Equilibrium Theory
a salient approach

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Proefschrift

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Chapter 1

Preliminaries

Introduction

While studying this thesis, the reader will notice that economical and mathematical theory are strongly entangled. The mathematical pillar is based on convex analysis and functional analysis. More precisely, the new concept of salient space is introduced and will be placed in its mathematical context. This concept is the underlying basis for the main goal of this thesis, which is a reformulation of existing models in equilibrium theory.

In this chapter, the historical interaction between economics and mathematics is described. Moreover, we briefly discuss the existence models which serve as a basis for our generalisation. At the end of this chapter we will motivate our salient approach.

1.1 Historical overview

In this section, we will give some historical background about the origin of general equilibrium theory. We will discuss the evolution from a descriptive problem formulation towards a formal mathematical model including equilibrium existence theorems. The overview is based, amongst others, upon information given in [8] and [13].

1.1.1 Towards the formalisation of models

In an economy, a multitude of agents produce, exchange and consume a large number of commodities. Their decisions are independent of each other and dictated by self-interest. Why is social chaos not the result? That question is central in economics.

Adam Smith [31] formulated it this way, according to his "descriptive problem formulation" of 1776:

> But the annual revenue of every society is always precisely equal to the exchangeable value of the whole annual produce of its industry, or rather is precisely the same thing with that exchangeable value. As every individual, therefore, endeavours as much as he can both to employ his capital in the support of domestic industry, and so to direct that industry that its produce may be of the greatest value; every individual necessarily labours to render the annual revenue of the society as great as he can. He generally, indeed, neither intents to promote the public interest, nor knows how much he is promoting it. By preferring the support of domestic to that of foreign industry, he intents only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for the society that was no part of it. By pursuing his own interest he frequently promotes that of the society more effectually that when he really intends to promote it.

Adam Smith's insight later became the "First Theorem of Welfare Economics".

The first attempt to formalise this problem came from Condorcet. He introduced the concept of homo suffragans (an abstraction, a kind of "social form" stripped of all qualities except the "social" faculty of voting), suggested by the concept of mass-point in mechanics. From this he derived the notion of homo oeconomicus which is the center of modern economics. The concept of equilibrium was the main inspiration for these new developments. In fact, the theory of general economic equilibrium plays a central role throughout the history of mathematical economics.

In 1838, Cournot gave a definition of a market which is still used in economics today.

> Economists understand by the term "market", not any particular market place in which things are bought and sold, but the whole of any region in which buyers and sellers are in such free intercourse with one another that prices of the same goods tend to equality easily and quickly.

Furthermore, he was the first to clearly formulate that the purpose of mathematical economists is essentially theoretical: its aim was not to offer tools for numerical calculation and practical applications to the real economy, but to discover the general laws governing its evolution. He was the first to indicate how to apply mathematics to the study of economic problems without having to specify more than the general properties of the functions involved. Cournot's contribution played a fundamental role in the scientific training of Léon Walras.

Walras is undoubtedly of such central importance that he is considered to be the father of modern mathematical economics. Walras' scientific program was based on the idea of building a political economy in mathematical form, following the example of Isaac Newton's mechanics, or in his phrase, an "analytical economics" as a counterpart of analytical mechanics. At the core of Walras' theory lay the concept of general economic equilibrium (formulated by analogy with the concept of static equilibrium) and the law of price formation (which for him played a role similar to that of the law of universal gravitation in mechanics). He was not a mathematician and did not solve any of the main technical problems of his theory; he did, however, make an outstanding contribution to the formulation of the modern structure of the theory of general economic equilibrium. One of the main specific achievements of Walras (around 1875) was the formulation of the three major analytical problems of this theory: existence, uniqueness and global stability of the equilibrium. The last problem is the equivalent in formal terms of Adam Smith's metaphor of the "invisible hand"- the idea that the market is acted on by "forces" tending to maintain it in a state of equilibrium which is consistent with the complete independence of the actions of the individual economic agents. Walras himself perceived that the theory that he proposed would be vacuous without a mathematical argument in support of the existence of at least one equilibrium state. However, for more than half a century the equality of the number of equations and of the number of unknowns of the Walrasian model remained the only, unconvincing remark made in favor of the existence of a competitive equilibrium.

The mathematician Abraham Wald was the first to prove the existence, uniqueness and global stability of equilibrium (1934-1936). However, Wald's results were based on some strong hypotheses (in particular, all the commodities of economies were considered as "substitutes") in order to make use of the techniques of calculus. Making use of the results obtained by J.F. Nash (1950), K.J Arrow and G. Debreu [2] demonstrated in 1954 the theorem of existence of equilibrium for the Walras model under very general hypotheses; this was perhaps the most important achievement of the theory.

Following the "First Theorem of Welfare Economics", Vilfredo Pareto added at the beginning of the twentieth century the far deeper understanding that, conversely, with an efficient use of the resources of an economy there is associated a price system relative to which each consumer is in equilibrium. This, the "Second Theorem of Welfare Economics", was first proved by differential calculus. Later, it was proved by means of convex analysis by Arrow and Debreu. In that theorem, which ensues from the supporting hyperplane property of convex sets, prices no longer appear as an historical accident particular to a certain type of economic organization; they are intrinsically present in a state of the economy that is optimal with respect to the different preferences of consumers.

1.1.2 Mathematics behind the models

The first applications of mathematics in the social sciences took place in the context of what we now call population statistics, and were therefore strictly tied to the birth of statistics and the calculus of probability (first contributions: 1662, 1682). The most significant problems in this context concerned annuities, in connection with insurances and mortality rates. It is possible to distinguish a phase in which mathematics was conceived of mainly as merely a technical aid to research, from a phase in which mathematics served as a conceptual core of a well-defined methodology of research. This last phase took on definite shape in the course of the nineteenth century.

Up to at least the 1830s there was hardly any systematic cooperation between economics and mathematics. Changes in this relationship began with the discussions of price-theoretic models of German, Danish and Austrian economies which came from the mathematical school of marginal-utility theory of Walras and Pareto in the last third of the nineteenth century. This cooperation resulted in the use of a kind of mathematics different from the classical approach: the introduction of inequalities (rather than equations) and non-negative conditions for the variables, functional analysis, convex analysis (linear optimization) and topology.

However, the greatest modification to the course of mathematical economics, at least from the technical point of view, was the acknowledgment of the importance of fixed-point theorems. In his 1937 paper, von Neumann made use of a lemma (today called "Kakutani's Theorem") which was a generalisation to the case of multi-valued functions of L.E.J. Brouwer's Fixed-Point Theorem (1910) which is for continuous functions only. Making use of both theorems, the problem of existence of a Walrasian equilibrium could be solved in Arrow and Debreu (1954) and McKenzie (1954). In these papers only assumptions with respect to the primary concepts were made in order to show the existence of a Walrasian equilibrium. Besides Kakutani's Theorem, another generalisation of Brouwer's Fixed-Point Theorem was made by J.P. Schauder (1930). He proved Brouwer's result in an (infinite-dimensional) topological vector space setting.

1.1.3 Further mathematical overview

In the previous subsection, it was mentioned that technical progress on equilibrium existence theorems in the twentieth century came from fixed-point theorems in spaces of functions. More generally, several of the techniques of real-variable analysis had begun to merge by the end of the nineteenth century, eventually to be called "functional analysis".

Vector spaces were around long before the concept became explicit. In common with all algebraic structures in mathematics, vector spaces became axiomatised about a century ago. The importance of vector spaces lies not in the power of their theory, which is elementary, but in the widespread belief that linear problems are easy. Linear problems are simple, for example since for a linear ordinary differential equation, the sum of two solutions is again a solution and every solution is a sum of a finite number of basic ones. As this example indicates, the study of vector spaces generalises naturally to the infinite-dimensional case. But infinite sums of vectors require a discussion of their convergence if they are to make numerical sense, and so involve questions of topology. Problems which are not linear are usually much harder to deal with precisely because the sum of two solutions is, in general, not a solution. Passing to a linear simplification is often still not only a heuristic first step, but the only way known to generate solutions mathematically.

By introducing a coordinate system, a finite-dimensional vectors pace can be used to turn geometrical problems into algebraic ones and, conversely, to provide a geometrical interpretation of algebraic problems. The former case was extended by many writers to include most problems with a mechanical origin; the latter is helpful as an aid to thought and when approximate solutions are sought. Euclidean geometry is named after the Greek mathematician Euclid (ca 300 BC), who wrote what became the definitive account of the elementary part of the subject in his Elements. The parallel postulate, is the claim that, "if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles". The postulate seemed open to objection on two opposing grounds: it was not as obviously true as the other assumptions made by Euclid, and it seemed more like a result that could be proved on the basis of the other assumptions alone.

Hilbert proposed that the geometric terms "point", "line", "plane", and so forth be controlled by a system of axioms that determine what one may say about them, but which makes no attempt to say what they are. Hilbert's work led to the discovery of geometries that cannot be described in terms of coordinates at all. His approach to geometry therefore greatly enlarged its scope and took it beyond the domain of simple, continuous manifolds.

Hilbert's presentation came to have a decisive effect on many branches of mathematics. It was as if the pure mathematician's task was to provide axiomatic systems and check that they were self-consistent, which applied mathematicians, physicists and others could then use as they saw fit. This neatly defined a new relationship between pure and applied mathematics. Within pure mathematics, what was done for Euclidean geometry was done for other geometries. Hilbert showed the next year (1900) how non-Euclidean geometry can be obtained by changing just his version of the parallel postulate. Other mathematicians joined in, describing geometries which differed more and more in their nature from Euclidean geometry. The other systems of mathematical ideas were given axiomatic treatments, starting with the theories of groups and fields.

The essence of the development of functional analysis was the transfer of a number of concepts from n-dimensional Euclidean space $\mathbb{R}^n$ and the functions defined on it to infinite-dimensional "function spaces" of various types and their "operators" - concepts such as compactness, boundedness, convergence, distance, continuity, completeness, dimension, scalar product and linearity. To bring this about, a way was needed to pass from the finite to the infinite; but the form of this passage was the object of great concern and even strife among the early functional analysts. Often it was only through generalisation - through the increasingly axiomatic definition of the new spaces, where $\mathbb{R}^n$ was subordinated as a special case - that the relations of the original concepts, and their partial logical dependence or independence, became recognisable. Concepts such as that of convergence became diversified, while equivalent properties such as boundedness and compactness separated from each other.

In 1872 Cantor published a rigorous foundation for the real numbers, as Richard Dedekind did likewise. Cantor constructed the real numbers as equivalence classes of Cauchy sequences of rational numbers, while Dedekind used "cuts" in the rationals. Cantor's approach was later used to complete any metric space, while Dedekind's approach was employed to complete any partial order. Cantor next initiated the theory of infinite cardinal numbers.

Among the early works written to introduce, explain and evaluate Cantor's set theory, the most influential contribution was made by Felix Hausdorff, who introduced such set topological notions as Hausdorff dimension and Hausdorff measure. He succeeded in creating a comprehensive theory of topological spaces which may be taken to mark the beginning of the study of both topological and metric spaces.

Set theory and topology were both of fundamental importance in the development of the theory of functions and the birth of functional analysis. However, the breakthrough to axiomatic functional analysis was made by John von Neumann in work beginning in 1928 that showed the applicability of Hilbert spectral theory to quantum mechanics. Von Neumann extended the results to unbounded operators in Hilbert space which he had defined axiomatically in 1928. With his work, functional analysis was established as one of the most important fields of modern analysis and as an independent mathematical discipline.

1.2 The neoclassical models

As a starting of motivation, we present and discuss formulations of three models standardly used in mathematical economy, namely: the model of Arrow and Debreu of a pure exchange economy, their model of a private ownership economy with production, and the approach, developed by Drèze, in which price rigidities are introduced into the model of Arrow and Debreu of a pure exchange economy.

1.2.1 The Arrow-Debreu model of a pure exchange economy

In a pure exchange economy, one encounters exchange of commodities between agents. A commodity is anything that may be used or consumed. It may be a physical good such as bread, a service such as the use of a car, a contract such as a train ticket which allows use of a certain section of the railway system at a certain date, a license to build a house on a certain piece of land, or a lottery ticket which gives the right to certain prizes, dependent on some future events. A commodity is assumed to be completely homogeneous, i.e., one unit of it is completely indistinguishable from another in all respects; not only in terms of its physical characteristics, but also in terms of its location in time and space. A bar of chocolate now is different from a bar of chocolate to be received in a year time, as any child will be able to tell you. Cola in Eindhoven is different from the same brand of cola in Tilburg, particularly if you are working at Eindhoven University on a hot day, thinking about your supply of cola in the refrigerator at home, in Tilburg. Thus, a commodity is fully described by its physical characteristics and the time and place at which it is available.

Focusing attention on changes of dates, one obtains, as a particular case of this theory of commodities, a theory of saving, investment, capital, and interest. Similarly, focusing attention on changes of locations, one obtains, as another particular case, a theory of location, transportation, international trade and exchange.

In their model of a pure exchange economy, Arrow and Debreu assume that there exists a finite number of commodities, implying that a finite specification of physical characteristics, location, etc. suffices for the problems studied. This finite specification of the number of commodities excludes treatment of situations in which characteristics may vary continuously, even though such situations arise in a natural way, for example, in the context of quality choice of commodities. Another problem with the assumption of a finite number of commodities concerns the time specification of commodities; each model with an infinite time horizon, whether in discrete or continuous time, requires an infinite-dimensional commodity space. So, the assumption that the number of commodities is finite, is rather restrictive.

The number of commodities present in the Arrow-Debreu model of an exchange economy, is denoted by the natural number k_0. A commodity bundle is characterised as being a composition of these commodities only, where each commodity is present in a certain amount. Assuming perfect divisibility of the commodities, any nonnegative real number is possible to fix the amount of each commodity. In [9, page 30], Debreu considers trucks, which can be seen as an example of an indivisible commodity. There, also, Debreu presents an argumentation that also in this situation, the assumption of perfect divisibility is reasonable.

So, it is assumed that every commodity bundle is represented by a tuple of nonnegative numbers $(x_1, \dots x_{k_0}) \in \mathbb{R}_+^{k_0}$. In this representation, x_k denotes the amount of units of commodity k. Each of the tuples $e^1 = (1, \dots, 0)$, $e^2 = (0, 1, 0, \dots, 0)$,…, $e^{k_0} = (0, \dots, 0, 1)$ represents the bundles in which one unit of one particular commodity is present. These bundles form the natural basis to describe a commodity bundle. More precisely, a commodity bundle x is described uniquely by $x = \sum_{k=1}^{k_0} x_k e^k$ and the collection of commodity bundles can be seen as the positive orthant, or positive cone, of the vector space $\mathbb{R}^{k_0}$ with $\{e^1, \dots, e^{k_0}\}$ as its natural basis. The set $\mathbb{R}_+^{k_0}$ of all commodity bundles is called the commodity set and in this set, commodity bundles can be added and multiplied with a nonnegative scalar, using the addition and scalar multiplication defined on $\mathbb{R}^{k_0}$.

In the Arrow-Debreu model, commodity bundles are ordered in a natural way. The bundle x is smaller than or equal to the bundle y if y can be split into two commodity bundles, one of which equals x, in other words, if $y - x$ is also a commodity bundle. So, the ordering of bundles is precisely described by the Euclidean order relation $\leq_E$ on $\mathbb{R}^{k_0}$:

$$(x_1, \dots, x_{k_0}) \leq_E (y_1, \dots, y_{k_0}) \iff \forall k \in \{1, \dots, k_0\} : x_k \leq y_k$$
$$(x_1, \dots, x_{k_0}) \ll_E (y_1, \dots, y_{k_0}) \iff \forall k \in \{1, \dots, k_0\} : x_k < y_k.$$

The vector space $\mathbb{R}^{k_0}$ with the Euclidean order relation is a vector lattice, i.e., the partial order relation $\leq_E$ on $\mathbb{R}^{k_0}$ satisfies:

- reflexivity: $\forall x \in \mathbb{R}^{k_0} : x \leq_E x$,
- transitivity: $\forall x, y, z \in \mathbb{R}^{k_0} : (x \leq_E y \text{ and } y \leq_E z) \implies x \leq_E z$,
- anti-symmetry: $\forall x, y \in \mathbb{R}^{k_0} : (x \leq_E y \text{ and } y \leq_E x) \implies x = y$,
- translation-invariance: $\forall x, y, z \in \mathbb{R}^{k_0} : (x + z) \leq_E (y + z) \implies x \leq_E y$,
- scaling-invariance: $\forall x, y \in \mathbb{R}^{k_0}\ \forall \alpha > 0 : (\alpha x) \leq_E (\alpha y) \implies x \leq_E y$.

Furthermore, $\mathbb{R}^{k_0}$ with the Euclidean order relation $\leq_E$ is a lattice: every pair x, y of elements of $\mathbb{R}^{k_0}$ has a least upper bound and a greatest lower bound with respect to $\leq_E$. For example, the least upper bound of x and y is the vector $z \in \mathbb{R}^{k_0}_+ := \{v \in \mathbb{R}^{k_0} \mid 0 \leq_E v\}$, for every $k \in \{1, \ldots, k_0\}$ defined by $z_k := \max\{x_k, y_k\}$.

In the pure exchange economy model of Arrow and Debreu one recognises prices. The price p_k of a commodity k, $k \in \{1, \ldots, k_0\}$, is a real number which represents the value of one unit of the commodity. A commodity, for which the corresponding price is negative, is called non-desirable. In many variants of the Arrow-Debreu model, it is assumed that all commodities are desirable, whence all prices are nonnegative. (We remark that as a consequence of this assumption, commodities with uncertainty, such as assets, do not fit into the variant of the Arrow-Debreu model that is described here.) Thus, a price system or a price vector is characterised as being a point in the positive orthant $\mathbb{R}^{k_0}_+$ of the Euclidean space $\mathbb{R}^{k_0}$, $p = (p_1, \ldots p_{k_0})$. Hence, the value of a commodity bundle $x = (x_1, \ldots, x_{k_0})$, given a price system $p = (p_1, \ldots, p_{k_0})$, is equal to the (Euclidean) inner product of x and p, given by

$$p \cdot x := \sum_{k=1}^{k_0} p_k x_k.$$

It is assumed that the economy operates without the use of a commodity serving as medium of exchange, such as money. Prices serve to describe the rate at which commodities can be exchanged. Thus p_i/p_j gives the amount of commodity j that may be exchanged for one unit of commodity i. The bundles x and y are exchangeable at price system p if

$$p \cdot x = p \cdot y.$$

In the pure exchange economy model of Arrow and Debreu, one recognises agents. An economic agent is completely characterised by three features: an initial endowment, a consumption set and a preference relation there-upon. We shortly discuss these features.

Firstly, an agent is characterized by what he possesses, which is called his initial endowment. His initial endowment, which is a commodity bundle, gives an agent the means of exchange to make himself better off.

Secondly, for an agent there may be some bundles of the commodity space $\mathbb{R}^{k_0}$, which are excluded as consumption possibilities by physical or logical restrictions. The set of all consumption bundles which are possible for the agent is called his consumption set. Although, in reality, this consumption set may be a very restricted subset of $\mathbb{R}^{k_0}_+$, the simplifying assumption that the consumption set of each agent is equal to the set of all possible commodity bundles, is often made.

Finally, the ultimate decision of an agent to choose a bundle out of the consumption set, depends on his tastes and aims, represented by his preferences. The basic concept involved with the preferences of an agent is the relation "is at least as preferred as", for which we write $\succeq$, i.e., for every $x, y \in \mathbb{R}^{k_0}_+$ the notation $x \succeq y$ indicates that commodity bundle x is at least as preferred as bundle y.

Usually, three fundamental axioms are imposed on the preference relation $\succeq$ that is defined on the consumption set of the agent. These axioms are often taken as a definition of a rational agent. Every preference relation on $\mathbb{R}^{k_0}_+$ is assumed to satisfy:

- reflexivity: $\forall x \in \mathbb{R}^{k_0}_+ : x \succeq x$,
- transitivity: $\forall x, y, z \in \mathbb{R}^{k_0}_+ : (\ x \succeq y \text{ and } y \succeq z\ ) \implies x \succeq z$,
- completeness: $\forall x, y \in \mathbb{R}^{k_0}_+ : x \succeq y \text{ or } y \succeq x$.

From the binary relation "is at least as preferred as", denoted by $\succeq$, we can derive two related relations on $\mathbb{R}^{k_0}_+ \times \mathbb{R}^{k_0}_+$ as follows. Let $x, y \in \mathbb{R}^{k_0}_+$, then we say that x is regarded indifferent to y, written by $x \sim y$ if x is at least as preferred as y and y is at least as preferred as x. Furthermore, x is strictly preferred to y, written by $x \succ y$, if x is at least as preferred as y and y is not at least as preferred as x. Hence, x is strictly preferred to y if and only if x is at least as preferred as y and x is not regarded indifferent to y. A preference relation $\succeq$ which satisfies the three stated properties is a complete pre-ordering on $\mathbb{R}^{k_0}_+$ (cf. [9, page 8]). The indifference relation $\sim$, when derived from a preference relation which satisfies the above three properties, defines an equivalence relation on $\mathbb{R}^{k_0}_+$.

In summary, an agent is characterised by the following:

- an initial endowment, being a commodity bundle,
- a consumption set, which is a subset of $\mathbb{R}^{k_0}_+$,

- a preference relation, defined on the set of all commodity bundles, that is reflexive, transitive and complete.

Henceforth, it is assumed that every consumption set is equal to the set of all commodity bundles.

In an exchange economy, every agent trades his initial endowment. The principal behavioral assumption being made, is that agents are price-takers. Given a price vector p, which determines the exchange value of his initial endowment, an agent is constrained to choose a commodity bundle of which the value is not higher than the value of his initial endowment. So, an agent with initial endowment $w \in \mathbb{R}_+^{k_0}$, chooses a commodity bundle out of the budget set $B(p, w)$, at price vector $p \in \mathbb{R}_+^{k_0}$, given by

$$B(p, w) := \{x \in \mathbb{R}_+^{k_0} \mid p \cdot x \leq p \cdot w\}.$$

This choice is based on his preferences. We define his demand set to be the equivalence class of all most preferable bundles from his budget set, or, explicitly, for an agent with preference relation $\succeq$ and initial endowment $w \in \mathbb{R}^{k_0}$, the demand set $D(p, w, \succeq)$ at price system $p \in \mathbb{R}_+^{k_0}$ is given by

$$D(p, w, \succeq) := \{x \in B(p, w) \mid \forall y \in B(p, w) : x \succeq y\}.$$

Thus, our formulation of the basic elements of an Arrow-Debreu model of a pure exchange economy reveals the following three primary concepts: a commodity set $\mathbb{R}_+^{k_0}$, where k_0 denotes the number of separate commodities, a price set $\mathbb{R}_+^{k_0}$, and a finite number i_0 of (economic) agents, where agent i, $i \in \{1, \ldots, i_0\}$, has an initial endowment $w_i \in \mathbb{R}_+^{k_0}$ and a preference relation $\succeq_i$ on $\mathbb{R}_+^{k_0} \times \mathbb{R}_+^{k_0}$.

We have seen, that at a given price vector $p \in \mathbb{R}_+^{k_0}$, two secondary concepts can be derived, indicating what an agent can choose and, given this, what he wants to choose. For every $i \in \{1, \ldots, i_0\}$, the budget set $B_i(p)$ of agent i is given by $B(p, w_i)$ and the demand set of agent i is given by $D(p, w_i, \succeq_i)$.

Now, in this model of a pure exchange economy, a Walrasian equilibrium constitutes of a price vector $p_{\text{eq}} \in \mathbb{R}_+^{k_0}$, the equilibrium price vector, and a choice d_i, $i \in \{1, \ldots, i_0\}$, in each of the demand sets $D(p_{\text{eq}}, w_i, \succeq_i)$, respectively, such that the total demand is smaller than or equal to the total endowment, i.e.,

$$\sum_{i=1}^{i_0} d_i \leq_E \sum_{i=1}^{i_0} w_i.$$

Arrow and Debreu showed that in the model, described here, an equilibrium price vector (and therefore also a Walrasian equilibrium) exists, under the following additional (mathematical) assumptions:

A) For every $i \in \{1, \ldots, i_0\}$, the preference relation $\succeq_i$ on $\mathbb{R}_+^{k_0}$ is

1) monotonous: $\forall\, x, y \in \mathbb{R}_+^{k_0} : x \leq_E y$ implies $y \succeq_i x$,

2) strictly convex: $\forall x, y \in \mathbb{R}_+^{k_0}\ \forall \tau \in (0,1) : x \succeq_i y$ and $x \neq y$ imply $\tau x + (1 - \tau)y \succ_i y$,

3) continuous: $\forall y \in \mathbb{R}_+^{k_0}$ the sets $\{x \in \mathbb{R}_+^{k_0} \mid x \succeq_i y\}$ and $\{x \in \mathbb{R}_+^{k_0} \mid y \succeq_i x\}$ are closed in $\mathbb{R}_+^{k_0}$.

B) The total initial endowment is strictly positive: $w_{\text{total}} := \sum_{i=1}^{i_0} w_i \gg_E 0$.

The proof of the existence of an equilibrium price vector implies that in the equilibrium situation

$$\sum_{i=1}^{i_0} d_i = \sum_{i=1}^{i_0} w_i.$$

Furthermore, an equilibrium price vector is strictly positive, i.e., it gives positive value to every commodity bundle $x \in \mathbb{R}_+^{k_0} \setminus \{0\}$.

The monotony of the preference relations states that "at least as much of everything is at least as good". If the agents can costlessly dispose of unwanted goods of a commodity bundle, this assumption is trivial. An assumption, weaker than monotony is called "non-saturation". It states that $\forall i \in \{1, \ldots, i_0\}\ \forall x \in \mathbb{R}_+^{k_0}\ \exists y \in \mathbb{R}_+^{k_0} : y \succ_i x$. In some descriptions of the Arrow-Debreu model the monotony assumption is replaced by the non-saturation assumption, and at the same time Assumption B is replaced by the stronger "for all $i \in \{1, \ldots, i_0\}$, the initial endowment w_i of agent i is strictly positive".

The assumption that all preference relations are strictly convex implies that the agents prefer averages to extremes, but, other than that, it has little economic content. (This assumption can be weakened to convex preferences, in which case the analysis of the model will involve correspondences instead of functions.)

The combination of the monotony and the strict convexity property implies that the preferences are strictly monotonous, i.e., $\forall x, y \in \mathbb{R}_+^{k_0} : (x \leq_E y \text{ and } x \neq y) \Longrightarrow y \succ x$. In words, it means that every commodity is extremely desirable. As a consequence, in a Walrasian equilibrium, the total demand is equal to the total initial endowment.

The continuity assumption for the preferences rules out certain discontinuous behaviour; it states that if $(x_n)_{n \in \mathbb{N}}$ is a sequence of commodity bundles that are all at least as good as a bundle y, and if this sequence converges to some bundle x, then

x is at least as good as y.

Finally, Assumption B states that every commodity should be present in the economy. (If this is not the case, this economy can be remodelled with $\mathbb{R}_+^{k_1}$, with $k_1 < k_0$ representing the set of all present commodity bundles.) This assumption is connected to the so-called "minimum income hypothesis", since it guarantees that at every price vector $p \in \mathbb{R}_+^{k_0} \setminus \{0\}$, there is at least one agent of which the initial endowment has a nonzero value.

As a closing remark to this section, we mention that the intuition behind the existence of an equilibrium price vector is the original belief that "demand and price move in opposite directions" (sometimes called "the law of demand", cf. [22]). This belief says that if at a certain price vector $p \in \mathbb{R}_+^{k_0}$, the demand of a specific commodity k, $k \in \{1, \ldots, k_0\}$, is larger than the available quantity of that commodity in the total initial endowment, an increase in the price of commodity k will reduce the demand. Since in every Arrow-Debreu model of a pure exchange economy it is assumed that a price vector can be any element of $\mathbb{R}_+^{k_0}$, prices are "flexible", i.e., they can adjust to a price vector where for every commodity the supply is equal to the total initial endowment.

1.2.2 Neoclassical model of a private ownership economy

In order to obtain a model of a private ownership economy with production from the above introduced model of a pure exchange economy, we have to introduce a fourth primary concept: a finite number of firms. Furthermore, we extend the third primary concept concerning the agents. We maintain the other two primary concepts: the commodity set $\mathbb{R}_+^{k_0}$, where k_0 denotes the number of separate commodities, and the price set $\mathbb{R}_+^{k_0}$. Thus far, the third primary concept was a finite number i_0 of (economic) agents, each characterised by two features: an initial endowment and a preference relation. Their role was to choose a consumption bundle.

The role of a firm is to choose (and carry out) a production plan. A production plan is a specification of the quantities of all the inputs and all the outputs; outputs are represented by positive numbers, inputs by negative numbers. With this convention, a production plan is represented by an element $y \in \mathbb{R}^{k_0}$. For instance, in an economy with four commodities, the production plan $(-1, 2-3, 1) \in \mathbb{R}^4$, expresses that one unit of commodity 1 and three units of commodity 3 are needed to produce two units of commodity 2 and one unit of commodity 4. The profit of executing this production plan, at price vector $(p_1, p_2, p_3, p_4) \in \mathbb{R}_+^4$, is equal to $2p_2 + p_4 - p_1 - 3p_3$, i.e., the value of the outputs minus the value of the inputs. In general, the profit of production plan $y \in \mathbb{R}^{k_0}$ at price vector $p \in \mathbb{R}_+^{k_0}$ is equal to the Euclidean inner product of y and p:

$$p \cdot y = \sum_{k=1}^{k_0} p_k y_k.$$

Note that the profit of a production plan can be negative. In this situation we speak of a loss. In a private ownership economy with production (henceforth simply called "private ownership economy"), the firms are "owned" by the agents; it is assumed that the profit of every firm is divided amongst the agents through the concept of shares.

A given production plan may be technologically possible or technologically impossible for a firm. The set $Y \subset \mathbb{R}^{k_0}$ of all production plans which are possible for the firm, is called his production set. In some models of a private ownership economy, the definition of production set includes several properties. For instance, in [1] a non-empty subset $Y \subset \mathbb{R}^{k_0}$ is a production set if and only if

- Y is closed,
- Y is convex,
- $\mathbb{R}_+^{k_0} \cap Y = \{0\}$,
- Y is bounded from above, i.e., there exists some $a \in \mathbb{R}_+^{k_0}$, satisfying $y \leq_E a$ for all $y \in Y$.

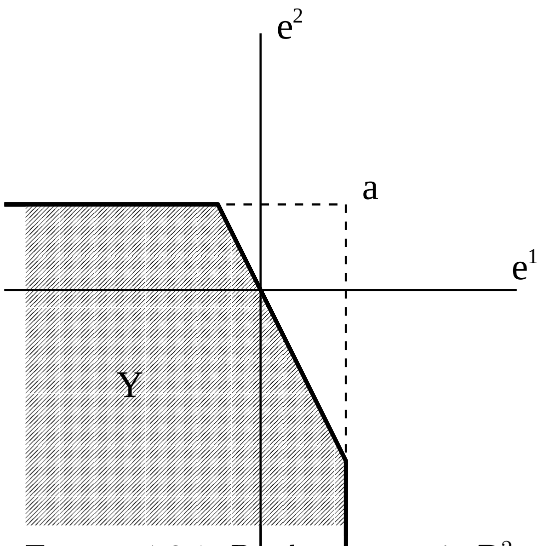

FIGURE 1.2.1: Production set in $\mathbb{R}^2$

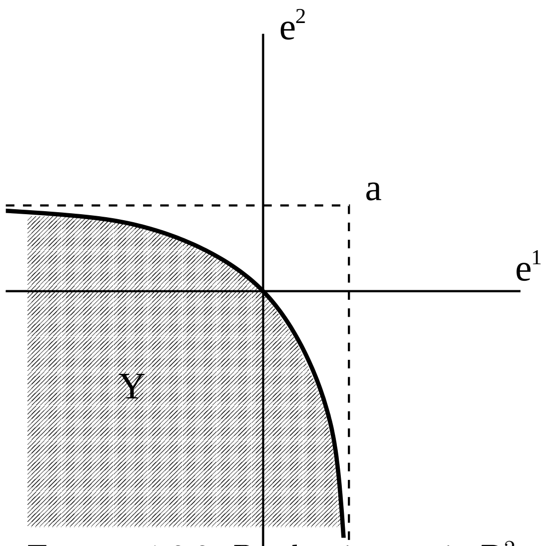

FIGURE 1.2.2: Production set in $\mathbb{R}^2$

Here, the third property implies that every possible production plan with zero input has zero output. Furthermore, it implies that the firm can always choose production plan 0, i.e., can choose not to produce. The combination of the second and the third property implies that if y is a possible production plan, then so is τy, where $\tau \in [0, 1]$; in economical terms: non-increasing returns to scale prevail (cf. [9, page 40]). For typical examples of a production set in the situation of two commodities, see Figures 1.2.1 and 1.2.2.

Another assumption concerning production, which is sometimes made, is that a production set $Y \subset \mathbb{R}^{k_0}$ satisfies $-(\mathbb{R}^{k_0}_+) \subset Y$, i.e., a production plan that has all its outputs equal to zero, is possible. In other words, it is possible for a firm to dispose of all commodities. Related to this assumption is the following: a production set $Y \subset \mathbb{R}^{k_0}$ satisfies $(Y - \mathbb{R}^{k_0}_+) \subset Y$, i.e., if a production plan is possible, so is one where input is larger and output is smaller (in absolute value). One, or both of these assumptions are often referred to as the "free disposal properties".

In general, a firm will have many ways of producing a certain combination of outputs from inputs. A production plan $y \in Y$ is efficient for the production set Y if there is no plan $z \in Y \setminus \{y\}$ such that $y \leq_E z$, i.e., if there is no production plan which produces at least as much output from at most the same input.

