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Probability Theory and Related Fields

On the Relationship between the Almost Sure Stability of Weighted Empirical Distributions and Sums of Order Statistics

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Summary. We shall disclose a relationship between the almost sure stability of weighted empirical distribution functions and sums of order statistics. First we obtain an extension of a theorem due to Csáki on the almost sure stability of the standardized uniform empirical distribution function. This result is then shown to be an essential tool to derive a characterization of the almost sure stability of the sum of k_n upper order statistics from a sample of n independent observations from a distribution with positive support in the domain of attraction of a non-normal stable law, where $1 \leqq k_n \leqq n$ and $k_n \to \infty$ as $n \to \infty$.

1. Introduction and Statements of Main Results

We begin with a result on the almost sure behavior of the standardized uniform empirical distribution function, which is an extension of a theorem due to Csáki (1975, 1982). For this let $U_1, U_2, \ldots,$ be a sequence of independent uniform $(0, 1)$ random variables and for each $n \geqq 1$ let $U_{1,n} \leqq \ldots \leqq U_{n,n}$ and G_n denote the order statistics and right continuous empirical distribution function based on the first n of these random variables.

Theorem 1. *Let l_n be any sequence of positive constants, k be a fixed positive integer and $0 \leqq \nu \leqq \frac{1}{2}$.*

If $l_n \uparrow$ and

$$\sum_{n=1}^{\infty} \frac{1}{n l_n^{k/(1-\nu)}} < \infty, \tag{1.1}$$

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then

$$\lim_{n\to\infty} \sup_{U_{k,n}\leqq t\leqq 1} n^{\nu}|G_n(t)-t|/(t^{1-\nu} l_n)=0 \quad \text{a.s.} \tag{1.2}$$

If $n l_n^{1/(1-\nu)}\uparrow$ *and*

$$\sum_{n=1}^{\infty} \frac{1}{n l_n^{k/(1-\nu)}}=\infty, \tag{1.3}$$

then

$$\limsup_{n\to\infty} \sup_{U_{k,n}\leqq t\leqq 1} n^{\nu}|G_n(t)-t|/(t^{1-\nu} l_n)=\infty \quad \text{a.s.} \tag{1.4}$$

The case $k=1$ yields almost immediately that for any such sequence l_n

$$\limsup_{n\to\infty} \sup_{0\leqq t\leqq 1} n^{\nu}|G_n(t)-t|/(t^{1-\nu} l_n)=0 \text{ or } \infty \quad \text{a.s.}$$

according as (1.1) or (1.3) holds with $k=1$, which for $\nu=1/2$ is the original Csáki (1975, 1982) result. The case $\nu=0$ can be shown to be equivalent to Theorem 2 of Shorack and Wellner (1978). The statement for arbitrary $0\leqq\nu\leqq 1/2$ is Theorem 2 of Mason (1981).

Theorem 1 is likely to have a wide variety of applications in probability theory. Here we shall show that it is closely connected to the almost sure stability of sums of order statistics. For this purpose let F be a distribution function in the domain of attraction of a non-normal stable law with positive support, i.e.

$$F(0-)=0 \quad \text{and} \quad 1-F(x)=x^{-a}\tilde{L}(x), \qquad x>0, \tag{1.5}$$

for some $0<a<2$ and function $\tilde{L}$ which is slowly varying at infinity. Let

$$Q(s)=\inf\{x: F(x)\geqq s\}, \qquad 0<s\leqq 1,$$

with $Q(0)=Q(0+)$ denote the inverse or quantile function of F. Then (1.5) is equivalent to, for the same $0<a<2$,

$$Q(0+)\geqq 0 \quad \text{and} \quad Q(1-s)=s^{-1/a}L(s), \qquad 0<s<1, \tag{1.6}$$

where L is slowly varying at zero, cf. de Haan (1975), Corollary 1.2.1.5 or Seneta (1976), Lemma 1.8.

Let $X_1, X_2, \ldots,$ be independent random variables with common distribution function F as in (1.5), and for each $n\geqq 1$ let $X_{1,n}\leqq\ldots\leqq X_{n,n}$ denote the order statistics based on the first n of these random variables. Also let k_n denote any sequence of integers such that $1\leqq k_n\leqq n$ and $k_n\to\infty$. From S. Csörgő and Mason (1986) it follows that

$$Q(1-1/n)^{-1}\left\{\sum_{i=1}^{k_n} X_{n+1-i,n}-n\mu_n\right\}\xrightarrow{\mathscr{D}}\Delta_a, \tag{1.7}$$

where Δ_a denotes a completely asymmetric stable random variable of index a, and

$$\mu_n = \begin{cases} 0, & \text{if } 0<a<1 \\ \int_{1-k_n/n}^{1-1/n} Q(s)\,ds, & \text{if } a=1 \\ \int_{1-k_n/n}^{1} Q(s)\,ds, & \text{if } 1<a<2. \end{cases}$$

(Unless otherwise specified all convergence statements are meant to hold as $n \to \infty$.)

