers play Cournot), and only after these positions have become known, firms choose their outputs. In the context of bargaining it means that parties can let their partners know whether or not they give up the option to wait until the surplus is observed.

It turns out that, in the bargaining context, this second timing model restores the support for the Nash bargaining solution. We show that for sufficiently small uncertainty, there is a unique subgame perfect equilibrium in which the parties split the surplus almost evenly. Hence, if parties can only credibly communicate that they will not wait (until uncertainty resolves), but not what they will demand, the Nash bargaining solution reemerges as a good approximation.

In both versions of the model commitment occurs in equilibrium, even though this entails a risk of breakdown of negotiations. This is because players are given the opportunity of an irrevocable commitment which they use to their advantage, at the expense of efficiency. Schelling (1966, chp. 3) already suggested that if commitments are not certain to cause an impasse, they might constitute a viable bargaining strategy, even when disagreement is costly. Accordingly, that uncertainty together with parties striving for commitments may imply the risk of impasse has been shown by Schelling (1956), Crawford (1982), and Muthoo (1996). In the latter two papers commitments are revokable at a known (Muthoo (1996)) or uncertain (Crawford (1982)) cost, while the size of the surplus is known. In this paper we assume irrevocable commitments, but the size of the surplus is uncertain at the time when commitments are available, in accordance with the noisy Nash demand game. The conclusion, however, remains: in the presence of uncertainty commitment opportunities may cause inefficiencies.

As a by-product we find a caveat to market models of endogenous timing under duopoly. With large uncertainty results become highly conditional. Depending on the timing of commitments and demands, risk aversion, and (the hazard rate of) the cumulative distribution almost everything can happen. This caveat also applies to duopoly models with endogenous timing. While in these models uncertainty and risk aversion are not a driving force, the shape of the (inverse) demand function and cost functions play a formally equivalent role.

The remainder of the paper is organized as follows. Section 2 introduces the modified Nash demand game and derives its equilibria. Section 3 considers the alternative model of endogenous timing. Section 4 collects some observations on large uncertainty and risk aversion. Section 5 concludes.

# 2 Modified Noisy Nash Demand Game

There are two bargaining parties, players $i = 1, 2$, who share a surplus that becomes available if they strike a deal. The size of the surplus is initially unknown. But both parties hold a common prior, described by a cumulative distribution function $F$ of the surplus size $z \geq 0$. Accordingly, we will refer to the game, that will now be described, as the $F$-*demand game*.

Under this symmetric uncertainty, each party $i = 1, 2$ chooses (simultaneously with the other) either to make a demand $x_i \geq 0$ or not to make a demand. After players have made their choices, a chance move governed by $F$ publicly reveals the size $z$ of the surplus. If both parties made demands $x_1$ and $x_2$, then player $i$ obtains $x_i$ units of the surplus if $x_1 + x_2 \leq z$; otherwise, i.e. if $x_1 + x_2 > z$, the demands are incompatible and no deal is struck. This is like in the Nash demand game. If one party, say, player $i$, made a demand $x_i$, and the other not, then player $i$ gets $x_i$ units of the surplus and the other party $j = 3 - i$ collects the residual $z - x_i$ if $x_i \leq z$; otherwise, i.e. if $x_i > z$, no deal is struck and both parties obtain zero. Finally, if neither party made a demand, then players divide the surplus in a prespecified way such that each player gets a positive amount. More precisely, player $i$ obtains $\beta_i z$, where $\beta_i > 0$ for $i = 1, 2$ and $\beta_1 \equiv \beta = 1 - \beta_2$.

This rules out that in the absence of uncertainty one party takes the whole surplus. Since most explicit models of non-cooperative bargaining under certainty result in such an interior $\beta$, this is taken as a "reduced form" of an (unmodelled) ensuing bargaining game under certainty.

Risk attitudes are taken into account by representing the players' preferences by twice continuously differentiable, strictly increasing, and (weakly) concave utility functions $u_i$, i.e., $u_i'(x) > 0$ and $u_i''(x) \leq 0$, for all $x \geq 0$, for $i = 1, 2$, where $x$ denotes what player $i$ receives from the surplus. Normalize utility such that $u_i(0) = 0$, for $i = 1, 2$. Strict concavity ($u_i'' < 0$) models risk aversion, linearity ($u_i'' = 0$) models risk neutrality.

To ensure that parties are willing to engage in the game we assume that $F(z) = 0$, for all $z < 0$, and that $F$ is twice continuously differentiable for all $z > 0$ in the interior of the support of $F$. (The lower bound of the support could be positive.) Since $F(z)$ is a distribution function, it is nondecreasing, right-continuous (even at the boundary of its support), and approaches 1 as $z$ goes to infinity. $F$ has a *density* $f = F'$ if it is continuous everywhere. If $F$ has density $f$, the *hazard rate* is $h(z) = f(z)/[1 - F(z)]$. Where needed, we will adopt the standard assumption of a monotone hazard rate, i.e. that $h$ is nondecreasing. The result in the next subsection is independent of any assumption on the hazard rate.