We remark that $\mathbb{R}^{k_0}$ can be regarded as the product of the positive cone $\mathbb{R}^{k_0}_+$ and the negative cone $-(\mathbb{R}^{k_0}_+)$ by corresponding to each production plan $x \in \mathbb{R}^{k_0}$, the pair (x^-, x^+) with output vector x^+ and input vector x^-, for every $k \in \{1, \ldots, k_0\}$, defined by $(x^+)_k := \max\{0, x_k\}$ and $(x^-)_k := \max\{(-x)_k, 0\}$. So, to each $x \in Y$ there is associated a unique pair $(x^+, x^-) \in \mathbb{R}^{k_0}_+ \times \mathbb{R}^{k_0}_+$, and thus Y can be seen as a subset $\tilde{Y}$ of $\mathbb{R}^{k_0}_+ \times \mathbb{R}^{k_0}_+$. We emphasize that the natural lattice structure (cf. Definition 2.2.18) of $\mathbb{R}^n$ with positive cone $\mathbb{R}^{k_0}_+$ enables to regard Y this way.

Similar to the situation for the agent, the principal behavioural assumption being made is that firms are price-takers. A firm treats a price vector as given and chooses a production plan in his production set which maximises profits. It turns out that if a price vector is strictly positive and if a profit maximising production plan exists, then this production plan is efficient. Indeed, if $y, z \in Y$ satisfy $y \leq_E z$ and $y \neq z$, then for every strictly positive price vector $p \in \mathbb{R}^{k_0}_+$, we find $p \cdot y < p \cdot z$.

In general, the four primary concepts of a private ownership economy are a commodity set $\mathbb{R}^{k_0}_+$, where k_0 denotes the number of separate commodities; a price set $\mathbb{R}^{k_0}_+$; a finite number j_0 of firms, where firm j, $j \in \{1, \ldots, j_0\}$, is characterised by a production set $Y_j \subset \mathbb{R}^{k_0}$; and a finite number i_0 of agents, where agent i, $i \in \{1, \ldots, i_0\}$, is characterised by an initial endowment $w_i \in \mathbb{R}^{k_0}_+$, a preference relation $\succeq_i$ on $\mathbb{R}^{k_0}_+$, and a set of real numbers $\{\theta_{ij} \mid j \in \{1, \ldots, j_0\}\}$, where for every $j \in \{1, \ldots, j_0\}$, θ_{ij} represents the share of agent i in the profit of firm j. It is assumed that $0 \leq \theta_{ij} \leq 1$

holds for all $i \in \{1, \ldots, i_0\}$ and all $j \in \{1, \ldots, j_0\}$, and that for all $j \in \{1, \ldots, j_0\}$:

$$\sum_{i=1}^{i_0} \theta_{ij} = 1.$$

For every $i \in \{1, \ldots, i_0\}$, we denote the vector $(\theta_{i1}, \ldots \theta_{ij_0})$ by θ_i.

At a given price vector $p \in \mathbb{R}_+^{k_0}$, the following secondary concepts can be derived. For every $j \in \{1, \ldots, j_0\}$ the supply set $S_j(p, Y_j)$ of firm j consists of all profit maximising production plans in his production set Y_j, i.e.,

$$S(p, Y_j) := \{y \in Y_j \mid \forall z \in Y_j : p \cdot y \geq p \cdot z\}.$$

At given executed production plans $y_j \in Y_j$, for every $j \in \{1, \ldots, j_0\}$, and a given price vector $p \in \mathbb{R}_+^{k_0}$, the income $\mathcal{K}(p, w_i, \theta_i; y_1, \ldots, y_{j_0})$ of agent i, $i \in \{1, \ldots, i_0\}$, is determined by the value of his initial endowment and the shares in the profits of the firms:

$$\mathcal{K}(p, w_i, \theta_i; y_1, \ldots, y_{j_0}) := p \cdot w_i + \sum_{j=1}^{j_0} \theta_{ij}(p \cdot y_j).$$

That is, the agents take prices and (profit of) production as given, and choose the most preferable element of the set of bundles available to them through their income. In this situation, the budget set $B(p, w_i, \theta_i; y_1, \ldots, y_{j_0})$ of agent i, and his demand set $D(p, w_i, \theta_i, \succeq_i; y_1, \ldots, y_{j_0})$ are given by

$$B(p, w_i, \theta_i; y_1, \ldots, y_{j_0}) = \{x \in \mathbb{R}_+^{k_0} \mid p \cdot x \leq \mathcal{K}(p, w_i, \theta_i; y_1, \ldots, y_{j_0})\},$$

and

$$\begin{aligned} &D(p, w_i, \theta_i, \succeq_i; y_1, \ldots, y_{j_0}) \\ &\quad = \{x \in B(p, w_i, \theta_i; y_1, \ldots, y_{j_0}) \mid \forall z \in B(p, w_i, \theta_i; y_1, \ldots, y_{j_0}) : x \succeq_i z\}. \end{aligned}$$

In this model of a private ownership economy, a Walrasian equilibrium constitutes of a price vector $p_{\text{eq}} \in \mathbb{R}_+^{k_0}$, a choice $s_j \in S(p_{\text{eq}}, Y_j)$, $j \in \{1, \ldots, j_0\}$, and a choice $d_i \in D(p_{\text{eq}}, w_i, \succeq_i, s_1, \ldots, s_{j_0})$, $i \in \{1, \ldots, i_0\}$ such that, after production, the total demand is smaller than or equal to the total supply, i.e.,

$$\sum_{i=1}^{i_0} d_i \leq_E \sum_{i=1}^{i_0} w_i + \sum_{j=1}^{j_0} s_j.$$

A price vector satisfying the above assumptions, is called an equilibrium price vector.

Arrow and Debreu showed that in this model an equilibrium price vector (and therefore also a Walrasian equilibrium) exists, under the following additional (mathematical) assumptions:

A) For every $j \in \{1, \ldots, j_0\}$, the set Y_j is strictly convex.

B) For every $i \in \{1, \ldots, i_0\}$, the preference relation $\succeq_i$ on $\mathbb{R}_+^{k_0}$ is

1) monotonous: $\forall\, x, y \in \mathbb{R}_+^{k_0} : x \leq_E y$ implies $y \succeq_i x$,

2) strictly convex: $\forall x, y \in \mathbb{R}_+^{k_0}\ \forall \tau \in (0,1) : x \succeq_i y$ and $x \neq y$ imply $\tau x + (1 - \tau)y \succ_i y$,

3) continuous: $\forall y \in \mathbb{R}_+^{k_0}$ the sets $\{x \in \mathbb{R}_+^{k_0} \mid x \succeq_i y\}$ and $\{x \in \mathbb{R}_+^{k_0} \mid y \succeq_i x\}$ are closed in $\mathbb{R}_+^{k_0}$.

C) The total initial endowment is strictly positive: $w_{\text{total}} := \sum_{i=1}^{i_0} w_i \gg_E 0$.

Similar to the model of Section 1.2.1, in this model, the proof of the existence of an equilibrium price vector implies that such a price vector is strictly positive, and, in the corresponding Walrasian equilibrium, the total demand is equal to the total supply.

Assumption A implies that the efficiency frontier $\{y \in Y \mid (\{y\} + \mathbb{R}_+^{k_0}) \cap Y = \{y\}\}$ contains no line segments. Figure 1.2.2 shows a production set Y which is strictly convex. The production set Y of Figure 1.2.1 does not satisfy Assumption A.

1.2.3 The Drèze model with price rigidities and rationing

In [10], Drèze introduces price regulations and price rigidities into the Arrow-Debreu model of a pure exchange economy, as described in Section 1.2.1. He assumes that the set of price vectors, considered in the model, is a strict subset P of $\mathbb{R}_+^{k_0}$. Since the proofs of the equilibrium existence theorems of Arrow and Debreu are explicitly based on the assumption that a price vector can be any element of $R_+^{k_0}$, these theorems are not applicable in this situation; it may very well be possible that the equilibrium price vector, of which existence is proved in the above mentioned equilibrium existence theorems, is an element of $\mathbb{R}_+^{k_0} \setminus P$. In other words, it is possible that for every price vector in P, the demand of the agents cannot be realised with the present initial endowments. In these situations, the allocation of commodities is regulated by using a rationing scheme for each agent. Drèze models a rationing scheme as a pair (L, l) of k_0-dimensional vectors, where, for example, L_k, $k \in \{1, \ldots, k_0\}$, denotes the maximum amount of commodity k, that an agent may demand on top of his initial endowment of that commodity. Similarly, the vector l is used as a lower bound, as will be explained below.

The primary concepts of the model are: the commodity set $\mathbb{R}_+^{k_0}$, where k_0 denotes the finite number of separate commodities that are present in the economy; the price set

$$P = \{p \in \mathbb{R}_+^{k_0} \mid f(p) = 1 \text{ and } \overline{p} \geq_E p \geq_E \underline{p}\},$$

to which the prices are restricted (here $\overline{p}, \underline{p} \in \mathbb{R}_+^{k_0}$ and $f : \mathbb{R}_+^{k_0} \to \mathbb{R}$ is a normalisation rule); a finite number i_0 of agents, where each agent i, $i \in \{1, \ldots, i_0\}$ is characterised by his initial endowment $w_i \in \mathbb{R}_+^{k_0}$ and a preference relation $\succeq_i$ on $\mathbb{R}_+^{k_0}$; and rationing scheme set $\mathbb{R}_+^{k_0} \times -(\mathbb{R}_+^{k_0})$.

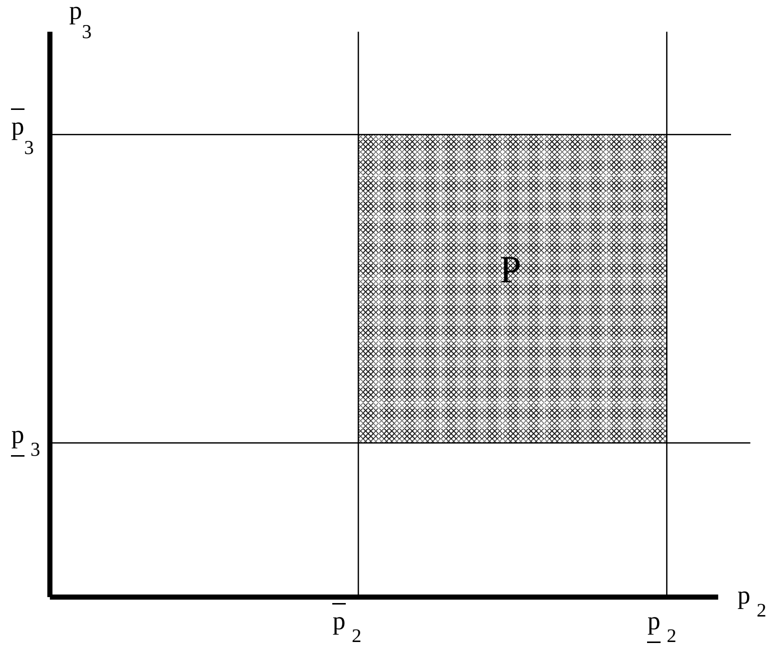

FIGURE 1.2.3: The set P with $p_1 = f(p) = 1$ and $k_0 = 3$

Figure 1.2.3, shows an example of a price set P which satisfies the above conditions.

Actually, Drèze also assumes that each agent i, $i \in \{1, \ldots, i_0\}$ has a consumption set $X_i \subset \mathbb{R}_+^{k_0}$ that is closed, convex and satisfies $\forall x \in X_i : \{x\} + \mathbb{R}_+^{k_0} \subset X_i$. Here, we consider the special case where for all $i \in \{1, \ldots, i_0\}$, the consumption set X_i is equal to $\mathbb{R}_+^{k_0}$.

With these primary concepts, the following secondary concepts are derived. Given a price vector $p \in P$ and a rationing scheme $(L, l) \in \mathbb{R}_+^{k_0} \times -(\mathbb{R}_+^{k_0})$, the (constrained) budget set $B(p, w_i, L, l)$ of agent i, $i \in \{1, \ldots, i_0\}$, is defined by

$$B(p, w_i, L, l) := \{x \in \mathbb{R}_+^{k_0} \mid p \cdot (x - w_i) \leq 0 \text{ and } L \geq_E x - w_i \geq_E l\}.$$

This indicates that for every commodity $k \in \{1, \ldots, k_0\}$, this agent is not allowed to ask more of commodity k than $L_k + (w_i)_k$, and he is not allowed to ask less than $l_k + (w_i)_k$. Note, that for all $p \in P$ and for all $(L, l) \in \mathbb{R}_+^{k_0} \times -(\mathbb{R}_+^{k_0})$, the initial endowment w_i is an element of the budget set. Figure 1.2.4 shows an example of a constrained budget set in $\mathbb{R}^2_+$.

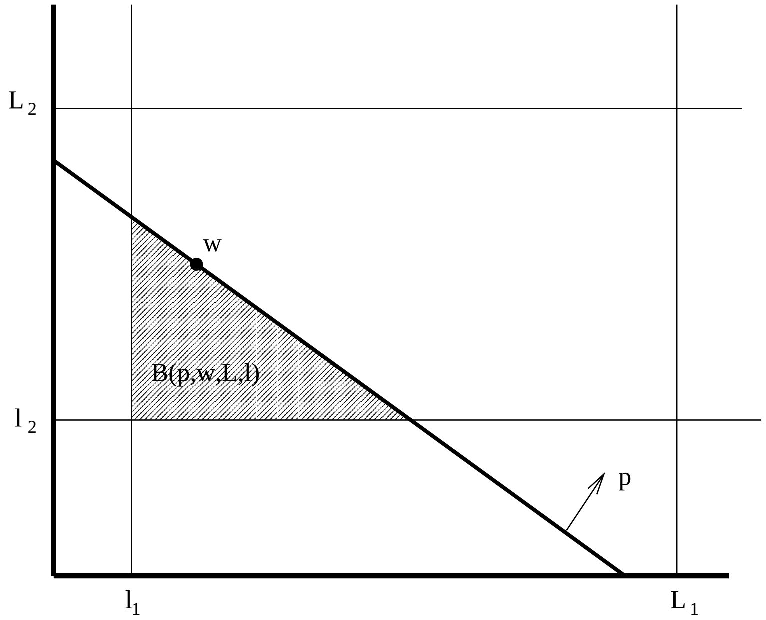

Figure 1.2.4: A constrained budget set ($k_0 = 2$)

The (constrained) demand set $D(p, w_i, \succeq_i, L, l)$ is given by

$$D(p, w_i, \succeq_i, L, l) = \{x \in B(p, w_i, L, l) \mid \forall z \in B(p, w_i, L, l) : x \succeq_i z\}.$$

For every $d \in D(p, w_i, \succeq_i, L, l)$ and for every $k \in \{1, \dots, k_0\}$, we say that agent i is constrained in his demand of commodity k if $(d_i)_k = L_k + (w_i)_k$. Similarly, we say that agent i is constrained in his supply of commodity k if $(d_i)_k = l_k + (w_i)_k$.

In this setting, an equilibrium under price restrictions and rationing (or constrained equilibrium) constitutes of $p_{\text{eq}} \in P$, $(L_{eq}, l_{eq}) \in \mathbb{R}^{k_0}_+ \times -(\mathbb{R}^{k_0}_+)$, and a choice d_i, $i \in \{1, \dots, i_0\}$, in each of the demand sets $D(p_{\text{eq}}, w_i, \succeq_i, L_{eq}, l_{eq})$, respectively, such that the total demand is equal to total initial endowment, i.e.,

$$\sum_{i=1}^{i_0} d_i = \sum_{i=1}^{i_0} w_i.$$

Since quantity rationing may be used to eliminate the difference between $\sum_{i=1}^{i_0} d_i$ and $\sum_{i=1}^{i_0} w_i$ in a Walrasian equilibrium (where $\sum_{i=1}^{i_0} d_i \leq_E \sum_{i=1}^{i_0} w_i$), this condition is stated in equality.

Furthermore, for every $k \in \{1, \dots, k_0\}$, the following properties have to be satisfied:

a.1) if $\exists i \in \{1, \ldots, i_0\} : (d_i)_k - (w_i)_k = L_k$ then $\forall i \in \{1, \ldots, i_0\} : (d_i)_k - (w_i)_k > l_k$,

a.2) if $\exists i \in \{1, \ldots, i_0\} : (d_i)_k - (w_i)_k = l_k$ then $\forall i \in \{1, \ldots, i_0\} : (d_i)_k - (w_i)_k < L_k$,

b.1) if $p_k < \overline{p}_k$ then $\forall i \in \{1, \ldots, i_0\} : (x_i)_k - (w_i)_k < L_k$,

b.2) if $p_k > \underline{p}_k$ then $\forall i \in \{1, \ldots, i_0\} : (x_i)_k - (w_i)_k > l_k$.

Conditions a.1 and a.2 are related to the assumed "market transparency", and state that rationing may affect either supply or demand, but may not affect simultaneously both supply and demand of a commodity. As a consequence, trivial equilibria like $L = l = 0$, are excluded. Conditions b.1 and b.2 state that no quantity rationing is allowed unless price rigidities are binding. The intuition behind this is that it is not "necessary" to introduce binding rationing schemes for a commodity if the price of this commodity is still "flexible". Only when a price is at its upper or lower bound, rationing of the corresponding commodity is introduced.

For this model, Drèze [10] proved existence of an equilibrium under price rigidities and rationing, under the following extra (mathematical) assumptions:

A) For every $i \in \{1, \ldots, i_0\}$, the preference relation $\succeq_i$ on $\mathbb{R}_+^{k_0}$ is

  1) monotonous: $\forall\, x, y \in \mathbb{R}_+^{k_0} : x \leq_E y$ implies $y \succeq_i x$,

  2) strictly convex: $\forall x, y \in \mathbb{R}_+^{k_0}\ \forall \tau \in (0, 1) : x \succeq_i y$ and $x \neq y$ imply $\tau x + (1 - \tau)y \succ_i y$,

  3) continuous: $\forall y \in \mathbb{R}_+^{k_0}$ the sets $\{x \in \mathbb{R}_+^{k_0} \mid x \succeq_i y\}$ and $\{x \in \mathbb{R}_+^{k_0} \mid y \succeq_i x\}$ are closed in $\mathbb{R}_+^{k_0}$.

B) For every $i \in \{1, \ldots, i_0\}$, the initial endowment w_i of agent i, is strictly positive.

Assumption A is similar to the assumption made on the preference relations of the agents in the Arrow-Debreu model of a pure exchange economy. Assumption B implies that at every $p \in P$, the initial endowment of every agent has nonzero value. Clearly, this is a stronger assumption than the corresponding assumption, related to the minimum income hypothesis, that is made in the Arrow-Debreu model of a pure exchange economy.

1.3 Motivation

One of the goals of this thesis is to present models of economies in which we leave the neoclassical idea that commodities always occur separately, and, instead, assume just the existence of commodity bundles. Here, the term commodity bundle is given a new interpretation; commodity bundles are not merely looked upon as a list of a finite number of different commodities, but instead will be regarded to represent a more complicated, possibly inextricable entanglement of characteristics and properties.

We represent the collection of all exchangeable objects, being it separate commodities, bundles of commodities or other objects, by a set C. Every element x of C represents "something which can be exchanged or traded". For a lack of a better or more precise terminology, we refer to the elements of C as "exchangeable objects", "bundles of exchange", "tradeable objects" or "bundles of trade". We are aware of the fact that thus far, these objects fall under the neoclassical definition of commodity, however we choose not to use the term "commodity" in order to emphasise that there is a difference between the nature of these exchangeable objects and the familiar term "commodity" as used in the neoclassical models. In the following we will try to explain the difference, among other things, with the help of an example.

1.3.1 Example. Consider a model of a pure exchange economy in which three commodities (in the neoclassical sense) are present: commodity a, b and c, and assume that trade can only take place in the following proportions:

- $1 : 1 : 2$,
- $1 : 2 : 1$,
- $2 : 1 : 1$,
- $1 : 2 : 2$.

For example, let a, b and c represent carrot, cabbage and leek. Then this example can describe the situation in which these vegetables are not sold separately, but only in four fixed combinations: mix for macaroni, mix for spaghetti, mix for Chinese noodles and mix for vegetable soup. Since the vegetables in these mixes are cut, sliced or grated, it is not possible to rearrange proportions during trade.
In this situation, the set of all exchangeable objects is represented by the set C, given by

$$C = \{(x_a, x_b, x_c) \in \mathbb{R}^3_+ \mid \exists \alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0 : \left\{ \begin{array}{l} \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 = x_a \\ \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 = x_b \\ 2\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 = x_c. \end{array} \right\} \}.$$

Note, that there are at least two different ways to represent commodity bundle $(12, 12, 12)$: we can choose $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3, 3, 3, 0)$ or $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 0, 4, 4)$.

A natural way to introduce an order relation $\leq_C$ on C is the following: for two exchangeable objects (x_a, x_b, x_c) and (y_a, y_b, y_c), we would like $(x_a, x_b, x_c) \leq_C (y_a, y_b, y_c)$ if and only if there exists $(z_a, z_b, z_c) \in C$ such that $(x_a, x_b, x_c) + (z_a, z_b, z_c) = (y_a, y_b, y_c)$. We remark that $x \leq_C y$ implies that (y_a, y_b, y_c) contains at least as much carrot, cabbage and leek as (x_a, x_b, x_c). However, since $C \neq \mathbb{R}^3_+$, the converse is not true.

There are two neoclassical approaches to model the above situation. The first one restricts the commodity bundle set $\mathbb{R}^3_+$ to the set C, described above. The disadvantage of this approach is that the Euclidean structure on $\mathbb{R}^3$ does not represent the natural order relation $\leq_C$ on the restricted commodity bundle set. Another disadvantage concerns the set of all possible price systems. We will come back to this in Example 4.2.2, where we show that our set of all possible price systems is larger than the neoclassical $\mathbb{R}^3_+$.
The second neoclassical approach to the above situation is to consider each of the fixed combinations $(1, 1, 2), (1, 2, 1), (2, 1, 1)$ and $(1, 2, 2)$, as a separate commodity, and model the set of commodity bundles by $\mathbb{R}^4_+$. However, in this approach the possibility of a double representation of commodity bundles, for example commodity bundle $(12, 12, 12)$ as derived above, has vanished. This is a disadvantage, especially if the preferences of the agents are based upon their appreciation of the commodities a, b and c, and the preference relations in the model have to be transformed to preference relations on $\mathbb{R}^4_+$. Furthermore, the Euclidean order relation $\leq_E$ of $\mathbb{R}^4_+$ does not represent $\leq_C$. Finally, we want to mention that in case the set C of all exchangeable objects is represented by the set S of Figure 2.1.3 (cf. page 36), this second approach is not possible. $\diamond$

The above example shows that a model based upon a set representing all exchangeable objects, rather than on separate commodities, can describe situations in which fixed links between different commodities are present, for instance an economy in which only fixed, prescribed combinations of commodities can be exchanged. Examples are pre-packed offers, or special products received when purchasing a large amount of a commodity, examples which are frequently observed in e.g. supermarkets or drugstores. Furthermore, this model can describe a situation in which the preferences of the agents are expressed in terms of the characteristics of the different commodities (cf. the work of Lancaster, [21]). We come back to this on page 103.

In our models, we require that the set C, consisting of all exchangeable objects, satisfies certain properties. First of all, if x and y are elements of C, i.e., if x and y are tradeable objects, then the collection consisting of x and y is also a tradeable object. When we denote the collection consisting of x and y, by $x + y$, we arrive at the mathematical condition that C has to be closed under the operation $+$. Clearly, this implies that if $x \in C$, then $2x \in C$, where $2x$ denotes $x + x$. In fact, for every $n \in \mathbb{N}$, we find that $x \in C$ implies $nx \in C$. The second requirement the set C has to meet goes a step further; we assume that if x is an exchangeable object, then so is αx for every $\alpha \geq 0$. If there is a justification for this assumption then it is exactly the same as the justification of the perfect divisibility requirement for the neoclassical models. This assumption implies that there is an element in C which represents $0x$, i.e., which represents the exchangeable object "nothing". We call this element the "null object". Thirdly, we assume that elements of C cannot cancel each other out, i.e., if x and y are exchangeable objects, then the object $x + y$ acquired by joining x and y can only be equal to the null object if both x and y are equal to the null object. Finally, we require an ordering of the elements of C which represents the natural ordering of exchangeable objects: a tradeable object x is smaller than or equal to a tradeable object y if and only if there is a tradeable object z such that $x + z = y$, i.e., if and only if y can be split up into two tradeable objects of which one is equal to x.

Summarising, (denoting the null element by v, and denoting the order relation by $\leq$) we find that we want the set C to satisfy

- $\forall x, y \in C : x + y \in C$;
- $\forall x \in C \; \forall \alpha \geq 0 : \alpha x \in C$;
- $\exists v \in C \; \forall x, y \in C : x + y = v \implies x = y = v$;
- $\forall x, y \in C : x \leq y \iff \exists z \in C : x + z = y$.

In the field of convex analysis, if a subset C of a vector space V satisfies the above conditions, then C is a pointed convex cone in V. In order to emphasise our focus on the set C, representing the set of all objects of trade, and not on the vector space surrounding it, we give an axiomatic introduction to it in the following chapter. Considering the pointedness of C, which represents that C does not have a linear subset other than $\{v\}$, we call the axiomatisation of a set satisfying the above described five conditions, a salient space. To our knowledge, this is a new concept that cannot be traced back to literature.

Since, in a salient space based model, we do not presume the availability of separate commodities, we cannot speak of the price of a commodity, but only of the value of an exchangeable object. So, to continue the set-up of a salient space model, price systems have to be objects that assign a nonnegative value to every element of C. Clearly, if p denotes a price system, then αp, which assigns to every element of C, α times the value that p assigns to it, is also a possible price system. The zero price system which assigns to every element of C the value zero, is also a price system.

Furthermore, if both p and q are price systems, then $p+q$, which assigns to every element of C the sum of the values that p and q assign to it, is also a price system. In case $p+q$ turns out to be the zero price system, then this is only possible if both p and q are equal to the zero price system. Finally, there is a natural order relation on price systems: p is considered to be greater than or equal to q if and only if for every $x \in C$, the value that p assigns to x is greater than or equal to the value that q assigns to x. Summarising we find that the set of all possible price systems can also be modelled by a salient space. In Chapter 4 we show that that set of all possible price systems, is, in a natural way, represented by the adjoint set of C, a concept which will be defined in the following chapter.

Apart from the definition of salient space, Chapter 2 contains algebraic considerations and concepts, such as the definition of order unit, salient mapping, and salient basis. A partial order relation is introduced, which is intrinsically related to every salient space, and we show that this partial order relation does not necessarily have a lattice structure. Furthermore, we introduce the concept of salient pairing and the adjoint set of a salient space. In Chapter 3 we give topological considerations regarding salient spaces; we introduce semi-norms and semi-metrics and investigate the topologies they generate. All these concepts will play a role in the application to the economic models of Chapter 4.

As far as we know, our approach to salient spaces is new, albeit that some of our ideas are related to well-explored concepts in literature. Clearly, concepts of linear dependent subset, linear dimension and the lattice structure of a salient space are derived from the corresponding vector space concepts. Where possible, we give the original vector space definition or give a way to find it by means of a reference. The idea behind the adjoint of a salient space is a combination of the concepts of dual vector space (cf. e.g. [7]) and of the polar of a cone (cf. [32], [6] and [7]). Furthermore, the concepts of salient basis and extreme set of a salient space generalise the corresponding concepts for polyhedral cones (cf. [32] and [6]).

However, we would like to remark that most often, pointed convex cones in literature are treated as a subset of a vector space; either they are a subset of a finite-dimensional Euclidean space, or they are the positive cone of a partially ordered vector space and are considered to be equivalent with this order relation.

For this reason, we decided not to write the following two chapters merely as a sum up of the mathematical tools, needed in the other chapters. But, apart from this, we also consider the investigation of algebraic and topological aspects of the concept of salient space as a goal in itself. As a result of this approach we are able to introduce several new concepts, such as salient basis, salient metric and salient topology, that do not have a vector space origin. These, and other concepts can be found in the following two chapters. More precisely, in Chapter 2, we discuss the algebraic features of the concept of salient space, and in Chapter 3 we introduce topology into this setting. Furthermore, since the positive cone of a partially ordered vector space is an example of a salient space, our analysis of this new concept yields some side-results concerning partially ordered vector spaces. These theorems can also be found in Chapter 2 and 3.

The main goal in this thesis is the introduction of salient space-based models for exchange economies and corresponding equilibrium existence proofs. In Chapter 4 we use the concept of salient space (and related concepts) as the basis of our models. Model A describes a pure exchange economy, Model B introduces price rigidities into this setting, whereas both Model C and Model D introduce production into Model A, and each describe a different private ownership economy. The final section of Chapter 4 states six equilibrium existence theorems: two for Model A, one for Model B, two for Model C, and one for Model D. Finally, Chapter 5 is devoted completely to the proofs of these theorems.

Chapter 2

Salient Spaces

Introduction

This chapter contains the axiomatic introduction and the study of the mathematical concept of salient space. In Chapter 1, we already discussed that this concept will play a key role throughout this thesis. Indeed, it is especially designed for usage as a building block in the models presented in Chapter 4. More precisely, these models are constructed around the set of all "bundles of trade", which is represented, in a natural way, by this novel concept of salient space.

A salient space is a set in which an addition is defined in such a way that the set is a semi-group, and a scalar multiplication is defined over the nonnegative reals. The axiomatic introduction of salient space resembles the one of vector space; the main difference is that for a vector space multiplication is allowed over the set of reals, where for a salient space scalar multiplication is restricted to the set of nonnegative real numbers. Another difference is that every vector space is an addition group, whereas each salient space is an addition semi-group. As a consequence, not every element of a salient space has an inverse with respect to addition. More specifically, one of the axioms of a salient space states that only one element, called the vertex, of a salient space has an inverse. Thus, vector space concepts, such as the definition of linear combination, which explicitly (or implicitly) make use of the vector space operation called subtraction, do not directly apply to salient spaces. In this chapter we will show that several definitions of such concepts can be adapted to salient space related concepts, in such a way that the minus sign is circumvented. On the other hand, we also introduce several properties of salient spaces which do not have a vector space related counterpart.

More specifically, in Section 2.1 we give the axiomatic introduction of the concept of salient space, we give some notions which are closely related to this new concept, such as for example salient independence and salient basis, and we describe the construction of the vector space which is reproduced by a salient space. Also, we recall vector space concepts such as linearly dependent set and internal point, and we derive the salient space related definitions of these concepts. In Section 2.2, we explore the partial order relation which is closely connected to any salient space, and examine under which conditions a salient space has a lattice structure. In Section 2.3, we define a pairing between two salient spaces and introduce the concept of an adjoint of a salient space. This adjoint space will play the role of the price set in most of our models (cf. Chapter 4). Finally, in Section 2.4, we investigate the connection between extreme sets of a salient space and the concept of a salient basis.

Although it turns out that each pointed convex cone in a vector space is a salient space, and conversely, each salient space induces a vector space for which the salient space is a positive cone, we feel that this new concept allows for a better description of the set of all commodity bundles in a model of a pure exchange economy or a model of a private ownership economy. One of the reasons we feel this way is due to the fact that every finite-dimensional real vector space is isomorphic with some real finite-dimensional Euclidean space, where this is not the case for a finite-dimensional salient space.

2.1 Salient space

We start with the formal introduction of the new mathematical concept of salient space and some other new concepts which are closely related to it. Throughout this thesis, the notation $\mathbb{R}_+$ will be used to denote the set $\{\alpha \in \mathbb{R} \mid \alpha \geq 0\}$ of nonnegative reals.

2.1.1 Definition (salient space, sum, vertex, scalar product). A salient space is a set S with the following properties:

- To every $s_1, s_2 \in S$ there corresponds an element, $s_1 + s_2$, in S, called the sum of s_1 and s_2, in such a way that
  - **a)** addition is commutative: $\forall s_1, s_2 \in S : s_1 + s_2 = s_2 + s_1$,
  - **b)** addition is associative: $\forall s_1, s_2, s_3 \in S : s_1 + (s_2 + s_3) = (s_1 + s_2) + s_3$,
  - **c)** there exists an element $v \in S$, called the vertex of S, such that $\forall s_1, s_2 \in S : s_1 + s_2 = v \iff s_1 = s_2 = v$,
  - **d)** for every $s_1 \in S$ the mapping $\text{add}_{s_1} : S \to S$, defined by $\text{add}_{s_1}(s) := s + s_1$, is injective: $\forall s_1, s_2, s_3 \in S : s_1 + s_2 = s_1 + s_3 \implies s_2 = s_3$.
- To every $s \in S$ and every $\alpha \in \mathbb{R}_+$ there corresponds an element, αs, in S, called the (scalar) product of α and s, in such a way that

e) multiplication over $\mathbb{R}_+$ is associative: $\forall s \in S\ \forall \alpha_1, \alpha_2 \in \mathbb{R}_+ : \alpha_1(\alpha_2 s) = (\alpha_1\alpha_2)s$,

f) $1s = s$,

g) multiplication over $\mathbb{R}_+$ is distributive with respect to the addition: $\forall s_1, s_2 \in S\ \forall \alpha \in \mathbb{R}_+ : \alpha(s_1 + s_2) = \alpha s_1 + \alpha s_2$,

h) multiplication over $\mathbb{R}_+$ is distributive with respect to scalar addition: $\forall s \in S\ \forall \alpha_1, \alpha_2 \in \mathbb{R}_+ : (\alpha_1 + \alpha_2)s = \alpha_1 s + \alpha_2 s$.

We observe that Condition 2.1.1.c implies that the mapping add_{s_1} can only be surjective if $s_1 = v$. Lemma 2.1.3, below, shows that add_v indeed is surjective and that the mapping add_{s_1} is surjective if and only if $s_1 = v$.

2.1.2 Lemma. *The vertex of a salient space S is unique.*

Proof.
Suppose both v and w are vertices of S, then from $w + w = w$ it immediately follows that $v + w + w = v + w$. Applying Condition 2.1.1.d, we get $v + w = v$ and, because v is a vertex of S, $w = v$ follows from Condition 2.1.1.c. □

2.1.3 Lemma. *The vertex v of a salient space S satisfies the following three properties:*

a) $\forall \alpha > 0 \ : \alpha v = v$,
b) $\forall x \in S \ : x + v = x$,
c) $\forall x \in S \ : 0x = v$.