When $a \geqq 2$ in (1.5), S. Csörgő and Mason (1986) demonstrate that these sums of k_n upper order statistics, when properly centered and normalized converge in distribution to a standard normal random variable. The corresponding almost sure behavior for the case $a>2$ is completely described in Haeusler and Mason (1987). They show, in fact, more generally that for each fixed integer $k \geqq 1$ and sequence k_n converging to infinity fast enough the sums

$$\sum_{i=k}^{k_n} X_{n+1-i,n}, \tag{1.8}$$

when properly centered and normalized obey a law of the iterated logarithm, whereas, for all other such sequences k_n one has a stability result. In the sequel we shall show that in the case $a<2$ the almost sure behavior of these sums is different. It turns out that one now has a stability result for the sums in (1.8) for all sequences k_n of integers with $1 \leqq k_n \leqq n$ and $k_n \to \infty$, but never a law of the iterated logarithm.

For any sequence of non-decreasing constants b_n converging to infinity set

$$a_n = b_n Q(1-1/n). \tag{1.9}$$

Obviously by (1.7), we have for any sequence of positive integers $1 \leqq k_n \leqq n$ and $k_n \to \infty$ and sequence b_n as above that

$$a_n^{-1} \left\{ \sum_{i=1}^{k_n} X_{n+1-i,n} - n\mu_n \right\} \to 0, \tag{1.10}$$

in probability. Theorem 1 will be an important tool in characterizing those sequences b_n for which the convergence in probability to zero in (1.10) can be replaced by almost sure convergence to zero. Our results on this problem are contained in

Theorem 2. *Assume F is of the form (1.5) for some $0<a<2$. Let k_n be a sequence of positive integers such that $1 \leqq k_n \leqq n$ and $k_n \to \infty$ and let a_n be defined by (1.9)* *for some sequence of constants* $b_n \uparrow \infty$. *Then the following three statements are equivalent for any fixed integer* $k \geqq 1$:

$$\sum_{n=1}^{\infty} n^{k-1}(1-F(a_n))^k < \infty, \tag{1.11}$$

$$a_n^{-1} X_{n+1-k,n} \to 0 \quad a.s., \tag{1.12}$$

and there exists a sequence of constants c_n *such that*

$$a_n^{-1}\left\{\sum_{i=k}^{k_n} X_{n+1-i,n} - c_n\right\} \to 0 \quad a.s. \tag{1.13}$$

If (1.13) *is true, then one can choose* $c_n = n\mu_n$.

In addition, the following three statements are equivalent for any fixed integer $k \geqq 1$:

$$\sum_{n=1}^{\infty} n^{k-1}(1-F(a_n))^k = \infty, \tag{1.14}$$

$$\limsup_{n\to\infty} a_n^{-1} X_{n+1-k,n} = \infty \quad a.s., \tag{1.15}$$

and for any sequence of constants c_n

$$\limsup_{n\to\infty} a_n^{-1}\left|\sum_{i=k}^{k_n} X_{n+1-i,n} - c_n\right| = \infty \quad a.s. \tag{1.16}$$

Our Theorem 2 demonstrates that the almost sure behavior of the sums of order statistics in (1.8) is governed by the largest order statistic $X_{n+1-k,n}$ appearing in this sum. Assume that a_n is such that (1.16) holds for the whole sum $\sum_{i=1}^{k_n} X_{n+1-i,n}$ (i.e., $k=1$) and any sequence of centering constants c_n; then it can be inferred from the first part of the theorem whether it is possible to delete a fixed number of the upper extremes $X_{n,n}, \ldots, X_{n+1-k,n}$ from this sum to obtain a stability result as described by (1.13) or not. In the case that this is possible, the exact number can be determined from (1.11) and (1.14).

A similar phenomenon has been noticed before in the context of trimmed sums formed by deleting the top k largest observations in absolute value. For investigations along this line based on the methods of classical probability, the reader is referred to Mori (1976, 1977) and Maller (1984). Our method, on the other hand, utilizes a connection between weighted uniform empirical distributions and sums of order statistics. It is particularly suited to analysing the behavior of sums of the form (1.8), which cover the case of sums of extreme values (e.g., $k_n \to \infty$, but $k_n/n \to 0$ as $n \to \infty$) and complete sums (i.e., $k_n = n$ for all $n \geqq 1$) simultaneously.

The proof of Theorem 1 is given in Sect. 2 and the proof of Theorem 2 in Sect. 3.

2. Proof of Theorem 1

For convenient reference later on we record the following facts:

Fact 1 (Mori (1976); see also Kiefer (1972)). For any sequence of positive constants a_n with $a_n \downarrow$

$$P(U_{k,n} \leqq a_n \text{ i.o.}) = 0 \text{ or } 1$$

according as the series

$$\sum_{n=1}^{\infty} n^{k-1} a_n^k < \infty \text{ or } = \infty,$$

with the same assertion holding with $U_{k,n}$ replaced by $1 - U_{n+1-k,n}$.

