Computational Aspects of the open-loop Nash equilibrium in Linear Quadratic Games

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Abstract

In this paper we consider open-loop Nash equilibria of the linear-quadratic differential game. In Engwerda (1997) both necessary and sufficient conditions for existence of a solution for as well the finite-planning horizon case as well the infinite-planning horizon case were presented. Here we will consider computational aspects of this problem. In particular we consider convergence aspects of the finite-planning horizon solution if the planning horizon expands. An algorithm is presented to calculate all equilibria of the infinite-planning horizon case. Furthermore sufficient conditions on the system parameters are presented, which guarantee the existence of a unique solution for both the finite as the infinite horizon problem.

Keywords: Linear quadratic differential games, open-loop Nash equilibria, solvability conditions, Riccati equations

1 Introduction

The last decade there has been an increasing interest to study several problems in economics using a dynamic game theoretical setting. In particular in the area of environmental economics and macro-economic policy coordination this is a very natural framework to model problems (see e.g. de Zeeuw et al. (1991), Mäler (1992), Kaitala et al. (1992) and Dockner et al. (1985), Tabellini (1986), Fershtman et al. (1987), Petit (1989), Levine et al. (1994), van Aarle et al. (1995), Neck et al. (1995), Douven et al (1996)). Particularly in macro-economic policy coordination problems, the open-loop Nash strategy is often used as a benchmark to evaluate different control strategies. In Engwerda (1997) several aspects of open-loop Nash equilibria are studied of the standard linear-quadratic differential game as considered by Starr and Ho in (1969). Both necessary and sufficient conditions for existence of a unique solution for the finite-planning horizon case are given, and it is shown that there exist situations where the set of associated Riccati differential equations has no solution, whereas the problem does have an equilibrium. Furthermore, conditions are given under which this strategy converges if the planning horizon expands, and a detailed study of the infinite planning horizon case is given. In particular it is shown that, in general, the infinite horizon problem has no unique equilibrium and that the limit of the above mentioned converged strategy may be not an equilibrium for the infinite planning horizon problem.

In this paper we focus on the computational aspects of this problem. The aim of this paper is on the one hand to make clear that infinite horizon open-loop Nash equilibria can be relatively easily calculated. On the other hand it presents conditions on the system parameters from which one can conclude a priori existence of open-loop equilibrium strategies.

The outline of the paper is as follows. In section two we start by stating the problem analysed in this paper and recall some basic results. In section three, we present a numerical algorithm to verify existence and to calculate solution(s). We will see that the algorithm resembles a computational algorithm to calculate the solution of the linear quadratic regulator problem using the Hamiltonian approach which traces back to MacFarlane (1963) and Potter (1966) (see also e.g. Kučera (1991)). In section four we consider conditions on the system parameters which guarantee existence of a solution. The paper ends with some concluding remarks.

2 Preliminaries

In this paper we consider the problem where two parties (henceforth called players) try to minimize their individual quadratic performance criterion. Each player controls a different set of inputs to a single system, described by a differential equation of arbitrary order. We assume that both players have to formulate their strategy already at the moment the system starts to evolve and this strategy can not be changed once the system runs. So, the players have to minimize their performance criterion based on the information that they only know the differential equation and its initial state. We are looking now for combinations of pairs of strategies of both players which are secure against any attempt by one player to unilaterally alter his strategy. That is, for those pairs of strategies which are such that if one player deviates from his strategy he will only lose. In the literature on dynamic games this problem is well-known as the open-loop Nash non-zero-sum linear quadratic differential game (see e.g. Starr and Ho (1969), Simaan and Cruz (1973), Abou-Kandil and Bertrand (1986) or Başar and Olsder (1995) and the references quoted in this book). Formally the system we consider is as follows:

$$\dot{x} = Ax + B_1u_1 + B_2u_2,\ x(0) = x_0, \tag{1}$$

where x is the n-dimensional state of the system, u_i is an m_i-dimensional (control) vector player i can manipulate, x_0 is the initial state of the system, A, B_1, and B_2 are constant matrices of appropriate dimensions, and $\dot{x}$ denotes the time derivative of x.
The performance criterion player $i = 1, 2$ aims to minimize is:

$$J_i(u_1, u_2) := \frac{1}{2}x(t_f)^T K_{if} x(t_f) + \frac{1}{2}\int_0^{t_f} \{x(t)^T Q_i x(t) + u_i(t)^T R_i u_i(t)\} dt,$$

in which matrix R_i is positive definite, K_{if} is semi-positive definite, Q_i is semi-positive definite and additionally is positive definite w.r.t. the controllability subspace $< A, B_i >,\ i = 1, 2$.
Note that usually in literature each player's performance criterion also includes a cross term, penalizing the control efforts of the other player. Since, however, this cross term does not play a role in the analysis of open-loop Nash equilibria, we dropped this term here.
From Engwerda (1997) we recall the following results.