Proof.
a) We prove that αv is a vertex of S for all $\alpha > 0$, then by the preceding lemma $\alpha v = v$. Consider the following equivalent assertions:
$x + y = \alpha v \iff \frac{1}{\alpha}x + \frac{1}{\alpha}y = v \iff (\frac{1}{\alpha}x = v) \wedge (\frac{1}{\alpha}y = v) \iff (x = \alpha v) \wedge (y = \alpha v)$.
b) Let $x \in S$ and define $y := x + v$. Then $y + y = 2y = 2(x + v) = 2x + v = x + (x + v) = x + y$. Applying Condition 2.1.1.d yields $y = x$.
c) Let $x \in S$, then by Property 2.1.3.b and the distributiveness of scalar multiplication over $\mathbb{R}_+$, we get $0x + 0x = (0 + 0)x = 0x = 0x + v$. So, Condition 2.1.1.d yields $0x = v$. □

From Property 2.1.3.b together with Conditions 2.1.1.c and 2.1.1.d, we conclude that $(S, +)$ is a semi-group with zero-element v (cf. [12]). Since in a salient space, scalar multiplication is defined only over $\mathbb{R}_+$ and due to Condition 2.1.1.c, $(S, +)$ is not a group, but a semi-group.

2.1.4 Example. For every $n \in \mathbb{N}$, the positive orthant $\mathbb{R}^n_+$ of the finite-dimensional, Euclidean inner product space $\mathbb{R}^n$ is a salient space, with the zero-vector as vertex and with the addition and scalar multiplication over $\mathbb{R}_+$ taken from $\mathbb{R}^n$. $\diamond$

2.1.5 Example. The set $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\} \cup \{(0, 0)\}$ is a salient space, with $(0, 0)$ as vertex and with the addition and scalar multiplication from $\mathbb{R}^2$. $\diamond$

2.1.6 Example. Let $n \in \mathbb{N}$. An $n \times n$ matrix M is said to be positive if M is symmetric and satisfies

$$\forall x \in \mathbb{R}^n : \langle Mx, x\rangle \geq 0.$$

Here, $\langle ., .\rangle$ denotes the standard inner product of $\mathbb{R}^n$. Let S be the set of all positive $n \times n$ matrices, then S is a salient space with addition and scalar multiplication over $\mathbb{R}_+$ defined as usual, and with the zero-matrix as vertex. $\diamond$

2.1.7 Definition (salient mapping, salient homomorphism, salient isomorphism, isomorphic salient spaces). Let S and T be two salient spaces. A mapping $\mathcal{L} : S \to T$ is salient if for all $s, s_1, s_2 \in S$ and for all $\alpha \in \mathbb{R}_+$:

$$\begin{cases} \mathcal{L}(s_1 + s_2) = \mathcal{L}(s_1) + \mathcal{L}(s_2) \\ \mathcal{L}(\alpha s) = \alpha\mathcal{L}(s). \end{cases}$$

A salient mapping $\mathcal{L}$ from S into T is a salient homomorphism if $\mathcal{L}$ is injective. A surjective salient homomorphism $\mathcal{L}$ from S onto T is a salient isomorphism. Two salient spaces S and T are isomorphic if there is a salient isomorphism $\mathcal{L} : S \to T$.

We observe that if $\mathcal{L} : S \to T$ is a salient homomorphism between the salient spaces S and T, then $\mathcal{L}(S)$ is a salient space, and S and $\mathcal{L}(S)$ are isomorphic. Note that $\mathcal{L}(v)$ is the vertex of T.

2.1.8 Definition (salient subspace). A subset T of a salient space S is a salient subspace of S, if T, endowed with the addition and scalar multiplication over $\mathbb{R}_+$ of S, is a salient space.

Without proof, we give the following characterisation for salient subspaces of a salient space.

2.1.9 Proposition. *A subset T of a salient space S is a salient subspace of S if and only if $\forall t_1, t_2 \in T\ \forall \alpha \in \mathbb{R}_+ : t_1 + t_2 \in T$ and $\alpha t_1 \in T$, i.e. if and only if T is closed under summation and scalar multiplication over $\mathbb{R}_+$.*

As a consequence of this proposition, the intersection of two salient subspaces of a salient space S is a salient subspace of S.

2.1.10 Definition (salient span, finitely generated salient space salient space, ray). Let S be a salient space. For a subset A of S, the salient span of A, denoted by $\mathrm{sal}(A)$, is the intersection of all salient subspaces of S, that contain A. If there is a finite set F such that $\mathrm{sal}(F) = S$, then S is a finitely generated salient space. For every $s \in S \setminus \{v\}$, the ray generated by s, denoted by $\mathrm{ray}(s)$, is the set $\{\alpha s \mid \alpha \in \mathbb{R}_+\}$. For a subset A of $S \setminus \{v\}$, the set consisting of the rays of all the elements of A, denoted by $\mathrm{ray}(A)$, is the set $\{\mathrm{ray}(a) \mid a \in A\}$.

Note that by definition, $\mathrm{sal}(\emptyset) = \{v\}$, $\mathrm{ray}(s) = \mathrm{sal}(\{s\})$ for every $s \in S \setminus \{v\}$, and that for every subset A of a salient space S, $\mathrm{sal}(A)$ is the "smallest" salient subspace of S that contains A.

2.1.11 Example. For every $n \in \mathbb{N}$, the salient space $\mathbb{R}^n_+$ is finitely generated by the set $\{e^i \mid i \in \{1, \ldots, n\}\}$. ◇

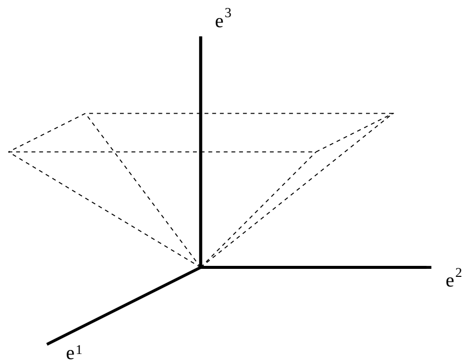

FIGURE 2.1.1: $\mathrm{sal}(A_0)$ of Example 2.1.12

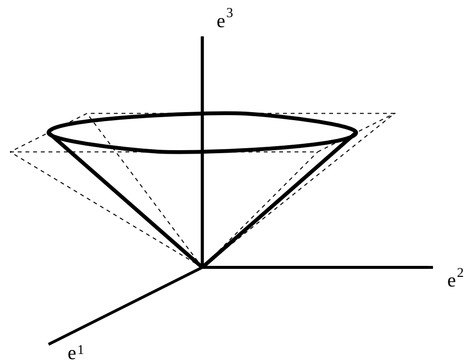

FIGURE 2.1.2: $\mathrm{sal}(A_0)$ and $\mathrm{sal}(A_1)$ of Example 2.1.12

2.1.12 Example. Consider the set

$$A_0 = \{(1, 1, 1), (1, -1, 1), (-1, 1, 1), (-1, -1, -1)\} \subset \mathbb{R}^3.$$

Then $S := \mathrm{sal}(A_0)$ is a finitely generated salient space in $\mathbb{R}^3$ (cf. Define

$$A_1 := \{(x_1, x_2, 1) \in \mathbb{R}^3 \mid (x_1)^2 + (x_2)^2 \leq 1\},$$

then $\mathrm{sal}(A_1)$ is a salient subspace of S. However, $\mathrm{sal}(A_1)$ is not finitely generated (cf. Figure 2.1.2 and the proof of Theorem 3.3.10). ◇

The proof of the following proposition is similar to the proof of Theorem 2.3 of [23], stating that the convex hull of a set A in a vector space consists of all (finite) convex combinations of the elements of A.

2.1.13 Proposition. *Let A be a subset of a salient space S, then for every $a \in \mathrm{sal}(A)$, there is a finite set $F \subset A$ such that $a \in \mathrm{sal}(F)$. Hence,*

$$\mathrm{sal}(A) = \{s \in S \mid \exists n \in \mathbb{N}\ \exists a_1, \ldots, a_n \in A\ \exists \lambda_1, \ldots, \lambda_n \in \mathbb{R}_+ : s = \sum_{i=1}^{n} \lambda_i a_i\}.$$

Henceforth, we call an element of $\mathrm{sal}(A)$, a *salient combination* or *positive combination* of A.

Without proof we state the following lemma concerning isomorphic salient spaces and generating sets.

2.1.14 Proposition. *Let S and T be two salient spaces and let $\mathcal{L} : S \to T$ be a salient isomorphism. If A is a subset of S, satisfying $\mathrm{sal}(A) = S$ then $\mathrm{sal}(\mathcal{L}(A)) = T$.*

The above proposition implies that for two isomorphic salient spaces S and T, the salient space S is finitely generated if and only if T is finitely generated. Hence, for example, $\mathrm{sal}(A_0)$ and $\mathrm{sal}(A_1)$ of Example 2.1.12 are not isomorphic.

Following [20], we introduce the following notation, for the sum of two subsets of a salient space S. Let $A, B \subset S$. Then the set $A + B$ is given by

$$A + B := \{s \in S \mid \exists a \in A\ \exists b \in B : s = a + b\}.$$

For every fixed $s_0 \in S$ the set $s_0 + A$ is given by

$$s_0 + A := \{s \in S \mid \exists a \in A : s = s_0 + a\}.$$

2.1.15 Definition (saliently independent set, salient basis). Let S be a salient space. For every $B \subset S \setminus \{v\}$, the set $\mathrm{ray}(B)$ is saliently independent if

$$\forall b_0 \in B\ \forall F \subseteq B, F \text{ finite}: \ \mathrm{ray}(b_0) \subset \mathrm{sal}(F) \implies \exists f \in F : \mathrm{ray}(b_0) = \mathrm{ray}(f).$$

A saliently independent set $\mathrm{ray}(B)$ is a salient basis for S if $\mathrm{sal}(B) = S$.

Note that a set $\mathrm{ray}(B)$ is saliently independent if and only if for all $b_0 \in B$ and for all finite sets $F \subseteq B$:

$$(\ \forall f \in F : f \notin \mathrm{ray}(b_0)\ ) \implies (\ \mathrm{ray}(b_0) \cap \mathrm{sal}(F) = \{0\}\ ).$$

Furthermore, we emphasise that if $\text{ray}(B)$ is a salient basis of a salient space S, then, by definition, $B \subset S \setminus \{v\}$.

We deliberately chose the set $\text{ray}(B)$ of rays, rather than the set B itself, to denote a salient basis, for the following reasons. Firstly, when we use the set B to denote a salient basis for S, we also have to introduce a certain equivalence relation between the many salient bases of S. Indeed, replacement of an arbitrary element b_0 of B by λb_0, with $\lambda > 0$, results in a salient basis which is equal to $\text{ray}(B)$. Clearly, the use of rays does not involve this equivalence problem. Secondly, for the reader who is acquainted with extreme sets, this approach raises the question whether the extreme rays of a salient space S form a saliently independent set. More precisely, in Section 2.4 we will show that if S has a salient basis then this basis is equal to the set of extreme rays. And thirdly, this approach emphasises the difference between a salient basis and a maximal linearly independent set in S (cf. page 39 and 40), which proves to be an entirely different concept. For one, a salient space can have many different linear bases (even without considering the obvious equivalence due to the scaling of elements), where the following proposition shows that a salient basis of a salient space is unique. Another difference is that every salient space has a maximal linearly independent set, while Example 2.1.18 shows that not every salient space has a salient basis. Finally, we mention that if $\text{ray}(B)$ is a salient basis of a salient space S, and if $s \in S$, then the nonnegative function $\mathcal{F} : B \to \mathbb{R}_+$, for which the set $\{b \in B \mid \mathcal{F}(b) > 0\}$ is finite and which satisfies

$$s = \sum_{b \in B} \mathcal{F}(b) b,$$

does not have to be unique (cf. Example 2.1.17).

2.1.16 Proposition. *If a salient space S has a salient basis, then this salient basis is unique.*

Proof.
Let S be salient space, let both $\text{ray}(A)$ and $\text{ray}(B)$ be a salient basis of S. Since $\text{sal}(B) = S$, we find that for every $a \in A$, there is a finite set $F_a \subset B$ such that

$$\text{ray}(a) = \sum_{b \in F_a} \text{ray}(b).$$

Since A is a saliently independent set, we find $\exists f \in F_a : \text{ray}(f) = \text{ray}(a)$. $\square$

2.1.17 Example. Consider the salient space S of Example 2.1.12, where S is finitely generated by $A_0 \subset \mathbb{R}^3$. The set $\text{ray}(A_0)$ consisting of the rays generated by the elements of A_0, is a salient basis for S. Observe that S is also generated by the set $A_0 \cup \{(0,0,1)\}$ and that $A_0 \cup \{(0,0,1)\}$ is not a salient basis. Furthermore, $\{(x_1, x_2, 1) \in A_1 \mid (x_1)^2 + (x_2)^2 = 1\}$ is a salient basis for $\mathrm{sal}(A_1)$. Finally, we observe that $(0,0,1) = \frac{1}{2}(1,1,1) + \frac{1}{2}(-1,-1,1)$ and $(0,0,1) = \frac{1}{2}(1,-1,1) + \frac{1}{2}(-1,1,1)$. Hence the representation of an element of S as a nonnegative combination of the elements of A_0 is not unique. ◇

2.1.18 Example. The salient space $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\} \cup \{(0,0)\}$ of Example 2.1.5 does not have a salient basis. ◇

Without proof we state the following lemma concerning isomorphic salient spaces, saliently independent sets and salient bases.

2.1.19 Lemma. *Let S and T be two isomorphic salient spaces with respect to the salient isomorphism $\mathcal{L} : S \to T$. If A is a saliently independent set of S, then $\mathcal{L}(A)$ is a saliently independent set of T. Furthermore, if A is a salient basis of S, then $\mathcal{L}(A)$ is a salient basis of T.*

2.1.20 Example. Consider Example 2.1.6 with $n = 2$, i.e., S is the salient space of all real, positive 2×2 matrices. Note that a matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \text{ is positive if and only if } \begin{cases} m_{12} = m_{21} \\ m_{11}m_{22} \geq (m_{12})^2 \\ m_{11} \geq 0 \text{ and } m_{22} \geq 0. \end{cases}$$

When we identify $M \in S$ with $(m_{11}, m_{12}, m_{22}) \in \mathbb{R}^3$, we find that S is isomorphic with the salient space $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geq 0, x_3 \geq 0 \text{ and } x_1 x_3 \geq (x_2)^2\}$. Clearly, the salient space S is not finitely generated. ◇

In Section 2.4 and in Theorem 3.3.10, we will return to the concept of salient basis. Section 2.4 deals with the connection between extreme rays of a salient space and the concept of salient basis. One of the conclusions is that if a salient space S has a salient basis, then this basis is equal to the set of all extreme rays of S. In Theorem 3.3.10, we prove that every finite-dimensional reflexive salient space has a salient basis (cf. page 59 for the definition of a reflexive salient space).

We started this section with an axiomatic introduction of the concept salient space which resembles the vector space axioms. We have seen that every salient space is an addition semi-group with zero-element v. Due to Condition 2.1.1.c, the semi-group $(S,+)$ is not a group. However, the semi-group $(S,+)$ can be extended to a group in a similar way as the semi-group $\mathbb{N} \cup \{0\}$ extends to the group $\mathbb{Z}$. Hereto, define the equivalence relation $\sim$ on the product set $S \times S$, for all $s_1, s_2, t_1, t_2 \in S$, by:

$$(s_1, s_2) \sim (t_1, t_2) \quad :\Longleftrightarrow \quad s_1 + t_2 = t_1 + s_2.$$

Then, the collection $V[S] := (S \times S)/_\sim$ of all equivalent classes

$$[(s_1, s_2)] := \{(t_1, t_2) \in S \times S \mid (t_1, t_2) \sim (s_1, s_2)\},$$

is an addition group where the addition of equivalence classes $[(s_1, s_2)]$ and $[(t_1, t_2)]$, for every $s_1, s_2, t_1, t_2 \in S$, is unambiguously defined by

$$[(s_1, s_2)] + [(t_1, t_2)] := [(s_1 + t_1, s_2 + t_2)].$$

By defining the scalar product of equivalence class $[(s_1, s_2)]$ and α, for every $s_1, s_2 \in S$ and every $\alpha \in \mathbb{R}$ by

$$\alpha[(s_1, s_2)] := \begin{cases} [(\alpha s_1, \alpha s_2)] & \text{if } \alpha \geq 0 \\ [((-\alpha)s_2, (-\alpha)s_1)] & \text{if } \alpha < 0, \end{cases}$$

$V[S]$ becomes a real vector space. We call $V[S]$ the *vector space reproduced by the salient space* S.

With $V_+[S]$ we denote the salient space

$$\{[(s_1, s_2)] \in V[S] \mid \exists s \in S : [(s_1, s_2)] = [(s, 0)]\}.$$

Note that $V_+[S]$ is isomorphic with S, since $\mathcal{J}_S : S \to V_+[S]$, for every $s \in S$ defined by $\mathcal{J}_S(s) := [(s, 0)]$, is a salient isomorphism. Throughout this chapter $\mathcal{J}_S$ will denote the isomorphism just described.

For every subset A of a salient space S, the vector space $V[\mathrm{sal}(A)]$ reproduced by the salient span of A, is equal to the linear span of the set $\mathcal{J}_S(\mathcal{L}(A))$ in the vector space $V[S]$. We will denote this linear span by $\mathrm{span}_{V[S]}(\mathcal{L}(A))$. So, we find that $\mathrm{sal}(\mathcal{J}_S(\mathcal{L}(A))) \subset \mathrm{span}_{V[S]}(A)$. As a result, if A is a finite set in S, then $V[\mathrm{sal}(A)]$ is a finite-dimensional subspace of $V[S]$. The converse is, in general, not true since not every salient space S, for which $V[S]$ is finite-dimensional, is finitely generated. For example, the vector space reproduced by the salient space S of Figure 2.1.3, is three-dimensional, while S is not finitely generated.

Clearly, every salient subspace T of a salient space S satisfies $V[T] \subseteq V[S]$. Note that, as the two salient spaces of Figure 2.1.2 show $T = S$ is not a necessary condition for $V[T] = V[S]$.

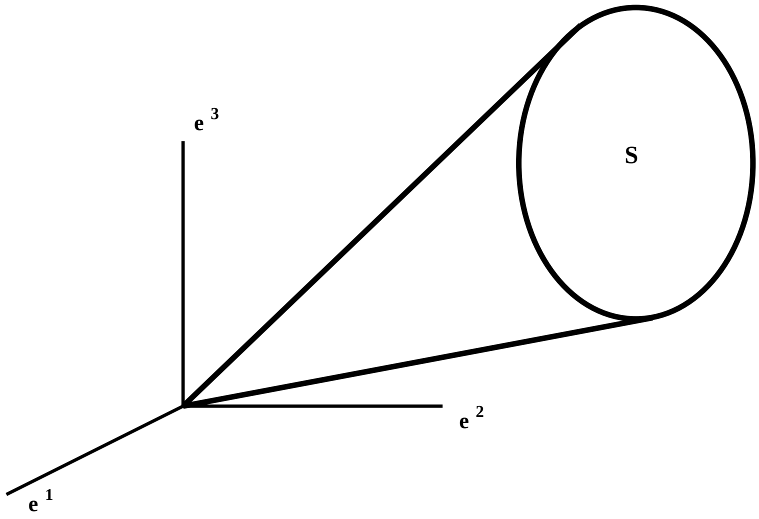

FIGURE 2.1.3: Salient space in $\mathbb{R}^3$

2.1.21 Definition (cone, convex cone, pointed convex cone). A set K in a real vector space V is called a cone if K is closed under scalar multiplication over $\mathbb{R}_+$, i.e., if $\forall k \in K\ \forall \alpha \in \mathbb{R}_+ : \alpha k \in K$. A convex cone is a cone K which is closed under addition, i.e., $\forall k_1, k_2 \in K : k_1 + k_2 \in K$. A pointed convex cone is a convex cone K satisfying $\forall k \in K \setminus \{0\} : -k \notin K$.

Clearly, a salient space S is a pointed convex cone in $V[S]$. Furthermore, every pointed convex cone K in a vector space V can be regarded as a salient space, using the addition and scalar multiplication of V. In this situation, $V[K]$ equals the linear span of K, hence $V[K]$ is a subspace of V.

We recall the salient isomorphism $\mathcal{J}_S$ between a salient space S and the salient space $V_+[S]$. The linear span of $\mathcal{J}_S(S)$ in the vector space $V[S]$ is equal to the vector space $V[S]$, i.e., the salient space $\mathcal{J}_S(S)$ is a total set in $V[S]$. The vertex $\mathcal{J}_S(v)$ of $V_+[S]$ coincides with the origin of $V[S]$, and henceforward we shall denote the vertex of a salient space by 0.

2.1.22 Example. For every $n \in \mathbb{N}$, the vector space $V[\mathbb{R}^n_+]$, reproduced by the salient space $\mathbb{R}^n_+$, equals $\mathbb{R}^n$. ◇

2.1.23 Example. The salient space $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\} \cup \{(0,0)\}$ of Example 2.1.5, satisfies $V[S] = \mathbb{R}^2$. ◇

2.1.24 Example. Let S be the salient space of all positive $n \times n$ matrices of Example 2.1.6. Then the vector space $V[S]$ generated by S is the set of all symmetric $n \times n$ matrices. $\diamond$

2.1.25 Definition (extension of a salient mapping). Let S and T be two salient spaces and let $\mathcal{L} : S \to T$ be a salient mapping. The extension $\mathcal{L}^{\text{ext}} : V[S] \to V[T]$ of $\mathcal{L}$ to a linear mapping on $V[S]$ is, for every $s_1, s_2 \in S$, given by

$$\mathcal{L}^{\text{ext}}([(s_1, s_2)]) := [(\mathcal{L}(s_1), \mathcal{L}(s_2))].$$

We observe that if $\mathcal{L} : S \to T$ is a salient homomorphism (salient isomorphism) between salient spaces S and T, then $\mathcal{L}^{\text{ext}}$ is a homomorphism (isomorphism) from $V[S]$ into $V[T]$.

Let V be a vector space and let A be an arbitrary subset of V. We recall (e.g. from [7, Chapter IV.1]) that an element a_0 of A is *internal point* of A if

$$\forall v \in V \; \exists \varepsilon > 0 \; \forall \tau \in (0, \varepsilon) : a_0 + \tau v \in A.$$

In the following definition, we give the salient version of the concept of internal point.

2.1.26 Definition (saliently internal point). Let A be a subset of a salient space S. Then an element $a_0 \in A$ is a saliently internal point of A if

$$\forall s \in S \; \exists \varepsilon > 0 \; \forall \tau \in (0, \varepsilon) : a_0 + \tau s \in A.$$

2.1.27 Example. Consider the salient space S and the set $A \subset S$ of Figure 2.1.4. Then a_0, a_1 and a_2 are saliently internal points of A, where a_3 is not. $\diamond$

Note that, since every salient space is convex, an element s_0 of salient space S is saliently internal point of S if and only if $\forall s \in S \; \exists \varepsilon > 0 : s_0 + \varepsilon s \in S$, i.e., every element of S is saliently internal point of S.

2.1.28 Definition (interior, boundary of a salient space). Let S be a salient space. Then $s_0 \in S$ is an element of the interior of S, denoted by $\text{int}(S)$, if

$$\forall s \in S \; \exists \varepsilon > 0 : s_0 \in \varepsilon s + S.$$

The boundary $\text{bd}(S)$ of the salient space S is given by $S \setminus \text{int}(S)$.

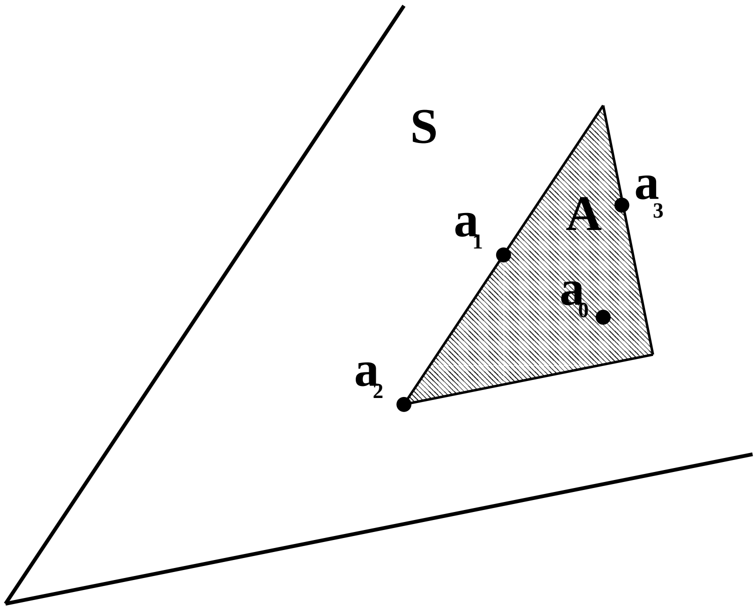

FIGURE 2.1.4: Saliently internal points

It is not difficult to check that $\operatorname{int}(S) \cup \{0\}$ is a salient subspace of S.

Without proof we state the following proposition concerning isomorphic salient spaces, saliently internal points and interior elements.

2.1.29 Proposition. *Let S and T be two isomorphic salient spaces with respect to the salient isomorphism $\mathcal{L} : S \to T$, and let $A \subset S$. Then $a_0 \in A$ is saliently internal point of A if and only if $\mathcal{L}(a_0)$ is saliently internal point of $\mathcal{L}(A) \subset T$. Furthermore, $\mathcal{L}(\operatorname{int}(S)) = \operatorname{int}(T)$.*

The previous proposition implies that for every saliently internal point a_0 of a subset A of a salient space S, the element $[(a_0, 0)]$ is a saliently internal point of the subset $\mathcal{J}_S(A)$ of the pointed convex cone $V_+[S]$. The following proposition implies that every element of $\operatorname{int}(V_+[S])$ is an internal point of $V_+[S] \subset V[S]$ in accordance with the above stated vector space definition of internal point.

2.1.30 Proposition. *Let S be a salient space. Then $s_0 \in \operatorname{int}(S)$ if and only if*

$$\forall s_1, s_2 \in S\ \exists \varepsilon > 0\ \forall \tau \in (0, \varepsilon) : [(s_0, 0)] + \tau[(s_1, s_2)] \in V_+[S].$$

Proof.
Clearly, $\forall s \in S\ \exists \varepsilon > 0 : s_0 \in \varepsilon s + S$ is equivalent with $\forall s_1, s_2 \in S\ \exists \varepsilon > 0 : s_0 + \varepsilon s_1 \in \varepsilon s_2 + \varepsilon s_1 + S \subset \varepsilon s_2 + S$. Since the set $\varepsilon s_2 + S$ is convex and contains s_0, we find $\forall s_1, s_2 \in S\ \exists \varepsilon > 0\ \forall \tau \in (0, \varepsilon) : s_0 + \tau s_1 \in \varepsilon s_2 + S \subseteq \tau s_2 + S$. Hence, $\forall s_1, s_2 \in S\ \exists \varepsilon > 0\ \forall \tau \in (0, \varepsilon) : [(s_0, 0)] + \tau[(s_1, s_2)] \in V_+[S]$. □

We conclude this section by presenting the salient version of a linearly dependent set in a salient space S and stating its relationship with the definition of a linearly dependent set in $V[S]$. Furthermore, we show that every salient space S has a maximal independent set and that this set is a basis for the vector space reproduced by S.

2.1.31 Definition (linearly dependent set, linearly independent set). Let A be a subset of a salient space S. Then the set A is linearly dependent if $0 \in A$ or if there is a non-empty, finite subset F of A such that

$$\mathrm{sal}(F) \cap \mathrm{sal}(A \setminus F) \neq \{0\}.$$

The set A is linearly independent if A is not linearly dependent, i.e., if $0 \notin A$ and if every non-empty, finite subset F of A satisfies

$$\mathrm{sal}(F) \cap \mathrm{sal}(A \setminus F) = \{0\}.$$

We remark that we could easily have introduced the notion of linear independence for a set of rays in S. However, we choose the above definition to emphasise the contrast between the notions of salient independence and linear independence (cf. page 41). Also, Definition 2.1.31 facilitates the comparison between linear independence in salient spaces and linear independence in vector spaces, as can be seen in the following lemma.

2.1.32 Lemma. *Let A be a subset of a salient space S. Then*

$$A \textit{ is linearly dependent in } S \iff A \textit{ is linearly dependent in } V[S].$$

Proof.
The above lemma is obviously true in case $0 \in A$, hence, throughout this proof, we assume $0 \notin A$.
If A is linearly dependent in S, there is $s \in S \setminus \{0\}$ and there is a non-empty, finite set $F \subset A$ such that $s \in \mathrm{sal}(F) \cap \mathrm{sal}(A \setminus F)$. Clearly, $s \in \mathrm{span}_{V[S]}(F) \cap \mathrm{span}_{V[S]}(A \setminus F)$, and so A is linearly dependent in $V[S]$.
For the converse, assume A is linearly dependent in $V[S]$. Then

$$\exists n \in \mathbb{N}\ \exists a_1, \ldots, a_n \in A, \text{ with } a_i \neq a_j\ (i \neq j)\ \exists \lambda_1, \ldots \lambda_n \in \mathbb{R} \setminus \{0\} : \sum_{i=1}^{n} \lambda_i a_i = 0.$$

Since S is pointed, we may as well assume that there is $k \in \mathbb{N}$ such that $1 < k < n$ and $\forall i \in \{1, \ldots, k\} : \lambda_i < 0$ and $\forall i \in \{k+1, \ldots, n\} : \lambda_i > 0$. Now, if $s = \sum_{i=1}^{k} (-\lambda_i) a_i$ then $s \neq 0$ and $s = \sum_{i=k+1}^{n} \lambda_i a_i$, i.e., $s \neq 0$ and $s \in \mathrm{sal}(\{a_1, \ldots, a_k\}) \cap \mathrm{sal}(\{a_{k+1}, \ldots, a_n\})$. □

2.1.33 Corollary. *Let A be a subset of a salient space S. Then*

$$A \text{ is linearly independent in } S \iff A \text{ is linearly independent in } V[S].$$

For every salient space S, the family $\mathcal{C}$ of linearly independent sets can be partially ordered by inclusion (cf. [20, Section 4.1] or Definition 2.2.1 for the definition of a partially ordered set): for all $A_1, A_2 \in \mathcal{C}$ define

$$A_1 \leq A_2 \quad :\iff \quad A_1 \subseteq A_2.$$

A totally ordered family or chain is a partially ordered set such that every two elements of the set are comparable. An upper bound of a subset $\mathcal{C}_0 \subset \mathcal{C}$ is an element $U \in \mathcal{C}$ such that $\forall A \in \mathcal{C}_0 : A \leq U$. A maximal element of $\mathcal{C}$ is an element $M \in \mathcal{C}$ such that $\forall A \in \mathcal{C} : \quad M \leq A \implies M = A$.

2.1.34 Proposition. *Let S be a salient space. Then there exists a maximal linearly independent subset in S.*

Proof.
To prove the proposition, we use Zorn's Lemma. So, let $\mathcal{C}_0$ be a chain in the family $\mathcal{C}$ of linearly independent subsets of S. We show that $\sup(\mathcal{C}_0) := \bigcup_{A \in \mathcal{C}_0} A$ is an upper bound for this chain. Clearly $\forall A \in \mathcal{C}_0 : A \subseteq \sup(\mathcal{C}_0)$, so we only have to prove that $\sup(\mathcal{C}_0)$ is linearly independent. Let F be a non-empty, finite subset of $\sup(\mathcal{C}_0)$. Let $x \in \operatorname{sal}(F) \cap \operatorname{sal}(\sup(\mathcal{C}_0) \setminus F)$, then there is a finite set $G \subset \sup(\mathcal{C}_0) \setminus F$ such that $x \in \operatorname{sal}(F) \cap \operatorname{sal}(G)$. Since $\mathcal{C}_0$ is a chain, there is an $A_0 \in \mathcal{C}_0$ such that $F \cup G \subseteq A_0$. Since $F \cap G = \emptyset$ we find $x \in \operatorname{sal}(F) \cap \operatorname{sal}(A_0 \setminus F)$ and since A_0 is linearly independent, we conclude $x = 0$. □

For two maximal linearly independent sets M_1 and M_2, each element of M_1 can be associated with a finite subset of M_2. So, the cardinality of M_1 is not greater than the cardinality of M_2. Interchanging the role of M_1 and M_2, we find that they have the same cardinality.

2.1.35 Definition (linear dimension of a salient space). Let S be a salient space, and let M be a maximal linearly independent set of S. The linear dimension of S, denoted by $\operatorname{lin}\dim(S)$, is equal to the cardinality of M.

As a result, a salient space S is *finite-dimensional* if every maximal linearly independent set of S is finite.

2.1.36 Lemma. *Let S be a salient space and consider the salient isomorphism $\mathcal{J}_S : S \to V_+[S]$. Then, every maximal linearly independent set M of S satisfies*

$$\mathrm{span}_{V[S]}(\mathcal{J}_S(M)) = V[S].$$

Proof.
Since M is maximal, we find that for every $s \in S \setminus \{0\} : M \cup \{s\}$ is linearly dependent, i.e., for every $s \in S \setminus \{0\}$ there is a finite subset $F \subseteq M$ such that $\mathrm{sal}(F \cup \{s\}) \cap \mathrm{sal}(M \setminus F) \neq \{0\}$. So, $\exists f \in \mathrm{sal}(F)$ $\exists m \in \mathrm{sal}(M \setminus F)$ such that $s + f = m$. Hence, $[(s, 0)] = [(m, f)]$, which is an element of $\mathrm{span}_{V[S]}(\mathcal{L}(M))$. □

Corollary 2.1.33 and Lemma 2.1.36 imply that for a salient space S, every maximal linearly independent set is a basis for $V[S]$. However, in general, M is too small to fully describe S; clearly, $\mathrm{sal}(M) = S$ does not have to hold, since this would imply that every salient space in a finite-dimensional vector space is finitely generated (see Figure 2.1.3 on page 36 for a counterexample).

2.1.37 Example. Consider the salient spaces $\mathrm{sal}(A_0)$ and $\mathrm{sal}(A_1)$ of Example 2.1.12. Every set of three linearly independent elements of A_1 forms a maximal linearly independent set for both $\mathrm{sal}(A_0)$ and $\mathrm{sal}(A_1)$. ◇

Without proof we state the following lemma concerning isomorphic salient spaces, linearly dependent sets and linear dimension.

