Optimal Investments with Increasing Returns to Scale: a Further Analysis

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Abstract. This paper considers a capital accumulation model that was previously analyzed by Barucci (1998). The specific feature of the model is that revenue is a convex function of the capital stock. We extend Barucci's work by giving a full analytical characterization of the case where a saddle point with a positive capital stock level exists. Furthermore we also analyze the other cases.

1 Introduction

In this paper subject of study is a standard capital accumulation model of the firm, where the objective is to maximize the discounted profit stream. The profit rate equals the difference between the revenue and the costs of investment. Revenue is obtained by selling goods on the market. The firm needs a capital stock to produce these goods. The higher the capital stock it owns , the more goods the firm produces, which in turn leads to a higher revenue. The firm can increase capital stock by investing. Technically spoken, this model is an optimal control model with one state variable, the capital stock, and one control variable, the investment rate.

The study of this framework goes back to the sixties, and started out with Eisner and Strotz (1963). In this contribution the revenue function was assumed to be concave and investment costs were convex. Using standard methods of control theory it is easily shown that optimal firm behavior subscribes convergence to a unique long run equilibrium at which marginal revenue equals marginal costs. Later it was recognized (Rothschild (1971)) that arguments could be found in favor of a (partly) concave shape of the investment cost function. The problems (chattering controls!) that then occur in the maximization problem were subject of study in Davidson and Harris (1981) and Jorgensen and Kort (1993).

On the other hand it can also be the case that the revenue function is convexly shaped for some intervals of capital stock values. Such a scenario was studied in Dechert (1983) and again Davidson and Harris (1981). From these contributions it can be concluded that partly convex revenue functions can lead to multiple equilibria. It then depends on the initial level of the capital stock to which of the equilibria it is optimal for the firm to converge to. In this sense we can speak of history dependent equilibria. Barucci (1998) studies the case where the revenue function is strictly convex troughout. He considered a framework where both the revenue function and the investment cost function are quadratic. As a result the isoclines, on which state, control, and co-state variables are constant, are linearly shaped, so that exactly one steady state exists. This means that multiple equilibria are ruled out. Barucci (1998) identifies the case where a saddle point equilibrium occurs for a positive level of the capital stock. He shows that convergence to this saddle point is the optimal policy.

Fascinated by the fact that such a simple optimal solution exists for the model with a fully convex revenue function, in this paper Barucci's framework is studied once again. We extend Barucci (1998) by (1) determining a full analytical characterization of the case with the saddle point with positive capital stock, and (2) by determining which other cases are also possible if the parameter values are different.

The contents of this paper is as follows. The model is formulated in Section 2. After establishing the necessary optimality conditions in Section 3, the equilibrium and its stability properties are studied in Section 4. In Section 5 all possible cases are studied, while some ideas for future research are outlined in Section 6.

2 Model formulation

Following Barucci (1998), the model we consider is the following:

$$\max_u \int_0^\infty e^{-\rho t}\left[r(k) - c(u)\right] dt\,, \tag{1}$$

$$\dot{k} = u - \mu k\,, \quad k(0) = k_0, \tag{2}$$

where k denotes the capital stock and u is investment. The revenue function is given by $r(k)$ while the investment costs are $c(u)$. The discount rate is ρ while μ denotes the depreciation rate.

Although Barucci did not impose this constraint, for economic reasons we assume that investments are irreversible:

$$u \geq 0\,. \tag{3}$$

In order to be able to obtain a full analytical solution, like Barucci (1998) we assume quadratic revenue and costs functions:

$$r(k) = ak + bk^2\,, \;\; c(u) = cu + du^2. \tag{4}$$

We require all parameters a, b, c, d, μ, and ρ to be positive. Hence, as already explained in the Introduction, the revenue function exhibits increasing returns to scale.