Fact 2 (Csáki (1977)). For any $\alpha \in (0, \infty)$

$$\limsup_{n \to \infty} \sup_{(\log n)^\alpha / n \leqq t \leqq 1} (n/\log\log n)^{1/2} |G_n(t) - t| / t^{1/2} \leqq 2 \quad \text{a.s.}$$

For any positive integer k and $0 \leqq \nu \leqq 1/2$, let

$$\Delta_{n,k} = \sup_{U_{k,n} \leqq t \leqq 1} n^\nu |G_n(t) - t| / t^{1-\nu}.$$

We are now ready to prove the first part of Theorem 1. Choose any $0 \leqq \nu \leqq 1/2$. First assume that (1.3) holds and $n l_n^{1/(1-\nu)} \uparrow$. Since $n l_n^{1/(1-\nu)} \uparrow$ it follows from Fact 1 that for every $\varepsilon > 0$

$$P(U_{k,n} \leqq \min\{\varepsilon/(n l_n^{1/(1-\nu)}), k/(2n)\} \text{ i.o.}) = 1. \tag{2.1}$$

We have

$$\Delta_{n,k}/l_n \geqq n^\nu (k/n - U_{k,n}) / (l_n U_{k,n}^{1-\nu}).$$

Now (2.1) implies that for every $\varepsilon > 0$

$$\limsup_{n \to \infty} \Delta_{n,k}/l_n \geqq k/(2\varepsilon^{1-\nu}) \quad \text{a.s.}$$

Letting $\varepsilon \downarrow 0$ proves (1.4).

Next we assume that (1.1) holds and $l_n\uparrow$. We first consider the most difficult case $\nu=1/2$. Observe that we may assume without loss of generality that $l_n \leqq \log n$ for all large enough n. Set $b_n=(\log n)^{1/k}/n$ and write

$$\Delta_n^{(1)}=\begin{cases}\sup\limits_{U_{k,n}\leqq t\leqq b_n} n^{1/2}|G_n(t)-t|/t^{1/2} & \text{if } U_{k,n}\leqq b_n\\ 0 & \text{otherwise.}\end{cases}$$

We shall show that

$$\lim_{n\to\infty} \Delta_n^{(1)}/l_n=0 \quad \text{a.s.} \tag{2.2}$$

Let

$$\Delta_n^{\pm}=\begin{cases}\sup\limits_{U_{k,n}\leqq t\leqq b_n} \pm n^{1/2}(G_n(t)-t)/t^{1/2} & \text{if } U_{k,n}\leqq b_n\\ 0 & \text{otherwise.}\end{cases}$$

To establish (2.2) it is enough to prove that

$$\limsup_{n\to\infty} \Delta_n^{\pm}/l_n\leqq 0 \quad \text{a.s.} \tag{2.3$\pm$}$$

Notice that whenever l_n satisfies (1.1), then for any $\varepsilon>0$, εl_n also satisfies (1.1). Thus to verify (2.3+) it suffices to show that (1.1) implies

$$\limsup_{n\to\infty} \Delta_n^{+}/l_n\leqq 1 \quad \text{a.s.} \tag{2.4}$$

Define the following events:

$$A_n=\{\Delta_n^+\geqq l_n\} \quad \text{and} \quad C_n=A_n\cap A_{n-1}^c.$$

According to the Borel-Cantelli lemma we need to prove

$$\sum_{n=2}^{\infty} P(C_n)<\infty \tag{2.5}$$

and

$$\lim_{n\to\infty} P(A_n)=0. \tag{2.6}$$

For any integer $0\leqq i\leqq n$ define $x_{i,n}$ to be the $\min(y_{i,n}, b_n)$ where $y_{i,n}$ is the solution of the equation

$$ny+n^{1/2}\,l_n\,y^{1/2}=i. \tag{2.7}$$

Define $k_n=1+[(\log n)^{1/k}+l_n(\log n)^{1/(2k)}]$, where $[x]$ denotes the integer part of x. Notice that $l_n\uparrow$ along with (1.1) implies

$$\lim_{n\to\infty} l_n^2/(\log n)^{1/k}=\infty, \tag{2.8}$$

which in turn implies that

$$\lim_{n\to\infty} k_n/l_n^2 = 0. \tag{2.9}$$

Also note that trivially we have

$$x_{k_n,n} = b_n. \tag{2.10}$$

To establish (2.5) and (2.6) we require the following lemmas whose statements are assumed to hold for all large enough n.

Lemma 2.1.

$$\{\Delta_{n-1}^+ < l_{n-1}\} \subset \{(n-1)\, G_{n-1}(t) < tn + n^{1/2}\, l_n\, t^{1/2} : x_{k-1,n} < t \leqq b_n\}. \tag{2.11}$$

Proof. Let $\omega \in \{\Delta_{n-1}^+ < l_{n-1}\}$.

Case 1 ($U_{k,n-1} > b_n$).
In this case, we have $(n-1)\, G_{n-1}(t) \leqq k-1$ for all $0 < t \leqq b_n$, which implies that $(n-1)\, G_{n-1}(t) < tn + n^{1/2}\, l_n\, t^{1/2}$ for all $x_{k-1,n} < t \leqq b_n$.