2.1.38 Lemma. *Let S and T be two isomorphic salient spaces with respect to the salient isomorphism $\mathcal{L} : S \to T$. If A is a linearly dependent set of S, then $\mathcal{L}(A)$ is a linearly dependent set of T. Furthermore,* $\mathrm{lin\ dim}(\mathcal{L}(S)) = \mathrm{lin\ dim}(T)$.

2.1.39 Example. The salient space $\mathrm{sal}(A_0)$ of Example 2.1.12 is not isomorphic with $\mathbb{R}^4_+$ (where each element of A_0 is isomorphic with a unit vector of $\mathbb{R}^4$), since $\mathrm{lin\ dim}(\mathrm{sal}(A_0)) = 3 \neq 4 = \mathrm{lin\ dim}(\mathbb{R}^4_+)$. ◇

The above example implies that if a finitely generated salient space S is isomorphic with $\mathbb{R}^n_+$, for certain $n \in \mathbb{N}$, then S has a salient basis consisting of n elements.

2.2 Partial order relation and lattice structure

In Section 2.1 the axiomatic introduction of the concept of salient space was given and some concepts closely related to it were discussed, such as saliently dependent set, salient basis and saliently internal point. Furthermore, we have seen that every salient space S reproduces the vector space $V[S]$, in which the salient space $V_+[S]$, which is isomorphic with S, is a pointed convex cone.

In this section we concentrate on some well known vector space concepts related to a partial order relation, and give the definition of their salient space related counterparts. From [20, Section 4.1] and [1] we recall the following definitions.

2.2.1 Definition (partially ordered set). A partially ordered set $(M, \leq)$ is a set M on which there is defined a partial ordering, that is, a binary relation $\leq$, satisfying

- reflexivity: $\forall m \in M : m \leq m$,
- anti-symmetry: $\forall m_1, m_2 \in M$: if $m_1 \leq m_2$ and $m_2 \leq m_1$, then $m_1 = m_2$,
- transitivity: $\forall m_1, m_2, m_3 \in M$: if $m_1 \leq m_2$ and $m_2 \leq m_3$, then $m_1 \leq m_3$.

2.2.2 Definition (partially ordered vector space). A partially ordered vector space is a partially ordered set $(V, \leq)$, where V is a vector space over $\mathbb{R}$ and the partial order relation $\leq$ on V satisfies:

- translation-invariance: $\forall v_1, v_2, v_3 \in V$: if $v_1 \leq v_2$, then $v_1 + v_3 \leq v_2 + v_3$,
- scaling-invariance: $\forall v_1, v_2 \in V\ \forall \alpha \in \mathbb{R}_+$: if $v_1 \leq v_2$, then $\alpha v_1 \leq \alpha v_2$.

In order to arrive at the definition of a partially ordered salient space, we give the following lemma, which introduces a way to define a partial order relation on a salient space S, which is closely connected with S.

2.2.3 Lemma. *Let K be a cone in a vector space V. Define the order relation $\leq_K$ on V by*

$$x_1 \leq_K x_2 \quad :\Longleftrightarrow \quad x_2 - x_1 \in K,$$

then $\leq_K$ is reflexive, transitive, and anti-symmetric if and only if K is non-empty, convex, and pointed, respectively.

Proof.
Suppose $\leq_K$ is reflexive, then $\forall x \in V : x \leq_K x$ or $0 = x - x \in K$. So, K is non-empty.
Suppose K is non-empty, then $0 \in K$ because K is closed under multiplication over $\mathbb{R}_+$. Let $x \in V$, then $x \leq_K x$ because $x - x = 0 \in K$.

Suppose $\leq_K$ is transitive. Let $k_1, k_2 \in K$ and $\tau \in (0,1)$. Since K is a cone, we find $\tau k_1 \in K$ and $(1-\tau)k_2 \in K$, i.e., $0 \leq_K \tau k_1$ and $(\tau - 1)k_2 \leq_K 0$. The order relation $\leq_K$ is transitive, so $(\tau-1)k_2 \leq_K \tau k_1$ and hence $\tau k_1 + (1-\tau)k_2 \in K$.
Suppose K is convex and suppose $x_1 \leq_K x_2$ and $x_2 \leq_K x_3$ for some $x_1, x_2, x_3 \in V$. From $x_2 - x_1 \in K$ and $x_3 - x_2 \in K$ we find $\frac{1}{2}(x_2 - x_1) + \frac{1}{2}(x_3 - x_2) = \frac{1}{2}(x_3 - x_1) \in K$. Hence, we conclude $x_1 \leq_K x_3$.

Suppose $\leq_K$ is anti-symmetric and $x \in V$ satisfies $x \in K$ and $-x \in K$. Then, we find $0 \leq_K x$ and $0 \leq_K -x$, i.e., $x = 0$. We conclude that K is pointed.
Suppose K is pointed and $x_1 \leq_K x_2$ and $x_2 \leq_K x_1$ for some $x_1, x_2 \in V$. Then $x_2 - x_1 \in K$ and $x_1 - x_2 \in K$. The cone K is pointed so $x_1 - x_2 = 0$ and we conclude $x_1 = x_2$. So $\leq_K$ is anti-symmetric. □

With the help of the above lemma, the following proposition is easy to prove.

2.2.4 Proposition. *Let V be a vector space and K a pointed convex cone in V, then $(V, \leq_K)$ is a partially ordered vector space. Let $(V, \leq)$ be a partially ordered vector space, then $V_+ := \{x \in V \mid 0 \leq x\}$ is a pointed convex cone.*

For a partially ordered vector space $(V, \leq)$ we call the pointed convex cone V_+, defined by $V_+ = \{x \in V \mid 0 \leq x\}$ the *positive cone* of V. We remark that the partially order relation $\leq$ on V satisfies for every $x_1, x_2 \in V$:

$$x_1 \leq x_2 \quad \Longleftrightarrow \quad \exists k \in V_+ : x_1 + k = x_2.$$

We continue by defining a partial order relation on a salient space S, which can be extended to a partial order relation on $V[S]$. It turns out that, with respect to this order relation, the salient space $V_+[S]$ is equal to the positive cone $(V[S])_+$ of $V[S]$.

2.2.5 Definition (partial order relation on a salient space). On a salient space S the partial order relation $\leq_S$ is, for elements $s_1, s_2 \in S$, given by

$$\begin{aligned} s_1 \leq_S s_2 \quad &:\Longleftrightarrow \quad s_2 \in s_1 + S, \\ s_1 <_S s_2 \quad &:\Longleftrightarrow \quad s_2 \in s_1 + (S \setminus \{0\}). \end{aligned}$$

Note that $s_1 \leq_S s_2$ is equivalent with $s_2 + S \subseteq s_1 + S$.

2.2.6 Definition (partially ordered salient space). A partially ordered salient space is a salient space S which is a partially ordered set with respect to the partial order relation $\leq_S$.

The partial order relation $\leq_S$, defined on S, can be extended to a partial order relation $\leq_S$ on $V[S]$, thus constructing the partially ordered vector space $(V[S], \leq_S)$.

2.2.7 Definition (partial order relation on $\boldsymbol{V[S]}$). Let S be a salient space. On the vector space $V[S]$, reproduced by S, the partial order relation $\leq_S$ is, for every $[(s_1, s_2)], [(t_1, t_2)] \in V[S]$, given by

$$[(s_1, s_2)] \leq_S [(0,0)] \quad :\Longleftrightarrow \quad s_1 \leq_S s_2$$

and

$$[(s_1, s_2)] \leq_S [(t_1, t_2)] \quad :\Longleftrightarrow \quad [(s_1 + t_2, s_2 + t_1)] \leq_S [(0,0)].$$

Note that this partial order relation on $V[S]$ satisfies for all $s_1, s_2, t_1, t_2 \in S$:

$$[(s_1, s_2)] \leq_S [(t_1, t_2)] \quad \Longleftrightarrow \quad \exists s \in S : [(s_1, s_2)] + [(s, 0)] = [(t_1, t_2)],$$

and

$$[(s_1, s_2)] \leq_S [(t_1, t_2)] \quad \Longleftrightarrow \quad [(t_2, t_1)] \leq_S [(s_2, s_1)].$$

Furthermore, the positive cone $(V[S])_+ = \{[(s_1, s_2)] \in V[S] \mid [(0,0)] \leq_S [(s_1, s_2)]\}$ is isomorphic with S, and therefore equal to the salient space $V_+[S]$. Indeed, for every $[(s_1, s_2)] \in (V[S])_+$, we find that $\exists s \in S : [(s_1, s_2)] = [(s, 0)]$.

2.2.8 Definition (order set, order unit). A subset U of a salient space S is an order set for S if

$$\forall s \in S \; \exists u \in \mathrm{sal}(U) : s \leq_S u.$$

An element u of a salient space S is an order unit for S if $\{u\}$ is an order set of S.

We remark that $u \in S$ is an order unit if and only if

$$\forall s \in S \; \exists \lambda > 0 : s \leq_S \lambda u.$$

Clearly, if a salient space S has a subset A such that $S = \mathrm{sal}(A)$, i.e., if S is generated by A, then A is an order set of S. This implies that every finitely generated salient space has a finite order set.

Without proof, we state the following lemma, which implies that every finitely generated salient space has an order unit.

2.2.9 Lemma. *Let U be a finite order set of a salient space S, then $\sum_{u \in U} u$ is an order unit for S.*

2.2.10 Proposition. *Let S be a finite-dimensional salient space. Then S has a finite order set.*

Proof.
Set $n := \text{lin dim}(S)$. Since $\text{span}_{V[S]}(V_+[S]) = V[S]$, there is a set $U = \{u_1, \dots, u_n\} \subset S$ such that $\{\mathcal{J}_S(u_1), \dots \mathcal{J}_S(u_n)\}$ is a linear basis of $V[S]$. Let $s \in S$, then there are $\alpha_1, \dots \alpha_n \in \mathbb{R}$ such that

$$\mathcal{J}_S(s) = \sum_{i=1}^{n} \alpha_i \mathcal{J}_S(u_i).$$

This implies

$$\mathcal{J}_S(s) \leq_S \sum_{i=1}^{n} |\alpha_i| \mathcal{J}_S(u_i).$$

We conclude

$$s \leq_S \sum_{i=1}^{n} |\alpha_i| u_i \text{ and } \sum_{i=1}^{n} |\alpha_i| u_i \in \text{sal}(U).$$

□

2.2.11 Corollary. *Every finite-dimensional salient space has an order unit.*

Clearly, if u is an order unit for a salient space S, then for every $s \in S$, the element $u + s$ is also an order unit for S. Since, by definition, $s_0 \in S$ is an order unit if and only if $\forall s \in S \; \exists \lambda > 0 : s_0 \in \frac{1}{\lambda} s + S$, we find the following proposition.

2.2.12 Proposition. *Let S be a salient space and let $u \in S$. Then u is an order unit in S if and only if $u \in \text{int}(S)$.*

2.2.13 Example. For every $n \in \mathbb{N}$, the order relation of $\mathbb{R}^n$, induced by the salient space $\mathbb{R}^n_+$, equals the Euclidean order relation of $\mathbb{R}^n$. Hence, every strictly positive element of $\mathbb{R}^n_+$ serves as order unit, and, conversely, every order unit is a strictly positive element of $\mathbb{R}^n_+$. ◇

2.2.14 Example. On the salient space $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\} \cup \{(0,0)\}$ of Example 2.1.5, the order relation $\leq_S$ is, for every $(x_1, x_2), (y_1, y_2) \in S$, given by

$$(x_1, x_2) \leq_S (y_1, y_2) \iff \begin{cases} (x_1, x_2) = (y_1, y_2) \\ \text{or} \\ x_1 < y_1. \end{cases}$$

Every element of $S \setminus \{(0,0)\}$ is an order unit for S. ◇

2.2.15 Example. Let S be the salient space of all real, positive $n \times n$ matrices of Example 2.1.6. Then the identity matrix I is an order unit for S, since $\forall A \in S : A \leq_S \lambda_{\max} I$, where $\lambda_{\max}$ is the largest eigenvalue of A. $\diamond$

The following lemma, concerning isomorphic salient spaces and order units, is a direct consequence of Proposition 2.1.29 and Proposition 2.2.12.

2.2.16 Proposition. *Let S and T be two isomorphic salient spaces with respect to the salient isomorphism $\mathcal{L} : S \to T$. If u is an order unit of S, then $\mathcal{L}(u)$ is an order unit for T.*

The following lemma shows the bounding properties in $V[S]$, of an order unit of a salient space S.

2.2.17 Lemma. *Let u be an order unit for S, and let $[(s_1, s_2)] \in V[S]$. Then*

$$\exists \lambda \geq 0 : -\lambda[(u,0)] \leq_S [(s_1, s_2)] \leq_S \lambda[(u,0)].$$

Proof.
Since u is an order unit for S, we find $\begin{cases} \exists \lambda_1 \geq 0 : s_1 \leq_S \lambda_1 u \\ \exists \lambda_2 \geq 0 : s_2 \leq_S \lambda_2 u \end{cases}$.
Define $\lambda := \max\{\lambda_1, \lambda_2\}$, then $\begin{cases} s_1 \leq_S s_2 + \lambda u \\ s_2 \leq_S s_1 + \lambda u \end{cases}$. $\square$

In [1], Aliprantis, Brown and Burkinshaw generalise the Arrow-Debreu model of a pure exchange economy (as described in Section 1.2.1) by replacing the set $\mathbb{R}_+^{k_0}$ representing all commodity bundles, by a vector lattice or Riesz space. After the following short introduction (cf. [34], [1]) of some lattice related concepts regarding a partially ordered set, we give the definition of a vector lattice, and investigate some properties concerning salient spaces and lattice structures. In particular, we will see that the order relation associated with every salient space does not necessarily have a lattice structure. Hence, a generalisation of the Arrow-Debreu model using a vector lattice as the basic concept is incomparable with a generalisation based on the concept of salient space.

2.2.18 Definition (upper bound, least upper bound, lower bound, greatest lower bound, lattice, vector lattice, Riesz space). Let A be a subset of a partially ordered set $(M, \leq)$. An upper bound for the set $A \subset M$ is an element $u \in M$ satisfying $\forall a \in A : a \leq u$. A least upper bound for the set A is an upper bound u satisfying $u \leq v$ for every upper bound v of A. A lower bound for the set A is an element $l \in M$ satisfying $\forall a \in A : l \leq a$. A greatest lower bound for the set A is a lower bound l satisfying $k \leq l$ for every lower bound k of A. A partially ordered set $(M, \leq)$ is a lattice if every pair $\{m_1, m_2\}$ of elements of M has a least upper bound and a greatest lower bound. For every $m_1, m_2 \in M$ the least upper bound and greatest lower bound of $\{m_1, m_2\}$ is denoted by $m_1 \vee m_2$, and $m_1 \wedge m_2$, respectively. A partially ordered vector space $(V, \leq)$ that is a lattice, is called a vector lattice or a Riesz space.

2.2.19 Lemma. *Let S be salient space, let $s_1, s_2, s_3 \in S$ and consider the partially ordered set $(S, \leq_S)$. If $s_2 \vee s_3$ exists in S, then $(s_1 + s_2) \vee (s_1 + s_3)$ exists in S and*

$$(s_1 + s_2) \vee (s_1 + s_3) \quad = \quad s_1 + (s_2 \vee s_3).$$

Proof.
Clearly, if $s_2 \vee s_3$ exists in S, then $s_1 + (s_2 \vee s_3)$ is an upper bound of the set $\{s_1+s_2, s_1+s_3\}$. Furthermore, suppose $u \in S$ is an upper bound of $\{s_1+s_2, s_1+s_3\}$, then $s_1 + s_2 \leq u$ and $s_1 + s_3 \leq u$ implies that there are $p, q \in S$ such that

$$s_1 + s_2 + p = u \text{ and } s_1 + s_3 + q = u.$$

Amongst others, this implies that s_2+p is an upper bound of the set $\{s_2, s_3\}$, hence, $s_1 + (s_2 \vee s_3) \leq u$. We conclude that $s_1 + (s_2 \vee s_3)$ is the least upper bound of $\{s_1 + s_2, s_1 + s_3\}$. □

2.2.20 Proposition. *Let S be a salient space, satisfying that every pair of elements of S has a least upper bound, with respect to the partial order relation $\leq_S$. Then $(S, \leq_S)$ is a lattice.*

Proof.
We need to prove that every pair of elements of S has a greatest lower bound in S. To this end, let $s_1, s_2 \in S$, and let $s_1 \vee s_2$ be the least upper bound of the set $\{s_1, s_2\}$. There are $t_1, t_2 \in S$ such that

$$s_1 + t_1 = (s_1 \vee s_2) \text{ and } s_2 + t_2 = (s_1 \vee s_2).$$

Since $t_1 \leq_S (s_1 \vee s_2)$ and $t_2 \leq_S (s_1 \vee s_2)$, we find $(t_1 \vee t_2) \leq_S (s_1 \vee s_2)$, hence there is $u \in S$ such that

$$(t_1 \vee t_2) + u = (s_1 \vee s_2).$$

We will prove that u is the greatest lower bound of the set $\{s_1, s_2\}$. Clearly, $t_1+u \leq_S (s_1 \vee s_2) = s_1 + t_1$ implies that $u \leq_S s_1$. Similarly, we can prove that $u \leq_S s_2$, hence u is a lower bound of $\{s_1, s_2\}$. Suppose $b \in S$ is a lower bound of $\{s_1, s_2\}$, and reconsider $t_1, t_2 \in S$ as defined above. Then we find $b + t_1 \leq_S (s_1 \vee s_2)$ and $b + t_2 \leq_S (s_1 \vee s_2)$. Hence, Lemma 2.2.19 implies

$$b + (t_1 \vee t_2) = (b + t_1) \vee (b + t_2) \leq_S (s_1 \vee s_2) = u + (t_1 \vee t_2),$$

i.e., $b \leq_S u$, and we conclude $u = s_1 \wedge s_2$. □

The following proposition is illustrated by Figure 2.2.1.

2.2.21 Proposition. *Let S be a salient space. Then $(S, \leq_S)$ is a lattice if and only if*

$$\forall s_1, s_2 \in S \, \exists t \in S : (s_1 + S) \cap (s_2 + S) = t + S.$$

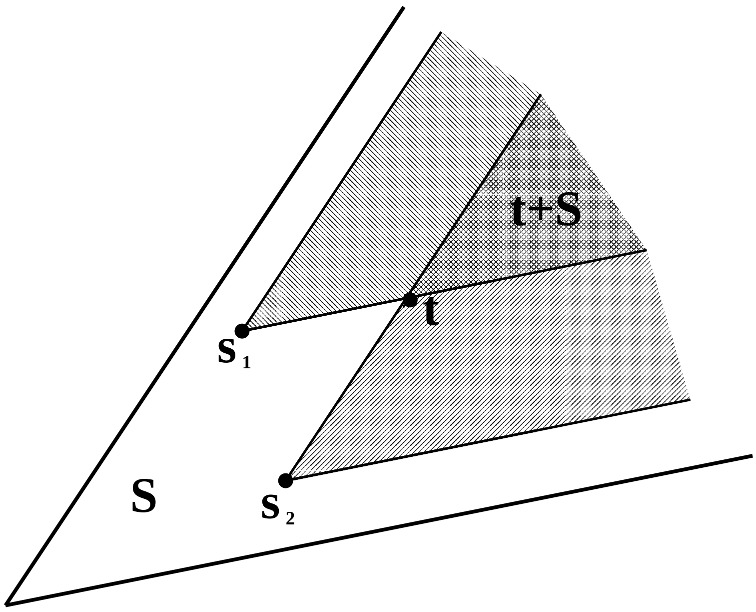

FIGURE 2.2.1: Lattice

Proof.
Suppose $(S, \leq_S)$ is a lattice. Let $s_1, s_2 \in S$ and let $t \in S$ satisfy t is the least upper bound of $\{s_1, s_2\}$. Then $s_1 \leq_S t$ and $s_2 \leq_S t$ implies $t + S \subseteq (s_1 + S) \cap (s_2 + S)$. Suppose $\exists u \in ((s_1 + S) \cap (s_2 + S)) \setminus (t + S)$, then u is an upper bound of $\{s_1, s_2\}$ satisfying $\neg(t \leq_S u)$. This is in contradiction with t being the least upper bound of $\{s_1, s_2\}$, hence, we conclude $t + S = (s_1 + S) \cap (s_2 + S)$.
For the converse, let $s_1, s_2 \in S$. We find that for every $s \in (s_1 + S) \cap (s_1 + S)$, the element s is an upper bound of $\{s_1, s_2\}$. By assumption $\exists t \in S : (s_1 + S) \cap (s_2 + S) = t + S$. Clearly, this implies that t is an upper bound of $\{s_1, s_2\}$ and that for every $u \in S$ for which u is an upper bound of $\{s_1, s_2\}$ there is $s \in S$ such that $t + s = u$. By Lemma 2.2.20, $(S, \leq_S)$ is a lattice. □

2.2.22 Example. The positive orthant $\mathbb{R}^n_+$ with the Euclidean order relation is a lattice. ◇

2.2.23 Example. Since every triple of vectors in the interior of some half-space of $\mathbb{R}^2$ satisfies that at least one of the vectors is a nonnegative combination of the other two, every pointed convex cone S in $\mathbb{R}^2$ is generated by at most two linearly independent vectors. In case S is generated by exactly two linearly independent vectors, these generators form a basis of $\mathbb{R}^2$. Thus, S is isomorphic with $\mathbb{R}^2_+$ with the Euclidean order relation and therefore has a lattice structure. ◇

The following two examples show that not every salient space has a lattice structure.

2.2.24 Example. Recall the salient space $S = \mathrm{sal}(A_0)$ of Example 2.1.12, with $A_0 = \{(1,1,1),(1,-1,1),(-1,1,1),(-1,-1,-1)\} \subset \mathbb{R}^3$. We prove that S does not have a lattice structure by showing that the pair $\{s,t\} \subset S$, where $s = (0,-1,1)$ and $t = (0,1,1)$, does not have a least upper bound. First, note that in order for an element $u \in S$ to qualify as an upper bound of the set $\{s,t\}$, u has to satisfy

$$\begin{cases} (u_1, u_2+1, u_3-1) \in S \\ (u_1, u_2-1, u_3-1) \in S. \end{cases}$$

Note (cf. Figure2.1.1), that for every $n \in \mathbb{N}$, the intersection with S and the hyperplane $x_3 = n$, results in a square with sides of length $2n$. Hence, the difference of two units in the x_2-coordinate of (u_1, u_2+1, u_3-1) and (u_1, u_2-1, u_3-1) implies $u_3 - 1 \geq 1$, i.e., $u_3 \geq 2$. Secondly, since both $(1,0,2)$ and $(-1,0,2)$ are upper bounds for the set $\{s,t\}$, in order to qualify as the least upper bound, the element u has to satisfy

$$\begin{cases} (1-u_1, -u_2, 2-u_3) \in S \\ (-1-u_1, -u_2, 2-u_3) \in S. \end{cases}$$

The difference of two units in the x_1-coordinate implies $2 - u_3 \geq 1$, i.e., $u_3 \leq 1$. We conclude that the set $\{s,t\}$ does not have a least upper bound. ◇

2.2.25 Example. (cf. [14, §72]) Let $n \in \mathbb{N}$. Then the salient space of all positive $n \times n$ matrices is not a lattice. Consider the salient space S of all positive 2×2 matrices of Example 2.1.20. The partial order relation $\leq_S$ on S is given by $M \leq_S N :\Longleftrightarrow N - M \in S$.
We show that $(S, \leq_S)$ does not have a lattice structure, by showing that $\{A, B\} \subset S$, with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

does not have a least upper bound. Suppose $U \in S$ is an upper bound of $\{A, B\}$. Then $U - A \in S$ and $U - B \in S$ imply that

$$U = \begin{pmatrix} 1+\varepsilon & \theta \\ \theta & 1+\delta \end{pmatrix} \text{ with } \begin{cases} \varepsilon \geq 0, \delta \geq 0 \\ \theta^2 \leq \varepsilon(1+\delta) \\ \theta^2 \leq \delta(1+\varepsilon). \end{cases}$$

Choosing, for example, $\varepsilon = \delta = 1$ and $\theta = \sqrt{2}$, results in the matrix

$$V = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix},$$

which is an upper bound of $\{A, B\}$. By choosing $\varepsilon = \delta = \theta = 0$, we find that I is also an upper bound of $\{A, B\}$. Furthermore, $U \leq_S I$ implies $U = I$. However, I is not the least upper bound of $\{A, B\}$ since $\neg(I \leq_S V)$.

Let $n \in \mathbb{N}$. We recall that an $n \times n$ matrix M is positive if and only if M is symmetrical and satisfies $\forall x \in \mathbb{R}^n : \langle Mx, x \rangle \geq 0$. Since we can replace the 2×2 matrices A and B by

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array}\right) \text{ and } \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I \end{array}\right),$$

respectively, and repeat the above reasoning, this implies that the salient space of all real, positive $n \times n$ matrices is not a lattice. $\diamond$

Without proof, we state the following lemma concerning isomorphic salient spaces and lattices.

2.2.26 Proposition. *Let S and T be two isomorphic salient spaces, and let $\mathcal{L} : S \to T$ be a salient isomorphism. Then $(T, \leq_T)$ is a lattice, if and only if $(S, \leq_S)$ is a lattice. Furthermore, the salient isomorphism $\mathcal{L}$ satisfies for all $s_1, s_2 \in S$: $\mathcal{L}(s_1 \vee s_2) = \mathcal{L}(s_1) \vee \mathcal{L}(s_2)$.*

2.2.27 Proposition. *Let S be a salient space for which the partially ordered set $(S, \leq_S)$ is a lattice. Then $(V[S], \leq_S)$ is a vector lattice.*

Proof.
Let $[(s_1, s_2)], [(t_1, t_2)] \in V[S]$. Since $(S, \leq_S)$ is a lattice, there are $p_s, p_t \in S$ such that

$$p_s + s_2 \; = \; (s_2 \vee t_2) \text{ and } p_t + t_2 \; = \; (s_2 \vee t_2).$$

We shall prove that

$$[(\, (s_1 + p_s) \vee (t_1 + p_t) \, , \, (s_2 \vee t_2) \,)]$$

is the least upper bound of the pair $\{[(s_1, s_2)], [(t_1, t_2)]\}$. Firstly, we see that

$$s_1 + (s_2 \vee t_2) \; = \; s_1 + p_s + s_2 \; \leq_S \; (s_1 + p_s) \vee (t_1 + p_t) + s_2$$

means

$$[(s_1, s_2)] \; \leq_S \; [(\,(s_1 + p_s) \vee (t_1 + p_t)\,,\; (s_2 \vee t_2)\,)].$$

By symmetry, we can prove that

$$[(t_1, t_2)] \; \leq_S \; [(\,(s_1 + p_s) \vee (t_1 + p_t)\,,\; (s_2 \vee t_2)\,)].$$

Hence, $[(\,(s_1+p_s)\vee(t_1+p_t)\,,\, s_2\vee t_2\,)]$ is an upper bound of the set $\{[(s_1, s_2)], [(t_1, t_2)]\}$. Suppose $[(b_1, b_2)]$ is an upper bound of $\{[(s_1, s_2)], [(t_1, t_2)]\}$, then

$$\begin{cases} p_s + s_1 + b_2 \; \leq_S \; p_s + s_2 + b_1 \; = \; b_1 + (s_2 \vee t_2) \\ p_t + t_1 + b_2 \; \leq_S \; p_t + t_2 + b_1 \; = \; b_1 + (s_2 \vee t_2) \end{cases}$$

and Lemma 2.2.19 imply that

$$b_2 + (p_s + s_1) \vee (p_t + t_1) \; = \; (p_s + s_1 + b_2) \vee (p_t + t_1 + b_2) \; \leq_S \; b_1 + (s_2 \vee t_2).$$

We conclude that

$$[(\,(p_s + s_1) \vee (p_t + t_1)\,,\; (s_2 \vee t_2)\,)] \; \leq_S \; [(b_1, b_2)].$$

Next, we have to prove that the greatest lower bound of the set $\{[(s_1, s_2)], [(t_1, t_2)]\}$ exists. We show that

$$[(\,(s_1 \vee t_1)\,,\; (s_2 + q_s) \vee (t_2 + q_t)\,)]$$

is the greatest lower bound of the pair $\{[(s_1, s_2)], [(t_1, t_2)]\}$, where $q_s, q_t \in S$ satisfy

$$s_1 + q_s \; = \; (s_1 \vee t_1) \;\text{ and }\; t_1 + q_t \; = \; (s_1 \vee t_1).$$

Firstly, we see that

$$(s_1 \vee t_1) + s_2 \; = \; s_1 + q_s + s_2 \; \leq_S \; s_1 + (q_s + s_2) \vee (q_t + t_2)$$

means

$$[(\,(s_1 \vee t_1)\,,\; (s_2 + q_s) \vee (t_2 + q_t)\,)] \; \leq_S \; [(s_1, s_2)].$$

Similarly, we can prove that

$$[(\,(s_1 \vee t_1)\,,\; (s_2 + q_s) \vee (t_2 + q_t)\,)] \; \leq_S \; [(t_1, t_2)].$$

Hence, $[(\,(s_1\vee t_1)\,,\,(s_2+q_s)\vee(t_2+q_t)\,)]$ is a lower bound of the set $\{[(s_1, s_2)], [(t_1, t_2)]\}$. Suppose $[(b_1, b_2)]$ is a lower bound of $\{[(s_1, s_2)], [(t_1, t_2)]\}$, then

$$\begin{cases} b_1 + s_2 + q_s \; \leq_S \; b_2 + s_1 + q_s \; = \; b_2 + (s_1 \vee t_1) \\ b_1 + t_2 + q_t \; \leq_S \; b_2 + t_1 + q_t \; = \; b_2 + (s_1 \vee t_1) \end{cases}$$

and Lemma 2.2.19 imply that

$$b_1 + (s_2 + q_s) \vee (t_2 + q_t) \;=\; (b_1 + s_2 + q_s) \vee (b_1 + t_2 + q_t) \;\leq_S\; b_2 + (s_1 \vee t_1).$$

So, we conclude that $[(b_1, b_2)] \;\leq_S\; [(\,(s_1 \vee t_1)\,,\,(s_2 + q_s) \vee (t_2 + q_t)\,)]$. □

If a partially ordered vector space $(V, \leq)$ is a vector lattice, we can define the following concepts (cf. [1, page 88]). With each element $v \in V$, its *positive part* v^+, its *negative part* v^- and its *absolute value* $|v|$ are defined by the formulas

$$v^+ := v \vee 0, \quad v^- := (-v) \vee 0 \quad \text{and} \quad |v| := v \vee (-v).$$

The following identities hold:

$$v = v^+ - v^- \quad \text{and} \quad |v| = v^+ + v^-.$$

Furthermore, the absolute value function satisfies the triangle inequality, i.e., for each pair $v_1, v_2 \in V$, we have

$$|v_1 + v_2| \leq |v_1| + |v_2|.$$

Finally, we want to mention the following relation: $\forall v_1, v_2 \in V$:

$$(-v_1) \vee (-v_2) = -(v_1 \wedge v_2).$$

2.2.28 Remark. According to the proof of Proposition 2.2.27, we find that for every $[(s_1, s_2)] \in V[S]$:

$$[(s_1, s_2)]^+ = [(s_1 \vee s_2, s_2)] \;\text{ and }\; [(s_1, s_2)]^- = [(s_1 \vee s_2, s_1)].$$

Furthermore, the proof of Proposition 2.2.27 implies that $\forall v_1, v_2 \in V[S] : (-v_1) \vee (-v_2) = -(v_1 \wedge v_2)$. Indeed, for every $[(s_1, s_2)], [(t_1, t_2)] \in V[S]$, we can derive

$$\begin{aligned}
&(-[(s_1, s_2)]) \vee (-[(t_1, t_2)]) \\
&= [(s_2, s_1)] \vee [(t_2, t_1)] \\
&= [((s_2 + q_s) \vee (t_2 + q_t), s_1 \vee t_1)] \\
&= -[(s_1 \vee t_1, (s_2 + q_s) \vee (t_2 + q_t))] \\
&= -([(s_1, s_2)] \wedge [(t_1, t_2)]),
\end{aligned}$$

where q_s and q_t are as defined in the proof of Proposition 2.2.27. ◇

The concluding theorem of this section is a consequence of Proposition 2.2.20, Proposition 2.2.27 and Proposition 2.2.4.

2.2.29 Theorem. *Let $(V, \leq)$ be a partially ordered vector space. If every pair of elements of the positive cone $V_+ = \{v \in V \mid 0 \leq V\}$ has a least upper bound, then $(V, \leq)$ is a vector lattice.*

In the following example we show that although a partially ordered vector space may not be a vector lattice, it is possible that for every $v \in V$, the positive part v^+ and the negative part v^- exist in V_+. In Proposition 3.1.35 we will see that in a partially ordered vector space, this property simplifies the construction of a salient semi-metric from a semi-norm on the salient space V_+.