3 Necessary conditions

Hamiltonian:

$$H = ak + bk^2 - cu - du^2 + q[u - \mu k]. \tag{5}$$

maximum principle:

$$\partial H/\partial u = 0 \quad \text{i.e.} \quad u = \frac{q-c}{2d}. \tag{6}$$

If (3) is imposed, then (6) holds only for $u > 0$ i.e. for $q > c$. Otherwise we have

$$u = 0 \ \text{ if } \ q \le c. \tag{6a}$$

The adjoint equation is

$$\dot{q} = \rho q - \partial H/\partial k = (\rho + \mu)q - a - 2bk\ . \tag{7}$$

From (5), i.e., $q = 2du + c$ and (6) we get:

$$\dot{u} = \frac{\dot{q}}{2d} = (\rho + \mu)u + \frac{(\rho+\mu)c - a}{2d} - \frac{b}{d}k\ . \tag{8}$$

4 Equilibrium and its Stability

The (unbounded) linear DE-system (2) and (8) has the following unique equilibrium:

$$\bar{k} = \frac{1}{2}\frac{c(\rho+\mu)-a}{b-d\mu(\rho+\mu)} \qquad \bar{u} = \mu\bar{k} = \frac{\mu}{2}\frac{c(\rho+\mu)-a}{b-d\mu(\rho+\mu)}. \tag{9}$$

On the other hand, the original canonical system (2) and (7) has the unique equilibrium

$$\bar{k} = \frac{1}{2}\frac{c(\rho+\mu)-a}{b-d\mu(\rho+\mu)} \qquad \bar{q} = \frac{bc - ad\mu}{b-d\mu(\rho+\mu)} \tag{10}$$

which is the same as Barucci's result on p. 794 except for a sign error in the second formula.

For economic reasons only positive equilibria $\bar{k}$ make sense. Then also $\bar{u} = \mu\bar{k}$ is positive. From (5) it follows that then also $\bar{q} = 2d\bar{u} + c$ is positive.

Proposition 1: *The unique equilibrium is in the relevant region (k > 0, u > 0, first quadrant), iff the sign of* $c(\rho+\mu)-a$ *equals the sign of* $b - d\mu(\rho+\mu)$.

The Jacobian of the linear DE-system (2) and (7) is

$$J = \begin{bmatrix} -\mu & 1 \\ -\frac{b}{d} & \rho+\mu \end{bmatrix} \ \text{ and } \ \det J = -\mu(\rho+\mu) + \tfrac{b}{d},$$

so that the equilibrium is a saddle point iff

$$\mu d(\rho+\mu) > b \tag{11}$$

as was also found by Barucci (1998); see (i) on p. 794.

The $\dot{k}=0$-isocline is

$$u=\mu k \tag{12}$$

and the $\dot{u}=0$-isocline is

$$u=-\frac{(\rho+\mu)c-a}{2d(\rho+\mu)}+\frac{b}{d(\rho+\mu)}k\ . \tag{13}$$

Comparing this with (11) it follows that the $\dot{k}=0$-isocline is steeper than the $\dot{u}=0$-isocline iff the equilibrium is a saddle point.

5 Solution in the 4 different cases

From Proposition 1 we obtain, that the signs of the expressions $c(\rho+\mu)-a$ and $b-d\mu(\rho+\mu)$ are crucial for the outcome of the model. Consequently we can distinguish four different cases.

5.1 Case 1: $c(\rho+\mu)-a$ < 0 and $b-d\mu(\rho+\mu)$ < 0

In this case, from (9) we get a positive equilibrium, which is, by (11), a saddle point.

The phase diagram looks as follows:

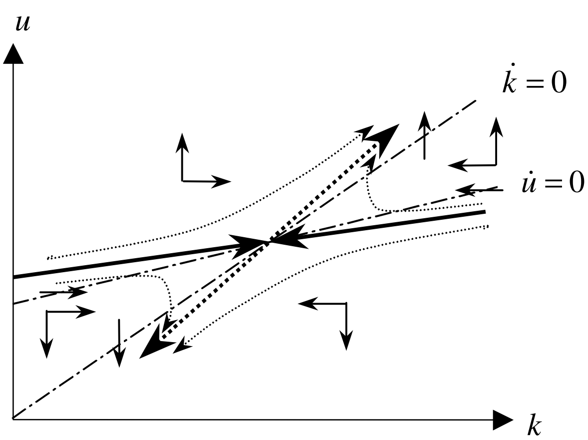

Fig. 1. The phase diagram in Case 1 where $c(\rho+\mu)-a$ < 0 and $b-d\mu(\rho+\mu)$ < 0.

This clearly illustrates the reverse accelerator feature of the stable investment path as expressed in Dechert (1983).