## 2.1 Strict Equilibria

The analysis of the $F$-demand game is fairly easy. The key insight is that, whenever the other party demands some $y \geq 0$ such that $F(y) < 1$ before uncertainty resolves, then to wait is strictly better than any positive demand. The intuition behind this is simple. By waiting one can always collect the residual. By also making a demand one cannot get more, but may enlarge the circumstances under which no deal is struck. And this is independent of how large the uncertainty is and how risk averse a player is.

Formally, for any demand $y \geq 0$ stated by the other party before uncertainty resolves, if player $i$ waits, she obtains $w_i = \int_y^\infty u_i(z-y)\,dF(z)$. If she demands $x \geq 0$, she obtains

$$v_i(x,y) = u_i(x)\left[1 - F(x+y)\right] \tag{1}$$

irrespective of whether the other party demands a positive amount $y > 0$ or waits and, therefore, currently demands nothing, $y = 0$. It follows that, for any demand $x \geq 0$ by player $i$,

$$\int_y^\infty u_i(z-y)\,dF(z) \geq \int_y^{x+y} u_i(z-y)\,dF(z) + v_i(x,y) \geq v_i(x,y) \tag{2}$$

If $F(y) < 1$ the first inequality is strict, and if, moreover, $x > 0$ the second inequality is strict. Hence, if the opponent demands $y > 0$ such that $F(y) < 1$, it never pays to also make a demand.

Yet, if $F(y) = 1$, both weak inequalities in (2) are equalities and waiting is as good as any demand. In particular, demanding $x$ such that $F(x) = 1$ is a best reply. It follows that, as in Nash's (1953) non-cooperative bargaining model, there are always inefficient equilibria, where both parties demand $x_i$ such that $F(x_i) = 1$ for $i = 1,2$. But these equilibria are weak (non-strict), because every strategy is a best reply. Since alternative strict equilibria exist, we henceforth ignore these inefficient equilibria.

Given that in a strict equilibrium it is never the case that both parties state positive demands, two possibilities remain. Either both parties wait, or one makes a demand and the other not.

If both parties wait, each player $i$ obtains

$$W_i(\beta) = \int_0^\infty u_i(\beta_i z)\;dF(z) \tag{3}$$

for $i = 1,2$. This could potentially be an equilibrium, if the uncertainty is large enough. But for small uncertainty it cannot.

To see this, assume that the distribution $F$ is concentrated in the following sense: there is some small $\varepsilon > 0$ such that, with $\bar{z} = \int_0^\infty z\, dF(z) > 0$ denoting the mean,

$$\varepsilon \geq \max\{F(\bar{z} - \varepsilon), 1 - F(\bar{z} + \varepsilon)\} \tag{4}$$

i.e., most of the mass is concentrated around the mean. Call a distribution *$\varepsilon$-concentrated* if it satisfies (4) for $\varepsilon > 0$.

Now observe that, by concavity of $u_i$, the payoff from both waiting is bounded from above by the utility from the respective share in the mean surplus, i.e. $W_i(\beta) \leq u_i(\beta_i \bar{z})$ for $i = 1, 2$. On the other hand, with a sufficiently concentrated distribution, and given that the opponent waits, player $i$ can obtain

$$V_i(0) = \max_{x \geq 0} v_i(x, 0) \tag{5}$$

by deviating to an optimal demand. If $F$ satisfies (4) for small $\varepsilon$, then $v_i(x, 0) \geq (1 - \varepsilon) u_i(x)$ for all $x \leq \bar{z} - \varepsilon$. Therefore, $V_i(0) \geq (1 - \varepsilon) u_i(\bar{z} - \varepsilon)$, for all $\varepsilon > 0$ sufficiently small. As $\varepsilon$ goes to zero, the lower bound on $V_i(0)$ approaches $u_i(\bar{z})$. Since $u_i(\bar{z}) > u_i(\beta_i \bar{z}) \geq W_i(\beta)$ for all $\beta_i < 1$, by continuity, the unique best reply against an opponent, who decides to wait, is to demand approximately the expected surplus.[1] Hence, both parties waiting cannot be an equilibrium.

Moreover, if the opponent decides to wait, then it is clearly suboptimal to demand some $x$ such that $F(x) = 1$ or $x = 0$. This follows from (1), because, if $F(x) = 1$ or $x = 0$, then $v_i(x, 0) = 0$, while $v_i(x, 0) > 0$ for all $x > 0$ such that $F(x) < 1$. Combining this with (2) we have shown:

**Proposition 1** *If $F$ is $\varepsilon$-concentrated and $\varepsilon > 0$ is sufficiently small, then the $F$-demand game has precisely two strict equilibria. In each of those one party demands approximately the expected surplus, and the other party waits.*

There are also equilibria in *mixed* strategies. In those, parties randomize between waiting and (positive) demands. But no mixed equilibrium is strict. Hence, in the presence of two strict equilibria we view those as the preferable solutions.

[1] The precise shape of the optimal demand depends both on the utility function and the shape of $F$. It is given by the implicit solution to the first-order condition (6) at $y = 0$.