Let M denote the with this game associated Hamiltonian matrix

$$M := \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}.$$

Here $S_i := B_i R_i^{-1} B_i^T$. Then, using the following notation:

$$H(t_f) := W_{11}(t_f) + W_{12}(t_f)K_{1f} + W_{13}(t_f)K_{2f},$$

with $W(t_f) = (W_{ij}(t_f))\ \{i,j = 1,2,3;\ W_{ij} \in R^{n\times n}\} := \exp(Mt_f)$, and

$$P := \begin{pmatrix} I\ 0\ 0 & \\ 0\ 0\ 0 & \\ 0\ 0\ 0 & \end{pmatrix};\ Q := \begin{pmatrix} 0 & 0 & 0 \\ -K_{1f} & I & 0 \\ -K_{2f} & 0 & I \end{pmatrix},$$

we have

<u>Theorem 1:</u>
The two-player linear quadratic differential game (1) has a unique open-loop Nash equilibrium for every initial state if and only if matrix $H(t_f)$ is invertible. Moreover, the open-loop Nash equilibrium solution as well as the associated state trajectory can be calculated from the linear two-point boundary value problem

$$\dot{y}(t) = -My(t),\ \text{with } Py(0) + Qy(t_f) = (x_0^T\ 0\ 0)^T. \qquad \square$$

Next, consider the following set of coupled asymmetric Riccati-type differential equations:

$$\dot{K}_1 = -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2;\ K_1(t_f) = K_{1f} \qquad (2)$$

$$\dot{K}_2 = -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1;\ K_2(t_f) = K_{2f} \qquad (3)$$

Let $K_i(t)$ satisfy this set of Riccati equations and assume that player i uses the strategy

$$u_1(t) = -R_{11}^{-1} B_1^T K_1(t)\Phi(t,0)x_0 \qquad (4)$$

$$u_2(t) = -R_{22}^{-1} B_2^T K_2(t)\Phi(t,0)x_0, \qquad (5)$$

where $\Phi(t,0)$ is the solution of the transition equation $\dot{\Phi}(t,0) = (A - S_1K_1(t) - S_2K_2(t))\Phi(t,0)$; $\Phi(0,0) = I$.

Then, we have

Theorem 2:
If the set of Riccati equations (2,3) has a solution then the two-player linear quadratic differential game (1) has a unique open-loop Nash equilibrium for every initial state.
Moreover, the equilibrium strategies are then given by (4,5). $\square$

Moreover, it was shown in an example that there exist situations where the set of Riccati differential equations (2,3) does not have a solution, whereas there exists an open-loop Nash equilibrium for the game. This raises the question what the relationship is between solvability of the set of Riccati equations and existence of an equilibrium for the game. The next theorem provides an answer to this question

Theorem 3:
The following statements are equivalent:
1) For all $t_f \in [0, t_1]$ there exists a unique open-loop Nash equilibrium for the two-player linear quadratic differential game (1) defined on the interval $[0, t_f]$.
2) $H(t)$ is invertible for all $t_f \in [0, t_1]$.
3) The set of Riccati differential equations (2,3) has a solution on $[0, t_1]$. $\square$

The above theorem shows that for both computational purposes and for a better theoretical understanding of the open-loop problem it would be nice to have a global existence result for the set of Riccati differential equations (2,3). In section four we will present some sufficient conditions.
In our analysis of convergence properties of the equilibrium strategy and the infinite horizon case, the set of M-invariant subspaces plays a crucial role. Therefore we introduce a separate notation for this set:

$$\mathcal{M}^{inv} := \{\mathcal{T} \mid M\mathcal{T} \subset \mathcal{T}\}.$$

In particular the with (2,3) associated set of algebraic Riccati equations

$$\left.\begin{array}{l} 0 = -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2; \\ 0 = -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1; \end{array}\right\} \quad (ARE)$$

can be calculated directly from the following collection of M-invariant subspaces:

$$\mathcal{K}^{pos} := \left\{ \mathcal{K} \in \mathcal{M}^{inv} | \mathcal{K} \oplus Im \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = \mathbb{R}^{3n} \right\}.$$

Here the symbol $\oplus$ is used to denote the sum of subspaces.
Note that elements in the set $\mathcal{K}^{pos}$ can be calculated using the set of matrices

$$K^{pos} := \left\{ K \in \mathbb{R}^{3n \times n} | \mathrm{ImK} \oplus Im \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = \mathbb{R}^{3n} \right\}.$$

The exact result on how all solutions of (ARE) can be calculated is given in the next theorem. Here we use the notation $M|_{\mathcal{K}}$ to denote the restriction of the linear transformation induced by M to the subspace $\mathcal{K}$ (see e.g. Lancaster et al. (1985, p.142)). Furthermore we use the notation $\sigma(X)$ to denote the spectrum of a matrix X.

Theorem 4:
(ARE) has a real solution (K_1, K_2) if and only if $K_1 = YX^{-1}$ and $K_2 = ZX^{-1}$ for some $\mathcal{K} =: Im \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathcal{K}^{pos}$.
Moreover, if the control functions $u_i^*(t) = -R_{ii}^{-1} B_i^T K_i \Phi(t) x_0$ are used to control the system (1), the spectrum of the closed-loop matrix $A - S_1 K_1 - S_2 K_2$ coincides with $\sigma(-M|_{\mathcal{K}})$. □

The proof of this theorem is given in the appendix.
Note that every element of $\mathcal{K}^{pos}$ defines exactly one solution of (ARE). Furthermore, this set contains only a finite number of elements if the geometric multiplicities of all eigenvalues of M is one (see e.g. Lancaster and Tismenetsky (1985)). So, in that case we immediately conclude that (ARE) will have at most a finite number of solutions.