Case 2 ($U_{k,n-1} \leqq b_n$).
In this case, we have $(n-1)\, G_{n-1}(t) < t(n-1) + (n-1)^{1/2}\, l_{n-1}\, t^{1/2}$ for all $U_{k,n-1} \leqq t \leqq b_n$, which since $l_n \uparrow$ implies

$$(n-1)\, G_{n-1}(t) < tn + n^{1/2}\, l_n\, t^{1/2} \qquad \text{for } U_{k,n-1} \leqq t \leqq b_n. \tag{2.12}$$

Notice that if $U_{k,n-1} \leqq y_{k-1,n}$, then $(n-1)\, G_{n-1}(y_{k-1,n}) \geqq k$, which contradicts (2.12). Thus if (2.12) holds we must have

$$U_{k,n-1} > y_{k-1,n}. \tag{2.13}$$

Now (2.13) implies that

$$(n-1)\, G_{n-1}(t) < tn + n^{1/2}\, l_n\, t^{1/2} \qquad \text{for } y_{k-1,n} < t < U_{k,n-1}. \tag{2.14}$$

Combining (2.12) and (2.14) and replacing $y_{k-1,n}$ by $x_{k-1,n}$ yields

$$\omega \in \{(n-1)\, G_{n-1}(t) < tn + n^{1/2}\, l_n\, t^{1/2} : x_{k-1,n} < t \leqq b_n\}.$$

This completes the proof of Lemma 2.1.

Lemma 2.2.

$$\{\Delta_n^+ \geqq l_n\} \subset \bigcup_{i=k}^{k_n} \{n G_n(t) \geqq nt + n^{1/2}\, l_n\, t^{1/2} \ \textit{for some}\ x_{i-1,n} < t \leqq x_{i,n}\}. \tag{2.15}$$

Proof. Let $\omega \in \{\Delta_n^+ \geqq l_n\}$. We must have $U_{k,n} \leqq b_n$. Since $x_{k_n,n} = b_n$ there exists $1 \leqq i \leqq k_n$ such that

$$U_{k,n} \in (x_{i-1,n}, x_{i,n}]. \tag{2.16}$$

If (2.16) holds with $i<k$, then $nG_n(x_{k,n})\geqq k$, which implies that ω is an element of the set on the right-hand side of (2.15). If (2.16) holds with $i\geqq k$, then there must exist a t with $y_{k-1,n}=x_{k-1,n}<U_{k,n}\leqq t\leqq b_n$ and $nG_n(t)\geqq nt+n^{1/2}\,l_n\,t^{1/2}$, which also implies that ω is an element of the set on the right-hand side of (2.15). This completes the proof of Lemma 2.2.

Lemma 2.3.

$$\{\Delta_{n-1}^{+}<l_{n-1}\}\cap\{\Delta_n^{+}\geqq l_n\}\subset\bigcup_{i=k}^{k_n}\{(n-1)\,G_{n-1}(y_{i,n})\geqq i-1;\,U_n\leqq y_{i,n}\}. \tag{2.17}$$

Proof. Observe that for any $k\leqq i\leqq k_n$

$$\begin{aligned}&\{(n-1)\,G_{n-1}(t)<nt+n^{1/2}\,l_n\,t^{1/2}\colon x_{k-1,n}<t\leqq b_n\}\\&\quad\cap\{nG_n(t)\geqq nt+n^{1/2}\,l_n\,t^{1/2}\text{ for some }x_{i-1,n}<t\leqq x_{i,n}\}\\&\quad\subset\{nG_n(t)\geqq i\text{ for some }x_{i-1,n}<t\leqq x_{i,n}\text{ and }(n-1)\,G_{n-1}(x_{i,n})<i\}\\&\quad\subset\{(n-1)\,G_{n-1}(x_{i,n})=i-1,\,U_n\leqq x_{i,n}\}\\&\quad\subset\{(n-1)\,G_{n-1}(y_{i,n})\geqq i-1,\,U_n\leqq y_{i,n}\}.\end{aligned}$$

Lemma 2.3 now follows from Lemmas 2.1 and 2.2.

We shall now show that (2.5) holds. Applying Lemma 2.3 we have

$$P(C_n)\leqq\sum_{i=k}^{k_n}P(U_n\leqq y_{i,n})\,P((n-1)\,G_{n-1}(y_{i,n})\geqq i-1)\leqq\sum_{i=k}^{k_n}\binom{n-1}{i-1}y_{i,n}^{i}.$$

From Eq. (2.7) we see that

$$y_{i,n}\leqq i^2/(n\,l_n^2). \tag{2.18}$$

Thus

$$P(C_n)\leqq\sum_{i=k}^{k_n}\binom{n-1}{i-1}\left(\frac{i^2}{n\,l_n^2}\right)^{i}\leqq\frac{1}{n\,l_n^{2k}}\sum_{i=k}^{k_n}\left(\frac{k_n}{l_n^2}\right)^{i-k}\frac{i^{i+k+1}}{i!}.$$

Using the Stirling formula we obtain

$$1/m!\leqq(e/m)^m.$$

We now see that the last summation given above is

$$\leqq\frac{1}{n\,l_n^{2k}}\sum_{i=k}^{k_n}\left(\frac{e\,k_n}{l_n^2}\right)^{i-k}e^k\,i^{k+1},$$

which since by (2.9) $k_n/l_n^2\to 0$, is for all n sufficiently large

$$\leqq\frac{1}{n\,l_n^{2k}}\sum_{i=k}^{\infty}\left(\frac{1}{2}\right)^{i-k}e^k\,i^{k+1}=\frac{1}{n\,l_n^{2k}}\,C_k$$

with $C_k<\infty$. Therefore, since we assume (1.1) we have (2.5).