# 3 Endogenous Timing

In the $F$-demand game, strict equilibria predict an outcome that resembles ultimatum bargaining rather than the Nash bargaining solution. This is so, because any attempt to commit to a first-mover advantage entails a quantitative demand. This makes it preferable for the opponent to wait for the surplus to be revealed and collect the residual.

That a bargaining party can only commit to a demand by stating it, is certainly plausible for face-to-face bargaining. On the other hand, in face-to-face bargaining irrevocable commitments are somewhat implausible. The latter are more compelling in bargaining by delegates with rigid missions. Also, on some markets one party enjoys a first-mover advantage by convention. For instance, for certain goods with negotiable prices, it is common for the seller to state her ask price in opening the negotiations.

But, if the convention is, that the seller makes the first offer in price negotiations, this tells the buyer only that the seller will make the first proposal, but not what the ask price is. Likewise, if one party announces that she will send a delegate with a rigid mission, this informs the other party that she will be confronted with an offer, but not about the offer itself.

The game that we have analyzed above corresponds to a "game of action commitment" from the literature on endogenous timing (Hamilton and Slutsky (1990)). This literature has also introduced "games with observable delay" which can be used to model the situation where bargaining parties may, or may not, send delegates with rigid instructions. These later games are played by the following rules.

At the first stage, given symmetric uncertainty over the size of the surplus, the two parties announce (simultaneously) at which time (before or after uncertainty resolves) they intend to make an irrevocable demand; but they need not specify the amount they will demand if they choose to move before uncertainty resolves. Then, after the players' choices are revealed to both, the game moves to the second stage. If, at the first stage, both declared that they will not make demands before uncertainty resolves, the size of the surplus is revealed and (as above) player $i$ obtains $\beta_i z$ unity of the surplus, for $i = 1, 2$. If, at the first stage, both declared that they will make demands before uncertainty resolves, then the parties simultaneously choose their respective demands $x_1$ and $x_2$ before the surplus size is revealed. If these demands turn out to be compatible ($x_1 + x_2 \leq z$), then each player gets exactly what she demanded; otherwise (if $x_1 + x_2 > z$), no deal is struck. Finally, if one party, say, $i$, declared that she will make an early demand, and the other not, player $i$ states her demand before the surplus size is revealed; after the surplus is observed, player $3-i$ then decides whether or not to accept $i$'s demand, for $i = 1, 2$. Acceptance gives player $3-i$ the residual $z - x_i$, while rejection means that no deal is struck.

The game thus defined will be referred to as the *F-delegates game*. Note the essential difference with the $F$-demand game from the previous section. The present game has a richer subgame structure, as players receive intermediate information. In the subgame, where both players commit to state demands before uncertainty resolves, each party will state her demand knowing that the other party will simultaneously state her demand; similarly, in the subgame, where only one party commits, this party knows that she will be the only one stating a demand before uncertainty resolves. We will analyze this $F$-delegates game by imposing subgame perfection (Selten (1965)).

## 3.1 Subgames

First, consider the subgame, where both parties intend to make demands. (Hence, the parameter $\beta$ has no role to play in this subgame.) The payoff function to players $i = 1, 2$ in this subgame is again given by (1), where $x$ denotes player $i$'s demand and $y$ the demand by the opponent. Because $v_i(0, y) = 0$ and $v_i(x, y) = 0$ whenever $F(x + y) = 1$, and $v_i(x, y) > 0$ for all $x > 0$ such that $F(x + y) < 1$, it follows that for any $y$ with $F(y) < 1$ the maximum of $v_i$ with respect to $x$ is interior. Hence, for $F(y) < 1$ the first-order conditions

$$u_i'(x)\,[1 - F(x + y)] - u_i(x)f(x + y) = 0 \qquad (6)$$

must obtain at a payoff maximum. Under a *monotone hazard rate* the first-order condition (6) yields a well-behaved reaction function. Statements (a) and (b) below are driven by the strict concavity of $v_i$ under a monotone hazard rate. Statement (c) says, quite intuitively, that a more risk averse individual will make lower demands. (For a proof see the Appendix.)[2]

**Lemma 1** *If the hazard rate $h$ is nondecreasing and continuously differentiable, then*

[2]Note that the assumption of a monotone hazard rate implies that the support of $F$ must be connected (either a compact interval or the nonnegative reals).

*(a) the reaction function* $\xi_i(y) = \arg\max_{x \geq 0} v_i(x, y)$ *is a bounded continuously differentiable function with slope* $-1 < \xi_i'(y) \leq 0$,

*(b) the maximum* $V_i(y) = \max_{x \geq 0} v_i(x, y)$ *is a continuous and strictly decreasing function of* $y$, *and*

*(c) a global increase in the coefficient of absolute risk aversion,* $r_i(x) = -u_i''(x)/u_i'(x)$, *shifts the reaction function downwards.*

In analogy to duopoly models, such reaction functions give rise to a *unique* equilibrium, when both parties intend to make demands, due to a "single-crossing" property.