2.2.30 Example. Let $n \in \mathbb{N}$ and let S be the salient space of all real positive $n \times n$ matrices. Recall that $V[S]$ is the set of all real symmetric $n \times n$ matrices. In the above example we have seen that $(S, \leq_S)$ is not a lattice. However, we show that every element M of the partially ordered vector space $(V[S] \leq_S)$, the least upper bound (and therefore also the greatest lower bound) of the set $\{0, M\}$ exists in S. Indeed, let $M \in V[S]$, and let P be the projection onto the eigenspace corresponding with the positive eigenvalues of M. Similarly, let Q be the projection onto the eigenspace corresponding with the non-positive eigenvalues of M. Then, we can write $M = PMP + QMQ$. Define $M^+ := PMP$ and $M^- := -QMQ$. Clearly, $0 \leq_S M^+$ and $0 \leq_S M^-$, by construction. Since $M^- = M^+ - M$ this implies $M \leq_S M^+$ and $-M \leq_s M^-$. Hence, M^+ is an upper bound of $\{0, M\}$.
Suppose $U \in V[S]$ satisfies $0 \leq_S U$, $M \leq_S U$ and $U \leq_S M^+$, then U satisfies $0 \leq_S PUP$ and $0 \leq_S QUQ$. Since $U = PUP + PUQ + QUP + QUQ$, we find that $U \leq_S M^+$ implies $P(U - M)P + PUQ + QUP + QUQ \leq_S 0$. Thus we find $0 \leq_S P(U - M)P$ and $QUQ \leq_S 0$. Combined, we find $QUQ = 0$. Furthermore, $M \leq_S U$ implies $PUP = PMP$. Hence, $U = M^+ + PUQ + QUP \leq_S M^+$, i.e., $PUQ + QUP \leq_S 0$. This means that $\forall x \in \mathbb{R}^n : \langle (PUQ + QUP)x, x\rangle = 2\langle UQx, Px\rangle \geq 0$. Since $PQ = 0$, we find that $PUQ + QUP = 0$, and we conclude $U = M^+$. $\diamond$

2.3 Pairing and duality

Following [18, Section 16], we introduce the concept of pairing of two vector spaces.

2.3.1 Definition (bi-linear form and linear pairing).
A bi-linear form on the product of two vector spaces V and W, is a function $\mathcal{B} : V \times W \to \mathbb{R}$ such that for all $v, v_1, v_2 \in V$, all $w, w_1, w_2 \in W$ and all $\alpha_1, \alpha_2 \in \mathbb{R}$:

$$\begin{cases} \mathcal{B}(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 \mathcal{B}(v_1, w) + \alpha_2 \mathcal{B}(v_2, w) \\ \mathcal{B}(v, \alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 \mathcal{B}(v, w_1) + \alpha_2 \mathcal{B}(v, w_2). \end{cases}$$

A linear pairing is an ordered pair $\{\, V \,, \, W \,; \mathcal{B} \,\}$ of vector spaces together with a bi-linear form on their product.

2.3.2 Example. Let H be a Hilbert space (cf. [20]) with inner product $\langle .,. \rangle$, then $\{ H , H ; \langle .,. \rangle \}$ is a linear pairing. $\diamond$

2.3.3 Definition (adjoint of a vector space). Let V be a vector space. The adjoint of V, denoted by V^*, is the set of all linear functions $\mathcal{F} : V \to \mathbb{R}$.

Each linear pairing $\{ V , W ; \mathcal{B} \}$ defines a mapping from either of the two vector spaces into the adjoint of the other. The linear map $\mathcal{M} : W \to V^*$ carries a member $w \in W$ into the linear function $\mathcal{M}_w$ on V such that $\mathcal{M}_w(v) = \mathcal{B}(v, w)$ for all $v \in V$. Because of the definition of the bi-linear form, the map $\mathcal{M}$ is linear, and the image of each member of W is a linear function on V. Consequently, $\mathcal{M}(W) := \{\mathcal{M}_w \in V^* \mid w \in W\}$ is a linear subspace of the adjoint V^* of V.

On the other hand, if W is an arbitrary linear subspace of V^*, the canonical pairing of V and W is the bi-linear form on $V \times W$, defined by $\mathcal{B}_{\text{can}}(v, w) = w(v)$, for all $v \in V$ and $w \in W$.

In the general case, where W is not a linear subspace of V^*, we can identify an element $w \in W$ and its image $\mathcal{M}_w$. Nevertheless, it must be remembered that $\mathcal{M}_{w_1}$ and $\mathcal{M}_{w_2}$ may be equal for distinct elements w_1 and w_2 of W. In case both the mapping from V into W^* and the mapping from W into V^* are homomorphisms, we say that the pairing is *non-degenerate.*

Next, we show that the concept of non-degenerate pairing is strongly related to the concept of separating set. We recall that for a vector space V, a set $F \subset V^*$ is said to be separating the elements of a subset $A \subset V$ if

$$\forall a_1, a_2 \in A, a_1 \neq a_2 \; \exists f \in F : f(a_1) \neq f(a_2).$$

If A is a linear set, this narrows down to $\forall a \in A \setminus \{0\} \; \exists f \in F : f(a) \neq 0$.

2.3.4 Definition (separating set). Consider a linear pairing $\{ V , W ; \mathcal{B} \}$, and let A be a subset of W. The set A separates the elements of V with respect to the bi-linear form $\mathcal{B}$ (or in short: A separates V) if $\mathcal{M}(A) = \{\mathcal{M}_a \mid a \in A\}$ separates the elements of V, i.e., if for all $v \in V \setminus \{0\}$ there is an $a \in A$ such that $\mathcal{M}_a(v) \neq 0$.

2.3.5 Lemma. *The linear pairing $\{ V , W ; \mathcal{B} \}$ is non-degenerate if and only if V separates W, and W separates V, both with respect to the bi-linear form $\mathcal{B}$.*

Proof.
Recall that for two elements $\mathcal{F}_1$ and $\mathcal{F}_2$ in the adjoint V^* of a vector space V, we say that $\mathcal{F}_1 \neq \mathcal{F}_2$ if and only if $\forall v \in V : \mathcal{F}_1(v) \neq \mathcal{F}_2(v)$. We show that the linear map $\mathcal{M} : W \to V^*$, induced by $\mathcal{B}$, is a homomorphism if and only if V separates the elements of W. Clearly the linear mapping $\mathcal{M}$ is a homomorphism if and only if

$$\forall w_1, w_2 \in W : \ (\ w_1 \neq w_2\ ) \implies (\ \mathcal{M}_{w_1} \neq \mathcal{M}_{w_2}\ )$$

of which the latter is equivalent with $\exists v \in V : \mathcal{M}_{w_1}(v) \neq \mathcal{M}_{w_2}(v)$. Hence $\mathcal{M}$ is a homomorphism if and only if

$$\forall w_1, w_2 \in W\ (\forall v \in V : \mathcal{M}_{w_1}(v) = \mathcal{M}_{w_2}(v)) \implies (w_1 = w_2).$$

□

Adapting the above concepts to our salient space-setting, we obtain the following construction.

2.3.6 Definition (bi-salient form and salient pairing). A bi-salient form on the product of two salient spaces S and T, is a function $\mathcal{B} : S \times T \to \mathbb{R}_+$ such that for all $s, s_1, s_2 \in S$, all $t, t_1, t_2 \in T$ and all $\alpha_1, \alpha_2 \geq 0$:

$$\begin{cases} \mathcal{B}(\alpha_1 s_1 + \alpha_2 s_2, t) = \alpha_1 \mathcal{B}(s_1, t) + \alpha_2 \mathcal{B}(s_2, t) \\ \mathcal{B}(s, \alpha_1 t_1 + \alpha_2 t_2) = \alpha_1 \mathcal{B}(s, t_1) + \alpha_2 \mathcal{B}(s, t_2). \end{cases}$$

A salient pairing is an ordered triple $\{\, S\,,\ T\,; \mathcal{B}\,\}$ of salient spaces S and T together with a bi-salient form $\mathcal{B}$ on their product.

2.3.7 Example. Let $n \in \mathbb{N}$. Then $\left\{\, \mathbb{R}^n_+\,,\ \mathbb{R}^n_+\,; \langle .,. \rangle_E \,\right\}$ is a salient pairing where the bi-salient form is taken to be the Euclidean inner product $\langle .,. \rangle_E$. ◇

2.3.8 Example. $\left\{\, \mathbb{R}_+\,,\ \mathbb{R}^2_+\,; \mathcal{B}\,\right\}$ where for every $x \in \mathbb{R}_+$ and every $(y_1, y_2) \in \mathbb{R}^2_+$ the bi-salient form $\mathcal{B}$ is defined by $\mathcal{B}(x, (y_1, y_2)) := x y_2$ is a salient pairing. When we define the salient mapping $\mathcal{L} : \mathbb{R}_+ \to \mathbb{R}^2_+$, for every $x \in \mathbb{R}_+$ by $\mathcal{L}(x) = (0, x)$ then the salient spaces $\mathbb{R}_+$ and $\mathcal{L}(\mathbb{R}_+)$ are isomorphic. Note that replacing the bi-salient form $\mathcal{B}(x, (y_1, y_2))$ with the Euclidean inner product $\langle (0, x), (y_1, y_2) \rangle$, the triple $\left\{\, \mathcal{L}(\mathbb{R}_+)\,,\ \mathbb{R}^2_+\,; \langle .,. \rangle \,\right\}$ is again a salient pairing. ◇

2.3.9 Example. Let S be the salient space of all real, positive $n \times n$ matrices. Define the bi-salient form $\mathcal{B} : S \times S \to \mathbb{R}_+$, for every $A, B \in S$, by $\mathcal{B}(A, B) = \mathrm{tr}(AB)$. Here $\mathrm{tr}(AB)$ denotes the trace (cf. [14]) of the product of matrices A and B. Then $\{\, S\,,\ S\,; \mathrm{tr}\,\}$ is a salient pairing. ◇

2.3.10 Example. Let $S := \{(s_1, s_2, s_3) \in \mathbb{R}^3 \mid (s_1)^2 + (s_2)^2 \leq s_3\}$ and $T := \{(t_1, t_2, t_3) \in \mathbb{R}^3 \mid (t_2)^2 + (t_3)^2 \leq t_1\}$. Define the bi-salient form $\mathcal{B} : S \times T \to \mathbb{R}_+$ by

$$\mathcal{B}((s_1, s_2, s_3), (t_1, t_2, t_3)) := t_1 s_3 + t_2 s_1 + t_3 s_2,$$

then $\{ S , T ; \mathcal{B} \}$ is a salient pairing. We remark that the bi-salient form $\mathcal{B}$, contrary to the bi-salient forms of Examples 2.3.7 and 2.3.9, is not directly based on the inner product of a Hilbert space. Indeed, $s = (1, -1, 2) \in S$ satisfies $s_1 s_3 + s_2 s_1 + s_3 s_2 < 0$, where an inner product on a vector space V satisfies $\forall v \in V : \langle v, v \rangle \geq 0$. $\diamond$

2.3.11 Proposition. *If $\{ S , T ; \mathcal{B} \}$ is a salient pairing, then $\{ V[S] , V[T] ; \mathcal{B}^{\text{ext}} \}$ is a linear pairing, where for every $[(s_1, s_2)] \in V[S]$ and $[(t_1, t_2)] \in V[T]$, the bi-linear form $\mathcal{B}^{\text{ext}}$ is defined by*

$$\mathcal{B}^{\text{ext}}([(s_1, s_2)], [(t_1, t_2)]) := \mathcal{B}(s_1, t_1) - \mathcal{B}(s_1, t_2) - \mathcal{B}(s_2, t_1) + \mathcal{B}(s_2, t_2).$$

Furthermore, if $\{ S , T ; \mathcal{B} \}$ is non-degenerate, then $\{ V[S] , V[T] ; \mathcal{B}^{\text{ext}} \}$ is non-degenerate.

Proof.
Let $\{ S , T ; \mathcal{B} \}$ be a salient pairing. It is easy to check that the definition of $\mathcal{B}^{\text{ext}}$ is independent of the choice of the representatives (s_1, s_2) and (t_1, t_2), and that with this definition $\mathcal{B}^{\text{ext}}$ is a bi-linear form on the product $V[S] \times V[T]$. If $\mathcal{B}([(s_1, s_2)], [(t_1, t_2)]) = 0$ for every $[(s_1, s_2)] \in V[S]$, then

$$\forall s \in S : \mathcal{B}([(s, 0)], [(t_1, t_2)]) = \mathcal{B}(s, t_1) - \mathcal{B}(s, t_2) = 0,$$

and we conclude $t_1 = t_2$, or, in other words, $[(t_1, t_2)] = [(0, 0)]$. $\square$

2.3.12 Definition (salient function). A salient function is a salient mapping $\mathcal{F}$ from a salient space S into $\mathbb{R}_+$.

Recall from the definition of salient mapping that every salient function $\mathcal{F}$ on a salient space S satisfies for all $s, s_1, s_2 \in S$ and for all $\alpha \in \mathbb{R}_+$:

$$\begin{cases} \mathcal{F}(s_1 + s_2) = \mathcal{F}(s_1) + \mathcal{F}(s_2) \\ \mathcal{F}(\alpha s) = \alpha \mathcal{F}(s). \end{cases}$$

Furthermore, by Definition 2.1.25, we find that for every salient function $\mathcal{F} : S \to \mathbb{R}_+$ its extension $\mathcal{F}^{\text{ext}} : V[S] \to \mathbb{R}$ is, for every $s_1, s_2 \in S$, given by

$$\mathcal{F}^{\text{ext}}([(s_1, s_2)]) = \mathcal{F}(s_1) - \mathcal{F}(s_2).$$

2.3.13 Definition (adjoint of a salient space). Let S be s a salient space. The adjoint of S, denoted by S^*, is the set of all salient functions on S.

If addition and positive scalar multiplication are defined pointwise in S^*, then S^* is a salient space with the zero-function as its vertex. Clearly, if T is a salient subspace of S then $S^* \subseteq T^*$. Furthermore, note that for every $\mathcal{F} \in S^*$ the extension $\mathcal{F}^{\text{ext}} \in (V[S])^*$ of S, corresponds with $[(\mathcal{F}, 0)] \in V[S^*]$.

The partial order relation $\leq_{S^*}$ on the adjoint S^* of a salient space S satisfies, for every $\mathcal{F}, \mathcal{G} \in S^*$:

$$\begin{aligned}
\mathcal{F} \leq_{S^*} \mathcal{G} &\iff \forall s \in S : \mathcal{F}(s) \leq \mathcal{G}(s), \\
\mathcal{F} <_{S^*} \mathcal{G} &\iff (\forall s \in S : \mathcal{F}(s) \leq \mathcal{G}(s)) \wedge (\exists s \in S : \mathcal{F}(s) < \mathcal{G}(s)).
\end{aligned}$$

Besides a partial order relation on $V[S^*]$ (cf. Definition 2.2.5 and subsequent construction), the partial order relation $\leq_{S^*}$ on S^* also induces a partial order relation $\leq_*$ on $(V[S])^*$: for every $f, g \in (V[S])^*$ we define

$$\begin{aligned}
f \leq_* g \;&:\iff\; \forall s \in S : f([(s,0)]) \leq g([(s,0)]), \\
f <_* g \;&:\iff\; (\forall s \in S : f([(s,0)]) \leq g([(s,0)])) \wedge (\exists s \in S : f([(s,0)]) < g([(s,0)])).
\end{aligned}$$

Similar to the vector space situation, each salient pairing $\{\, S\,,\, T\,; \mathcal{B}\,\}$ defines a mapping from either of the two salient spaces into the set of all salient functions on the other. The salient map $\mathcal{M} : T \to S^*$, induced by $\mathcal{B}$, carries a member $t \in T$ into the salient function $\mathcal{M}_t$ on S such that $\mathcal{M}_t(s) = \mathcal{B}(s,t)$ for all $s \in S$. Because of the definition of the bi-salient form, the map $\mathcal{M}$ is salient, and the image of each member of T is a salient function on S. Consequently, $\mathcal{M}(T) := \{\mathcal{M}_t \in S^* \mid t \in T\}$ is a salient subspace of the adjoint S^* of S. If T is an arbitrary salient subspace of S^*, then S and T form a salient pairing with the canonical bi-salient form $\mathcal{B}_{\text{can}}$ on $S \times T$, for all $s \in S$ and $t \in T$, defined by $\mathcal{B}_{\text{can}}(s,t) = t(s)$. In the general case, where T is not a salient subspace of S^*, we can identify an element $t \in T$ and its image $\mathcal{M}_t$. In case both the mapping from S into T^* and the mapping from T into S^* are a salient homomorphism, we say that the pairing is *non-degenerate*.

2.3.14 Definition (separating set). Consider a salient pairing $\{\, S\,,\, T\,; \mathcal{B}\,\}$, and let A be a subset of T. The set A separates the elements of S, with respect to the bi-salient form $\mathcal{B}$, if $\mathcal{M}(A) := \{\mathcal{M}_a \mid a \in A\}$ separates the elements of S, i.e., if for all $s_1, s_2 \in S$, with $s_1 \neq s_2$, there is an $a \in A$ such that $\mathcal{M}_a(s_1) \neq \mathcal{M}_a(s_2)$.

Similar to the vector space situation, we find the following lemma.

2.3.15 Lemma. *A salient pairing* $\{\, S\,,\, T\,; \mathcal{B}\,\}$ *is non-degenerate if and only if* S *separates* T *and* T *separates* S*, both with respect to the bi-salient form* $\mathcal{B}$*.*

2.3.16 Corollary. *Since every salient space* S *separates the elements of its adjoint* S^**, the canonical pairing* $\{\, S\,,\, S^*\,; \mathcal{B}_{\text{can}}\,\}$ *is non-degenerate if* S^* *separates the elements of* S*.*

2.3.17 Example. The salient pairing $\left\{\, \mathbb{R}^n_+\,,\, \mathbb{R}^n_+\,; \langle .\,,.\rangle_E \,\right\}$ of Example 2.3.7 is non-degenerate, where $\left\{\, \mathbb{R}_+\,,\, \mathbb{R}^2_+\,; \mathcal{B}\,\right\}$ of Example 2.3.8 is not. ◇

In the following statements we further explore the connection between a salient pairing $\{\, S\,,\, T\,; \mathcal{B}\,\}$ and the properties of the vector spaces reproduced by S and T.

2.3.18 Lemma. *Let* $\{\, S\,,\, T\,; \mathcal{B}\,\}$ *be a salient pairing. Then a set* $A \subset T$ *separates the elements of* S *if and only if the collection* $A_{V[T]} := \{[(a_1, a_2)] \in V[T] \mid a_1, a_2 \in A\}$ *separates the elements of* $V[S]$*.*

Proof.
Let $s_1, s_2 \in S$. Consider the following sequence of equivalent statements

$$\begin{aligned}
&\forall a \in A : \mathcal{B}(s_1, a) = \mathcal{B}(s_2, a),\\
&\forall a_1, a_2 \in A : \mathcal{B}(s_1, a_1) + \mathcal{B}(s_2, a_2) = \mathcal{B}(s_2, a_1) + \mathcal{B}(s_1, a_2),\\
&\forall [(a_1, a_2)] \in A_{V[T]} : \mathcal{B}(s_1, a_1) + \mathcal{B}(s_2, a_2) - \mathcal{B}(s_2, a_1) - \mathcal{B}(s_1, a_2) = 0,\\
&\forall [(a_1, a_2)] \in A_{V[T]} : \mathcal{B}^{\text{ext}}([(s_1, s_2)], [(a_1, a_2)]) = 0,
\end{aligned}$$

where $\mathcal{B}^{\text{ext}}$ is defined in Proposition 2.3.11. Note that $s_1 \neq s_2$ is equivalent with $[(s_1, s_2)] \neq [(0, 0)]$. □

2.3.19 Corollary. *A salient pairing* $\{\, S\,,\, T\,; \mathcal{B}\,\}$ *is non-degenerate if and only if the linear pairing* $\{\, V[S]\,,\, V[T]\,; \mathcal{B}^{\text{ext}}\,\}$ *is non-degenerate.*

2.3.20 Example. Reconsider the two salient spaces S and T of Example 2.3.10. Let $\mathcal{J}_S$ be the salient isomorphism between S and $V_+[S]$, and let $\mathcal{J}_T$ be the salient isomorphism between T and $V_+[T]$. Since $\mathcal{J}_S\,(S)$ separates $V[T]$ and $\mathcal{J}_T\,(T)$ separates $V[S]$, we conclude that the salient pairing $\{\, S\,,\, T\,; \mathcal{B}\,\}$ is non-degenerate. ◇

2.3.21 Definition (partial order relations related to a salient pairing). Let $\{\, S\,,\, T\,; \mathcal{B}\,\}$ be a non-degenerate salient pairing. The partial order relation $\leq_{\mathcal{B}}$, induced by the salient space T on the salient space S is, for elements $s_1, s_2 \in S$, given by

$$s_1 \leq_{\mathcal{B}} s_2 \quad :\Longleftrightarrow \quad \forall t \in T : \mathcal{B}(s_1, t) \leq \mathcal{B}(s_2, t).$$

Similarly, the partial order relation $\leq_{\mathcal{B}}$ on T, is for all $t_1, t_2 \in T$ given by

$$t_1 \leq_{\mathcal{B}} t_2 \quad :\Longleftrightarrow \quad \forall s \in S : \mathcal{B}(s, t_1) \leq \mathcal{B}(s, t_2).$$

Note that $\leq_{\mathcal{B}}$, defined above, both on S and on T, is a partial order relation, since T separates S and S separates T.

2.3.22 Proposition. *Let $\{\, S\,,\, T\,; \mathcal{B}\,\}$ be a non-degenerate salient pairing. Then the partial order relation $\leq_{\mathcal{B}}$ on S satisfies:*

$$\forall s_1, s_2 \in S : \quad s_1 \leq_S s_2 \quad \Longrightarrow \quad s_1 \leq_{\mathcal{B}} s_2.$$

If a salient space and its adjoint form a non-degenerate salient pairing, then, by definition, there is a salient homomorphism $\mathcal{L}$ from S in the adjoint S^{**} of S^*, i.e., S is embedded in the salient space S^{**}. Hence, we can consider the following partial order relation on S, induced by S^{**}: for every $s_1, s_2 \in S$, we define

$$\begin{aligned} s_1 \leq_{S^{**}} s_2 \quad &:\Longleftrightarrow \quad \exists x \in S^{**} : s_1 + \mathcal{L}^{\leftarrow}(x) = s_2 \\ &\Longleftrightarrow \quad \forall \mathcal{F} \in S^* : (\mathcal{L}^{\leftarrow}(s_1))(\mathcal{F}) \leq (\mathcal{L}^{\leftarrow}(s_2))(\mathcal{F}) \\ &\Longleftrightarrow \quad \forall \mathcal{F} \in S^* : \mathcal{F}(s_1) \leq \mathcal{F}(s_2). \end{aligned}$$

In case the salient homomorphism from S into S^{**} turns out to be a salient isomorphism, i.e., in case all positive functions on S^* arise from elements of S, we say that the salient space S is *reflexive*. Hence, we find the following lemma.

2.3.23 Lemma. *Let S be a reflexive salient space. Then the partial order relation $\leq_S$ on S is equivalent with the partial order relation $\leq_{S^{**}}$ on S, i.e., $\forall s_1, s_2 \in S$:*

$$s_1 \leq_S s_2 \iff \forall \mathcal{F} \in S^* : \mathcal{F}(s_1) \leq \mathcal{F}(s_2).$$

Clearly, if S is reflexive then $V[S]$ is isomorphic with $V[S^{**}]$. One may wonder whether in this case $V[S]$ is also isomorphic with $(V[S])^{**}$. The underlying problem is under which conditions $V[S^*]$ is isomorphic with $(V[S])^*$. Clearly, the linear map $\mathcal{L} : V[S^*] \to (V[S])^*$, for every $[(\mathcal{F}_1, \mathcal{F}_2)] \in V[S^*]$ and every $[(s_1, s_2)] \in V[S]$ defined by

$$(\mathcal{L}([(\mathcal{F}_1, \mathcal{F}_2)]))\,([(s_1, s_2)]) := \mathcal{F}_1^{\text{ext}}([(s_1, s_2)]) - \mathcal{F}_2^{\text{ext}}([(s_1, s_2)]),$$

is a homomorphism from $V[S^*]$ into $(V[S])^*$.

One of the questions we try to answer in the next chapter, especially in Section 3.3, is under which condition, the homomorphism $\mathcal{L}$ is an isomorphism. The following counterexample shows, among other things, that $V[S^*]$ is not necessarily isomorphic to $(V[S])^*$.

2.3.24 Example. Consider the salient space $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\} \cup \{(0,0)\}$ of Example 2.1.5. The adjoint S^* of S satisfies $S^* = \{(0, f) \in \mathbb{R}^2 \mid f \geq 0\}$. Clearly, the natural pairing $\{\, S\,,\, S^*\,; \mathcal{B}_{\text{can}}\,\}$ is not non-degenerate, and S is not embedded in S^{**}. Furthermore, $V[S^*]$ is not isomorphic with $(V[S])^*$. ◇

The notion of non-degenerate pairing $\{\, S\,,\, S^*\,; \mathcal{B}_{\text{can}}\,\}$ also raises the question whether the adjoint of a salient space S contains enough elements to be able to separate the elements of S. The Hahn-Banach Theorem (cf. [7, Chapter III.6], [17, §6]) states that if X is a real vector space and f is a linear function from a linear manifold M of X into $\mathbb{R}$, satisfying $\forall m \in M : f(m) \leq q(m)$ where $q : X \to \mathbb{R}$ is a sub-linear function, then there exists an extension $\tilde{f} : X \to \mathbb{R}$ of f that remains dominated by q, i.e., $\forall x \in X : \tilde{f}(x) \leq q(x)$. Note that the linearity of $\tilde{f}$ implies that

$$\forall x \in X : -q(-x) \leq \tilde{f}(x) \leq q(x).$$

Hence, if we can find a non-trivial sub-linear function q on the vector space $V[S]$, satisfying $\forall s \in S : q([(s,0)]) \leq 0$, and a linear function $f \neq 0$ on a subspace of $V[S]$, then the Hahn-Banach Theorem implies that the linear function $-\tilde{f}$ acts positively on $V_+[S]$. Since $V_+[S]$ is isomorphic with S, and since $\tilde{f}$ is not equal to the zero-function, this would imply that the adjoint S^* of S is non-trivial, i.e. is not equal to $\{0\}$. In the following proposition we apply this construction to a partially ordered vector space, using an order unit of the positive cone.

2.3.25 Proposition. *Let $(V, \leq)$ be a partially ordered vector space, let V_+ be the positive cone of V and assume that V_+ has an order unit. Then there is a linear function $f \in V^*$ such that $\forall s \in V_+ : f(s) \geq 0$.*

Proof.
Let u be an order unit for V_+. Define the set $K := V_+ - u$ and observe that 0 is an internal point of K. Let $p_K : V \to \mathbb{R}_+$ be the Minkowski functional (cf. [7, page 106]) of K, i.e.,

$$\forall v \in V : p_K(v) = \inf\{\lambda \geq 0 \mid v \in \lambda K\}.$$

We remark that for every $\lambda > 0$ the set λK is equal to $V_+ - \lambda u$. Since $\lambda u \in V_+$ for all $\lambda \geq 0$, we find that $\forall \lambda > 0\; \forall s \in V_+ : s \in \lambda K$. Hence, $\forall s \in V_+ : p_K(s) = 0$. Let $x \in V$, and define the subspace X of V, by $X := \{\lambda x \mid \lambda \in \mathbb{R}\}$. We define the linear function $g : X \to \mathbb{R}$ for every $\lambda \in \mathbb{R}$ by $g(\lambda x) = \lambda p_K(x)$. By the Hahn-Banach Theorem, there is a linear function $\tilde{g} : V \to \mathbb{R}$ such that $\forall v \in V : \tilde{g}(v) \leq p_K(v)$ and $\tilde{g}(x) = p_K(x)$. When we define the function $f : V \to \mathbb{R}$ by $f := -\tilde{g}$, then we conclude $\forall s \in V_+ : f(s) \geq 0$. □

Proposition 2.3.25 has the following direct consequence for salient spaces.

2.3.26 Proposition. *If a salient space S has an order unit, then $S^* \neq \{0\}$.*

This proposition can also be proved in the following manner.

Proof.
Let u be an order unit for S. The set $U \subset V[S]$, defined by $U := \{\lambda[(u,0)] \mid \lambda \in \mathbb{R}\}$, is a subspace of $V[S]$. By Lemma 2.2.17, we find

$$\forall[(x_1, x_2)] \in V[S]\ \exists\lambda \geq 0 : -\lambda[(u,0)] \leq [(x_1, x_2)] \leq \lambda[(u,0)].$$

Thus, we can define the sub-linear function $q : V[S] \to \mathbb{R}$ by

$$q([(x_1, x_2)]) := \inf\{\lambda \in \mathbb{R} \mid [(x_1, x_2)] \leq \lambda[(u,0)]\}.$$

Define $f_u(\lambda[(u,0)]) := \lambda$, for every $\lambda \in \mathbb{R}$. With this definition, $f_u : U \to \mathbb{R}$ is a positive linear function on U satisfying $\forall\lambda \in \mathbb{R} : f_u(\lambda[(u,0)]) := q(\lambda[(u,0)])$. By the Hahn-Banach Theorem, there exists a linear function $\tilde{f}_u : V[S] \to \mathbb{R}$ such that on the set U, $\tilde{f}_u$ is equal to f_u, and $\forall[(x_1, x_2)] \in V[S] : -q([x_2, x_1]) \leq \tilde{f}_u([(x_1, x_2)]) \leq q([(x_1, x_2)])$. Hence, for all $s \in S : \tilde{f}_u([(0,s)]) \leq q([(0,s)]) \leq 0$. □

Recall that if u is an order unit for C, then so is $\lambda u + c$ for every $c \in C$ and every $\lambda \in \mathbb{R}_+$. The above construction implies that for every order unit u, the function $\tilde{f}_u$ satisfies $\tilde{f}_u(u) = 1$. Hence, existence of one order unit results in the existence of infinitely many positive linear functions on C.

In the previous sections, for example on page 44, we have seen that every salient space S is isomorphic with $V_+[S]$, and that $V_+[S] = (V[S])_+$. Since the adjoint S^* of S is a salient space, we find that S^* is isomorphic with $V_+[S^*] = (V[S^*])_+ = \{[(\mathcal{F}_1, \mathcal{F}_2)] \in V[S^*] \mid [(0,0)] \leq_{S^*} [(\mathcal{F}_1, \mathcal{F}_2)]\}$. Furthermore, the salient mapping $\mathcal{L} : S^* \to (V[S])^*_+ = \{f \in (V[S])^* \mid \forall s \in S : f([(s,0)]) \geq 0\}$, for every $\mathcal{G} \in S^*$ defined by $\mathcal{L}(\mathcal{G}) := \mathcal{G}^{\text{ext}}$, is a salient isomorphism.

2.3.27 Definition (self-adjoint salient space). A salient space S is self-adjoint through the bi-salient form $\mathcal{B}_{\text{can}}$ if there is a salient isomorphism $\mathcal{L}$ between S and its adjoint S^*. For every $s_1, s_2 \in S$ the bi-salient form $\mathcal{B}_{\text{can}}$ is given by $\mathcal{B}_{\text{can}}(s_1, s_2) = (\mathcal{L}(s_2))(s_1)$.

2.3.28 Example. The finitely generated salient space $\mathbb{R}^n_+$ is self-adjoint, since $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$. $\diamond$

2.3.29 Example. The salient space S of all real, positive $n \times n$ matrices is self-adjoint, where for all $A, F \in S$ the action of F on A is given by $\operatorname{tr}(AF)$. Indeed, for every positive linear function $\mathcal{F} : S \to \mathbb{R}_+$, there is a set $\{\alpha_{ij} \in \mathbb{R} \mid i \in \{1, \ldots, n\}, j \geq i\}$ such that for every $A \in S$, we have $\mathcal{F}(A) := \sum_{i,j \geq i} \alpha_{ij} a_{ij}$. Then the matrix F, defined by

$$\begin{cases} f_{ii} := \alpha_{ii} & \\ f_{ij} := \frac{1}{2}\alpha_{ij} & \text{if} \quad j > i \\ f_{ij} := \frac{1}{2}\alpha_{ji} & \text{if} \quad j < i, \end{cases}$$

satisfies $\operatorname{tr}(AF) = \sum_{i,j} a_{ij} f_{ji} = \sum_{i,j \geq i} \alpha_{ij} a_{ij} = \mathcal{F}(A)$. Left to prove that F is positive. Let $x \in \mathbb{R}^n$ and define the matrix X by $x_{ij} := x_i x_j$, then X is symmetric and satisfies $\forall y \in \mathbb{R}^n : \langle Xy, y \rangle = (\sum_i x_i y_i)^2 \geq 0$. We conclude that $X \in S$. Since $\mathcal{F} \in S^*$, we find $0 \leq \mathcal{F}(X) = \operatorname{tr}(XF) = \sum_{i,j} x_{ij} f_{ji} = \sum_{i,j} f_{ij} x_i x_j = \langle Fx, x \rangle$. $\diamond$

2.4 Extreme sets and salient bases

In this section, we show the connection between the concept of salient basis of a salient space S, and the concept of extreme ray of S. We start with the definition of extreme set.

2.4.1 Definition (extreme set). Let K be a convex set. A subset E of K is extreme if $\tau k_1 + (1-\tau) k_2 \in E$, with $\tau \in (0,1)$ and $k_1, k_2 \in K$ implies $k_1, k_2 \in E$.

In case an extreme set E consist of exactly one element, we call that element an *extreme point* of K. Both the sets $\emptyset$ and K are extreme in K, and are called the *trivial extreme sets.*

2.4.2 Proposition. *Let E be an extreme set of a convex set K and let L be a convex subset of K, then $E \cap L$ is an extreme set of L.*

Proof.
Let e be an element of $E \cap L$ and assume $e = \tau l_1 + (1-\tau) l_2$ for certain l_1 and l_2 in L, and $\tau \in [0,1]$. Since E is an extreme set of K both l_1 and l_2 belong to E. We conclude that $E \cap L$ is an extreme set of L. $\square$

2.4.3 Proposition. *Every extreme set of a salient space S is closed under multiplication over $\mathbb{R}_+$, i.e., is a cone in S.*

Proof.
Let E be an extreme set of S and let $e \in E$. Let $\lambda \geq 1$. Then $e = \frac{1}{\lambda}\lambda e + (1 - \frac{1}{\lambda})0$, where λe and $0 \in S$. Since E is extreme in S, we find λe and $0 \in E$. Let $0 \leq \lambda < 1$. Then $e = \frac{1}{2-\lambda}\lambda e + (1 - \frac{1}{2-\lambda})2e$, where λe and $2e \in S$. Since E is extreme in S, we find λe and $2e \in E$. $\square$

Note that this implies that every salient space has got exactly one extreme point, namely its vertex.