We now compute the trajectories $u(t)$ and $k(t)$ along the saddle point path and evaluate the objective function:

First, we have to get the eigenvalues:

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 + 4\mu(\rho+\mu) - \frac{4b}{d}}}{2}, \ \lambda_2 = \frac{\rho + \sqrt{\rho^2 + 4\mu(\rho+\mu) - \frac{4b}{d}}}{2}. \tag{14}$$

Then we need the eigenvector $[k^*, 1]'$ associated with the negative eigenvalue λ_1 and get the solution:

$$\begin{bmatrix} k(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \bar{k} \\ \bar{u} \end{bmatrix} + \frac{k_0 - \bar{k}}{k^*} \begin{bmatrix} k^* \\ 1 \end{bmatrix} e^{\lambda_1 t}. \tag{15}$$

The eigenvector is easily computed as follows:

$$\begin{bmatrix} -\mu - \lambda_1 & 1 \\ -\frac{b}{d} & \rho + \mu - \lambda_1 \end{bmatrix} \begin{bmatrix} k^* \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields:

$$k^* = \frac{1}{\mu + \lambda_1}. \tag{16}$$

From (15) and (16) we generate the following expressions:

$$k(t) = \bar{k} + \left(k_0 - \bar{k}\right) e^{\lambda_1 t}, \tag{17}$$

$$u(t) = \bar{u} + \left(k_0 - \bar{k}\right)(\mu + \lambda_1) e^{\lambda_1 t}. \tag{18}$$

Then, with $\alpha = k_0 - \bar{k}$ and $\beta = \mu + \lambda_1$ the profit rate is

$$\pi(k,u) = ak + bk^2 - cu - du^2$$

$$= \left[a\bar{k} + b\bar{k}^2 - c\bar{u} - d\bar{u}^2\right] + e^{\lambda_1 t}\left[a\alpha + 2b\alpha\bar{k} - c\alpha\beta - 2d\alpha\beta\bar{u}\right] + e^{2\lambda_1 t}\left[b\alpha^2 - d\alpha^2\beta^2\right]$$

Evaluating the objective function (1) using this expression, we get:

$$\Pi(k_0) = \int_0^\infty e^{-\rho t}\left[r(k) - c(u)\right] dt =$$

$$= \frac{a\bar{k} + b\bar{k}^2 - c\bar{u} - d\bar{u}^2}{\rho} + \frac{a + 2b\bar{k} - c\beta - 2d\beta\bar{u}}{\rho - \lambda_1}\alpha + \frac{b - d\beta^2}{\rho - 2\lambda_1}\alpha^2.$$

Using $\alpha = k_0 - \bar{k}$ we get the profit $\Pi(k_0)$ as a function of the initial state:

$$\Pi(k_0) = \left[\frac{a\bar{k} + b\bar{k}^2 - c\bar{u} - d\bar{u}^2}{\rho} - \frac{a + 2b\bar{k} - c\beta - 2d\beta\bar{u}}{\rho - \lambda_1}\bar{k} + \frac{b - d\beta^2}{\rho - 2\lambda_1}\bar{k}^2\right]$$
$$+\left[\frac{a + 2b\bar{k} - c\beta - 2d\beta\bar{u}}{\rho - \lambda_1} - 2\frac{b - d\beta^2}{\rho - 2\lambda_1}\bar{k}\right]k_0 + \frac{b - d\beta^2}{\rho - 2\lambda_1}k_0^2.$$

Apparently, profit $\Pi(k_0)$ is a quadratic function of k_0.It is clear, that the NPV of the profit is higher the larger the initial capital stock is. This means, that the coefficient of k_0^2 is positive, and that the unbounded minimum of $\Pi(k_0)$ occurs for k_0. ≤ 0.

Extensive numerical experiments have shown, that in Case 1 the NPV of the profit is always positive, although it seems difficult to derive this result from the above formulas analytically. If one wants to minimize $\Pi(0)$ under all the constraints that hold in Case 1 and under the positivity of the parameters, then one can make $\Pi(0)$ a positive number arbitrarily close to zero, while d becomes very large compared to all the other parameters and $\bar{k}$ approaches zero.

This confirms Barucci's (1998) result (Proposition 4.1) that approaching the equilibrium is always optimal in Case 1.