**Proposition 2** *If the hazard rate* $h$ *is nondecreasing and continuously differentiable, there is a unique point* $x^* = (x_1^*, x_2^*) \gg 0$ *such that* $\xi_i\left(x_{3-i}^*\right) = x_i^*$ *for* $i = 1, 2$.

**Proof.** By Lemma 1 reaction functions are bounded and nonincreasing. Hence, there is a rectangle $X = [0, \xi_1(0)] \times [0, \xi_2(0)]$ such that the product mapping $\xi = \xi_1 \times \xi_2$ is a continuous function from $X$ to itself. By Brouwer's fixed point theorem, there exists $x^* \in X$ such that $\xi(x^*) = x^*$. Since $x^*$ must be interior and corresponds to an intersection of the graphs of the reaction functions and since the slope of reaction functions is between zero and $-1$, it follows that $x^*$ is unique. ■

The payoffs from the unique equilibrium of the subgame where both parties intend to make demands are, therefore, given by $v_i(x^*)$ for $i = 1, 2$.

Now consider the subgame where only one party intends to make a demand, and the other not. By subgame perfection, the uncommitted party accepts, if the surplus exceeds the demand on the table (after the realization of the surplus). Therefore, the party that intends to demand, say $i$, will choose her demand so as to maximize $v_i(x, 0)$ at $x = \xi_i(0)$, yielding her payoff $V_i(0)$ (see (5)).

The party that waits will collect the residual, if it is nonnegative. Her expected payoff, therefore, is

$$w_i = \int_{\xi_{3-i}(0)}^{\infty} u_i\left(z - \xi_{3-i}(0)\right)\, dF(z) \tag{7}$$

where $i$ denotes the uncommitted party.

Finally, if neither party intends to make a demand, the surplus is split, as before, in proportions $\beta_1$ and $\beta_2$ respectively, after it is observed. The expected payoffs obtained by both waiting are, accordingly, given by $W_i(\beta)$ (see (3)), for $i = 1, 2$. This completes the derivation of equilibrium payoffs from subgames.

## 3.2 Unique Equilibrium

Now, replace subgames by equilibrium payoffs from the subgames. The *truncation* is a $2 \times 2$ game with strategies $t_i = 0$ if player $i$ intends to make a demand and $t_i = 1$ if not, for $i = 1, 2$. Table 1 represents the payoffs for the two players for all strategy constellations (see Proposition 2, (5), (7), and (3)).

| $t_1$ \ $t_2$ | 0 | 1 |
|---|---|---|
| 0 | $v_1(x^*)$, $v_2(x^*)$ | $V_1(0)$, $w_2$ |
| 1 | $w_1$, $V_2(0)$ | $W_1(\beta_1)$, $W_2(\beta_2)$ |

**Table 1** The truncation in matrix form

The difference to the modified noisy Nash demand game is that parties choose their demands conditional on whether or not the other party intends to make a demand. This introduces a force towards symmetry. In particular, condition (2) has no bearing on the solution any more. The key comparison now is between $v_i(x^*)$ and $w_i$. With a sufficiently concentrated distribution, the former yields approximately half the surplus and the latter almost nothing.

**Proposition 3** *If $F$ is $\varepsilon$-concentrated, $\varepsilon > 0$ is sufficiently small, and $F$ satisfies the conditions of Lemma 1, then the $F$-delegates game has a unique subgame perfect equilibrium, $(t_1, t_2) = (0, 0)$. In this equilibrium, payoffs correspond approximately to the Nash bargaining solution.*

**Proof.** First, we show that $x^* = \xi(x^*)$ is an equilibrium of the subgame, where both parties intend to make demands, if and only if $x^*$ maximizes the Nash product

$$U(x_1, x_2) = u_1(x_1)\, u_2(x_2)\, [1 - F(x_1 + x_2)] \tag{8}$$

Let $\bar{x} \in \arg\max_{x \geq 0} U(x)$. Then $\bar{x} \gg 0$ and $U(\bar{x}) \geq U(x_1, \bar{x}_2)$ and $U(\bar{x}) \geq U(\bar{x}_1, x_2)$ for all $x_1 \geq 0$ and all $x_2 \geq 0$ imply that $v_1(\bar{x}_1, \bar{x}_2) \geq v_1(x_1, \bar{x}_2)$ and $v_2(\bar{x}_2, \bar{x}_1) \geq v_2(x_2, \bar{x}_1)$ for all $x_1 \geq 0$ and all $x_2 \geq 0$. Therefore, $\bar{x} = \xi(\bar{x})$. Since the fixed point of $\xi$ is unique by Proposition 2, it follows that $\bar{x} = x^*$. Conversely, suppose that $x^* = \xi(x^*)$ holds. By interiority the first order conditions (6) imply that $\partial U / \partial x_i = 0$ for $i = 1, 2$ at $x = x^*$. Since $U$ is concave in $x = (x_1, x_2)$ and also in each variable $x_i$ separately, for $i = 1, 2$, this implies that $x^* \in \arg\max_{x \geq 0} U(x)$. Hence, $x = \xi(x)$ if and only if $x$ maximizes $U(x)$, irrespective of how concentrated $F$ is.