2.4.4 Proposition. *Let E be a non-empty, convex, extreme set of a salient space S. Then E is a salient subspace of S.*

Proof.
In order to prove this proposition, we only have to prove that E is closed under addition. Let $e_1, e_2 \in E$. Since E is convex, we find $\frac{1}{2}e_1 + \frac{1}{2}e_2 \in E$. By Proposition 2.4.3, we find that $2(\frac{1}{2}e_1 + \frac{1}{2}e_2)) = e_1 + e_2 \in E$. $\square$

2.4.5 Example. Consider the salient space S of Example 2.1.12, generated by the finite set A_0. For every element $a_0 \in A_0$ the ray $\{s \in S \mid \exists\lambda \geq 0 : s = \lambda a_0\}$ is extreme in S. Moreover, the set $E = \text{ray}(A_0)$ is extreme in S. Note that E is not convex. We also observe that, for example, $\text{sal}(A_0) = \text{sal}(A_0 \cup \{(0, 0, 1)\})$ and that the ray generated by $(0, 0, 1)$ is not an extreme ray of $\text{sal}(A_0)$. $\diamond$

Without proof we state the following lemma concerning isomorphic salient spaces and extreme sets.

2.4.6 Lemma. *Let S and T be two isomorphic salient spaces with respect to the salient isomorphism $\mathcal{L} : S \to T$. If E is an extreme set of S, then $\mathcal{L}(E)$ is an extreme set of T.*

Next, we focus on the relationship between a salient basis of a salient space S and extreme rays of S. Recall (Definition 2.1.15) that a salient basis is a saliently independent collection $\text{ray}(B)$ of rays, such that $B \subset S \setminus \{0\}$ and $\text{sal}(B) = S$. Also, recall that it is possible that a salient space does not have a salient basis.

2.4.7 Lemma. *Let S be a salient space, generated by the set $S_0 \subset S \setminus \{0\}$. Let $E \neq \{0\}$ be a non-empty extreme set of S. Then, $E \cap S_0 \neq \emptyset$.*

Proof.
Let $e \in E \setminus \{0\}$. Since the set S_0 generates S, there is a finite set $F \subset S_0$, with at least two elements, such that $e \in \text{sal}(F)$. By assumption, $e \neq 0$, hence

$$\exists f_0 \in F \; \exists \alpha > 0 \; \exists f \in \text{sal}(F \setminus \{f_0\}) : e = \alpha f_0 + f.$$

Proposition 2.4.3 states that E is closed under multiplication over $\mathbb{R}_+$, hence, without loss of generality, we may assume $\alpha = \frac{1}{2}$. Since E is an extreme set of S, we find that $e = \frac{1}{2} f_0 + \frac{1}{2}(2f)$ implies $f_0 \in E$. □

2.4.8 Lemma. *Let S be a salient space, let E be a non-empty set in $S \setminus \{0\}$, and assume that* ray(E) *is equal to the set of all extreme rays of S. Then* ray(E) *is saliently independent.*

Proof.
Let $e \in E$, and let $F \subseteq E$ be a finite set, such that $\text{ray}(e) \subset \text{sal}(F)$. Since $e \neq 0$, we find

$$\exists f_0 \in F \; \exists \alpha > 0 \; \exists f \in \text{sal}(F \setminus \{f_0\}) : e = \alpha f_0 + f.$$

Since ray(e) is extreme ray of S and since ray(e) is closed under multiplication over $\mathbb{R}_+$, we conclude $\text{ray}(e) = \text{ray}(f_0)$. □

The following property states that if a salient space has a salient basis, then this basis equals the set of all extreme rays. In Theorem 3.3.10, we will see that every reflexive finite-dimensional salient space has a salient basis.

2.4.9 Proposition. *Let S be a salient space and let* ray(A) *be a salient basis for S. Let $s_0 \in S \setminus \{0\}$. Then*

$$\text{ray}(s_0) \subset \text{ray}(A) \quad \Longleftrightarrow \quad \text{ray}(s_0) \textit{ is extreme in } S.$$

Proof.
Assume $\text{ray}(s_0) \subset \text{ray}(A_0)$. Let $s_1, s_2 \in S$, and assume that for certain $\tau \in (0,1)$ we have $\tau s_1 + (1-\tau)s_2 \in \text{ray}(s_0)$. In case $\tau s_1 + (1-\tau)s_2 = 0$, we conclude $s_1 = s_2 = 0$ and we are done. Next, consider the case where there is $\alpha > 0$ such that $s_0 = \alpha(\tau s_1 + (1-\tau)s_2)$. Since $\text{sal}(A) = S$, there are finite sets $F_1 \subseteq A$ and $F_2 \subseteq A$ such that $s_1 \in \text{sal}(F_1)$ and $s_2 \in \text{sal}(F_2)$. Define $\widehat{F_1} := \{f \in F_1 \mid f \notin \text{ray}(s_0)\}$ and $\widehat{F_2} := \{f \in F_2 \mid f \notin \text{ray}(s_0)\}$. Then,

$$\exists f_1 \in \widehat{F_1} \; \exists f_2 \in \widehat{F_2} \; \exists \lambda_1, \lambda_2 \geq 0 : s_1 = f_1 + \lambda_1 s_0 \text{ and } s_2 = f_2 + \lambda_2 s_0$$

implies that

$$s_0 = \alpha\,(\,\tau f_1 + (1-\tau) f_2)\,) + \alpha\,(\,\tau\lambda_1 + (1-\tau)\lambda_2\,)s_0.$$

Since S is pointed, we find that $\alpha\,(\,\tau\lambda_1 + (1-\tau)\lambda_2\,) \leq 1$. Now, suppose $\alpha\,(\,\tau\lambda_1 + (1-\tau)\lambda_2\,) < 1$, then we find $(1-\alpha\,(\,\tau\lambda_1 + (1-\tau)\lambda_2\,))s_0 \in \mathrm{sal}(\widehat{F_1} \cup \widehat{F_2})$ which is in contradiction with the construction of $\widehat{F_1}$ and $\widehat{F_2}$ and the assumption that $\mathrm{ray}(s_0) \subset \mathrm{ray}(A)$. We conclude that $\alpha\,(\,\tau\lambda_1 + (1-\tau)\lambda_2\,) = 1$, i.e., $\alpha\,(\,\tau f_1 + (1-\tau)f_2\,) = 0$. Since $\alpha > 0$ and $\tau \in (0,1)$, we find $f_1 = f_2 = 0$. Hence, both s_1 and s_2 are elements of $\mathrm{ray}(s_0)$.
The converse is a direct consequence of Lemma 2.4.7. □

We conclude this section with the following theorem concerning finitely generated salient spaces.

2.4.10 Theorem. *If S is a finitely generated salient space, then so is S^*.*

The proof of the above proposition makes use of the following lemma, which can be found in [32].

2.4.11 Lemma. *Let $n \in \mathbb{N}$ en let $K \subset \mathbb{R}^n$ be a finitely generated, pointed convex cone. Let $\langle .,. \rangle_E$ denote the Euclidean inner product on $\mathbb{R}^n$. Then the polar set K° of K, defined by $K^\circ := \{y \in \mathbb{R}^n \mid \forall z \in K : \langle y, z\rangle \leq 0\}$ is a finitely generated, pointed convex cone in $\mathbb{R}^n$.*

Proof of Theorem 2.4.10
Since the salient space S is finitely generated there is a finite set $F \subset S$ such that $S = \mathrm{sal}(F)$. Furthermore, the dimension of S is finite. Define $n := \dim(V[S])$ and let $\Phi : V[S] \to \mathbb{R}^n$ be an isomorphism. Define the isomorphism $\Psi : \mathbb{R}^n \to (V[S])^*$ by

$$\forall [(s_1, s_2)] \in V[S]\ \forall x \in \mathbb{R}^n : (\Psi(x))\,([(s_1,s_2)]) = \langle \Phi([(s_1,s_2)]), x\rangle_E .$$

Note that $\Phi(S) = \mathrm{sal}(\Phi(F))$, so $\Phi(S)$ is a finitely generated salient space in $\mathbb{R}^n$. Lemma 2.4.11 implies that $(\Phi(S))^\circ = \{x \in \mathbb{R}^n \mid \forall y \in \Phi(S) : \langle x, y\rangle_E \leq 0\}$ is finitely generated. So, for S^* we find:

$$\begin{aligned} S^* &= \{\mathcal{F} \in (V[S])^* \mid \forall s \in S : \mathcal{F}(s) \geq 0\} \\ &= \{\mathcal{F} \in (V[S])^* \mid \forall s \in S : \langle \Phi([(s,0)]), \Psi^{-1}(\mathcal{F})\rangle_E \geq 0\} \\ &= \{\mathcal{F} \in (V[S])^* \mid \forall x \in \Phi(S) : \langle x, \Psi^{-1}(\mathcal{F})\rangle_E \geq 0\} \\ &= \Psi(\{y \in \mathbb{R}^n \mid \forall x \in \Phi(S) : \langle x, y\rangle_E \geq 0\}) \\ &= -\Psi((\Phi(S))^\circ). \end{aligned}$$

Hence, $S^* = -\Psi((\Phi(S))^\circ)$, which implies that S^* is finitely generated. □

Chapter 3

Salient Spaces and Topology

Introduction

In order to be able to prove the equilibrium existence theorems stated in Section 4.8, we need to define continuity of salient mappings. Hence, we like to have a natural, intuitive construction of a topology on a salient space. More precisely, since the models of Chapter 4 are constructed around the concept of a salient pairing, we are looking for a topology on a salient space S which is not only compatible with the salient structure of S, but is also, in some way, induced by a salient space T and a bi-salient form $\mathcal{B}$, where $\{\, S\,,\, T\,; \mathcal{B}\,\}$ is a salient pairing.

The first section of this chapter starts with an intuitive way of introducing a topology on a salient space, and continues with a way of constructing a topology by means of semi-metrics. Furthermore, we give a method for constructing semi-metrics on a salient space S from semi-norms on S. A special example will be a collection of semi-metrics, and thus a topology, induced by a salient pairing. In Section 3.2, we consider continuity and convergence, related to the topologies described in Section 3.1. Finally, in Section 3.3 we discuss some aspects of finite-dimensional salient spaces. In particular, we prove that every reflexive salient space has a salient basis, as defined in Definition 2.1.15, and we give an adaption of Brouwer's Fixed Point Theorem which is related to salient spaces.

3.1 Salient topology

We start with a definition of a topology for a salient space which is compatible with its structure.

3.1.1 Definition (salient topology, bounded set). Let S be a salient space and let τ be a topology on S. The topology τ is salient if the mappings $(s_1, s_2) \to s_1 + s_2$

and $(\alpha, s) \to \alpha s$ are continuous from $S \times S$ into S, and from $\mathbb{R}_+ \times S$ into S, respectively, where $S \times S$ and $\mathbb{R}_+ \times S$ carry the product topology. A (salient) topology is Hausdorff if for every $s_1, s_2 \in S$, with $s_1 \neq s_2$, there are open neighbourhoods O_1, O_2 of s_1 and s_2, respectively, such that $O_1 \cap O_2 = \emptyset$. A set $A \subset S$ is bounded with respect to topology τ if for every open set $O \in \tau$, with $0 \in O$, there is $\alpha \in \mathbb{R}_+$ such that $A \subset \alpha O$.

Since topology on vector spaces is a well-explored topic in mathematics, the construction in the following example is a straightforward way to obtain a salient topology.

3.1.2 Example. Let S be a salient space, and let τ be a linear topology on the vector space $V[S]$, i.e., $(V[S], \tau)$ is a topological vector space (cf. [7, section IV.1]). When we endow $V_+[S]$ with the relative topology induced by τ, and subsequently derive the corresponding topology on S such that the canonical isomorphism $\mathcal{J}_S : S \to V_+[S]$ becomes a homeomorphism, then the resulting topology on S is a salient topology. $\diamond$

The question arises whether the other way around is also possible: when starting with a salient space S with a salient topology τ, is it possible to extend τ to a linear topology on $V[S]$? If this question could be answered affirmatively, then a finite-dimensional salient space carries only one salient topology. In this chapter, we partly go into the question under which kind of conditions a salient topology can be extended.

Furthermore, since we regard the salient space, rather than the vector space reproduced by it, to be the essential mathematical concept of this thesis, another goal we try to achieve in this section is the construction of a salient topology on a salient space S, without making use of the underlying linear structure of $V[S]$.

Due to the absence of the inverse with respect to addition, the direct construction of a salient topology by means of a suitable norm or a collection of semi-norms on a salient space, as is common for vector spaces, is not possible. Instead, we will use a metric or a collection of semi-metrics on a salient space S to define a topological structure on S, and give extra conditions (homegeneity of degree 1 and translation-invariance) on each (semi-)metric in order to guarantee that the generated topology is indeed a salient topology as described above. This idea is inspired by replacing the arbitrary linear topology τ of Example 3.1.2 with a locally convex topology, generated by a collection P of semi-norms on $V[S]$ (cf. [7, Chapter IV]). Before we show the construction of a salient topology from a collection of semi-metrics, we first give the formal definition of a (semi-)metric on a salient space.

3.1.3 Definition (semi-metric and metric on a salient space). Let S be a salient space. A semi-metric on S is a function $d : S \times S \to \mathbb{R}_+$ with the following properties:

- $\forall s \in S : d(s,s) = 0$,
- symmetry: $\forall s_1, s_2 \in S : d(s_1, s_2) = d(s_2, s_1)$,
- triangle inequality: $\forall s_1, s_2, s_3 \in S : d(s_1, s_2) \leq d(s_1, s_3) + d(s_3, s_2)$.

A metric on S is a semi-metric d satisfying $d(s_1, s_2) = 0$ if and only if $s_1 = s_2$.

If π is a semi-norm on the vector space $V[S]$, then $d_\pi : S \times S \to \mathbb{R}_+$, for every $s_1, s_2 \in S$ defined by $d_\pi(s_1, s_2) := \pi([(s_1, s_2)])$, is a semi-metric on S. Indeed, d is symmetric since

$$d_\pi(s_1, s_2) = \pi([(s_1, s_2)]) = \pi(-[(s_2, s_1)]) = \pi([(s_2, s_1)]) = d_\pi(s_2, s_1),$$

and the triangle inequality follows from

$$\begin{aligned} d_\pi(s_1, s_2) &= \pi([(s_1, s_2)]) \\ &= \pi([(s_1, s_3)] + [(s_3, s_1)]) \\ &\leq \pi([(s_1, s_3)]) + \pi([(s_3, s_2)]) \\ &= d_\pi(s_1, s_3) + d_\pi(s_3, s_2). \end{aligned}$$

Note that if π is a norm on $V[S]$, then d_π is a metric on S.

It is not difficult to derive that the semi-metric d_π satisfies the following two properties:

- homogeneity of degree 1: $\forall s_1, s_2 \in S \; \forall \alpha \in \mathbb{R}_+ : d_\pi(\alpha s_1, \alpha s_2) = \alpha d_\pi(s_1, s_2)$,
- translation-invariance: $\forall s_1, s_2, s_3 \in S : d_\pi(s_1 + s_3, s_2 + s_3) = d_\pi(s_1, s_2)$.

Furthermore, let $s \in S$, let $\varepsilon > 0$ and let π be a semi-norm on $V[S]$, then defining

$$B_\pi([(s,0)], \varepsilon) := \{[(t_1, t_2)] \in V[S] \mid \pi([(s + t_2, t_1)]) < \varepsilon, \}$$

implies that the set

$$B_{d_\pi}(s, \varepsilon) = \{t \in S \mid d_\pi(s, t) < \varepsilon\}$$

satisfies

$$\begin{aligned} \mathcal{J}_S(B_{d_\pi}(s, \varepsilon)) &= \{[(t, 0)] \in V[S] \mid \pi([(s, t)]) < \varepsilon\} \\ &= V_+[S] \cap B_\pi([(s, 0)], \varepsilon). \end{aligned}$$

So, if τ_P is the locally convex topology on $V[S]$, induced by the collection P of semi-norms on $V[S]$, then the corresponding topology on S, as derived in Example 3.1.2, is equal to the topology induced by the collection $D_P := \{d_\pi \mid \pi \in P\}$.

Hence, a salient topology can be generated by a (specific) collection of semi-metrics, of which each semi-metric is translation-invariant and homogeneous of degree 1.

3.1.4 Example. Let $\{\, S \,,\, T \,; \mathcal{B} \,\}$ be a non-degenerate salient pairing. Each element $t \in T$ induces a semi-norm π_t on $V[S]$ by defining, for every $[(s_1, s_2)] \in V[S]$:

$$\pi_t([(s_1, s_2)]) := |\mathcal{B}(s_1, t) - \mathcal{B}(s_2, t)|.$$

Since T separates S, we find

$$(\forall t \in T : \pi_t([(s_1, s_2)]) = 0) \iff s_1 = s_2,$$

so T induces a Hausdorff locally convex topology on $V[S]$. Note that this is the "smallest" topology on $V[S]$ in which every element of T induces a continuous semi-norm on $V[S]$. We denote this topology by $w(V[S], T)$. Above, we have seen that this implies that T induces a salient topology on S. This topology is generated by the collection $\{d_{\pi_t} \mid t \in T\}$, for every $t \in T$ and every $s_1, s_2 \in S$, is defined

$$d_{\pi_t}(s_1, s_2) := \pi_t([(s_1, s_2)]) = |\mathcal{B}(s_1, t) - \mathcal{B}(s_2, t)|.$$

We denote this topology, which is homeomorphic with the restriction of $w(V[S], T)$ to $V_+[S]$, by $w(S, T)$.

We remark that this construction of semi-metrics on S is not compatible with the order relation on T. Indeed, if $t, u \in T$ satisfy $t \leq_T u$, then this does not necessarily imply that $\forall s_1, s_2 \in S : d_{\pi_t}(s_1, s_2) \leq d_{\pi_u}(s_1, s_2)$. $\diamond$

Next, we concentrate on semi-metrics which are homogeneous of degree 1 and translation-invariant.

3.1.5 Definition (salient semi-metric and salient metric). A salient (semi-)-metric $d : S \times S \to \mathbb{R}_+$ is a (semi-)metric with the following properties:

- homogeneity of degree 1: $\forall s_1, s_2 \in S \; \forall \alpha \in \mathbb{R}_+ : d(\alpha s_1, \alpha s_2) = \alpha d(s_1, s_2)$,
- translation-invariance: $\forall s_1, s_2, s_3 \in S : d(s_1 + s_3, s_2 + s_3) = d(s_1, s_2)$.

3.1.6 Lemma. *Let $d : S \times S \to \mathbb{R}_+$ be a salient semi-metric on a salient space S. Then d satisfies, for all $\alpha, \beta \in \mathbb{R}_+$ and for all $s, s_1, s_2, t_1, t_2 \in S$:*

$$\begin{array}{ll} a) & \text{if } s_1 + t_2 = s_2 + t_1 \text{ then } d(s_1, t_1) = d(s_2, t_2), \\ b) & d(s_1 + s_2, t_1 + t_2) \leq d(s_1, t_1) + d(s_2, t_2), \\ c) & d(\alpha s, \beta s) = |\alpha - \beta| d(s, 0). \end{array}$$

Proof.
Let $s_1, s_2, t_1, t_2 \in S$ and $\alpha, \beta \in \mathbb{R}_+$. Then a) follows, since $s_1 + t_2 = t_1 + s_2$ implies $d(s_1, t_1) = d(s_1 + t_2, t_1 + t_2) = d(t_1 + s_2, t_1 + t_2) = d(s_2, t_2)$. Furthermore, a), combined with the symmetry of d, implies c). Finally, since the semi-metric d satisfies the triangle inequality, b) follows from $d(s_1 + s_2, t_1 + t_2) \leq d(s_1 + s_2, t_1 + s_2) + d(t_1 + s_2, t_1 + t_2) = d(s_1, t_1) + d(s_2, t_2)$. □

We remark that, when using salient semi-metrics to describe the topology on a salient space S, properties b) and c) of the above lemma imply the continuity of addition and scalar multiplication over $\mathbb{R}_+$, respectively.

3.1.7 Definition (directed set of salient semi-metrics). Let D be a collection of salient semi-metrics on a salient space S. On D, the partial order relation $\leq_D$, is, for semi-metrics $d_1, d_2 \in D$, given by

$$d_1 \leq_D d_2 \quad :\Longleftrightarrow \quad \forall s_1, s_2 \in S : d_1(s_1, s_2) \leq d_2(s_1, s_2).$$

The collection D is directed if

$$\forall d_1, d_2 \in D \; \exists d \in D : d_1 \leq_D d \text{ and } d_2 \leq_D d.$$

As announced above, we can use a specific collection of salient semi-metrics to generate a salient topology.

3.1.8 Definition ($\boldsymbol{\tau_D}$). Let S be a salient space and let D denote a collection of salient semi-metrics on S, which has the property that for every $s_1, s_2 \in S$:

$$(\forall d \in D : d(s_1, s_2) = 0) \implies s_1 = s_2.$$

This collection induces a Hausdorff salient topology τ_D on S, with sub-basis neighbourhood system (cf. [30])

$$\{B_d(s, \varepsilon) \mid d \in D, s \in S, \varepsilon > 0\},$$

where, for every $d \in D$, $s \in S$, and $\varepsilon > 0$:

$$B_d(s, \varepsilon) := \{t \in S \mid d(s, t) < \varepsilon\}.$$

The corresponding topology on $V_+[S]$, such that the canonical isomorphism $\mathcal{J}_S$ is a homeomorphism, is also denoted by τ_D.

3.1.9 Corollary. *Let S be a salient space and let $d : S \times S \to \mathbb{R}_+$ be a salient metric on S. Then the topology generated by d on S is a salient topology.*

3.1.10 Lemma. *Let S be a salient space and let D be a collection of salient semi-metrics on S. Then there is a collection $\widetilde{D}$ of salient semi-metrics on S which is directed and satisfies $\tau_{\widetilde{D}} = \tau_D$.*

Proof.
Define $\widetilde{D}$ to be the collection of all finite (pointwise) sums of elements of D. □

3.1.11 Remark. If D is a countable set, i.e. if $D = \{d_n \mid n \in \mathbb{N}\}$, then $\widetilde{D}$ can be defined as

$$\widetilde{D} := \{\sum_{i=1}^{n} d_i \mid n \in \mathbb{N}\}.$$

◇

The above described process of constructing a salient semi-metric d_π on a salient space S from a semi-norm π on $V[S]$, can be reversed.

3.1.12 Definition ($\boldsymbol{\pi_d}$). Let $d : S \times S \to \mathbb{R}_+$ be a salient semi-metric on a salient space S. Then the mapping $\pi_d : V[S] \to \mathbb{R}_+$ is, for every $[(s_1, s_2)] \in V[S]$, given by

$$\pi_d\left([(s_1, s_2)]\right) := d\left(s_1, s_2\right).$$

Note that every semi-norm π on $V[S]$ satisfies $\pi_{d_\pi} = \pi$, and that every salient semi-metric d on S satisfies $d_{\pi_d} = d$.

3.1.13 Proposition. *Let d be a salient semi-metric on a salient space S. Then the mapping $\pi_d : V[S] \to \mathbb{R}_+$ is a semi-norm on $V[S]$. Furthermore, d is a salient metric on S if and only if π_d is a norm on $V[S]$.*

Proof.
Let $s_1, s_2, t_1, t_2 \in S$. The mapping π_d is defined independently of the choice of representatives: if $[(s_1, s_2)] = [(t_1, t_2)]$, i.e., if $s_1 + t_2 = s_2 + t_1$, then $\pi_d\left([(s_1, s_2)]\right) = d\left(s_1, s_2\right) = d\left(s_1 + t_1, s_2 + t_1\right) = d\left(s_1 + t_1, s_1 + t_2\right) = d\left(t_1, t_2\right) = \pi_d\left([(t_1, t_2)]\right)$. Furthermore, π_d satisfies the triangle inequality:

$$\begin{aligned}
\pi_d\left([(s_1, s_2)] + [(t_1, t_2)]\right) &= d\left(s_1 + t_1, s_2 + t_2\right) \\
&\leq d\left(s_1 + t_1, s_2 + t_1\right) + d\left(s_2 + t_1, s_2 + t_2\right) \\
&= d\left(s_1, s_2\right) + d\left(t_1, t_2\right) \\
&= \pi_d\left([(s_1, s_2)]\right) + \pi_d\left([(t_1, t_2)]\right).
\end{aligned}$$

The rest of the proof is left to the reader. □

A collection D of salient semi-metrics on S, induces a collection $P_D = \{\pi_d \mid d \in D\}$ of semi-norms on $V[S]$, where P_D satisfies

$$(\forall \pi \in P_D : \pi([(s_1, s_2)]) = 0) \implies [(s_1, s_2)] = [(0, 0)],$$

if and only if D satisfies

$$(\forall d \in D : d\,(s_1, s_2) = 0) \implies s_1 = s_2.$$

So, if the collection D generates a Hausdorff salient topology τ_D on S, then the collection P_D generates a Hausdorff locally convex topology τ_{P_D} on $V[S]$, with subbasis neighbourhood system

$$\{B_{\pi_d}([(s_1, s_2)], \varepsilon) \mid d \in D, [(s_1, s_2)] \in V[S], \varepsilon > 0\},$$

where, for every $d \in D$, $s_1, s_2 \in S$, and $\varepsilon > 0$:

$$B_{\pi_d}([(s_1, s_2)], \varepsilon) := \{[(t_1, t_2)] \in V[S] \mid \pi_d\,([(s_1 + t_2, s_2 + t_1)]) < \varepsilon\}.$$

We remark that the topology τ_D on $V_+[S]$ is the relative topology of τ_{P_D}, when restricted to $V_+[S]$.

It is well-known that a topological vector space is a locally convex space if and only if there is a basis, consisting of convex, balanced and absorbing sets, for the neighbourhood system at 0. This basis induces the collection of semi-norms that in turn induces the locally convex topology. One can wonder, whether a salient space with a salient topology τ also contains an element of which the neighbourhood system induces a set of salient semi-metrics that generates τ. However, we are more interested in a different way of constructing salient semi-metrics.

We show a way to construct a salient semi-metric on a salient space S from a semi-norm on S. Since in a salient pairing $\{\, S\, ,\, T\,; \mathcal{B}\,\}$ each element of T, through $\mathcal{B}$, induces a semi-norm on S, this results in a way of constructing a salient topology on S without directly using the well-known vector space concepts of topology.

In comparison with the salient topology of Example 3.1.4, we will see that the new method results in a topology which is compatible with the order relation on T. Furthermore, we will see that the new construction allows for a nice introduction of continuity. More precisely, it turns out that if T has an order unit, then this order unit induces a metric on S and hence the topology (and the continuity of functions) can be described using one metric only.

Above, we have seen that every salient metric $d : S \times S \to \mathbb{R}_+$ on a salient space S induces a norm on the vector space $V[S]$. Thus, restricting this norm to the positive cone $V_+[S]$ of $V[S]$, and subsequently applying the salient isomorphism $\mathcal{J}_S$, we find a norm on the salient space S. In the following, we generalise this idea to a salient space setting. We start with the definition, some properties and some examples of a semi-norm on a salient space. Thereafter, we give the construction of a salient metric from a semi-norm on a salient space.

3.1.14 Definition (semi-norm and norm on a salient space). Let S be a salient space. A function $\varphi : S \to \mathbb{R}_+$ is a semi-norm on S if for all $s, s_1, s_2 \in S$ and for all $\alpha \in \mathbb{R}_+$:

$$\begin{cases} \varphi(s_1 + s_2) \leq \varphi(s_1) + \varphi(s_2) \\ \varphi(\alpha s) = \alpha \varphi(s). \end{cases}$$

A semi-norm φ on S is a norm on S if for all $s \in S$:

$$\varphi(s) = 0 \implies s = 0.$$

Let $\varphi : S \to \mathbb{R}_+$ be a semi-norm on a salient space S. Then $T := \{s \in S \mid \varphi(s) = 0\}$ is a salient subspace of S. In case $T \neq \{0\}$, define the equivalence relation $\sim_T$ on S by

$$s_1 \sim_T s_2 \quad := \quad \exists t_1, t_2 \in T : s_1 + t_1 = s_2 + t_2,$$

and the equivalence class $[s_1]$ of s_1 by

$$[s_1] := \{s_2 \in S \mid s_2 \sim_T s_1\},$$

then the salient quotient space $S/T = \{[s] \mid s \in S\}$ is a salient space where addition and multiplication over $\mathbb{R}_+$ are given by $[s_1] + [s_2] := [s_1 + s_2]$ and $\alpha[s] := [\alpha s]$, respectively. The equivalence class $[0]$ is the vertex of S/T. Furthermore, $\tilde{\varphi} : S/T \to \mathbb{R}_+$, defined by $\tilde{\varphi}([s]) := \varphi(s)$, is a norm on S/T.

3.1.15 Example. The function $\varphi : \mathbb{R}^3_+ \to \mathbb{R}_+$, for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$ defined by $\varphi(x) := x_1$, is a semi-norm on $\mathbb{R}^3_+$. In this situation, $T = \{x \in \mathbb{R}^3_+ \mid x_1 = 0\}$ and $[x] = \{y \in \mathbb{R}^3_+ \mid y_1 = x_1\}$. Clearly, $\tilde{\varphi}([x]) = x_1 = 0$ implies $[x] = [0]$. Note that $\mathbb{R}^3_+/T$ is equivalent with $\mathbb{R}_+$. ◇

3.1.16 Definition (monotonous semi-norm). A semi-norm $\varphi : S \to \mathbb{R}_+$, on a salient space S is monotonous, with respect to the order relation $\leq_S$, if

$$\forall s_1, s_2 \in S : (s_1 \leq_S s_2) \implies (\varphi(s_1) \leq \varphi(s_2)).$$

3.1.17 Example. Let $\{ S , T ; \mathcal{B} \}$ be a salient pairing and let t be an element of T. Then $\mathcal{M}_t$, as defined on page 57 is a monotonous semi-norm on S. In particular, every element of the adjoint S^* of S acts as a monotonous semi-norm on S. $\diamond$

3.1.18 Example. Let S be a salient space and let $u \in S$ be an order unit of S. Then $\varphi_u : S \to \mathbb{R}_+$, for every $s \in S$ defined by

$$\varphi_u (s) := \inf\{\lambda \mid s \leq_S \lambda u\},$$

is a semi-norm on S. Note that φ_u is monotonous with respect to the order relation $\leq_S$. $\diamond$

3.1.19 Example. The trace, denoted by tr, is a norm on the salient space S of all real, positive $n \times n$ matrices. Clearly, $\mathrm{tr} : S \to \mathbb{R}_+$ is a semi-norm on S. Let $A \in S$. If $\mathrm{tr}(A) = 0$, then every diagonal element of A equals 0. Consider entry a_{ij} of A, where $i \neq j$. Let $\alpha, \beta \in \mathbb{R}$, and define the vector $x \in \mathbb{R}^n$ by

$$x_k := \begin{cases} \alpha & \text{if} \quad k = i \\ \beta & \text{if} \quad k = j \\ 0 & \text{if} \quad k \in \{1, \dots, n\} \setminus \{i, j\} \end{cases} .$$

The matrix A is positive, so $\langle Ax, x\rangle = 2a_{ij}\alpha\beta \geq 0$. Since this must hold for all $\alpha, \beta \in \mathbb{R}$, we conclude $a_{ij} = 0$. We conclude that tr is a norm on S. Note, that tr satisfies $\forall A, B \in S : A \leq_S B \Longrightarrow \mathrm{tr}(A) \leq \mathrm{tr}(B)$, i.e., the norm tr is monotonous. $\diamond$

3.1.20 Definition (partial ordering of semi-norms on a salient space). Let S be a salient space. Every collection P of semi-norms on S can be ordered pointwise, by defining

$$\varphi_1 \leq \varphi_2 \quad :\Longleftrightarrow \quad \forall s \in S : \varphi_1 (s) \leq \varphi_2 (s) .$$

Next, we construct a salient semi-metric from a semi-norm on a salient space S. Since every salient semi-metric d on a salient space S induces a semi-norm π_d on $V[S]$, this implies a way of constructing a semi-norm on $V[S]$ from a semi-norm on S. We will investigate whether this semi-norm on $V[S]$ is an extension of the semi-norm on S.

Let S be a salient space, and let $d : S \times S \to \mathbb{R}_+$ be a salient semi-metric on S. In Definition 3.1.12, we have seen that d induces the semi-norm π_d on $V[S]$, of which the restriction to the set $V_+[S]$ is a semi-norm on S. The question arises whether we can reverse this process, i.e., if $\varphi : S \to \mathbb{R}_+$ is a semi-norm on a salient space S, does there exist a salient semi-metric d_φ on S, induced by φ, such that $\forall s \in S : \varphi(s) = d_\varphi(s, 0)$. If this is the case, then $\pi_{d_\varphi} : V[S] \to \mathbb{R}_+$ would be an extension of φ to a semi-norm on $V[S]$.

To this end, observe that a salient semi-metric d, induced by a semi-norm φ, satisfies

$$d(s_1, s_2) \leq \varphi(s_1) + \varphi(s_2).$$

Furthermore, if $s_1 + t_1 = s_2 + t_2$ then d satisfies

$$d(s_1, s_2) = d(s_1 + t_1, s_2 + t_1) = d(s_2 + t_2, s_2 + t_1) = d(t_2, t_1).$$

So, we find that for all $s_1, s_2 \in S$, the semi-metric d has to satisfy

$$d(s_1, s_2) \leq \inf\{\varphi(t_1) + \varphi(t_2) \mid t_1, t_2 \in S \text{ with } s_1 + t_1 = s_2 + t_2\}.$$

This leads us to the following definition.

3.1.21 Definition (d_φ). Let $\varphi : S \to \mathbb{R}_+$ be semi-norm on a salient space S. Then the mapping $d_\varphi : S \times S \to \mathbb{R}_+$ is, for every $s_1, s_2 \in S$, given by

$$d_\varphi(s_1, s_2) := \inf\{\varphi(q) + \varphi(r) \mid q, r \in S \text{ with } s_1 + q = s_2 + r\}.$$

The following proposition shows, among other things, that for every semi-norm φ on S, the mapping d_φ is a salient semi-metric on S.