5.2 Case 2: $c(\rho + \mu) - a < 0$ and $b - d\mu(\rho + \mu) > 0$

In this case the equilibrium is not in the first quadrant and it is not a saddle point:

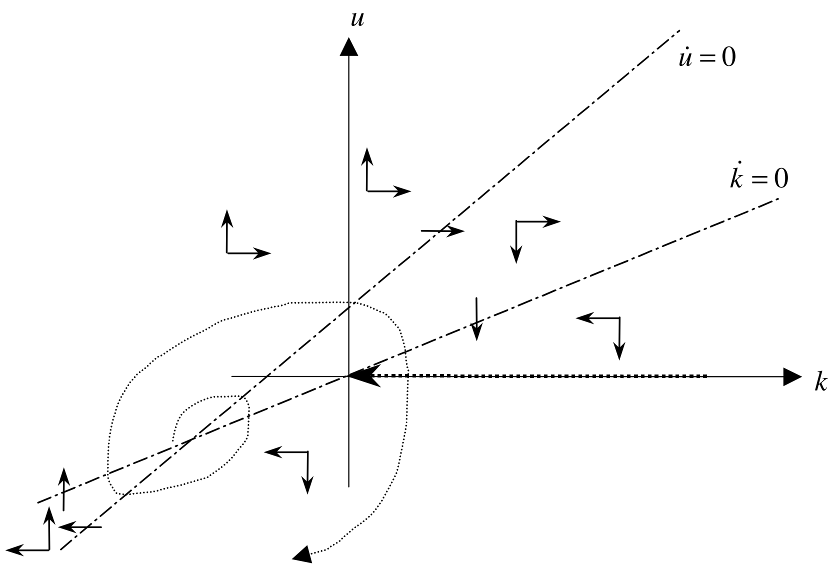

Fig. 2. The phase diagram in Case 2 where $c(\rho + \mu) - a < 0$ and $b - d\mu(\rho + \mu) > 0$.

The equilibrium with negative k and u is an unstable focus.

If (3) is imposed, $u = 0$ and $k \to 0$ could be expected to be optimal when looking at the figure. However, for economic reasons it is clear that this is not true. Case 2 is characterized by large values of the parameters a and b occurring in the revenue function. In this case, approaching $k = 0$ is certainly not optimal.

In fact, no optimal solution exists, since the objective is unbounded. This will now be verified by showing analytically that constant or proportional investment rates can yield arbitrarily high values of (1)

5.2.1 Constant Investment

We first consider a constant investment policy

$$u(t) = u^* \text{ for all } t.$$

Then the capital stock develops according to

$$k(t) = \frac{u^*}{\mu} + \left(k_0 - \frac{u^*}{\mu}\right)e^{-\mu t}.$$

Evaluating the profit using these expressions, we get:

$$\pi(k,u) = ak + bk^2 - cu - du^2$$

$$= \left[a\frac{u^*}{\mu} + b\frac{u^{*2}}{\mu^2} - cu^* - du^{*2} \right] + e^{-\mu t}\left[\left(k_0 - \frac{u^*}{\mu} \right) + 2b\frac{u^*}{\mu}\left(k_0 - \frac{u^*}{\mu} \right) \right]$$

$$+ e^{-2\mu t} b\left(k_0 - \frac{u^*}{\mu} \right)^2 .$$

so that the objective function (1) becomes:

$$\Pi(k_0) = \int_0^\infty e^{-\rho t}\left[r(k) - c(u) \right] dt =$$

$$= \frac{a\frac{u^*}{\mu} + b\frac{u^{*2}}{\mu^2} - cu^* - du^{*2}}{\rho} + \frac{\left(k_0 - \frac{u^*}{\mu} \right) + 2b\frac{u^*}{\mu}\left(k_0 - \frac{u^*}{\mu} \right)}{\rho + \mu} + \frac{b\left(k_0 - \frac{u^*}{\mu} \right)^2}{\rho + 2\mu} .$$

The terms with u^{*2} are

$$\frac{\frac{b}{\mu^2} - d}{\rho} - \frac{2b\frac{1}{\mu^2}}{\rho + \mu} + \frac{b\frac{1}{\mu^2}}{\rho + 2\mu} = \frac{2b - d(\rho + \mu)(\rho + 2\mu)}{\rho(\rho + \mu)(\rho + 2\mu)} .$$

Thus, the objective (1) can be made arbitrarily large, if the constant u^* is chosen large enough, provided that

$$b > \frac{d}{2}(\rho + \mu)(\rho + 2\mu) = d\left(\mu^2 + \tfrac{3}{2}\rho\mu + \tfrac{1}{2}\rho^2 \right).$$

Barucci shows that the objective is unbounded for

$$b > d\left(\mu^2 + \rho\mu \right)$$

which is a weaker condition.