Now, we consider the question how the open-loop equilibrium solution changes when the planning horizon t_f tends to infinity. To study convergence properties of the equilibrium solution for the game, it seems reasonable to require that problem (1) has a properly defined solution for every finite planning horizon. Therefore we make the following well-posedness assumption (see theorem 1)

$$H(t_f) \text{ is invertible for all } t_f < \infty. \tag{6}$$

Of course, this assumption is difficult to verify in practice. It stresses once more the need to find general conditions under which the set of Riccati differential equations (2,3) will have a solution on $(0, \infty)$.
To derive general convergence results we define first.

Definition 5:
M is called dichotomically separable if there exist subspaces V_1 and V_2 such that $M\,V_i \subset V_i, i = 1, 2,\ V_1 \oplus V_2 = \mathbb{R}^{3n}$, where dim $V_1 = n$, dim $V_2 = 2n$, and moreover $Re\, \lambda > Re\, \mu$ for all $\lambda \in \sigma(M|_{V_1}), \mu \in \sigma(M|_{V_2})$. $\square$

Theorem 6:
Assume that the well-posedness assumption (6) holds.
If M is dichotomically separable and Span $\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \oplus V_2 = \mathbb{R}^{3n}$, then

$$K_1(0, t_f) \to Y_0 X_0^{-1}, \text{ and } K_2(0, t_f) \to Z_0 X_0^{-1}.$$

Here X_0, Y_0, Z_0 are defined by (using the notation of definition 5) $V_1 =:$ Span$(X_0^T\ Y_0^T\ Z_0^T)^T$. $\square$

Next we consider the case that the performance criterion player $i = 1, 2$ likes to minimize is given by:

$$\lim_{t_f \to \infty} J_i(u_1, u_2),$$

The information structure is similar to the finite-planning horizon case. Each player only knows the initial state of the system and has to choose a control for the entire infinite time horizon. So, the actions are now described as functions of time, where time runs from zero to infinity. Since we only like to consider those outcomes of the game that yield a finite cost to both players, we restrict ourselves to consider only control functions belonging to the following set

$$U := \{ \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, t \in [0, \infty) | \lim_{t_f \to \infty} J_i(u_1, u_2) < \infty,\ i = 1, 2.\}$$

Note that a necessary condition for this set to be nonempty is that in the system both (A, B_1) and (A, B_2) are stabilizable. From now on, we will assume that the system satisfies these stabilizability conditions.
Moreover, we require that our open-loop equilibrium strategies allow for a feedback synthesis. That is, the closed-loop dynamics of the game can be described by: $\dot{x}(t) = Fx(t);\ x(0) = x_0$ for some constant matrix F.
In the appendix we prove that:

<u>Theorem 7:</u>
The infinite-planning horizon two-player linear quadratic differential game has for every initial state an open-loop Nash equilibrium strategy $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ if and only if there exist K_1 and K_2 that are solutions of the algebraic Riccati equations (ARE) satisfying the additional constraint that the eigenvalues of $A_{cl} := A - S_1K_1 - S_2K_2$ are all situated in the left half complex plane.
In that case, the strategy

$$u_i(t) = -R_{ii}^{-1} B_i^T K_i \Phi(t, 0) x_0, i = 1, 2,$$

where $\Phi(t, 0)$ satisfies the transition equation $\dot{\Phi}(t, 0) = A_{cl}\Phi(t, 0);\ \Phi(0, 0) = I$, is an open-loop Nash equilibrium strategy.
Moreover, the costs obtained by using this strategy for the players are $x_0^T M_i x_0,\ i = 1, 2$, where M_i is the unique positive semi-definite solution of the Lyapunov equation

$$A_{cl}^T M_i + M_i A_{cl} + Q_i + K_i^T S_i K_i = 0, \tag{7}$$

□

Combination of the results from theorem 4, 6 and 7 yields then

<u>Corollary 8:</u>
Assume that the planning horizon t_f in the differential game (1) tends to infinity and the following conditions are satisfied

1. $H(t_f)$ is invertible for all t_f;

2. M is dichotomically separable;

3. Span $\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \oplus V_2 = \mathbb{R}^{3n}$

4. $Re\ \lambda > 0, \forall \lambda \in \sigma(M|_{V_1})$.