3.1.22 Proposition. *Let S be a salient space and let $\varphi : S \to \mathbb{R}_+$ be a semi-norm on S. Then $d_\varphi : S \times S \to \mathbb{R}_+$ is a salient semi-metric on S, satisfying:*

$$d_\varphi(s, 0) \leq \varphi(s).$$

Proof.
It can be easily checked that d_φ is symmetric and satisfies $\forall s \in S : d_\varphi(s, s) = 0$. So, in order to complete the proof that d_φ is a semi-metric, we prove that the triangle inequality holds. Let $s_1, s_2, s_3 \in S$, then

$$\begin{aligned}
&d_\varphi(s_1, s_2) + d_\varphi(s_2, s_3) \\
&= \inf\{\varphi(q) + \varphi(r) \mid q, r \in S \text{ with } s_1 + q = s_2 + r\} \\
&\quad + \inf\{\varphi(\widehat{q}) + \varphi(\widehat{r}) \mid \widehat{q}, \widehat{r} \in S \text{ with } s_2 + \widehat{q} = s_3 + \widehat{r}\} \\
&= \inf\{\varphi(q) + \varphi(r) + \varphi(\widehat{q}) + \varphi(\widehat{r}) \mid q, \widehat{q}, r, \widehat{r} \in S,\ s_1 + q = s_2 + r, s_2 + \widehat{q} = s_3 + \widehat{r}\} \\
&\geq \inf\{\varphi(q + \widehat{q}) + \varphi(r + \widehat{r}) \mid q, \widehat{q}, r, \widehat{r} \in S,\ s_1 + q + s_2 + \widehat{q} = s_2 + r + s_3 + \widehat{r}\} \\
&= \inf\{\varphi(q + \widehat{q}) + \varphi(r + \widehat{r}) \mid q, \widehat{q}, r, \widehat{r} \in S \text{ with } s_1 + q + \widehat{q} = s_3 + r + \widehat{r}\} \\
&= d_\varphi(s_1, s_3).
\end{aligned}$$

Since the translation invariance and homogeneity of degree 1 are easily checked by the reader, we find that d_φ is a salient semi-metric.
Since, for every $s \in S$, we can always choose $q = 0$ and $r = s$, it follows immediately that $d_\varphi(s, 0) \leq \varphi(s)$. □

We remark that even in case $\varphi : S \to \mathbb{R}_+$ is a norm on a salient space S, the above construction of d_φ only implies that d_φ is a semi-metric and not that d_φ a metric on S. We will come back to this observation on page 88.

3.1.23 Proposition. *Let $\{ S , T ; \mathcal{B} \}$ be a salient pairing. Every $t \in T$ induces a semi-metric $d_{\mathcal{M}_t}$ on S. The semi-norm on $V[S]$, generated by $d_{\mathcal{M}_t}$ is an extension of the semi-norm $\mathcal{M}_t : S \to \mathbb{R}_+$. Furthermore, if the pairing $\{ S , T ; \mathcal{B} \}$ is non-degenerate and if $t_0 \in \mathrm{int}(T)$ then the salient semi-metric, generated by t_0 is a metric on S.*

Proof.
Let $t \in T$. The function $\mathcal{M}_t : S \to \mathbb{R}_+$ is a semi-norm on S, so $d_{\mathcal{M}_t}$ is indeed a salient semi-metric on S. Since $d_{\mathcal{M}_t}$ satisfies

$$\begin{aligned} d_{\mathcal{M}_t}(s, 0) &= \inf\{\mathcal{B}(q, t) + \mathcal{B}(r, t) \mid s + q = r\} \\ &= \inf\{\mathcal{B}(s, t) + 2\mathcal{B}(q, t) \mid s + q = r\} \\ &= \mathcal{B}(s, t), \end{aligned}$$

we conclude that the semi-norm on $V[S]$, generated by $d_{\mathcal{M}_t}$ is an extension of the semi-norm $\mathcal{M}_t : S \to \mathbb{R}_+$.
Let $t_0 \in \mathrm{int}(T)$. Let $s_1, s_2 \in S$ and suppose $d_{\mathcal{M}_t}(s_1, s_2) = 0$. Then, there are sequences $(q_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ in S, such that $\lim_{n\to\infty} \mathcal{B}(q_n, t_0) = \lim_{n\to\infty} \mathcal{B}(r_n, t_0) = 0$ and $\forall n \in \mathbb{N} : s_1 + q_n = s_2 + r_n$. For every $t \in T$, we find $\lim_{n\to\infty} \mathcal{B}(q_n, t) = \lim_{n\to\infty} \mathcal{B}(r_n, t) = 0$, i.e., $\forall t \in T : \mathcal{B}(s_1, t) = \lim_{n\to\infty}(\mathcal{B}(s_1, t) + \mathcal{B}(q_n, t)) = (\mathcal{B}(s_2, t) + \mathcal{B}(r_n, t)) = \mathcal{B}(s_2, t)$. Since T separates S, we conclude $s_1 = s_2$. □

3.1.24 Corollary. *Let S be a salient space. Every $\mathcal{F} \in S^*$ induces a salient semi-metric $d_\mathcal{F}$ on S, satisfying $d_\mathcal{F}(s, 0) = \mathcal{F}(s)$. The action of $\pi_{d_\mathcal{F}}$ on $V_+[S]$ is equivalent to the action of $\mathcal{F}$ on S. If S^* separates S and if $\mathcal{F} \in \mathrm{int}(S^*)$, then $d_\mathcal{F}$ is a salient metric on S.*

The question arises whether the observed statement $\forall s \in S : d_\mathcal{F}(s, 0) = \mathcal{F}(s)$, or in other words "$\pi_{d_\mathcal{F}}$ is an extension of the semi-norm $\mathcal{F}$ to $V[S]$", not only holds for elements of S^* but for every semi-norm on S. Before we answer this question, we first define the salient topology on a salient space S, induced by a salient space T through a non-degenerate pairing $\{ S , T ; \mathcal{B} \}$.

3.1.25 Definition (topology $\tau(S,T)$). Let $\{\, S\,,\, T\,; \mathcal{B}\,\}$ be a non-degenerate salient pairing. Every element $t \in T$ induces the semi-metric $d_{\mathcal{M}_t}$, on S. With $\tau(S,T)$, we denote the locally convex topology on S, induced by the collection $\{d_{\mathcal{M}_t}\,(|, t) \in T\}$.

We remark that the collection $\{d_{\mathcal{M}_t} \mid t \in T\}$ is a directed set and that, since T separates S, the topology $\tau(S,T)$ is Hausdorff. If the salient space T has an order set U, then the topology $\tau(S,T)$ is generated by the collection $\{d_u \mid u \in U\}$. In Lemma 2.2.9 we have seen that if T has a finite order set, then T has an order unit u. In this situation, the topology $\tau(S,T)$ is generated by the salient metric d_u, i.e, the topology $\tau(V[S],T)$ is generated by the norm π_u. If $t,u \in \text{int}(T)$ satisfy $t \leq_T u$, then we find $d_{\mathcal{M}_t} \leq_D d_{\mathcal{M}_u}$, where $D = \{d_{\mathcal{M}_t} \mid t \in T\}$ and $\leq_D$ is as defined in Definition 3.1.7. Since for all $t_0, t_1 \in \text{int}(T)$ there are $\mu, \lambda > 0$ such that $\mu t_0 \leq_T t_1 \leq_T \lambda t_0$, we conclude that all the salient metrics, generated by elements of $\text{int}(T)$, are equivalent.

Without proof, we state the following proposition.

3.1.26 Proposition. *Let $\{\, S\,,\, T\,; \mathcal{B}\,\}$ be a non-degenerate salient pairing. A subset $A \subset S$ is $\tau(S,T)$-bounded if and only if*

$$\forall t \in T\ \exists \alpha \geq 0\ \forall a \in A : \mathcal{B}(a,t) \leq \alpha,$$

i.e., if

$$\exists u \in \text{int}(T)\ \exists \alpha \geq 0\ \forall a \in A : \mathcal{B}(a,u) \leq \alpha.$$

In Corollary 3.1.24 we saw that for every $\mathcal{F}$ in the adjoint of a salient space S, the function $\pi_{d_{\mathcal{F}}}$ is an extension of $\mathcal{F}$ to a semi-norm on $V[S]$. More generally, for a non-degenerate salient pairing $\{\, S\,,\, T\,; \mathcal{B}\,\}$, we have seen that for every function $\mathcal{M}_t : S \to \mathbb{R}_+$, with $t \in T$, the semi-metric on $V[S]$, generated by $d_{\mathcal{M}_t}$ is an extension of $\mathcal{M}_t$.

Below, we investigate whether every semi-norm on a salient space S can be extended to a semi-norm on $V[S]$, i.e. whether for all $s \in S : d_\varphi\,(s,0) = \varphi\,(s)$. If this is the case, we have a nice way of describing the continuity of salient functions; a way which is similar to the vector space situation where continuity of a linear function is equivalent to continuity of that function in 0. Indeed, if S is a salient space, endowed with salient topology τ_D, where D is a directed collection of salient semi-metrics, and if $V[S]$ is endowed with the topology τ_{P_D} as defined on page 73, then a salient function $\mathcal{L} : S \to \mathbb{R}_+$ is τ_D-continuous if and only if $|\mathcal{L}(s)| \leq \pi_d([(s,0)]) = d(s,0)$ for a certain salient semi-metric $d \in D$. If this semi-metric is generated from a semi-norm φ on S, then $d_\varphi(s,0) = \varphi(s)$ would imply that continuity of a salient function is determined by the collection of semi-norms that generates the set D. We remark that this implies that every $\mathcal{M}_t$ is continuous with respect to $\tau(S,T)$. In case we cannot prove the desired property, the question remains if continuity with respect to the semi-norms is necessary or sufficient.

We start the investigation with some new notation.

3.1.27 Definition ($\boldsymbol{\psi_d}$). Let d be a salient semi-metric on a salient space S. Then the mapping $\psi_d : S \to \mathbb{R}_+$ is, for every $s \in S$, given by

$$\psi_d\left(s\right) := d\left(s, 0\right).$$

The fact that ψ_d is equivalent with the restriction of π_d to $V_+[S]$, implies the following proposition.

3.1.28 Proposition. *Let d be a salient semi-metric on a salient space S, then the mapping $\psi_d : S \to \mathbb{R}_+$ is a semi-norm on S. Furthermore, d and ψ_d satisfy*

$$\forall s_1, s_2 \in S : d\left(s_1, s_2\right) \leq \psi_d\left(s_1\right) + \psi_d\left(s_2\right).$$

Note, that if d is a metric on S then ψ_d is a norm on S.

The question now translates to the following. When we start with a semi-norm $\varphi : S \to \mathbb{R}_+$ on a salient space S, derive the salient semi-metric d_φ and define the semi-norm $\psi_{d_\varphi} : S \to \mathbb{R}_+$, for every $s \in S$, by $\psi_{d_\varphi}(s) := d_\varphi(s,0)$, is ψ_{d_φ} equal to φ? Clearly, $\psi_{d_\varphi} \leq \varphi$. The following proposition shows that monotony of φ is a sufficient condition to guarantee that $\varphi \leq \psi_{d_\varphi}$. Note that this condition is true for the semi-norms considered in Proposition 3.1.23 en Corollary 3.1.24

3.1.29 Proposition. *Let $\varphi : S \to \mathbb{R}_+$ be a monotonous semi-norm on a salient space S. Then the semi-norm ψ_{d_φ} on S satisfies $\psi_{d_\varphi} = \varphi$.*

Proof.
Let $s \in S$. For all $x, y \in S$, satisfying $s + x = y$, we find $s \leq_S x + y = s + 2x$. Hence, $\varphi(s) \leq \varphi(x+y) \leq \varphi(x) + \varphi(y)$, and we find $\varphi(s) \leq d_\varphi(s,0)$. □

We remark that if $\varphi : S \to \mathbb{R}_+$ is a monotonous semi-norm on a salient space S, then for all $s_1, s_2, s_3 \in S$, we find that $s_1 = s_2 + s_3$ implies $d_\varphi(s_1, s_2) = d_\varphi(s_3, 0) = \varphi(s_3)$.

3.1.30 Example. Let S be the salient space of all real, positive $n \times n$ matrices. In Example 3.1.19, we have seen that $\mathrm{tr} : S \to \mathbb{R}_+$ is a norm on S, which is monotonous with respect to the partial order relation $\leq_S$. Let $A, B \in S$, and note that the set $\{M_1, M_2 \in S \mid A + M_1 = B + M_2 \text{ and } \mathrm{tr}(M_1 + M_2) \leq \mathrm{tr}(A + B)\}$ is compact in S. The above proposition implies that $d_{\mathrm{tr}} : S \times S \to \mathbb{R}_+$, defined by

$$d_{\mathrm{tr}}(A, B) := \min\{\mathrm{tr}(M_1 + M_2) \mid M_1, M_2 \in S \text{ and } A + M_1 = B + M_2\},$$

is a salient semi-metric on S, satisfying $d_{\mathrm{tr}}(A, 0) = \mathrm{tr}(A)$, i.e., $\psi_{d_{\mathrm{tr}}} = \mathrm{tr}$. Proposition 3.1.13 implies that $\pi_{d_{\mathrm{tr}}} : V[S] \to \mathbb{R}_+$, defined by $\pi_{d_{\mathrm{tr}}}([(A, B)]) := d_{\mathrm{tr}}(A, B)$, is a semi-norm on the set of all symmetrical $n \times n$ matrices, satisfying $\pi_{d_{\mathrm{tr}}}([(A, 0)]) = \mathrm{tr}(A)$. $\diamond$

Proposition 3.1.31 and Example 3.1.33 show that monotony of the semi-norm $\varphi : S \to \mathbb{R}_+$, with respect to the order relation $\geq_S$ of a salient space S, is not a necessary condition to guarantee that $\psi_{d_\varphi} = \varphi$.

3.1.31 Proposition. *Let S be a salient space and let π be a semi-norm on $V[S]$. Define the norm $\varphi : S \to \mathbb{R}_+$, for every $s \in S$, by $\varphi(s) := \pi([(s, 0)])$. Then $\psi_{d_\varphi} = \varphi$.*

Proof.
Let $s \in S$. Then $\forall q \in S : \varphi(s) = \pi([(s + q, q)]) \leq \pi([(s + q, 0)]) + \pi([(0, q)]) = \varphi(s + q) + \varphi(q)$. Hence, $\varphi(s) \leq \inf\{\varphi(q) + \varphi(r) \mid q, r \in S \text{ with } s + q = r\} = d_\varphi(s, 0)$. $\square$

We want to emphasise that in the above situation, π_{d_φ} is not necessarily equal to π (cf. Example 3.1.34).

3.1.32 Corollary. *Let V be a partially ordered vector space, with positive cone V_+. Let $\| \cdot \|$ be a norm on V. Let the salient metric $d : V_+ \times V_+ \to \mathbb{R}_+$, for every $v_1, v_2 \in V_+$, be defined by $d(v_1, v_2) := \| v_1 - v_2 \|$. Then $\forall v \in V_+ : \psi_d(v) = \| v \|$.*

3.1.33 Example. Let $S \subset \mathbb{R}^2$ be the salient space defined by

$$S := \left\{ x \in \mathbb{R}^2 \mid \begin{array}{c} 2x_1 + x_2 \geq 0 \\ x_1 + 2x_2 \geq 0 \end{array} \right\},$$

and let $\varphi : S \to \mathbb{R}_+$ be defined by $\varphi(x) := \sqrt{(x_1)^2 + (x_2)^2}$. In this situation, φ is not monotonous with respect to $\leq_S$: choose $x = (-1, 2)$ and $y = (2, -1)$, then $x, y \in S$ and $\varphi(x) = \varphi(y) = \sqrt{5}$ while $\varphi(x + y) = \varphi((1, 1)) = \sqrt{2}$. Although Proposition 3.1.29 is not applicable, Proposition 3.1.31 implies that $\psi_{d_\varphi} = \varphi$. $\diamond$

The following example shows that if the norm φ on S is derived from a norm $\| \, . \, \|$ on $V[S]$, restricted to the pointed convex cone $V_+[S]$, then it is possible that $\| \, . \, \| \neq \pi_{d_\varphi}$. However, on $V_+[S]$, the norms are equal since they are both an extension of $\varphi : S \to \mathbb{R}_+$.

3.1.34 Example. Consider the norm $\| \, . \, \|_\infty$ on $\mathbb{R}^2$. This norm induces a semi-norm φ on the salient space $\mathbb{R}^2_+$, by defining for every $x = (x_1, x_2) \in \mathbb{R}^2_+$: $\varphi(x) := \| \, x \, \|_\infty = \max\{x_1, x_2\}$. Choose $x = (2,1)$ and $y = (1,2)$, then

$$\| \, [(x,y)] \, \|_\infty = \| \, [(\, (2,1) \, , \, (1,2) \,)] \, \|_\infty = \| \, [(\, (1,0) \, , \, (0,1) \,)] \, \|_\infty = 1.$$

However, $\pi_{d_\varphi}([(x,y)]) = \inf\{\varphi(v) + \varphi(w) \mid v, w \in \mathbb{R}^2_+ \text{ with } x + v = y + w\} = \| \, (1,0) \, \|_\infty + \| \, (0,1) \, \|_\infty = 2$. Hence, $\| \, . \, \|_\infty \neq \pi_{d_\varphi}$, or put differently, $d_\varphi(s_1, s_2) \neq \| \, s_1 - s_2 \, \|_\infty$ for certain $s_1, s_2 \in S$. $\diamond$

The following proposition implies that in case the partially ordered set $(S, \leq_S)$ is a lattice, then the infimum in the definition of the salient semi-metric d_φ from a semi-norm $\varphi : S \to \mathbb{R}_+$ is attained and so both d_φ and π_{d_φ} can be made explicit.

3.1.35 Proposition. *Let S be a salient space, let $\varphi : S \to \mathbb{R}_+$ be a monotonous semi-norm on S, and let $\mathcal{J}_S$ denote the salient isomorphism between S and $V_+[S]$. Let $[(s_1, s_2)]$ be an element of the partially ordered vector space $(V[S], \leq_S)$, such that the positive part $[(s_1, s_2)]^+$ and the negative part $[(s_1, s_2)]^-$ exist in $V_+[S]$. Then*

$$d_\varphi(s_1, s_2) = \varphi\left(\mathcal{J}_S^{-1}\left([(s_1, s_2)]^+\right)\right) + \varphi\left(\mathcal{J}_S^{-1}\left([(s_1, s_2)]^-\right)\right).$$

Proof.
Let $q, r \in S$ satisfy $s_1 + q = s_2 + r$, then $[(r, 0)]$ is an upper bound of the set $\{[(s_1, s_2)], [(0,0)]\}$, and $[(q, 0)]$ is an upper bound of the set $\{[(s_2, s_1)], [(0,0)]\}$. Hence, we find $[(s_1, s_2)]^+ \leq [(r, 0)]$ and $[(s_1, s_2)]^- \leq [(q, 0)]$, which is equivalent with $\mathcal{J}_S^{-1}([(s_1, s_2)]^+) \leq_S r$ and $\mathcal{J}_S^{-1}([(s_1, s_2)]^-) \leq_S q$. By the monotony of $\varphi : S \to \mathbb{R}_+$, with respect to the partial order relation $\leq_S$, we find $\varphi\left(\mathcal{J}_S^{-1}([(s_1, s_2)]^+)\right) \leq \varphi(r)$ and $\varphi\left(\mathcal{J}_S^{-1}([(s_1, s_2)]^-)\right) \leq \varphi(q)$, which implies

$$\varphi\left(\mathcal{J}_S^{-1}\left([(s_1, s_2)]^+\right)\right) + \varphi\left(\mathcal{J}_S^{-1}\left([(s_1, s_2)]^-\right)\right) \leq \varphi(q) + \varphi(r).$$

We conclude that $\pi_{d_\varphi}([(s_1, s_2)]) = \varphi\left(\mathcal{J}_S^{-1}([(s_1, s_2)]^+)\right) + \varphi\left(\mathcal{J}_S^{-1}([(s_1, s_2)]^-)\right)$. $\square$

Note, that if, in the above proposition, $\varphi : S \to \mathbb{R}_+$ is a norm on S, then π_{d_φ} is a norm on $V[S]$.

3.1.36 Example. Let $(V, \leq)$ be a partially ordered vector space such that for every $v \in V$ there are $v^+, v^- \in V_+$ such that $v = v^+ - v^-$. Let $\varphi : V_+ \to \mathbb{R}_+$ be a semi-norm on the positive cone of V. Then

$$\forall v \in V : \pi_{d_\varphi}(v) = \varphi\left(v^+\right) + \varphi\left(v^-\right).$$

$\diamond$

3.1.37 Example. Consider the salient space $\mathbb{R}^n_+$ and let $\varphi : \mathbb{R}^n_+ \to \mathbb{R}_+$ be the p-norm, $p \in (0,1)$, on $\mathbb{R}^n_+$ (cf. [20]), for every $x \in \mathbb{R}^n_+$ defined by

$$\varphi(x) := \left(\sum_{j=1}^{n} (x_j)^p\right)^{\frac{1}{p}}.$$

Then $\pi_{d_\varphi} : \mathbb{R}^n \to \mathbb{R}_+$ is defined by

$$\pi_{d_\varphi}(x) = \left(\sum_{j=1}^{n} (x_j^+)^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} (x_j^-)^p\right)^{\frac{1}{p}},$$

where $\forall j \in \{1, \ldots, n\} : x_j^+ := \max\{x_j, 0\}$ and $x_j^- := \max\{0, -x_j\}$. Note that π_{d_φ} restricted to $\mathbb{R}^n_+$ is equal to φ. $\diamond$

The following example shows that in Proposition 3.1.35, the requirement that S is a lattice, indeed is not necessary.

3.1.38 Example. Let S be the salient space of all real, positive $n \times n$ matrices. In Example 3.1.30 we have seen that $\pi_{d_{\text{tr}}} : V[S] \to \mathbb{R}_+$, is a semi-norm on the set $V[S]$ of all symmetrical $n \times n$ matrices, satisfying $\pi_{d_{\text{tr}}}([(A,0)]) = \text{tr}(A)$. Let $M \in V[S]$. In Example 2.2.30, we have seen that although S does not have a lattice structure, the least upper bound M^+ of the set $\{0, M\}$ and the least upper bound M^- of the set $\{0, -M\}$ exist in S. Hence, $\pi_{d_{\text{tr}}}(M) = \text{tr}(M^+ + M^-)$. $\diamond$

Let S be a salient space which is a lattice and let $s_1, s_2 \in S$. In Remark 2.2.28, we showed that $[(\,(s_1 \vee s_2)\,,\, s_2\,)]$ is equal to the positive part of $[(s_1, s_2)]$, and that $[(\,(s_1 \vee s_2)\,,\, s_1\,)]$ is equal to the negative part of $[(s_1, s_2)]$. Hence, we find the following corollary.

3.1.39 Corollary. *Let S be a salient space, which is a lattice, let $\varphi : S \to \mathbb{R}_+$ be a monotonous semi-norm on S, and let $s_1, s_2 \in S$. Let $p, q \in S$ satisfy*

$$s_2 + p \;=\; (s_1 \vee s_2) \;\; \textit{and} \;\; s_1 + q \;=\; (s_1 \vee s_2).$$

Then

$$d_\varphi(s_1, s_2) = \varphi(p) + \varphi(q).$$

3.2 Continuity of salient mappings

Let S and T be salient spaces, each with a salient topology generated by a collection of salient semi-metrics. In this section, we focus on the continuity of salient mappings $\mathcal{L} : S \to T$. We start with the definition of continuity with respect to a single semi-metric d_S on S and a single semi-metric d_T on T. This approach is directly applicable to situations in which the topology of both S and T is generated by one metric. Furthermore, we will see that this approach easily leads to the definition of continuity with respect to several semi-metrics and continuity with respect to semi-norms on S and T. Also, we will derive the relationship with the continuity of the extended mapping $\mathcal{L}^{\text{ext}} : V[S] \to V[T]$.

3.2.1 Definition (continuity with respect to salient semi-metrics). Let S and T be two salient spaces, and let $d_S : S \times S \to \mathbb{R}_+$ and $d_T : T \times T \to \mathbb{R}_+$ be two salient semi-metrics. Then a salient mapping $\mathcal{L} : S \to T$ is continuous with respect to π_{d_S} and π_{d_T} if

$$\forall s_1 \in S\, \forall \varepsilon > 0\, \exists \delta > 0\, \forall s_2 \in S : \; d_S\,(s_1, s_2) < \delta \implies d_T\,(\mathcal{L}(s_1), \mathcal{L}(s_2)) < \varepsilon.$$

The salient mapping $\mathcal{L}$ is uniformly continuous with respect to d_S and d_T if

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall s_1, s_2 \in S : \; d_S\,(s_1, s_2) < \delta \implies d_T\,(\mathcal{L}(s_1), \mathcal{L}(s_2)) < \varepsilon.$$

The following lemma shows that continuity of the linear mapping $\mathcal{L}^{\text{ext}} : V[S] \to V[T]$, is determined from the action of $\mathcal{L}^{\text{ext}}$ on the positive cone $V_+[S]$.

3.2.2 Lemma. *Let S and T be two salient spaces, let $d_S : S \times S \to \mathbb{R}_+$ and $d_T : T \times T \to \mathbb{R}_+$ be two salient semi-metrics, and let $\mathcal{L} : S \to T$ be a salient mapping. Then $\mathcal{L}^{\text{ext}} : V[S] \to V[T]$ is continuous with respect to the semi-norms π_{d_S} and π_{d_T} if and only if $\mathcal{L} : S \to T$ is uniformly continuous with respect to d_S and d_T. Furthermore, if one of the above conditions holds then*

$$\exists \kappa > 0\, \forall s_1, s_2 \in S : \pi_{d_T}\left(\mathcal{L}^{\text{ext}}([(s_1, s_2)])\right) \leq \kappa \pi_{d_S}\left([(s_1, s_2)]\right).$$

Proof.
Clearly,

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall s_1, s_2 \in S : \pi_{d_S}\left([(s_1, s_2)]\right) < \delta \implies \pi_{d_T}\left(\mathcal{L}^{\text{ext}}(\,[(s_1, s_2)]\,)\right) < \varepsilon$$

is equivalent with

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall s_1, s_2 \in S : d_S\,(s_1, s_2) < \delta \implies d_T\,(\mathcal{L}(s_1), \mathcal{L}(s_2)) < \varepsilon.$$

Furthermore, choose $\varepsilon = 1$ and take δ as indicated above. Then for all $\lambda > 0$:

$$\frac{\delta}{\lambda + \pi_{d_S}([(s_1, s_2)])} \pi_{d_T}\left(\mathcal{L}^{\text{ext}}([(s_1, s_2)])\right) = \pi_{d_T}\left(\mathcal{L}^{\text{ext}}\left(\frac{\delta[(s_1, s_2)]}{\lambda + \pi_{d_S}([(s_1, s_2)])}\right)\right) < 1,$$

which yields the required result. □

3.2.3 Corollary. *Let S and T be two salient spaces, and let $d_S : S \times S \to \mathbb{R}_+$ and $d_T : T \times T \to \mathbb{R}_+$ be two salient semi-metrics. Let $\mathcal{L} : S \to T$ be a salient mapping. Then $\mathcal{L}$ is uniformly continuous with respect to d_S and d_T, if and only*

$$\exists \kappa \in \mathbb{R}_+ \, \forall s_1, s_2 \in S : d_T(\mathcal{L}(s_1), \mathcal{L}(s_2)) \leq \kappa d_S(s_1, s_2).$$

The previous results have the following consequence for the continuity of salient mappings between salient spaces with salient topologies induced by a collection of salient semi-metrics.

3.2.4 Proposition. *Let S_1 and S_2 be two salient spaces, and let D_1 and D_2 be two separating collections of salient semi-metrics on S_1 and S_2, respectively. If the collection D_1 is a directed set (cf. Definition 3.1.7), then the salient mapping $\mathcal{L} : S_1 \to S_2$ is continuous if and only if for every $d_2 \in D_2$ there is $d_1 \in D_1$ such that $\mathcal{L}$ is continuous with respect to d_1 and d_2. The linear mapping $\mathcal{L}^{\text{ext}} : V[S_1] \to V[S_2]$ is continuous with respect to the topologies generated by the collections P_{D_1} and P_{D_2} if and only if $\forall d_2 \in D_2$ there is $d_1 \in D_1$ such that $\mathcal{L}$ is uniformly continuous with respect to d_1 and d_2.*

It is well known that a linear function on a vector space is continuous if and only if it is continuous in 0. Next, we explore the salient space related version of this property.

3.2.5 Proposition. *Let S and T be two salient spaces, let $\varphi_S : S \to \mathbb{R}_+$ and $\varphi_T : T \to \mathbb{R}_+$ be two semi-norms, and let $\mathcal{L} : S \to T$ be a salient mapping. If*

$$\exists \kappa \geq 0 \, \forall s \in S : \varphi_T(\mathcal{L}(s)) \leq \kappa \varphi_S(s),$$

then $\mathcal{L}$ is uniformly continuous with respect to the salient semi-metrics d_{φ_S} and d_{φ_T}.

Proof.
Suppose $\exists \kappa \geq 0 \, \forall s \in S : \varphi_T(\mathcal{L}(s)) \leq \kappa \varphi_S(s)$. Let $s_1, s_2 \in S$. Then

$$\begin{aligned}
& d_{\varphi_T}(\mathcal{L}(s_1), \mathcal{L}(s_2)) \\
&= \inf\{\varphi_T(\mathcal{L}(q)) + \varphi_T(\mathcal{L}(r)) \mid q, r \in S \text{ with } s_1 + q = s_2 + r\} \\
&\leq \kappa \inf\{\varphi_S(q) + \varphi_S(r) \mid q, r \in S \text{ with } s_1 + q = s_2 + r\} \\
&= \kappa d_{\varphi_S}(s_1, s_2).
\end{aligned}$$

□

Contrary to the proposition above, the next result gives a condition which is necessary and sufficient for uniform continuity of a salient mapping $\mathcal{L} : S \to T$.

3.2.6 Proposition. *Let S and T be two salient spaces, let $\varphi_S : S \to \mathbb{R}_+$ and $\varphi_T : T \to \mathbb{R}_+$ be two semi-norms, and let $\mathcal{L} : S \to T$ be a salient mapping. Then the following four statements are equivalent.*

a) *$\mathcal{L}$ is uniformly continuous with respect to d_{φ_S} and d_{φ_T},*

b) *$\mathcal{L}$ is continuous in 0 with respect to d_{φ_S} and d_{φ_T},*

c) $\exists \kappa \geq 0 \, \forall s \in S : \psi_T(\mathcal{L}(s)) \leq \kappa \psi_S(s)$,

d) $\exists \kappa \geq 0 \, \forall s \in S : \psi_T(\mathcal{L}(s)) \leq \kappa \varphi_S(s)$,

where ψ_S and ψ_T denote $\psi_{d_{\varphi_S}}$ and $\psi_{d_{\varphi_T}}$, respectively.

Proof.
"a $\Longrightarrow$ b" is obvious.
"b $\Longrightarrow$ c": Suppose $\mathcal{L}$ is continuous in 0, then

$$\exists \delta > 0 \, \forall s \in S : \; d_{\varphi_S}(s, 0) < \delta \implies d_{\varphi_T}(\mathcal{L}(s), 0) < 1.$$

Let $s \in S$, and let $\lambda > 0$. Then

$$d_{\varphi_S}\left(\frac{\delta s}{\lambda + \psi_S(s)}, 0\right) = \frac{\delta}{\lambda + \psi_S(s)} d_{\varphi_S}(s, 0) = \frac{\delta \psi_S(s)}{\lambda + \psi_S(s)} < \delta.$$

Hence, for all $\lambda > 0$, we find

$$1 > d_{\varphi_T}\left(\mathcal{L}\left(\frac{\delta s}{\lambda + \psi_S(s)}\right), 0\right) = \frac{\delta}{\lambda + \psi_S(s)} \psi_T(\mathcal{L}(s)).$$

This implies

$$\psi_T(\mathcal{L}(s)) \leq \lim_{\lambda \downarrow 0} \frac{\lambda + \psi_S(s)}{\delta} = \frac{1}{\delta} \psi_S(s).$$

"c $\Longrightarrow$ d" is implied by $\psi_S \leq \varphi_S$.
"d $\Longrightarrow$ a": Let $s_1, s_2 \in S$, and let $q, r \in S$ satisfy $s_1 + q = s_2 + r$. Then

$$\begin{aligned} & d_{\varphi_T}(\mathcal{L}(s_1), \mathcal{L}(s_2)) = d_{\varphi_T}(\mathcal{L}(q), \mathcal{L}(r)) \\ & \leq d_{\varphi_T}(\mathcal{L}(q), 0) + d_{\varphi_T}(\mathcal{L}(r), 0) = \psi_T(\mathcal{L}(q)) + \psi_T(\mathcal{L}(r)) \\ & \leq \kappa(\varphi_S(q) + \varphi_S(r)). \end{aligned}$$

Hence, $d_{\varphi_T}(\mathcal{L}(s_1), \mathcal{L}(s_2)) \leq \kappa \inf\{\varphi_S(q) + \varphi_S(r) \mid q, r \in S \text{ with } s_1 + q = s_2 + r\} = \kappa d_{\varphi_S}(s_1, s_2)$. □

The following statements are a direct consequence of Proposition 3.2.6.

3.2.7 Corollary. *Let S and T be two salient spaces, with a salient topology generated by the salient metrics d_{φ_S} and d_{φ_T}, respectively, where $\varphi_S : S \to \mathbb{R}_+$ and $\varphi_T : T \to \mathbb{R}_+$ are semi-norms. The following two statements are equivalent.*

- $\mathcal{L} : S \to T$ *is uniformly continuous,*
- $\mathcal{L} : S \to T$ *is continuous in* 0.

3.2.8 Corollary. *Let S and T be two salient spaces, let $\varphi_S : S \to \mathbb{R}_+$ and $\varphi_T : T \to \mathbb{R}_+$ be two semi-norms. Assume $\psi_T = \varphi_T$. Then a salient mapping $\mathcal{L} : S \to T$ is uniformly continuous with respect to d_{φ_S} and d_{φ_T} if and only if*

$$\exists \kappa \geq 0 \; \forall s \in S : \varphi_T \left(\mathcal{L}(s) \right) \leq \kappa \varphi_S \left(s \right).$$

Corollary 3.2.8 implies that, if S and T are two salient spaces with semi-norms $\varphi_S : S \to \mathbb{R}_+$ and $\varphi_T : T \to \mathbb{R}_+$, respectively, and if $\forall t \in T : d_{\varphi_T}(t, 0) = \varphi_T(t)$, then the set of all salient mappings which are continuous with respect to the salient metrics d_{φ_S} and d_{φ_T}, is a salient space.