5.2.2 Proportional Investment

We now consider a constant investment policy

$$u(t) = [\mu + \varepsilon]k(t) \text{ for all } t.$$

Then the capital stock develops according to

$$k(t) = k_0 e^{\varepsilon t} \quad \text{and} \quad u(t) = k_0[\mu + \varepsilon]e^{\varepsilon t} .$$

Evaluating the profit function using these expressions, we get:

$$\pi(k,u) = ak + bk^2 - cu - du^2$$

$$= k_0\left[a - c(\mu+\varepsilon)\right]e^{\varepsilon t} + k_0^2\left[b - d(\mu+\varepsilon)^2\right]e^{2\varepsilon t},$$

which is again used to evaluate the objective function (1):

$$\Pi(k_0) = \int_0^\infty e^{-\rho t}\left[r(k) - c(u)\right]dt = k_0\frac{a - c(\mu+\varepsilon)}{\rho-\varepsilon} + k_0^2\frac{b - d(\mu+\varepsilon)^2}{\rho - 2\varepsilon}$$

provided that $2\varepsilon < \rho$.

However (1) is infinite for $2\varepsilon \geq \rho$. In particular, it is +∞ if $b > d(\mu+\varepsilon)^2$ and it is -∞ if $b < d(\mu+\varepsilon)^2$.

So (1) is unbounded, provided that $b > d(\mu+\varepsilon)^2$ and $2\varepsilon \geq \rho$, i.e., if

$$b > d\left(\mu + \frac{\rho}{2}\right)^2 = d\left(\mu^2 + \rho\mu + \frac{\rho^2}{4}\right).$$

This lower bound is better than that obtained for constant investment, but still above the Barucci boundary.

5.3 Case 3: $c(\rho+\mu) - a > 0$ and $b - d\mu(\rho+\mu) < 0$

In this case the equilibrium is not in the first quadrant and it is a saddle point:

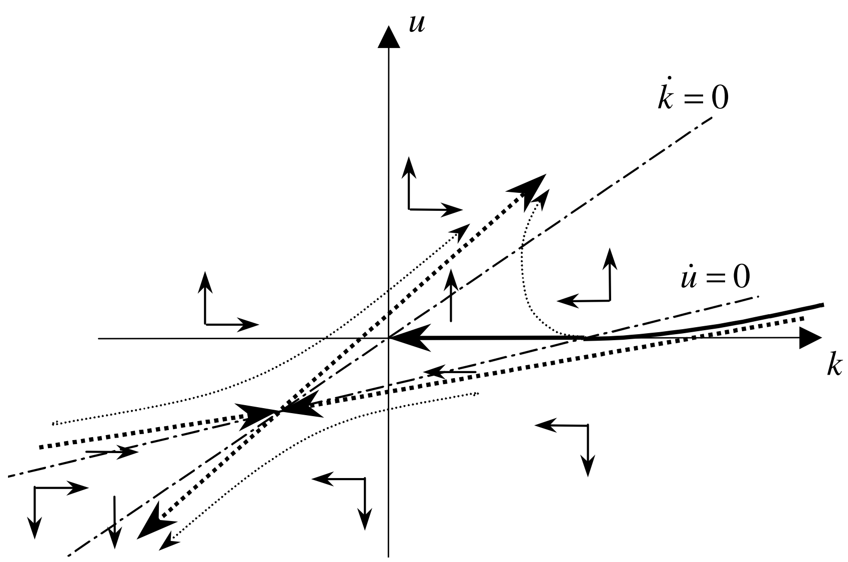

Fig. 3. The phase diagram in Case 3 where $c(\rho+\mu) - a > 0$ and $b - d\mu(\rho+\mu) < 0$.