If $F$ satisfies (4) for successively smaller $\varepsilon$'s, then it follows that $x^* = \xi(x^*)$ approaches the maximizer of $u_1(x_1)\, u_2(x_2)\, \chi(x_1 + x_2)$, where $\chi(z) = 1$ if $z \leq \bar{z}$ and $\chi(z) = 0$ for all $z > \bar{z}$. But this implies that $x^* \gg 0$, even in the limit as $\varepsilon \to 0$, because

$$\max_{x \geq 0} u_1(x_1)\, u_2(x_2)\, \chi(x_1 + x_2) \geq u_1(\bar{z}/2)\, u_2(\bar{z}/2) > 0$$

So, both parties end up with a positive share of the surplus in the equilibrium of the subgame $(t_1, t_2) = (0, 0)$ for all $\varepsilon \geq 0$.

Consequently, as $\varepsilon$ approaches zero, the payoff $v_i(x^*)$ approaches a positive number, for $i = 1, 2$. The payoff $w_i$ approaches zero, because $\xi_{3-i}(0) \to_{\varepsilon \to 0} \bar{z}$ and all the mass becomes concentrated at $z = \bar{z}$, for $i = 1, 2$. Finally, $V_i(0)$ approaches $u_i(\bar{z})$ as $\varepsilon$ goes to zero, and $W_i(\beta)$ approaches $u_i(\beta_i \bar{z}) < u_i(\bar{z})$. Therefore, by continuity, for all $\varepsilon$ sufficiently small, $t_i = 0$ constitutes a dominant strategy in the truncation, for $i = 1, 2$. ■

This establishes that with sufficiently precise information both players will intend to make demands and obtain approximately the payoffs from the Nash bargaining solution. But it is obtained by a somewhat more elaborate procedure, where players can choose whether to demand simultaneously or sequentially.

Nash's (1953) approach was to smooth the discontinuous payoff functions of the Nash demand game and, thereby, select among the equilibria. Uncertainty about the surplus is but one way to smooth. But, under this interpretation of smoothing, why should the parties not wait until they know (more about) the surplus? In Proposition 3 we argue that they will *choose not to wait*, because under sufficiently small uncertainty it is a dominant strategy to go for a first-mover advantage.

Again, the equilibrium is very strong, albeit in the truncation. In this case it is not only strict, but even in (strictly) dominant strategies (in the truncation). Moreover, there are no other (weaker) subgame perfect equilibria.

# 4 Large Uncertainty

For small uncertainty both models above yield equilibria where at least one party makes a demand before uncertainty resolves, thereby creating an inefficiency by a positive probability that no deal is struck. But if uncertainty is not negligible, risk aversion may induce players to avoid this risk of impasse.

More precisely, Lemma 1(c) states that more risk aversion shifts the reaction function (against a given demand by the other party) downwards. This implies smaller returns from an early demand to a more risk averse party. Hence, more uncertainty may create an incentive to wait and see.

In both models waiting is a best reply against an opponent, who also waits, if and only if $W_i(\beta) \geq V_i(0)$. Below a sufficient condition is given for this inequality to hold. Intuitively, the condition says that the probability of impasse at a unilateral commitment must be large enough to make waiting the more attractive strategy. (A proof is in the Appendix.)

**Proposition 4** *If $F$ is continuous, the hazard rate $h$ is nondecreasing, and*

$$F(\xi_i(0)) \geq \frac{1}{1+\beta_i} \tag{9}$$

*holds for $i = 1, 2$, then both parties waiting is an equilibrium, both in the $F$-demand game and in the $F$-delegates game.*

To see what Proposition 4 says, consider the natural case $\beta = 1/2$. Then the right hand side of (9) is $2/3$; the left hand side is the probability that the pie falls short of the optimal demand $\xi_i(0)$ at zero. Hence, condition (9) requires that the probability that the pie is smaller than the optimal demand $\xi_i(0)$ is at least $2/3$.

If this condition is satisfied for both parties, then in both bargaining scenarios it is an equilibrium that both parties wait. This avoids the risk that no deal is struck. But at the same time, more uncertainty may create more equilibria. To illustrate this claim, we turn to a specific example.

Consider distributions $F$ in the class $F(z) = c^a z^a$, for all $0 \leq z \leq 1/c$, for some $a > 0$ and $0 < c \leq 1$, with $F(z') = 0 = 1 - F(z'')$ for all $z' \leq 0$

and $z'' \geq 1/c$. Moreover, assume that both parties have *constant relative risk aversion*, i.e., preferences are represented by $u_i(x) = x^{b_i}$ for some $0 < b_i \leq 1$, for $i = 1, 2$. The coefficient of relative risk aversion is $1 - b_i$. If $b_i = 1$, player $i$ is risk neutral.