Then the unique open-loop Nash equilibrium solution converges to a (stationary) strategy

$$u_i^*(t) = -R_{ii}^{-1} B_i^T K_i \Phi(t,0) x_0, i = 1, 2,$$

where $\Phi(t,0)$ satisfies the transition equation $\dot{\Phi}(t,0) = (A - S_1K_1 - S_2K_2)\Phi(t,0)$; $\Phi(0,0) = I$.
In these equations the constant matrices $K_i, i = 1, 2$, can be calculated from the eigenspaces of matrix M (see theorem 6). Moreover, the strategies will stabilize the closed-loop system and are also equilibrium solutions to the infinite planning horizon game. □

3 Computational aspects and some illustrative examples

The above considerations yield the following numerical algorithm to verify whether this two-player game has an open-loop equilibrium. Moreover, if such an equilibrium exists, this algorithm immediately yields the appropriate control strategies. Note that the assumptions we made thus far on the system imply that the system should satisfy the following conditions: $Q_i \geq 0$, $R_i > 0$, (A, B_i) stabilizable and (Q_i, A) detectable. Provided these assumptions hold we have

Algorithm 9:

Step 1 : Calculate $M := \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$

Step 2 : Calculate the spectrum of matrix M.
If the number of positive eigenvalues (counted with algebraic multiplicities) is less than n, goto Step 5.

Step 3 : Calculate all M invariant subspaces $\mathcal{K} \in \mathcal{K}^{pos}$ for which $Re\lambda > 0$ for all $\lambda \in \sigma(M|_{\mathcal{K}})$. If this set is empty, goto Step 5.

Step 4 : Let $\mathcal{K}$ be an arbitrary element of the set determined in Step 3.
Calculate 3 nxn matrices X, Y and Z such that $Im \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathcal{K}$.
Denote $K_1 := YX^{-1}$ and $K_2 := ZX^{-1}$. Then
$u_i^*(t) := -R_i^{-1}B_i^T K_i \Phi(t)x_0$ is an open – loop Nash equilibrium strategy.
The spectrum of the corresponding closed – loop matrix $A - S_1K_1 - S_2K_2$ equals $\sigma(-M|_{\mathcal{K}})$.
If the set determined in step 3 contains more elements one can repeat this step to calculate different equilibria.

Step 5 : End of algorithm.

One remark we like to make here is that although the algorithm may yield infinitely many different solutions K_i, there are at most $\begin{pmatrix} 2n \\ n \end{pmatrix}$ different structures for the eigenvalues of the closed-loop system.
In the next example we illustrate the algorithm. In particular the example illustrates that there may be more than one equilibrium, even in case matrix A is stable:

<u>Example 10</u>
Let $A = \begin{pmatrix} -0.1 & 0 \\ 0 & -2 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0.1 \end{pmatrix}$, $Q_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $R_1 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, and $R_2 = 1$. Note that matrix A is stable, and thus (A, B_i), $i = 1, 2$, is stabilizable. Furthermore, both Q_i and R_i are positive definite. So, all assumptions we made on the system and the performance criteria are satisfied.

The first step in the algorithm is to calculate M. This yields:

$$M = \begin{pmatrix} 0.1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 2 & 0 & 0 \\ 1 & 0 & -0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 & -0.1 & 0 \\ 1 & 2 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

The second step is to calculate the spectrum of M. Numerical calculations show that $M = TJT^{-1}$, where J is a diagonal matrix with entries $\{-2; 2.2073; 1.0584; -2.0637; 0.1648; -1.4668\}$ and

$$T = \begin{pmatrix} 0 & 0.2724 & -0.6261 & -0.0303 & -0.1714 & 0.3326 \\ 0 & 0.7391 & 0.5368 & -0.0167 & 0.3358 & 0.0633 \\ 0 & 0.1181 & -0.5405 & 0.0154 & -0.6473 & -0.2433 \\ 0 & 0.0176 & 0.0176 & 0.0262 & 0.0155 & 0.0119 \\ 0 & 0.4384 & -0.0771 & 0.0239 & 0.6207 & -0.2897 \\ 1 & 0.4161 & 0.1463 & 0.9987 & 0.2311 & 0.8614 \end{pmatrix}.$$

We see that M has six different eigenvalues, three of them are positive. So, there are at most $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3$ different equilibrium strategies.
We proceed with step 3 of the algorithm. Introduce the following notation $T =: (T_1\ T_2\ T_3\ T_4\ T_5\ T_6)$. First consider $\mathcal{K}_1 := (T_2\ T_3)$. The first 2x2 block of this matrix is given by $\begin{pmatrix} 0.2724 & -0.6261 \\ 0.7391 & 0.5368 \end{pmatrix}$. This matrix is invertible. So, $\mathcal{K}_1$ is an element of $\mathcal{K}^{pos}$ for which $\sigma(M|_{\mathcal{K}_1}) = \{2.2073, 1.0584\}$. That is, it satisfies all conditions mentioned in step 3 of the algorithm. So, it is an appropriate element. In a similar way it can be verified that also $\mathcal{K}_2 := (T_2\ T_5)$ and $\mathcal{K}_3 := (T_3\ T_5)$ are appropriate elements. So, step 3 yields three M invariant subspaces satisfying the conditions.
In step 4 we calculate the actual equilibrium strategies. From step 3 we have that there are three different equilibrium strategies. We will calculate the equilibrium strategy resulting from $\mathcal{K}_3$. To that end we factorize $\mathcal{K}_3$ as