We show next that (2.6) holds. Observe that by Lemma 2.2 for all large enough n

$$P(A_n) \leqq \sum_{i=k}^{k_n} P(nG_n(y_{i,n}) \geqq i) \leqq \sum_{i=k}^{k_n} \binom{n}{i} y_{i,n}^i, \tag{2.19}$$

which by the same bound as just given is for all n sufficiently large $\leqq C_k/l_n^{2k}$. Since by (2.8) $l_n \to \infty$ as $n \to \infty$ we have (2.6). Therefore we have established (2.4) and hence (2.3+).

Next consider (2.3−). Notice that trivially

$$\Delta_n^-/l_n \leqq (nb_n)^{1/2}/l_n. \tag{2.20}$$

Applying (2.8) we see that the right-hand side of (2.20) converges to zero, so we also have (2.3−). Statements (2.3+) and (2.3−) imply (2.2).

Set

$$\Delta_n^{(2)} = \sup_{b_n \leqq t \leqq 1} n^{1/2} |G_n(t) - t| / t^{1/2}.$$

Combining Fact 2 and (2.8) implies

$$\lim_{n \to \infty} \Delta_n^{(2)}/l_n = 0 \quad \text{a.s.} \tag{2.21}$$

Obviously, statements (2.2) and (2.21) imply (1.2).

Having established that under the above assumptions (1.2) holds for the case $v = 1/2$, we are now ready to prove the general case. Choose any $0 \leqq v < 1/2$ and notice that

$$\begin{aligned} &\sup_{U_{k,n} \leqq t \leqq 1} n^v |G_n(t) - t| / (t^{1-v} l_n) \\ &\leqq \left(\frac{1}{n l_n^{1/(1-v)} U_{k,n}} \right)^{1/2 - v} \sup_{U_{k,n} \leqq t \leqq 1} n^{1/2} |G_n(t) - t| / (t^{1/2} l_n^{1/(2(1-v))}) \\ &\equiv D_n^{(1)} D_n^{(2)}. \end{aligned}$$

Assumption (1.1) holding for this choice of v implies by Theorem 1 (for the case $v = 1/2$) that $D_n^{(2)}$ converges to zero almost surely and by Fact 1 that

$$\limsup_{n \to \infty} D_n^{(1)} \leqq 1 \quad \text{a.s.}$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2

Since the two sequences of random variables $X_1, X_2, \ldots,$ and $Q(U_1), Q(U_2), \ldots,$ are equal in law and consequently the two processes $\{X_{i,n}: 1 \leqq i \leqq n, n \geqq 1\}$ and

$\{Q(U_{i,n}):\ 1 \leqq i \leqq n,\ n \geqq 1\}$ are equal in law, too, we shall assume without loss of generality that $X_{i,n} = Q(U_{i,n})$ for $1 \leqq i \leqq n$ and $n \geqq 1$.

We shall require the following technical lemmas:

Lemma 3.1. *Let x_n and y_n be two sequences of positive constants such that $x_n = o(y_n)$ and $y_n \to 0$. Then for any $\beta < 0$, $y_n^\beta L(y_n) = o(x_n^\beta L(x_n))$.*

Lemma 3.2. *For all $0 < \varepsilon < 1/a$ and $0 < \gamma < \infty$ there exists a $\delta > 0$ such that for all $0 < s \leqq t < \delta$*

$$\tfrac{1}{2}(t/s)^{-1/a-\gamma} \leqq \frac{Q(1-t)}{Q(1-s)} = \frac{t^{-1/a} L(t)}{s^{-1/a} L(s)} \leqq 2(t/s)^{-1/a+\varepsilon}.$$

Lemma 3.3. *Let x_n and b_n be two sequences of positive constants converging to infinity. Then for all $\delta > 0$ and all sufficiently large n*

$$b_n^{-\delta} \leqq \tilde{L}(b_n x_n)/\tilde{L}(x_n) \leqq b_n^\delta.$$

Lemma 3.4. $1 - F(Q(1-s)) \sim s$ *as* $s \downarrow 0$.

Lemmas 3.1, 3.2 and 3.3 are direct consequences of the Karamata representation theorem and Lemma 3.4 is contained in the results in Sect. 1.5 of Seneta (1976).

We are now prepared to begin the proof of Theorem 2. Since the sequence a_n is non-decreasing, Lemma 3 in Mori (1976) implies the equivalence of (1.11) and (1.12), and (1.14) and (1.15), cf. also Theorem 4 in Hall (1979). Next we shall demonstrate that (1.11), respectively (1.12), imply (1.13).