Distributions from the above exponential family are naturally ordered by stochastic dominance (Rothschild and Stiglitz (1970)). Independently of the parameter $c$, a distribution $F$ with parameter $a$ stochastically dominates another such distribution with parameter $a'$ if and only if $a \geq a'$. This holds for first-order and, therefore, also for second-order stochastic dominance. Hence, a higher parameter $a$ corresponds to an unambiguously more favorable distribution or, more intuitively, to higher stakes.

The parameter $c$ can be used to vary the moments of $F$. If $c = \frac{a}{1+a}$, the mean is fixed at 1 and the variance is a decreasing function of $a$, with infinite variance as $a$ goes to zero and zero variance as $a$ goes to infinity. If $c = (1+a)^{-1}\sqrt{a/(2+a)}$, the variance is fixed at 1 and the mean is an increasing function of $a$, with zero mean as $a$ goes to zero and infinite mean as $a$ goes to infinity.

In this example three of the four relevant payoffs can be calculated explicitly. If both parties wait, payoffs are given by

$$W_i(\beta) = ac^a \int_0^{1/c} (\beta_i z)^{b_i} z^{a-1} dz = \frac{a}{a+b_i} \beta_i^{b_i} c^{-b_i} \text{ for } i = 1, 2. \tag{10}$$

The distribution $F$ has a monotone (nondecreasing) hazard rate if and only if $a \geq 1$. Hence, by Proposition 2, if $a \geq 1$ there is a unique solution for the subgame, where both parties intend to make demands. If $0 < a < 1$, reaction functions may not be monotone, but direct computation shows that even in this case there is a unique $x^*$ such that $\xi_i\left(x_{3-i}^*\right) = x_i^*$ for $i = 1, 2$. Substituting this into $v_i$ from (5) yields expected payoffs

$$v_i(x^*) = \frac{a}{a+b_1+b_2} \left(\frac{b_i}{b_1+b_2}\right)^{b_i} \left(\frac{b_1+b_2}{a+b_1+b_2}\right)^{\frac{b_i}{a}} c^{-b_i} \text{ for } i = 1, 2. \tag{11}$$

It remains to solve the asymmetric cases. When choosing her demand against an opponent, who waits, player $i$ will maximize $v_i(x, 0)$ at

$$\xi_i(0) = \frac{1}{c} \left(\frac{b_i}{a+b_i}\right)^{\frac{1}{a}} \text{ for } i = 1, 2, \tag{12}$$

which yields her expected payoff

$$V_i(0) = \frac{a}{a+b_i}\left(\frac{b_i}{a+b_i}\right)^{\frac{b_i}{a}} c^{-b_i} \text{ for } i = 1, 2. \tag{13}$$

The party that waits, on the other hand, obtains the expected utility $w_i$ from the residual given by (7). The latter cannot be evaluated in terms of elementary functions when $b_i < 1$.

Now, consider the modified noisy Nash demand game. If the opponent waits, then, using (10) and (13), one obtains

$$V_i(0) < W_i(\beta) \text{ if and only if } b_i < \frac{a}{\beta_i^{-a} - 1} \equiv g(a, \beta_i), \tag{14}$$

for $i = 1, 2$. If $b_i < g(a, \beta_i)$ and $b_{3-i} > g(a, \beta_{3-i})$, then there is a *unique* strict equilibrium (rather than two, as in Proposition 1), where player $3 - i$ demands $x_{3-i} = \xi_{3-i}(0)$ and player $i$ waits for the residual, for $i = 1, 2$. Hence, if the modified noisy Nash demand game has a unique asymmetric equilibrium, the less risk averse party will tend to commit, and the more risk averse party will tend to wait.

If both parties are sufficiently risk averse ($b_i \approx 0$ for $i = 1, 2$) and $a > 0$, then both parties waiting is the unique equilibrium, because $g(a, \beta_i) > 0$. If both parties are sufficiently risk tolerant and the stakes $a$ are very high, so that $b_i > g(a, \beta_i)$ for $i = 1, 2$, then the modified noisy Nash demand game has again precisely two asymmetric (strict) equilibria, because $\lim_{a\to\infty} g(a, \beta_i) = 0 < b_i$ for $i = 1, 2$.

Next, consider the alternative model of endogenous timing. Here, that both parties wait, that one intends to make a demand and the other not, and that both intend to make demands can all be equilibria, depending on parameter values.

That in equilibrium both parties wait occurs if $a$ is small enough. For, $F(\xi_i(0)) = b_i/(a + b_i) > 1/(1 + \beta_i)$ if and only if $b_i > a/\beta_i$, from (12). Therefore, if $a$ is small enough, so that $b_i > a/\beta_i$ for $i = 1, 2$, then both parties waiting, $(t_1, t_2) = (1, 1)$, constitutes a subgame perfect equilibrium of the endogenous timing game by Proposition 4.

Asymmetric equilibria can obtain if $1 < a < 1.4$, risk aversion is small, and the split in the subgame, where both decided to wait, is sufficiently balanced, i.e. $\beta \approx 1/2$. Numerical computation shows that at $b_1 = b_2 = 1$ and $\beta = 1/2$ the difference $v_i(x^*) - w_i$ is negative if and only if $1 < a < 1.4$.