This case is characterized by small values of the parameters a and b occurring in the revenue function. In this case, approaching $k = 0$ is certainly optimal.

The question remains whether $u = 0$ throughout, or whether it looks like shown in the above figure.

5.3.1 Allowing for reversibility of investment

Here it makes sense to consider the scenario where constraint (3) is replaced by the state constraint $k \geq 0$. Then the optimal trajectory in the phase diagram would converge to the $k = 0$ within finite time. Note that disinvestment occurs here.

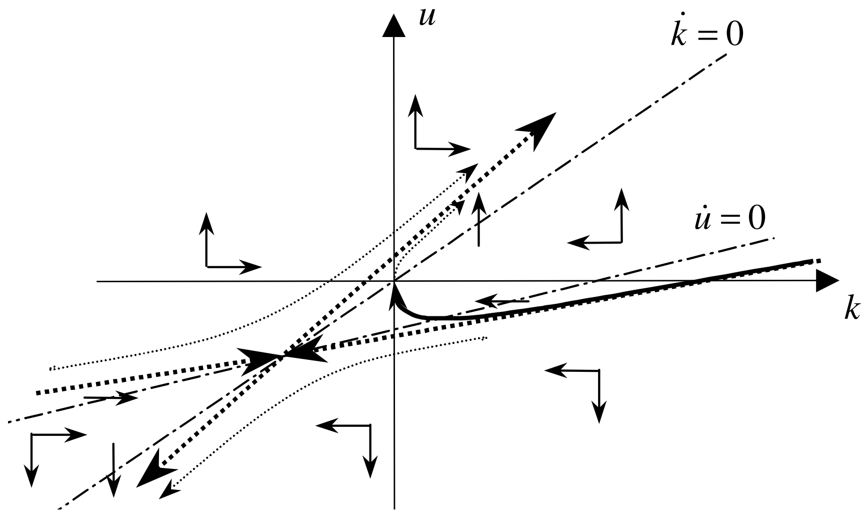

Fig. 3a. The phase diagram in Case 3 where investment is reversible.

The proof is the same as in Example 8.8 on p.219 in Feichtinger and Hartl (1986).

5.4 Case 4: $c(\rho + \mu) - a > 0$ and $b - d\mu(\rho + \mu) > 0$

In this case the equilibrium is in the first quadrant and it is not a saddle point:

Fehler! Es ist nicht möglich, durch die Bearbeitung von Feldfunktionen Objekte zu erstellen.

Fig. 4. The phase diagram in Case 4 where $c(\rho + \mu) - a > 0$ and $b - d\mu(\rho + \mu) > 0$.

The equilibrium is an unstable focus. Except for the fact that the equilibrium now occurs for positive values of *k* and *u* this case is identical to Case 2.

No optimal solution exists, and that the objective is unbounded. The calculations in Section 5.2 concerning constant and proportional investment, respectively, also apply to this case.

6 Ideas for future research

In this paper the standard capital accumulation model, but then with a strictly convex revenue function, was studied. Due to the quadratic specifications of the revenue and investment cost function, interesting results could be generated. A straightforward extension is to make the revenue function a third order polynomial in which k^3 is multiplied with a negative parameter. In this way a convex-concave revenue function arises which makes it possible to redo the calculations of Dechert (1983) and Davidson and Harris (1981). As already mentioned in the Introduction, they arrived at solutions with multiple equilibria. They could identify levels of the capital stock, which we now call DNS (Dechert Nishimura Skiba)-points, where the firm is indifferent concerning to which equilibrium it should converge. Hopefully, it is possible to generate additional insights concerning these DNS-points in case we study the model with such a third order polynomial as revenue function.

A second interesting extension is to combine the just described convex-concave revenue function with introducing adjustment costs on changes in the investment rate (see Jorgensen and Kort (1983), Section 3.4.2). In such a model investments will be introduced as a second state variable, and the rate of change of investment is the control variable. Hence, the resulting model now contains two state variables and one control variable. It can be expected that also here multiple steady states exist. Depending on the location in the (k,u)-plane, it is optimal to converge to one of these steady states. It would be interesting to study whether so-called DNS-curves (which are DNS-points in a one-state-variable-model) exist, on which the firm is indifferent concerning to which steady state to converge to.

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