follows

$$\mathcal{K}_3 = \begin{pmatrix} -0.6261 & -0.1714 \\ 0.5368 & 0.3358 \\ -0.5405 & -0.6473 \\ 0.0176 & 0.0155 \\ -0.0771 & 0.6207 \\ 0.1463 & 0.2311 \end{pmatrix} =: \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

where X, Y and Z are 2x2 matrices.
Then $K_1 := YX^{-1} = \begin{pmatrix} -0.5405 & -0.6473 \\ 0.0176 & 0.0155 \end{pmatrix} \begin{pmatrix} -0.6261 & -0.1714 \\ 0.5368 & 0.3358 \end{pmatrix}^{-1}$ and $K_2 := ZX^{-1} = \begin{pmatrix} -0.0771 & 0.6207 \\ 0.1463 & 0.2311 \end{pmatrix} \begin{pmatrix} -0.6261 & -0.1714 \\ 0.5368 & 0.3358 \end{pmatrix}^{-1}$. The corresponding open-loop Nash strategy is then $u_i^*(t) := -R_i^{-1}B_i^T K_i \Phi(t)x_0$. The spectrum of the corresponding closed-loop matrix $A - S_1K_1 - S_2K_2$ equals $\{-1.0584, -0.1648\}$. □

The next example illustrates the phenomenon that there exist situations in which the finite planning horizon game always has an equilibrium and, even stronger, this strategy converges if the planning horizon expands, whereas the corresponding infinite planning horizon game has no equilibrium strategy.

Example 11
Let $A = \begin{pmatrix} -1 & 0 \\ 0 & -5/22 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0.01 \end{pmatrix}$, $Q_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $R_1 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, and $R_2 = 1$. Note that (A, B_i), $i = 1, 2$, is stabilizable and both Q_i and R_i are positive definite. So, again, all assumptions we made on the system and the performance criteria are satisfied.
Next we calculate M and its spectrum. Numerical calculations show that

$M = TJT^{-1}$, where

$$J = \begin{pmatrix} \frac{-5}{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.7978 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.8823 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0319 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.4418 + 0.1084i & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.4418 - 0.1084i \end{pmatrix}, \text{ and}$$

$$T = \begin{pmatrix} 0 & 0.7271 & 0.3726 & 0.0410 & -0.0439 - 0.0222i & -0.0439 + 0.0222i \\ 0 & 0.1665 & 0.2013 & -0.1170 & 0.1228 + 0.0699i & 0.1228 - 0.0699i \\ 0 & 0.2599 & -0.4222 & 0.0423 & -0.0833 - 0.0236i & -0.0833 + 0.0236i \\ 0 & 0.0008 & -0.0012 & -0.0060 & -0.0033 - 0.0049i & -0.0033 + 0.0049i \\ 0 & 0.3194 & -0.6504 & -0.0786 & 0.1522 + 0.0558i & 0.1522 - 0.0558i \\ 1 & 0.5235 & -0.4684 & -0.9882 & -0.5289 - 0.8149i & -0.5289 + 0.8149i \end{pmatrix}.$$

With the following notation $T =: (T_1\ T_2\ T_3\ T_4\ T_5\ T_6)$ we see that M is dichotomically separable if we choose $V_1 := (T_2\ T_4)$ and $V_2 := (T_1\ T_3\ Re(T_5)\ Im(T_5))$. Note that $\sigma(M|_{V_1}) = \{1.7978, -0.0319\}$ and $\sigma(M|_{V_2}) = \{\frac{-5}{22}, -1.8823, -0.4418 \pm 0.1084i\}$.

Next, we choose $K_{1f} := K_{2f} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Numerical calculation shows that with these choices for the final cost, the determinant of $H(t)$ always differs from zero. That is, $H(t)$ is invertible for every positive t. So, the finite-planning horizon problem has a unique equilibrium for every t_f.

On the other hand, this implies that the well-posedness assumption (6) is satisfied. Since, moreover, M is dichotomically separable and Span $\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \oplus V_2 = \mathbb{R}^{3n}$, it is clear from theorem 6 that the equilibrium solution converges. This converged solution can be calculated from $(X_0^T\ Y_0^T\ Z_0^T)^T := (T_2\ T_4)$. The converged solutions of the Riccati equations are $K_1 := Y_0 X_0^{-1} = \begin{pmatrix} 0.2599 & 0.0423 \\ 0.0008 & -0.0060 \end{pmatrix} \begin{pmatrix} 0.7271 & 0.0410 \\ 0.1665 & -0.1170 \end{pmatrix}^{-1}$ and

$$K_2 := Z_0 X_0^{-1} = \begin{pmatrix} 0.3194 & -0.0786 \\ 0.5235 & -0.9882 \end{pmatrix} \begin{pmatrix} 0.7271 & 0.0410 \\ 0.1665 & -0.1170 \end{pmatrix}^{-1}.$$

The eigenvalues of the closed-loop system (1) using the converged open-loop strategies $u_i^*(t) = -R_{ii}^{-1} B_i^T K_i \Phi(t) x_0, i = 1, 2,$ are $\{-1.7978, 0.0319\}$. So, the

converged equilibrium solution is not a "stabilizing solution".
Next, use algorithm 6 to calculate equilibrium solution(s) for the infinite horizon game. We see that in step 2 the algorithm terminates. There exist no solutions to the infinite planning horizon game.
So we conclude that although the finite planning horizon game has always a solution and, even stronger, the corresponding equilibrium solution converges if the planning horizon expands, this converged strategy is not an equilibrium solution of the infinite-planning horizon game. □