Moreover, $.854 < g(a, 1/2) < 1$ for $1 < a < 1.4$, so $b_i \approx 1 > g(a, 1/2) \approx g(a, \beta_i)$ implies $V_i(0) > W_i(\beta)$ for $i = 1, 2$ from (14). Hence, by continuity, for sufficient risk tolerance and $\beta_i \approx 1/2$ for $i = 1, 2$ that one party intends to make a demand and the other not is a strict equilibrium of the truncation. (The existence of a third, mixed equilibrium follows.)

That it can be an equilibrium that both parties intend to make demands follows from Proposition 3, because distributions in the present class satisfy (4) for $\varepsilon > 0$ if $\varepsilon c + \varepsilon^{1/a} > a/(1+a) > (1-\varepsilon)^{1/a} - \varepsilon c$ which becomes $1 + \varepsilon c > 1 > 1 - \varepsilon c$ as $a \to \infty$.

Hence, in both models it entirely depends on parameters of preferences and the prior which bargaining behavior will prevail. When uncertainty is not negligible, “anything goes”.

This also applies to duopoly models of endogenous timing.

# 5 Conclusions

This paper considers two natural modifications of the Nash demand game with noise. In the first variant, players decide to commit to a demand before uncertainty resolves or wait until it has resolved. In particular, players are allowed to remain silent. The effect of this change is dramatic. For small uncertainty, the game has two strict equilibria, in each of which one party demands approximately the whole surplus, and the other party waits for the residual. This outcome resembles the subgame perfect equilibrium of ultimatum bargaining and is very different from the Nash bargaining solution.

In the second variant, players commit to whether or not they will make a demand, but demands have to be stated only after each player has been informed whether or not her opponent is also committed. In this variant, the support for the Nash bargaining solution is restored. The “game with observable delay”, where subgames correspond to intentions of players, yields a unique subgame perfect equilibrium, where both parties intend to make demands and divide the surplus evenly, provided uncertainty is small.

Both models, however, yield an inefficiency, because by stating demands prior to the resolution of uncertainty players risk a breakdown of negotiations. But, if uncertainty is small, this efficiency loss is also small.

If uncertainty is large, “anything goes”, i.e., the structure of equilibria depends entirely on preference parameters and the prior distribution. Though we conduct the analysis in the framework of bargaining, this caveat also applies to duopoly models of endogenous timing, that have a close enough payoff structure. In duopoly models of endogenous timing, the shape of the profit function plays a formally analogous role as risk aversion and the shape of $F$ do here. For instance, the example from the previous section, with $a = b_1 = b_2 = c = 1$, $\beta = 1/2$, and $F$ uniform on the unit interval, resembles a duopoly model with linear demand function, with slope $-1$ and intercept 1, and zero marginal costs. Therefore, by appropriately perturbing demand and cost functions, one may generate both the Cournot (1838) solution and the equilibrium of the Stackelberg (1951) game. Hence, the caveat that "anything goes" may also be of relevance for such oligopoly models.

# 6 Appendix

**Proof of Lemma 1:** (a) Differentiating (6) at a point where it equals zero with respect to $x$ yields

$$\begin{aligned}
& u_i''(x)\,[1-F(x+y)] - 2u_i'(x)f(x+y) - u_i(x)f'(x+y) \\
= \; & u_i''(x)\,[1-F(x+y)] - \frac{2u_i(x)f(x+y)^2}{1-F(x+y)} - u_i(x)f'(x+y) \\
\leq \; & u_i''(x)\,[1-F(x+y)] - \frac{2u_i(x)f(x+y)^2}{1-F(x+y)} + \frac{u_i(x)f(x+y)^2}{1-F(x+y)} \\
= \; & u_i''(x)\,[1-F(x+y)] - \frac{u_i(x)f(x+y)^2}{1-F(x+y)} < 0
\end{aligned}$$

because the assumption of a monotone hazard rate,

$$\frac{d}{dx}\left(\frac{f(x+y)}{1-F(x+y)}\right) = \frac{f'(x+y)}{1-F(x+y)} + \left(\frac{f(x+y)}{1-F(x+y)}\right)^2 \geq 0$$

is equivalent to $f(x+y)^2/[1-F(x+y)] \geq -f'(x+y)$ whenever $F(x+y) < 1$. Therefore, the maximum of $v_i(x,y)$ is unique and given by the solution $\xi_i$ of (6) with respect to $x$. By the implicit function theorem $\xi_i$ is a continuous function of the parameter $y$. Implicitly differentiating (6) yields

$$\frac{d\xi_i}{dy} = \frac{u_i'(\xi_i)f(\xi_i+y) + u_i(\xi_i)f'(\xi_i+y)}{u_i''(\xi_i)\,[1-F(\xi_i+y)] - 2u_i'(\xi_i)f(\xi_i+y) - u_i(\xi_i)f'(\xi_i+y)} \leq 0$$

because at a point where $u_i' = u_i f/(1-F)$ holds, $u_i' f + u_i f' = u_i f^2/(1-F) + u_i f' \geq u_i f^2/(1-F) - u_i f^2/(1-F) = 0$, by the monotone hazard rate assumption, and $\partial^2 v_i/\partial x^2 < 0$ by the second-order condition. That $d\xi_i/dy > -1$ follows from $u_i'(\xi_i)f(\xi_i + y) > 0$.