We shall first show when (1.11) holds that for some fixed integer $l \geqq k$

$$a_n^{-1}\left\{\sum_{j=l+1}^{k_n} X_{n+1-j,n} - n \int_{1-k_n/n}^{1-l/n} Q(s)\,ds\right\} \to 0 \quad \text{a.s.} \tag{3.1}$$

Notice that for any fixed integer $l \geqq 1$ that by two integrations by parts, where the usual conventions about integration with respect to the left continuous function Q are applied,

$$\begin{aligned}
&a_n^{-1}\left\{\sum_{j=l+1}^{k_n} X_{n+1-j,n} - n \int_{1-k_n/n}^{1-l/n} Q(s)\,ds\right\} \\
&= a_n^{-1}\left\{n \int_{U_{n-k_n,n}}^{U_{n-l,n}} Q(s)\,dG_n(s) - n \int_{1-k_n/n}^{1-l/n} Q(s)\,ds\right\} \\
&= a_n^{-1}\, n \int_{U_{n-k_n,n}}^{U_{n-l,n}} (s - G_n(s))\,dQ(s) - a_n^{-1}\, n \int_{1-l/n}^{U_{n-l,n}} (1 - s - l/n)\,dQ(s) \\
&\quad - a_n^{-1}\, n \int_{1-k_n/n}^{U_{n-k_n,n}} (1 - s - k_n/n)\,dQ(s) \\
&\equiv \Delta_{1,n} + \Delta_{2,n} - \Delta_{3,n}.
\end{aligned}$$

Our aim is to prove that for a sufficiently large integer l the terms $\Delta_{i,n}$, $i=1,2,3$, converge to zero with probability one.

Note that by Lemmas 3.3 and 3.4 for any $\delta>0$ we have for all large enough n

$$\begin{aligned}1-F(a_n)&=\frac{1-F(b_n Q(1-1/n))}{1-F(Q(1-1/n))}(1-F(Q(1-1/n)))\\&\sim b_n^{-a}\{\tilde{L}(b_n Q(1-1/n))/\tilde{L}(Q(1-1/n))\}\, n^{-1}\\&\geqq n^{-1}\, b_n^{-a-\delta}.\end{aligned} \tag{3.2}$$

Consequently with $\delta=2-a>0$ convergence of the series in (1.11) implies

$$\sum_{n=1}^{\infty} n^{-1}\, b_n^{-2k}<\infty \tag{3.3}$$

which since $b_n \uparrow \infty$ yields

$$(\log n)/b_n^{2k}\to 0. \tag{3.4}$$

Subsequently for all l sufficiently large we have

$$\sum_{n=1}^{\infty} n^{-1}\, b_n^{-(l+1)/2}\leqq \sum_{n=1}^{\infty} n^{-1}\, b_n^{-a(l+1)/4}<\infty. \tag{3.5}$$

Choose any positive integer l such that the right-hand series in (3.5) is finite. First consider the term $\Delta_{1,n}$. We see that

$$|\Delta_{1,n}|\leqq I_{1,n}\cdot I_{2,n}$$

where

$$I_{1,n}=\sup_{0\leqq s\leqq U_{n-l,n}} n^{1/2}|G_n(s)-s|/(b_n^{1/4}(1-s)^{1/2})$$

and

$$I_{2,n}=\int_0^{U_{n-l,n}}(1-s)^{1/2}\, dQ(s)/(b_n^{3/4}\, n^{-1/2}\, Q(1-1/n)).$$

Since the left-hand series in (3.5) is also finite, the obvious upper tail version of Theorem 1 implies that $I_{1,n}\to 0$ almost surely.

Next by integration by parts and applying Theorem 1.2.1 in de Haan (1975)

$$\int_0^{1-t}(1-s)^{1/2}\, dQ(s)\sim\frac{2}{2-a}\, t^{1/2-1/a}\, L(t)\quad\text{as}\quad t\downarrow 0.$$

Thus almost surely as $n\to\infty$

$$I_{2,n}\sim\frac{2}{2-a}\,\frac{(1-U_{n-l,n})^{1/2-1/a}\, L(1-U_{n-l,n})}{b_n^{3/4}\, n^{-1/2-1/a}\, L(1/n)}. \tag{3.6}$$

Applying Lemma 3.2 with $\gamma=1/a$ along with $Q(1-s)\uparrow$ as $s\downarrow 0$ we have almost surely for all n sufficiently large that the right-hand side of (3.6) is

$$\leqq \frac{4}{2-a}(n(1-U_{n-l,n})/b_n^{1/2})^{1/2}\{(b_n^{a/4}\, n(1-U_{n-l,n}))^{-2/a}+1\}.$$

By Theorem 2 of Kiefer (1972) we have for any fixed positive integer l

$$\limsup_{n\to\infty} n(1-U_{n-l,n})/\log\log n=1 \quad \text{a.s.,} \tag{3.7}$$

and by the finiteness of the right-hand series in (3.5) in combination with Fact 1 of Sect. 2

$$1/(b_n^{a/4}\, n(1-U_{n-l,n}))\to 0 \quad \text{a.s.} \tag{3.8}$$

Hence by (3.4), (3.7), and (3.8), $I_{2,n}\to 0$ almost surely, which by the preceding steps gives $\Delta_{1,n}\to 0$ almost surely.