4 Some sufficient conditions

In the previous sections we presented both necessary and sufficient conditions for existence of open-loop Nash equilibria in terms of the Hamiltonian matrix associated with this game, $M = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$. Now, these conditions are rather technical and they do not provide much insight into the question under which conditions on the system parameters one may expect that an equilibrium exists or, even more, under which conditions on the system parameters there will be a unique solution. From an application point of view these are rather relevant questions, and therefore we will consider these questions in this section in some more detail and present some preliminary results.
First, since the scalar case is often used in applications, we recall from Engwerda (1997) the following result (to stress the fact that we are dealing with the scalar case, the system parameters are put in lower case)

Theorem 12:
Assume that $s_1q_1 + s_2q_2 > 0$.
Then, the finite planning horizon open-loop Nash equilibrium solution converges to the (stationary) strategy:

$$u_i^*(t) = -\frac{1}{r_i} b_i k_i e^{(a - s_1k_1 - s_2k_2)} x_0,\ i = 1, 2$$

where $k_i = \frac{(a+\mu)q_i}{s_1q_1+s_2q_2}$, $i = 1, 2$, and $\mu = \sqrt{a^2 + s_1q_1 + s_2q_2}$.
Moreover, these strategies are the unique solution to the infinite-planning horizon open-loop problem. □

Another case in which we can conclude that there will be at least one solution is if, roughly spoken, either the weight matrices Q_i or the matrices S_i are proportional. That is assume that there exist matrices S and $C_i,\ i=1,2$, where C_1 is invertible, such that either one of the following two properties holds:

$$\begin{array}{rl} I. & 1) S_i = SC_i,\ i=1,2;\ 2) A^TC_i = C_iA^T,\ i=1,2;\ 3) S+S^T \geq 0; \\ & 4) C_1Q_1 + C_2Q_2 + (C_1Q_1+C_2Q_2)^T \geq 0;\ 5) C_1K_{1f} + C_2K_{2f} + (C_1K_{1f}+C_2K_{2f})^T \geq 0. \\ II. & 1) Q_i = C_iS,\ i=1,2;\ 2) A^TC_i = C_iA^T,\ i=1,2;\ 3) S+S^T \geq 0; \\ & 4) S_1C_1 + S_2C_2 + (S_1C_1+S_2C_2)^T \geq 0;\ 5) K_{1f}C_1 + K_{2f}C_2 + (K_{1f}C_1+K_{2f}C_2)^T \geq 0. \end{array}$$

Straightforward multiplication shows then that, e.g. under the assumption that condition I holds, we can factorize M as $M = VJV^{-1}$, where

$$J = \begin{pmatrix} -A & S & 0 \\ C_1Q_1 + C_2Q_2 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix} \text{ and } V = \begin{pmatrix} I & 0 & 0 \\ 0 & C_1^{-1} & -C_1^{-1}C_2 \\ 0 & 0 & I \end{pmatrix}. \quad (8)$$

Consequently,

$$H(t_f) = (I\ 0\ 0)e^{Mt_f} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} = (I\ 0\ 0)Ve^{Jt_f}V^{-1} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}$$

$$= (I\ 0\ 0)e^{Jt_f} \begin{pmatrix} I \\ C_1K_{1f} + C_2K_{2f} \\ K_{2f} \end{pmatrix} = (I\ 0)e^{J_1t_f} \begin{pmatrix} I \\ C_1K_{1f} + C_2K_{2f} \end{pmatrix},$$

where $J_1 := \begin{pmatrix} -A & S \\ C_1Q_1 + C_2Q_2 & A^T \end{pmatrix}$.

Now, similar to theorem 3 we have that matrix $(I\ 0)e^{J_1t_f} \begin{pmatrix} I \\ C_1K_{1f} + C_2K_{2f} \end{pmatrix}$ is invertible for all $t_f \in [0, t_1]$ if and only if the Riccati differential equation

$$\dot{K} = -A^TK - KA - (C_1Q_1 + C_2Q_2) + KSK;\ K(t_f) = C_1K_{1f} + C_2K_{2f} \quad (9)$$

has a solution on $[0, t_1]$. Feucht showed in (1994, lemma 7.1) that under the assumptions I.(3-5) this equation (9) has a solution. So, we conclude:

Theorem 14:
In case either one of the conditions I or II holds then the finite planning horizon game has a unique equilibrium. $\square$

Feucht also proved this theorem in (1994) without making the invertibility assumption on C_1. He showed the correctness of this theorem by a direct analysis of the set of Riccati equations (2, 3).
Note that by considering in condition II the case $C_1 = I$, $C_2 = \alpha I$, $\alpha > 0$, and S a semi-positive definite matrix we reobtain the case that $Q_1 = \alpha Q_2$ as studied by Abou-Kandil et al in (1986).
The next theorem shows that in the above mentioned particular case it is also possible to conclude that the infinite planning horizon game has at least one solution. The proof is given in the appendix.