The first order condition (6) holds if and only if $u_i'(x) - u_i(x)h(x+y) = 0$, where $h(z) = f(z)/[1 - F(z)]$ is the hazard rate. Since $u_i'(x)/u_i(x) \leq u_i'(0)/u_i(0)$ and $h(x)$ is nondecreasing, $\xi_i(0)$ is bounded and, by $\xi_i'(y) \leq 0$, the whole function $\xi_i$ is bounded.

(b) Next, by the envelope theorem, the derivative of the maximum is given by $\partial V_i(y)/\partial y = -u_i(\xi_i)f(\xi_i + y) < 0$, as required in the second claim.

(c) To see the third claim, let $r_i(x) = -u_i''(x)/u_i'(x)$ denote the coefficient of absolute risk aversion. With the appropriate normalization the utility function can be written as

$$u_i(x) = \int_0^x e^{-\int_0^\tau r_i(t)\ dt} d\tau$$

Now let $\hat{r}_i(x) > r_i(x)$ and set $r_i^\lambda(x) = \lambda\hat{r}_i(x) + (1-\lambda)r_i(x)$ for all $x \geq 0$ and all $0 \leq \lambda \leq 1$. Implicitly differentiating the first-order condition

$$1 - h(x+y)\int_0^x e^{\int_\tau^x r_i^\lambda(t)\ dt} d\tau = 0$$

(with the hazard rate substituted in) yields

$$\frac{dx}{d\lambda} = \frac{h(x+y)\int_0^x \int_\tau^x [r_i(t) - \hat{r}_i(t)]\, dt\ e^{\int_\tau^x r_i^\lambda(t)\ dt} d\tau}{h'(x+y)\int_0^x e^{\int_\tau^x r_i^\lambda(t)\ dt} d\tau + h(x+y)\left[1 + r_i^\lambda(x)\int_0^x e^{\int_\tau^x r_i^\lambda(t)\ dt} d\tau\right]} < 0$$

Hence, the reaction function $\xi_i$ is decreasing in the risk aversion coefficient. This completes the proof of Lemma 1.

**Proof of Proposition 4:** If $F$ is continuous, it can be written in terms of its hazard rate by

$$F(z) = 1 - e^{-\int_0^z h(x)\ dx}, \text{ for all } z \geq 0 \tag{15}$$

By concavity and monotonicity of $u_i$, integrating by parts, the monotone hazard rate, and (6)

$$\int_0^\infty u_i(\beta_i z)\ dF(z) = \int_0^\infty u_i(\beta_i z)\, h(z) e^{-\int_0^z h(t)\, dt}\ dz$$
$$\geq\ \beta_i \int_0^\infty u_i(z)\, h(z) e^{-\int_0^z h(t)\, dt}\ dz \geq \beta_i \int_0^{\xi_i(0)} u_i(z)\, h(z) e^{-\int_0^z h(t)\, dt}\ dz$$

$$
\begin{aligned}
&+ \quad \beta_i u_i(\xi_i(0)) e^{-\int_0^{\xi_i(0)} h(z)\,dz} = \beta_i \int_0^{\xi_i(0)} u_i'(z)\, e^{-\int_0^z h(t)\,dt}\, dz \\
&\geq \quad \beta_i u_i'(\xi_i(0)) \int_0^{\xi_i(0)} e^{-\int_0^z h(t)\,dt}\, dz = \beta_i u_i(\xi_i(0)) h(\xi_i(0)) \int_0^{\xi_i(0)} e^{-\int_0^z h(t)\,dt}\, dz \\
&\geq \quad \beta_i u_i(\xi_i(0)) \int_0^{\xi_i(0)} h(z)\, e^{-\int_0^z h(t)\,dt}\, dz = \beta_i u_i(\xi_i(0)) \left[ 1 - e^{-\int_0^{\xi_i(0)} h(z)\,dz} \right] .
\end{aligned}
$$

Since $V_i(0) = u_i(\xi_i(0)) \exp\left\{ -\int_0^{\xi_i(0)} h(z)\,dz \right\}$, it follows that (9) implies

$$
\begin{aligned}
\frac{\beta_i}{1+\beta_i} \quad &\geq \quad e^{-\int_0^{\xi_i(0)} h(z)\,dz} \Rightarrow \beta_i u_i(\xi_i(0)) F\left(\xi_i(0)\right) \geq u_i(\xi_i(0)) e^{-\int_0^{\xi_i(0)} h(z)\,dz} \\
&\Rightarrow \quad \int_0^{\infty} u_i\left(\beta_i z\right)\, dF(z) \geq u_i(\xi_i(0)) e^{-\int_0^{\xi_i(0)} h(z)\,dz} = V_i(0) .
\end{aligned}
$$

This is precisely the condition required for the statement of the Proposition.

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