We shall now show that for a sufficiently large l, $\Delta_{2,n}\to 0$ with probability one. Choose l such that the series in (3.5) are finite. Observe that

$$\begin{aligned}|\Delta_{2,n}| &\leqq a_n^{-1}\, n\,|U_{n-l,n}-(1-l/n)|\,|Q(U_{n-l,n})-Q(1-l/n)|\\ &\leqq I_{3,n}\cdot I_{4,n}\end{aligned} \tag{3.9}$$

where

$$I_{3,n}=2\max\{l,\, n(1-U_{n-l,n})\}/b_n^{1/2}$$

and

$$I_{4,n}=2\max\{Q(U_{n-l,n}),\, Q(1-l/n)\}/(b_n^{1/2}\, Q(1-1/n)).$$

Statements (3.4) and (3.7) imply $I_{3,n}\to 0$ almost surely and by nearly the same proof as just given, $I_{4,n}\to 0$ almost surely. Thus $\Delta_{2,n}\to 0$ with probability one.

Finally we must verify that $\Delta_{3,n}\to 0$ with probability one. For each integer $n\geqq 1$ set

$$l_n=k_n\vee[(\log\log n)^2] \quad \text{and} \quad m_n=k_n\wedge[(\log\log n)^2].$$

Since inequality (3.9) holds with 2 replaced by 3 and l replaced by k_n it will be enough to show that

$$D_{1,n}\equiv a_n^{-1}\, n\,|U_{n-l_n,n}-(1-l_n/n)|\,|Q(U_{n-l_n,n})-Q(1-l_n/n)|\to 0 \quad \text{a.s.} \tag{3.10}$$

and

$$D_{2,n}\equiv a_n^{-1}\, n\,|U_{n-m_n,n}-(1-m_n/n)|\,|Q(U_{n-m_n,n})-Q(1-m_n/n)|\to 0 \quad \text{a.s.} \tag{3.11}$$

For the proof of (3.10), we observe that $l_n/\log\log n\to\infty$, so that Theorem 2 of Einmahl and Mason (1988) implies that with probability one

$$\limsup_{n\to\infty}\frac{n\,|U_{n-l_n,n}-(1-l_n/n)|}{(l_n\log\log n)^{1/2}}\leqq 2. \tag{3.12}$$

Let A denote the event on which (3.12) holds. Then we have $1-U_{n-l_n,n}\sim l_n/n$ on A. Let n' be any subsequence of the sequence of positive integers such that $l_{n'}/n'\to c$ as $n'\to\infty$ for some $0\leqq c\leqq 1$. If $c=0$, then on A in view of (1.6)

$$Q(U_{n'-l_{n'},n'})/Q(1-l_{n'}/n')\to 1 \quad \text{as} \quad n'\to\infty,$$

hence for all large enough n' by (3.12)

$$D_{1,n'}\leqq\frac{(l_{n'}\operatorname{loglog} n')^{1/2}\,Q(1-l_{n'}/n')}{b_{n'}\,Q(1-1/n')}\leqq 2\,\frac{(\operatorname{loglog} n')^{1/2}}{b_{n'}},$$

where the last inequality follows from Lemma 3.2 with $\varepsilon=1/a-1/2$. This proves $D_{1,n'}\to 0$ as $n'\to\infty$ on A on account of (3.4). If $c>0$, then $Q(U_{n'-l_{n'},n'})$ and $Q(1-l_{n'}/n')$ are bounded on A along n', and by (3.12) for all large n'

$$\begin{aligned}\frac{n'\,|U_{n'-l_{n'},n'}-(1-l_{n'}/n')|}{a_{n'}}&\leqq\frac{(n'\operatorname{loglog} n')^{1/2}}{b_{n'}\,Q(1-1/n')}\\&=\frac{(\operatorname{loglog} n')^{1/2}}{b_{n'}}\left(\frac{1}{n'}\right)^{1/a-1/2}L^{-1}\left(\frac{1}{n'}\right)\to 0 \quad \text{as} \quad n'\to\infty\end{aligned}$$

on account of (3.4) and $1/a-1/2>0$. This proves $D_{1,n'}\to 0$ as $n'\to\infty$ on A again, which therefore holds for any subsequence n' for which $l_{n'}/n'$ converges. The proof of (3.10) is now completed by an elementary subsequence argument.

For the proof of (3.11) we note that since $k_n\to\infty$, for any fixed integer l for all n sufficiently large with $l\leqq m_n\leqq[(\operatorname{loglog} n)^2]\equiv\rho_n$

$$\begin{aligned}D_{2,n}\leqq\, & n\,|1-U_{n-\rho_n,n}|\,Q(U_{n-l,n})/(b_n\,Q(1-1/n))\\&+n\,|1-U_{n-\rho_n,n}|\,Q(1-l/n)/(b_n\,Q(1-1/n))\\&+m_n\,Q(U_{n-l,n})/(b_n\,Q(1-1/n))+m_n\,Q(1-l/n)/(b_n\,Q(1-1/n)).\end{aligned}$$

From (3.12) with l_n replaced by ρ_n we have $n\,|1-U_{n-\rho_n,n}|\leqq 2\,(\operatorname{loglog} n)^2$ with probability one for all large n, and hence

$$D_{2,n}\leqq\frac{3(\operatorname{loglog} n)^2}{b_n^{1/2}}\left(\frac{Q(U_{n-l,n})}{b_n^{1/2}\,Q(1-1/n)}+\frac{Q(1-l/n)}{b_n^{1/2}\,Q(1-1/n)}\right).$$

Statement (3.4) and $I_{4,n}\to 0$ with probability one imply $D_{2,n}\to 0$ almost surely. Hence we have established that $\Delta_{3,n}\to 0$ with probability one. This completes the proof of assertion (3.1).