Theorem 15:
In case either $S_1 = \alpha S_2$ or $Q_1 = \alpha Q_2$ ($\alpha > 0$), the infinite planning horizon game has at least one equilibrium. In case matrix A additionally is stable, the game has a unique equilibrium. $\square$

In particular we deduce by combining the results of these last two theorems 14 and 15 with corollary 8 that

Corollary 16:
Assume that the following conditions are satisfied:

1. matrix A is stable
2. either $S_1 = \alpha S_2$ or $Q_1 = \alpha Q_2$ ($\alpha > 0$)
3. Span $\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \oplus V_2 = \mathbb{R}^{3n}$

Then, both the finite and infinite planning horizon game have a unique equilibrium solution. Furthermore, if the planning horizon expands the finite planning horizon equilibrium converges to the solution of the infinite planning horizon game. $\square$

5 Concluding Remarks

In this paper we presented an algorithm from which easily can be deduced whether or not the infinite planning horizon linear quadratic game has an open-loop Nash equilibrium. Furthermore, the algorithm immediately gives the equilibrium strategies if they exist. A similar algorithm is also used in the study of the corresponding regulator problem. In fact various improvements have been suggested in literature to improve the numerical stability of that algorithm, which might also be relevant for our algorithm (see e.g. Laub (1979), Paige et al. (1981), Van Doorn (1981) and Mehrmann (1991)). Since, however, up to now usually game-theoretic analysis is restricted to small models (from a computational point of view), and nowadays good computer packages exist to compute eigenvectors, eigenvalues and the inverse of a matrix we view this as a subject that maybe in the future might be worth to elaborate.
A more important subject is, in our opinion, to find conditions on the system matrices from which one can conclude a priori whether or not the game will have a solution. That is to develop a better intuition on these games. We presented some sufficient conditions which, roughly spoken, say that if either the Q_i of R_i matrices are proportional then both the finite and infinite horizon game always have a solution. Moreover we showed that in case matrix A is additionally stable, the equilibrium strategy of the finite planning horizon game converges to the unique equilibrium strategy of the infinite planning horizon game. This, under a mild assumption that the cost on the final state in the finite planning horizon problem are chosen appropriately. It will be clear that this is only a preliminary result and that one may hope to find more general results.
We conclude this paper by noting that the results obtained here can be straightforwardly generalized to the N-player case.

Appendix

Proof of theorem 7:
The "$\Leftarrow$"-part of the theorem can be found in theorem 12 of Engwerda. In fact most of the $\Rightarrow$"-part was also proved in this theorem. Using the notation of that proof we recall that in particular it was shown that the

optimal strategies $\bar{u}_i(t),\ i = 1, 2$, satisfy

$$\bar{u}_i(t) = -R_{ii}^{-1} B_i^T z_{i,x_0}(t), \tag{10}$$

where $z_{i,x_0}(t) = \int_t^\infty e^{A^T(s-t)} Q_i x_{\bar{u}}(s) ds$. So, $x_{\bar{u}}(s)$ satisfies

$$\dot{x_{\bar{u}}}(t) = Ax_{\bar{u}}(t) - S_1 z_{1,x_0}(t) - S_2 z_{2,x_0}(t);\ x(0) = x_0. \tag{11}$$

Note that by assumption for arbitrary x_0, $x(t)$ and $z_{i,x_0}(t)$ converge to zero. Next, introduce the matrices $K_i = (z_{i,e_1}(0) ... z_{i,e_n}(0)),\ i = 1, 2$, and $X = (x_{\bar{u},e_1} ... x_{\bar{u},e_n})$ where e_i denotes the i^{th} unit vector in R^n and $x_{\bar{u},e_i}$ denotes the optimal trajectory corresponding with the initial state $x(0) = e_i$. From (11) it follows now immediately that X satisfies the equation

$$\dot{X}(t) = AX(t) - S_1 Z_1(t) - S_2 Z_2(t);\ X(0) = I, \tag{12}$$

with $Z_i = \int_t^\infty e^{A^T(s-t)} Q_i X(s) ds$. Now, due to our assumption on the considered control functions, the solution $X(s)$ of this equation is an exponential function. So, $X(s+t) = X(s)X(t)$, for any s, t. Using this, it is easily verified that the above differential equation can be rewritten as

$$\dot{X}(t) = AX(t) - S_1 K_1 X(t) - S_2 K_2 X(t);\ X(0) = I. \tag{13}$$

So, obviously $X(t) = e^{(A - S_1 K_1 - S_2 K_2)t}$ solves this equation (12). Furthermore, since due to our assumptions $X(t)$ converges to zero, it follows that matrix $A - S_1 K_1 - S_2 K_2$ is stable. Next we show that K_i solve (ARE). We have:

$$\begin{aligned}
-A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2 &= \\
-A^T \int_0^\infty e^{A^T s} Q_1 X(s) ds - Q_1 - \int_0^\infty e^{A^T s} Q_1 X(s) ds (A - S_1 K_1 - S_2 K_2) &= \\
-\int_0^\infty \frac{de^{A^T s}}{ds} Q_1 X(s) ds - Q_1 - \int_0^\infty e^{A^T s} Q_1 X(s)(A - S_1 K_1 - S_2 K_2) ds &= \\
-\int_0^\infty \frac{d(e^{A^T s} Q_1 X(s))}{ds} ds + \int_0^\infty e^{A^T s} Q_1 \frac{dX(s)}{ds} ds - Q_1 - \int_0^\infty e^{A^T s} Q_1 \frac{dX(s)}{ds} ds &= \\
Q_1 + \int_0^\infty e^{A^T s} Q_1 \frac{dX(s)}{ds} ds - Q_1 - \int_0^\infty e^{A^T s} Q_1 \frac{dX(s)}{ds} ds &= 0.
\end{aligned}$$

Similarly it can be shown that also $-A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1 = 0$.

Which proves the claim. $\square$

Proof of theorem 15:

Consider the factorization (8) of M. Since by assumption (A, B_i) is stabilizable, R_i is positive definite and (A, Q_i) is detectable (which follows immediately from our assumption on Q_i), there exist matrices $\bar{K}$ and $\bar{L}$ such that

$$A^T\bar{K} + \bar{K}A - \bar{K}S\bar{K} + (Q_1 + \alpha Q_2) = 0$$

and, dually,

$$A\bar{L} + \bar{L}A^T - \bar{L}(Q_1 + \alpha Q_2)\bar{L} + S = 0$$

which additionally satisfy, $\sigma(A - S\bar{K}) \subset C^-$ and $\sigma(-A^T + (Q_1 + \alpha Q_2)\bar{L}) \subset C^+$ (see e.g. Kailath (1980)).
From this it is easily verified that

$$J_1V = V\begin{pmatrix} -(A - S\bar{K}) & 0 \\ 0 & -(-A^T + (Q_1 + \alpha Q_2)\bar{L}) \end{pmatrix},$$

where $V = \begin{pmatrix} I & -\bar{L} \\ \bar{K} & I \end{pmatrix}$. So,

$$M = V_1V_2JV_2^{-1}V_1^{-1}, \tag{14}$$

where

$$V_1 := \begin{pmatrix} I & 0 & 0 \\ 0 & I & -\alpha I \\ 0 & 0 & I \end{pmatrix}; V_2 := \begin{pmatrix} I & -\bar{L} & 0 \\ \bar{K} & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

$$\text{and } J := \begin{pmatrix} -(A - S\bar{K}) & 0 & 0 \\ 0 & -(-A^T + (Q_1 + \alpha Q_2)\bar{L}) & 0 \\ (Q_2\ 0)\begin{pmatrix} I & -\bar{L} \\ \bar{K} & I \end{pmatrix} & & A^T \end{pmatrix}.$$

Next, consider matrix J. We first note that $J\begin{pmatrix} 0 \\ -\alpha I \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha I \\ I \end{pmatrix} A^T$.

Furthermore, assume that $J\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = -\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}(-A^T + (Q_1 + \alpha Q_2)\bar{L})$.

Then, in particular it follows that matrix X has to satisfy $-(A - S\bar{K})X =$

$-X(-A^T+(Q_1+\alpha Q_2)\bar{L})$.
Since $\sigma(A-S\bar{K})\cap\sigma(-A^T+(Q_1+\alpha Q_2)\bar{L})=\emptyset$, it follows that this Lyapunov equation has the unique solution $X=0$ (see e.g. Kailath (1980)). Now, assume

$$J\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} = -\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}(A-S\bar{K}).$$

Since every vector in this subspace $\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}$ is independent of the vectors from the other two invariant subspaces, it follows that X_0 is invertible. Furthermore, using the same arguments as above, it follows from the equation

$$-(-A^T+(Q_1+\alpha Q_2)\bar{L})Y_0 = -Y_0(A-S\bar{K}),$$

that $Y_0=0$.
So, we have that

$$J = V_3 J_3 V_3^{-1},$$

where $V_3 := \begin{pmatrix} X_0 & 0 & 0 \\ 0 & Y & -\alpha I \\ Z_0 & Z & I \end{pmatrix}$, and $J_3 := \begin{pmatrix} -(A-S\bar{K}) & 0 & 0 \\ 0 & -(-A^T+(Q_1+\alpha Q_2)\bar{L}) & 0 \\ 0 & 0 & A^T \end{pmatrix}$.
Substitution of J into (14) shows that

$$M\begin{pmatrix} X_0 \\ \bar{K}X_0-\alpha Z_0 \\ Z_0 \end{pmatrix} = -\begin{pmatrix} X_0 \\ \bar{K}X_0-\alpha Z_0 \\ Z_0 \end{pmatrix}(A-S\bar{K}).$$

Since X_0 is invertible, we conclude that the set mentioned in step 3 of algorithm 9 is nonempty. That is, the infinite horizon game has at least one equilibrium solution.
From the above arguments it is moreover clear, that in case matrix A is stable, there exist no more invariant subspaces corresponding with just unstable eigenvalues than the one we just constructed, which completes the proof. □

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