Routine arguments based on well known properties of regularly varying functions, cf. de Haan (1975), Seneta (1976), show that for all $0<a<2$ and all fixed integers $l\geqq 1$

$$a_n^{-1}\left(n\int_{1-k_n/n}^{1-l/n}Q(s)\,ds-n\mu_n\right)\to 0.$$

Since (1.12) implies that for each fixed integer $l \geqq k$

$$a_n^{-1} \sum_{i=k}^{l} X_{n+1-i,n} \to 0 \quad \text{a.s.},$$

this finishes the proof that (1.11), respectively (1.12), implies (1.13) with $c_n = n\mu_n$.

We shall demonstrate next that

$$\limsup_{n\to\infty} a_n^{-1} \left| \sum_{i=k}^{k_n} X_{n+1-i,n} - c_n \right| < \infty \quad \text{a.s.} \tag{3.13}$$

for some sequence of constants c_n implies (1.11). Our first step will be to show that the constants c_n can be replaced by $n\mu_n$. As remarked in S. Csörgő and Mason (1986) we have for each fixed integer $k \geqq 1$

$$Q(1-1/n)^{-1} \sum_{i=1}^{k-1} X_{n+1-i,n} \xrightarrow{\mathscr{D}} \sum_{i=1}^{k-1} (S_i)^{-1/a}$$

with $S_i = E_1 + \ldots + E_i$, $i = 1, \ldots, k-1$, where $E_1, \ldots, E_{k-1}$ are independent exponential random variables with mean one. Consequently, in probability,

$$a_n^{-1} \sum_{i=1}^{k-1} X_{n+1-i,n} \to 0.$$

Thus on account of (3.13) and (1.10) the sequence of constants $a_n^{-1}|n\mu_n - c_n|$ must be bounded, which in combination with (3.13) yields

$$\limsup_{n\to\infty} a_n^{-1} \left| \sum_{i=k}^{k_n} X_{n+1-i,n} - n\mu_n \right| < \infty \quad \text{a.s.} \tag{3.14}$$

Set for $n \geqq 2$

$$b_n' = \max\left\{ b_n, \max_{2 \leqq i \leqq n} Q\left(1 - \frac{1}{i(\log i)^{1/(2k)}}\right) \Big/ Q(1-1/i) \right\}$$

and $a_n' = b_n' \, Q(1-1/n)$. Observe that $b_n' \uparrow \infty$ and notice that by Lemma 3.4

$$1 - F(a_n') \leqq 1 - F\left(Q\left(1 - \frac{1}{n(\log n)^{1/(2k)}}\right)\right) \sim \frac{1}{n(\log n)^{1/(2k)}}.$$

This leads to

$$\sum_{n=1}^{\infty} n^{2k} (1 - F(a_n'))^{2k+1} < \infty,$$

which by the already established fact that (1.11) implies (1.13) gives

$$\lim_{n\to\infty} a_n'^{-1}\left\{\sum_{i=2k+1}^{k_n} X_{n+1-i,n}-n\mu_n\right\}=0 \quad \text{a.s.} \tag{3.15}$$

Since $a_n'\geqq a_n$, statements (3.14) and (3.15) and the assumed positivity of the random variables imply that

$$\limsup_{n\to\infty} a_n'^{-1} X_{n+1-k,n}<\infty \quad \text{a.s.}$$

By the equivalence of (1.11) and (1.12), and (1.14) and (1.15), this can only happen if

$$\sum_{n=1}^{\infty} n^{k-1}(1-F(a_n'))^k<\infty. \tag{3.16}$$

Thus by (3.2) with a_n replaced by a_n' we have for any $\delta>0$

$$\sum_{n=1}^{\infty} n^{-1} b_n'^{-k(a+\delta)}<\infty,$$

which because of $b_n'\uparrow\infty$ implies that

$$b_n'^{(a+\delta)k}/\log n\to\infty.$$

Applying Lemma 3.2 we obtain that for any $0<\delta<a$ there exists a constant $0<c_\delta<\infty$ such that for all $n\geqq 2$

$$\max_{2\leqq i\leqq n} Q\left(1-\frac{1}{i(\log i)^{1/(2k)}}\right)\Big/ Q(1-1/i)\leqq c_\delta(\log n)^{1/(2k(a-\delta))}. \tag{3.17}$$

Noticing that for a sufficiently small δ

$$(\log n)^{(a+\delta)/(2(a-\delta))}/\log n\to 0,$$

we see by (3.17) that necessarily $b_n'=b_n$ and hence $a_n'=a_n$ for all n sufficiently large. Therefore convergence of the series in (3.16) implies convergence of the series in (1.11). This completes the proof of Theorem 2.

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