By taking $T = \mathbb{R}_+$ with the canonical norm $\varphi_T : \mathbb{R}_+ \to \mathbb{R}_+$, for every $x \in \mathbb{R}_+$, defined by $\varphi_T(x) := x$, we find $\varphi_T = \psi_T$. Hence, we arrive at the following corollary of Proposition 3.2.6.

3.2.9 Corollary. *Let S be a salient space and let $\varphi_S : S \to \mathbb{R}_+$ be a semi-norm on S. Then a salient function $\mathcal{F} : S \to \mathbb{R}_+$ is uniformly continuous, with respect to d_{φ_S} if and only if*

$$\exists \kappa \geq 0 \; \forall s \in S : \mathcal{F}(s) \leq \kappa \varphi_S \left(s \right).$$

3.2.10 Example. Recall from Corollary 3.1.24, that for a salient space S, every $\mathcal{F} \in S^*$ can be regarded as a semi-norm on S, satisfying $\mathcal{F} = \psi_{d_{\mathcal{F}}}$. Hence, every $\mathcal{G} \in S^*$, satisfying $\mathcal{G} \leq_{S^*} \mathcal{F}$, is uniformly continuous with respect to the salient semi-metric $d_{\mathcal{F}}$. Clearly, if $\mathcal{F}$ is an order unit for S^*, then every $\mathcal{G} \in S^*$ is continuous with respect to $d_{\mathcal{F}}$. ◇

Propositions 3.2.4, 3.2.6 and 3.1.23 imply the following corollary.

3.2.11 Corollary. *Consider a non-degenerate salient pairing $\{ S , T ; \mathcal{B} \}$ with the salient topology $\tau(S,T)$ on S. Then the continuity of a salient mapping $\mathcal{F} : S \to \mathbb{R}_+$ is equivalent with each of the following statements*

a) $\exists t \in T : \mathcal{F}$ *is uniformly continuous with respect to* $d_{\mathcal{M}_t}$,

b) $\exists t \in T : \mathcal{F}$ *is continuous in 0 with respect to* $d_{\mathcal{M}_t}$*,*

c) $\exists t \in T \; \exists \kappa \geq 0 \; \forall s \in S : \mathcal{F}(s) \leq \kappa \mathcal{B}(s_0, t)$*,*

Hence, for every $t \in T$*, the salient function and semi-norm* $\mathcal{M}_t : S \to \mathbb{R}_+$ *is continuous with respect to* $\tau(S, T)$*.*

When, in the above example, we choose $T = S^*$, we find the first part of the following lemma.

3.2.12 Lemma. *Let* S *be salient space. Each* $\mathcal{F} \in S^*$ *is continuous with respect to* $\tau(S, S^*)$*. Furthermore,* $\forall \mathcal{F} \in S^* \; \forall s_1, s_2 \in S : |\mathcal{F}(s_1) - \mathcal{F}(s_2)| \leq d_{\mathcal{F}}(s_1, s_2)$*.*

Proof.
For every $t_1, t_2 \in S$, satisfying $s_1 + t_1 = s_2 + t_2$, we find

$$|\mathcal{F}(s_1) - \mathcal{F}(s_2)| = |\mathcal{F}(t_1) - \mathcal{F}(t_2)| \leq \mathcal{F}(t_1) + \mathcal{F}(t_2).$$

Hence, we conclude

$$|\mathcal{F}(s_1) - \mathcal{F}(s_2)| \leq \inf\{\mathcal{F}(t_1) + \mathcal{F}(t_2) \mid t_1, t_2 \in S \text{ with } s_1 + t_1 = s_2 + t_2\} = d_{\mathcal{F}}(s_1, s_2).$$

□

3.2.13 Corollary. *Let* S *be a salient space, let* $A \subset S^*$ *and let* $\mathcal{F}_0 \in S^*$ *with* $\forall \mathcal{F} \in A : \mathcal{F} \leq_S^* \mathcal{F}_0$*. Then, for every* $s_1, s_2 \in S$*:*

$$|\mathcal{F}(s_1) - \mathcal{F}(s_2)| \leq d_{\mathcal{F}_0}(s_1, s_2).$$

3.2.14 Definition (bounded salient mapping with respect to φ_S and φ_T). Let S and T be two salient spaces, let $\varphi_S : S \to \mathbb{R}_+$ and $\varphi_T : T \to \mathbb{R}_+$ be two semi-norms, and let $\varphi_T = \psi_T$. For a salient mapping $\mathcal{L} : S \to T$, define

$$\rho(\mathcal{L}) := \sup\{\varphi_T(\mathcal{L}(s)) \mid s \in S \text{ and } \varphi_S(s) = 1\}.$$

If $\rho(\mathcal{L}) < \infty$, then the salient mapping $\mathcal{L}$ is bounded with respect to φ_S and φ_T.

We remark that in case both $\varphi_S : S \to \mathbb{R}_+$ and $\varphi_T : T \to \mathbb{R}_+$ are norms, then ρ is a norm on the salient space of all salient mappings which are continuous with respect to the salient metrics d_{φ_S} and d_{φ_T}.

Combined with the above results, Lemma 3.2.2, implies the following.

3.2.15 Proposition. *Let S and T be two salient spaces, let $\varphi_S : S \to \mathbb{R}_+$ and $\varphi_T : T \to \mathbb{R}_+$ be two semi-norms, and let $\varphi_T = \psi_T$. Let $\mathcal{L} : S \to T$ be a bounded salient mapping with respect to d_{φ_S} and d_{φ_T}. Then $\rho(L) = \| \mathcal{L}^{\text{ext}} \|$, where $\| \mathcal{L}^{\text{ext}} \|$ is the norm of $\mathcal{L}^{\text{ext}}$, regarded as a bounded linear mapping from the vector space $V[S]$, with semi-norm φ_S, into the vector space $V[T]$, with semi-norm φ_T.*

The following theorem states that the continuity of a linear mapping $\mathcal{L} : V \to W$, between two partially ordered vector spaces, and satisfying $\mathcal{L}(V_+) \subseteq W_+$, is determined by its action on the positive cone V_+.

3.2.16 Theorem. *Let $(V, \leq_V)$ and $(W, \leq_W)$ be partially ordered vector spaces. Let $\varphi_V : V_+ \to \mathbb{R}_+$ and $\varphi_W : W_+ \to \mathbb{R}_+$ be norms which are extended to norms on V and W, by $\pi_{d_{\varphi_V}}$ and $\pi_{d_{\varphi_W}}$. Let $\mathcal{L} : V \to W$ be a linear mapping such that $\mathcal{L}(V_+) \subset W_+$. Then $\mathcal{L}$ is continuous with respect to the extended norms on V and W if and only if*

$$\exists \kappa \geq 0 \; \forall v \in V_+ : \| \mathcal{L}(v) \|_W \leq \kappa \; \| v \|_V \; .$$

Proof.
Use Lemma 3.2.2, Proposition 3.1.32 and Proposition 3.2.6. □

As mentioned directly after the proof of Proposition 3.1.22, in case $\varphi : S \to \mathbb{R}_+$ is a norm on a salient space S, the construction

$$d_\varphi(s_1, s_2) := \inf\{\varphi(q) + \varphi(r) \mid q, r \in S \text{ with } s_1 + q = s_2 + r\}$$

of the salient semi-metric d_φ does not guarantee that d_φ is a metric on S. Indeed, if for certain $s_1, s_2 \in S$ we have $d_\varphi(s_1, s_2) = 0$ we can conclude only that there are sequences $(q_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ in S such that $\lim_{n\to\infty} \varphi(q_n) = \lim_{n\to\infty} \varphi(r_n) = 0$ and $\forall n \in \mathbb{N} : s_1 + q_n = s_2 + r_n$. Considering all $\mathcal{F} \in S^*$, which are continuous with respect to d_φ, we see that $\mathcal{F}(s_1) = \mathcal{F}(s_2)$. This leads us to the following definitions and results.

3.2.17 Definition (T_φ and φ'). Let $\{ S , T ; \mathcal{B} \}$ be a salient pairing, and let $\varphi : S \to \mathbb{R}_+$ be a semi-norm on S. Then the subset T_φ of T is given by

$$T_\varphi := \{t \in T \mid \sup\{\mathcal{B}(s,t) \mid s \in S \text{ and } \varphi(s) \leq 1\} < \infty\},$$

Furthermore, the semi-norm $\varphi' : T_\varphi \to \mathbb{R}_+$ is, for every $t \in T_\varphi$, given by

$$\begin{aligned} \varphi'(t) &:= \sup\{\mathcal{B}(s,t) \mid s \in S \text{ and } \varphi(s) \leq 1\} \\ &= \inf\{\kappa \geq 0 \mid \forall s \in S : \mathcal{B}(s,t) \leq \kappa\varphi(s)\}. \end{aligned}$$

We remark that

$$\forall t \in T_\varphi \ \forall s \in S : \mathcal{B}(s,t) \leq \varphi'(t)\varphi(s) .$$

Clearly, T_φ is the set of all elements t of T for which $\mathcal{M}_t : S \to \mathbb{R}_+$ is (uniformly) d_φ-continuous. Note that $T_\varphi = \{t \in T \mid \varphi'(t) < \infty\}$, that the set T_φ is a salient subspace of T and that φ' is a monotonous semi-norm on T_φ, so $\varphi'(t) = d_{\varphi'}(t, 0)$, for all $t \in T_\varphi$.

3.2.18 Proposition. *Let $\{ S, T; \mathcal{B} \}$ be a salient pairing, let $\varphi : S \to \mathbb{R}_+$ be a semi-norm on S, and let T_φ separate the elements of S. Then d_φ is a salient metric on S, and π_{d_φ} is a norm on $V[S]$.*

Proof.
Let $s_1, s_2 \in S$ and suppose $d_\varphi(s_1, s_2) = 0$. Then, there are sequences $(q_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ in S, such that $\lim_{n\to\infty} \varphi(q_n) = \lim_{n\to\infty} \varphi(r_n) = 0$ and $\forall n \in \mathbb{N} : s_1 + q_n = s_2 + r_n$. For every $t \in T_\varphi$, we find $\lim_{n\to\infty} \mathcal{B}(q_n, t) = \lim_{n\to\infty} \mathcal{B}(r_n, t) = 0$, i.e., $\forall t \in T_\varphi : \mathcal{B}(s_1, t) = \lim_{n\to\infty}(\mathcal{B}(s_1, t) + \mathcal{B}(q_n, t)) = (\mathcal{B}(s_2, t) + \mathcal{B}(r_n, t)) = \mathcal{B}(s_2, t)$. Since T_φ separates S, we conclude $s_1 = s_2$. □

3.2.19 Example. Let S be a salient space for which S^* separates the elements of S. Let $\mathcal{F}_0$ be an order unit of S^* (cf. Corollary 2.2.11). Then the above proposition implies that $S^*_{\mathcal{F}_0} = S^*$ and $d_{\mathcal{F}_0} : S \times S \to \mathbb{R}_+$ is a salient metric on S, satisfying, among other things, $\forall s \in S : d_{\mathcal{F}_0}(s, 0) = \mathcal{F}_0(s)$. Furthermore, $\pi_{d_{\mathcal{F}_0}}$ is a norm on $V[S]$. Hence topology $\tau(S, S^*)$ is induced by the salient semi-metric $d_{\mathcal{F}_0}$. ◇

3.2.20 Proposition. *Let S be a salient space, and let $\varphi : S \to \mathbb{R}_+$ be a semi-norm on S. Endow $S^*_\varphi = \{\mathcal{F} \in S^* \mid \sup\{\mathcal{F}(s) \mid s \in S \text{ and } \varphi(s) \leq 1\} < \infty\}$ with the salient semi-metric d_{φ^*}. Here, the norm φ^* is defined by*

$$\varphi^*(\mathcal{F}) := \sup\{\mathcal{F}(s) \mid s \in S \text{ and } \varphi(s) \leq 1\}.$$

Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in S^*_φ, then there is $\mathcal{F} \in S^*_\varphi$ such that the sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ converges to $\mathcal{F}$, uniformly on the set $\{s \in S \mid \varphi(s) \leq 1\}$.

Proof.
Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in S^*_φ. Let $s \in S$, $n, m \in \mathbb{N}$ and let $\mathcal{G}, \mathcal{H} \in S^*_\varphi$ satisfy $\mathcal{F}_n + \mathcal{H} = \mathcal{F}_m + \mathcal{G}$. Then $\mathcal{F}_n(s) - \mathcal{F}_m(s) = \mathcal{G}(s) - \mathcal{H}(s) \leq (\varphi^*(\mathcal{G}) + \varphi^*(\mathcal{H}))\varphi(s)$ implies $|\mathcal{F}_n(s) - \mathcal{F}_m(s)| \leq (\varphi^*(\mathcal{G}) + \varphi^*(\mathcal{H}))\varphi(s)$. Hence, by the definition of d_{φ^*}, we find

$$\forall s \in S \ \forall n, m \in \mathbb{N} : |\mathcal{F}_n(s) - \mathcal{F}_m(s)| \leq d_{\varphi^*}(\mathcal{F}_n, \mathcal{F}_m)\, \varphi(s) ,$$

which implies

$$\forall s \in S, \varphi(s) \leq 1 \; \forall n, m \in \mathbb{N} : |\mathcal{F}_n(s) - \mathcal{F}_m(s)| \leq d_{\varphi^*}(\mathcal{F}_n, \mathcal{F}_m),$$

i.e, $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence on the set $\{s \in S \mid \varphi(s) \leq 1\}$. Since for every $s \in S$, the sequence $(\mathcal{F}(s))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, we can define $\mathcal{F} \in S^*$, for every $s \in S$, by $\mathcal{F}(s) := \lim_{n\to\infty} \mathcal{F}_n(s)$. Let $\varepsilon > 0$. We have to prove

$$\exists n_1 \in \mathbb{N} \; \forall n > n_1 \; \forall s \in S, \varphi(s) \leq 1 : |\mathcal{F}_n(s) - \mathcal{F}(s)| < \varepsilon.$$

Choose $n_0 \in \mathbb{N}$ such that

$$\forall n, m > n_0 \; \forall s \in S, \varphi(s) \geq 1 : |\mathcal{F}_n(s) - \mathcal{F}_m(s)| < \frac{\varepsilon}{2}.$$

Let $n > n_0$ and let $s \in S$ satisfy $\varphi(s) \leq 1$. Then

$$\exists n_1 \in \mathbb{N}, n_1 \geq n_0 \; \forall m > n_0 : |\mathcal{F}_m(s) - \mathcal{F}(s)| < \frac{\varepsilon}{2}.$$

Let $m > n_0$, then we find

$$|\mathcal{F}_n(s) - \mathcal{F}(s)| \leq |\mathcal{F}_n(s) - \mathcal{F}_m(s)| + |\mathcal{F}_m(s) - \mathcal{F}(s)| < \varepsilon.$$

Now, there is only left to prove that $\mathcal{F} \in S^*_\varphi$. This is implied by

$$\forall n \in \mathbb{N} \; \forall s \in S, \varphi(s) \leq 1 : \mathcal{F}(s) \leq |\mathcal{F}(s) - \mathcal{F}_n(s)| + \mathcal{F}_n(s)$$

and the fact that $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a sequence in S^*_φ. □

In the following two statements, we investigate the connection between reflexivity and closedness for finite-dimensional vector spaces. Further discussion regarding salient spaces of finite linear dimension can be found in Section 3.3.

3.2.21 Lemma. *Let S be a finite-dimensional salient space. Then S^* is complete with respect to topology $\tau(S^*, S)$.*

Proof.
Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence in S^*, which is Cauchy with respect to $\tau(S^*, S)$. Let $s \in S$, let $n, m \in \mathbb{N}$, and let $\mathcal{G}, \mathcal{H} \in S^*$ satisfy $\mathcal{F}_n + \mathcal{H} = \mathcal{F}_m + \mathcal{G}$. Then, $\mathcal{F}_n(s) - \mathcal{F}_m(s) = \mathcal{G}(s) - \mathcal{H}(s) \leq \mathcal{G}(s) + \mathcal{H}(s)$, implies $|\mathcal{F}_n(s) - \mathcal{F}_m(s)| \leq \mathcal{G}(s) + \mathcal{H}(s)$. Hence, $\forall s \in S \; \forall n, m \in \mathbb{N} : |\mathcal{F}_n(s) - \mathcal{F}_m(s)| \leq d_s(\mathcal{F}_n, \mathcal{F}_m)$, which implies that for every $s \in S$, the sequence $(\mathcal{F}_n(s))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}_+$. Let $\mathcal{F} : S \to \mathbb{R}_+$, be the element of S^* which, for every $s \in S$, satisfies $\mathcal{F}(s) = \lim_{n\to\infty} \mathcal{F}(s_n)$. Since S is finite-dimensional, $V[S^*]$ is finite-dimensional, and we can identify $V[S^{**}]$ and $(V[S^*])^*$.

Since $V[S^*]$ is complete with respect to $\tau(V[S^*], S)$, there is $[(\mathcal{F}_1, \mathcal{F}_2)] \in V[S^*]$ such that

$$\forall s \in S : \lim_{n\to\infty} \mathcal{F}_n(s) - \mathcal{F}_1(s) + \mathcal{F}_2(s) = 0.$$

Since $\lim_{n\to\infty} \mathcal{F}_n(s) - \mathcal{F}_1(s) + \mathcal{F}_2(s) = \mathcal{F}(s) - \mathcal{F}_1(s) + \mathcal{F}_2(s)$, and since S separates the elements of S^*, this implies that $[(\mathcal{F}_1, \mathcal{F}_2)] = [(\mathcal{F}, 0)]$, i.e., $\lim_{n\to\infty} \mathcal{F}_n = \mathcal{F}$. □

3.2.22 Corollary. *Let S be a finite-dimensional, reflexive salient space. Then S is complete with respect to topology $\tau(S, S^*)$.*

3.2.23 Theorem. *Let S be a salient space with finite linear dimension and for which S^* separates S. Then the following three statements are equivalent.*

a) *S is reflexive,*

b) *S is complete with respect to $\tau(S, S^*)$,*

c) *$V_+[S]$ is closed in $V[S]$ with respect to $\tau(V[S], S^*)$.*

Proof.
“ a $\Longrightarrow$ b ” This is Corollary 3.2.22,
“ b $\Longrightarrow$ c ” Due to the homeomorphism between S and $V_+[S]$, the pointed convex cone $V_+[S]$ is complete and since $V[S]$ is complete, $V_+[S]$ is closed in $V[S]$.
“ c $\Longrightarrow$ a ” Since S has a finite linear dimension, we find that $V[S]$, $(V[S])^*$ and $(V[S])^{**}$ have the same linear dimension as S. For convenience of notation, we identify $V[S]$ and $(V[S])^{**}$. Endow $V[S]$ with the topology $\tau(V[S], S^*)$. Note that the image $V_+[S^{**}]$ of S^{**} under the salient isomorphism $\mathcal{J}_{S^{**}}$ is a pointed convex cone in $(V[S])^{**} = V[S]$, such that $V_+[S] \subset V_+[S^{**}]$. Suppose S is not reflexive, i.e., suppose

$$\exists x \in S^{**}\ \exists \mathcal{F} \in S^* : x(\mathcal{F}) \neq \mathcal{F}(s).$$

Define $[(t_1, t_2)] := \mathcal{J}_{S^{**}}(x)$. Since $V_+[S]$ is convex and closed in $V[S]$ and since $[(t_1, t_2)] \notin V_+[S]$, the Strong Separation Theorem of Minkowski ([24, page 59]) implies that there is $f_0 \in (V[S])^*$ such that

$$f_0([(t_1, t_2)]) < 0 \text{ and } \forall [(s_1, s_2)] \in V_+[S] : f_0([(s_1, s_2)]) \geq 0.$$

Define $\mathcal{F}_0 \in S^*$, for every $s \in S$, by $\mathcal{F}_0(s) := f_0([(s, 0)])$, then $x \in S^{**}$ implies $x(\mathcal{F}_0) \geq 0$. Since $x(\mathcal{F}_0) = f_0([(t_1, t_2)]) < 0$, we arrive at a contradiction, and we conclude that S is reflexive. □

3.3 Finite-dimensional salient spaces

In this section, we derive some statements concerning salient spaces which are reflexive or have a finite linear dimension.

Consider a non-degenerate salient pairing $\{ S , T ; \mathcal{B} \}$. By Lemma 2.3.18, we find that in this case $V[S]$ separates the elements of $V[T]$ and, conversely, $V[T]$ separates $V[S]$. If the salient space S has a finite linear dimension, then $V[S]$ is also finite-dimensional and we conclude that the dimension of $V[T]$ equals the dimension of $V[S]$, i.e., $V[S]$ and $V[T]$ are isomorphic. Conversely, if S is a salient space with finite linear dimension, and if $\{ S , T ; \mathcal{B} \}$ is a salient pairing, such that $V[S]$ and $V[T]$ are isomorphic (finite-dimensional) vector spaces, then Lemma 2.3.18 implies that the pairing is non-degenerate. Summarising, we find the following proposition.

3.3.1 Proposition. *Let $\{ S , T ; \mathcal{B} \}$ be a salient pairing of which the salient space S has a finite linear dimension. Then*

$$\{ S , T ; \mathcal{B} \} \text{ is non-degenerate} \iff V[S] \text{ is isomorphic with } V[T].$$

Next, we consider the special case where $T = S^*$. So, let S be a salient space such that S^* separates S. Then the canonical pairing $\{ S , S^* ; \mathcal{B}_{\mathrm{can}} \}$ is non-degenerate.

For a finite-dimensional salient space S, the previous proposition states that S^* separates S if and only if the vector spaces $V[S]$ and $V[S^*]$ are isomorphic.

We are interested in the situation where $V[S^*] = (V[S])^*$, and we will see that the condition S^* separates S is necessary and sufficient.

3.3.2 Proposition. *Let S be a finite-dimensional salient space. Then*

$$S^* \text{ separates the elements of } S \iff V[S^*] = (V[S])^*.$$

Proof.
For every finite-dimensional salient space S, the vector space $V[S]$ is isomorphic with $(V[S])^*$. Proposition 3.3.1 completes the proof. □

3.3.3 Example. For the salient space $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\} \cup \{(0,0)\}$ of Examples 2.1.5 and 2.3.24, the vector space $V[S^*]$ is not isomorphic with $(V[S])^*$. Furthermore, S^* does not separate the elements of S. ◇

If a finite-dimensional salient space S is reflexive, then

$$\dim((V[S])^{**}) = \dim(V[S]) = \text{lin}\dim(S) = \text{lin}\dim(S^{**}) = \dim(V[S^{**}]).$$

As a consequence, we find $\dim(V[S^*]) = \dim(V[S])$, i.e., $\text{lin}\dim(S^*) = \text{lin}\dim(S)$, and we conclude that S^* separates S.

If S is a salient space for which S^* separates S, and which satisfies $V[S^*] = (V[S])^*$, then every linear function on $V[S]$ is continuous with respect to the locally convex topology $\tau(V[S], S^*)$. So, if S^* has a countable order set, Proposition 3.3.5 below implies that $V[S]$, and therefore also S, is finite-dimensional. Summarising, we find the following proposition.

3.3.4 Proposition. *Let S be a salient space which has a countable order set and for which S^* separates S. Then*

$$V[S^*] = (V[S])^* \iff S \text{ is finite-dimensional.}$$

3.3.5 Proposition. *Let V be an infinite-dimensional topological vector space, for which the topology is generated by a countable collection $\{p_n \mid n \in \mathbb{N}\}$ of semi-norms, satisfying $\{x \in V \mid \forall n \in \mathbb{N} : p_n(x) = 0\} = \{0\}$. Then there is an unbounded linear function $\mathcal{F} : V \to \mathbb{R}$.*

Proof.
Without loss of generality (cf. Remark 3.1.11), we may assume

$$\forall n \in \mathbb{N} : p_n \leq p_{n+1}.$$

Let $H = \{h_i \mid i \in I\}$ be a maximal linearly independent subset, or Hamel basis (cf. [7]), in V. For every $x \in V$, let the function $\mathcal{L}_x : I \to \mathbb{R}$ be defined such that the set $\{i \in I \mid \mathcal{L}_x(i) \neq 0\}$ is finite and $x = \sum_{i \in I} \mathcal{L}_x(i) h_i$. Let $\tilde{H} = \{h_{i_n} \mid n \in \mathbb{N}\}$ be a countable subset of H satisfying $\forall n \in \mathbb{N} : p_n(h_{i_n}) \neq 0$. Then $\forall n, m \in \mathbb{N} : \mathcal{L}_{h_{i_m}}(i_n) = \delta_{nm}$. Define the linear function $\mathcal{F} : V \to \mathbb{R}$, for every $x \in V$, by

$$\mathcal{F}(x) := \sum_{n \in \mathbb{N}} n \mathcal{L}_x(i_n) p_n(h_{i_n}).$$

We shall prove that $\mathcal{F}$ is unbounded, i.e., we prove that

$$\forall n \in \mathbb{N} \, \forall \alpha > 0 \, \exists x \in V : |\mathcal{F}(x)| > \alpha p_n(x).$$

Let $n \in \mathbb{N}$ and let $\alpha > 0$. Take $m \in \mathbb{N}$, such that $m > \max\{n, \alpha\}$. Then

$$|\mathcal{F}(h_{i_m})| = m p_m(h_{i_m}) \geq m p_n(h_{i_m}) > \alpha p_n(h_{i_m}).$$

□

3.3.6 Remark. The condition in the above proposition that the collection $\{p_n \mid n \in \mathbb{N}\}$ of semi-norms is countable is necessary. Indeed, consider the set $\mathbb{R}^{\mathbb{N}}$ of all real sequences, and define the subset $\Gamma \subset \mathbb{R}^{\mathbb{N}}$, consisting of all finite sequences, for every $x = (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ by

$$x \in \Gamma \quad :\Longleftrightarrow \quad \exists N \in \mathbb{N}\ \forall n > N : x_n = 0.$$

Every $a = (a_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ defines a semi-norm π_a on Γ in the following way:

$$\pi_a(x) := |\sum_{n=1}^{\infty} a_n x_n|.$$

Clearly, the set $\{\pi_a \mid a \in \mathbb{R}^{\mathbb{N}}\}$ is uncountable. Furthermore, every linear functional on Γ is continuous. Indeed, let $\mathcal{F} : \Gamma \to \mathbb{R}$ be a linear functional, then there is $f \in \mathbb{R}^{\mathbb{N}}$ such that $\forall x \in \Gamma : \mathcal{F}(x) = \pi_f(x)$. $\diamond$

Next, we present a salient space related characterisation of $\mathrm{int}(S)$ (cf. Definition 2.1.28) for finite-dimensional salient spaces.

3.3.7 Lemma. *Let S be a finite-dimensional salient space and let $s \in S$. Then $s \in \mathrm{int}(S)$ if and only if $\forall \mathcal{G} \in S^* \setminus \{0\} : \mathcal{G}(s) > 0$.*

Proof.
Let $s \in \mathrm{int}(S)$ and let $\mathcal{G} \in S^* \setminus \{0\}$. There is $s_1 \in S$ with $\mathcal{G}(s_1) > 0$. Since $s \in \mathrm{int}(S)$, there is $\varepsilon > 0$ and $s_2 \in S$ such that $s = \varepsilon s_1 + s_2$. We conclude $\mathcal{G}(s) > 0$. For the converse, suppose $s \in \mathrm{bd}(S) \setminus \{0\}$. Since S is a convex set, $\mathrm{int}(S)$ is a convex set. We remark that Corollary's 2.2.11 and 2.2.12 guarantee that $\mathrm{int}(S) \neq \emptyset$. By the Weak Separation Theorem of Minkowski ([24, p.60])

$$\exists\, \mathcal{F} \in (V[S])^* \setminus \{0\}\ \exists\, \alpha \in \mathbb{R} : \begin{cases} \forall\, \lambda \geq 0 : & \mathcal{F}(\lambda s) \leq \alpha \\ \forall\, u_0 \in \mathrm{int}(S) : & \mathcal{F}(u_0) \geq \alpha. \end{cases}$$

Choosing λ equal to 0, and choosing a sequence in $\mathrm{int}(S)$ converging to 0, we find $\alpha = 0$. As a consequence $\mathcal{F} \in S^* \setminus \{0\}$. By subsequently choosing λ equal to 1, we find $\mathcal{F}(s) \leq 0$. $\square$

Note, that as a consequence of this lemma, we find $(\mathrm{int}(S))^* = S^*$.

If S is a finite-dimensional salient space, for which S^* separates S, then the set $\mathrm{int}(S)$ coincides with the $\tau(V[S], S^*)$-interior of S. Hence, every element of the $\tau(V[S], S^*)$-interior of S is an order unit, with respect to the partial order relation $\leq_S$ on S. Furthermore, every element $\mathcal{F} \in S^*$ is continuous with respect to topology $\tau(S, S^*)$.

The following corollary summarises some statements which we will need in the coming chapters of this thesis.

3.3.8 Corollary. *Let S be a finite-dimensional, reflexive salient space. Then the following statements hold.*

a) *Let $\mathcal{F}_0 \in \text{int}(S^*)$ and let $(s_n)_{n\in\mathbb{N}}$ be a sequence in S. Then $(s_n)_{n\in\mathbb{N}}$ converges to 0 with respect to the topology $\tau(S,S^*)$ if and only if $\lim_{n\to\infty} \mathcal{F}_0(s_n) = 0$.*

b) *Let A be a subset of S and let $\mathcal{F}_0 \in \text{int}(S^*)$. Then A is $\tau(S,S^*)$-bounded if and only if the set $\{\mathcal{F}_0(a) \mid a \in A\}$ is bounded.*

c) *For all $\mathcal{F}_0 \in \text{int}(S^*)$, the sets $\{s \in S \mid \mathcal{F}_0(s) \leq 1\}$ and $\{s \in S \mid \mathcal{F}_0(s) = 1\}$ are $\tau(S,S^*)$-compact.*

3.3.9 Lemma. *Let S be a salient space for which S^* separates S and $\text{int}(S^*) \neq \emptyset$, let A be a subset of S, and let $u_0 \in \text{int}(S)$. Then A is $\tau(S,S^*)$-bounded if $\exists \lambda \geq 0 : A \subset \{s \in S \mid s \leq_S \lambda u_0\}$. If, in addition, S is reflexive and finite-dimensional, then A is $\tau(S,S^*)$-bounded if and only if $A \subset \{s \in S \mid s \leq_S \lambda u_0\}$, for some $\lambda \geq 0$.*

Proof.
Suppose $\exists \lambda \geq 0\ \forall a \in A : a \leq_S \lambda u_0$. Let $\mathcal{F}_0 \in \text{int}(S^*)$, then $\forall a \in A : \mathcal{F}_0(a) \leq \lambda \mathcal{F}_0(u_0)$, hence A is bounded.
Now, suppose $\forall \lambda \geq 0\ \exists a \in A : \neg(a \leq_S \lambda u_0)$, i.e., $\forall \lambda \geq 0\ \exists a \in A\ \exists \mathcal{F} \in \{\mathcal{G} \in S^* \mid \mathcal{G}(u_0) = 1\} : \mathcal{F}(a) > \lambda \mathcal{F}(u_0)$. Then, for every $n \in \mathbb{N}$ there is $a_n \in A$ and $\mathcal{F}_n \in \{\mathcal{G} \in S^* \mid \mathcal{G}(u_0) = 1\}$ such that $\frac{1}{n}\mathcal{F}_n(a_n) > \mathcal{F}_n(u_0)$. To prove the lemma, we show that the sequence $(a_n)_{n\in\mathbb{N}}$ is unbounded. Suppose $(a_n)_{n\in\mathbb{N}}$ is bounded. Since S is assumed to be finite-dimensional and reflexive, we may assume that the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent with limit $a \in S$, and the sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is convergent with limit $\mathcal{F} \in \{\mathcal{G} \in S^* \mid \mathcal{G}(u_0) = 1\}$. This implies $0 = \lim_{n\to\infty} \frac{1}{n}\mathcal{F}_n(a_n) \geq 1$. We conclude that A is unbounded. □

3.3.10 Theorem. *Let S be a finite-dimensional reflexive salient space. Then S has a salient basis.*

In the proof of this theorem, we use the following well known theorem.

3.3.11 Krein-Milman Theorem *(cf. [24, p.191])*
Each non-empty compact convex set K in a finite-dimensional vector space is the convex hull of its set of extreme points, i.e., $K = co(ext(K))$.

Proof of Theorem 3.3.10.
Let $\mathcal{F}_0 \in \text{int}(S^*)$ and define the set $K := \{s \in S \mid \mathcal{F}_0(s) = 1\}$. By Corollary 3.3.8.c, the set K is compact. Since K is also non-empty and convex, the Krein-Milman Theorem implies that $\text{co}(\text{ext}(K)) = K$. We shall prove that $E := \{\text{ray}(e) \mid e \in \text{ext}(K)\}$ is a saliently independent set.

Claim: For every e in $\text{ext}(K)$ the set $\text{ray}(e)$ is extreme in S.
Proof: Let $\mu \geq 0$. Take $\mu e = \tau s_1 + (1-\tau)s_2$ for some $s_1, s_2 \in S$ and $\tau \in (0,1)$. If $\mu = 0$ then $s_1 = s_2 = 0$ because 0 is an extreme point of S. So assume $\mu > 0$. If $s_1 = 0$ or $s_2 = 0$ there is nothing to prove, so we assume $s_1 \neq 0 \neq s_2$. Now, $e = \tau\frac{s_1}{\mu} + (1-\tau)\frac{s_2}{\mu} = \tau t_1 + (1-\tau)t_2$ where $t_1, t_2 \in S$. We shall prove that t_1 and t_2 (and therefore also s_1 and s_2) are elements of $\text{ray}(e)$.

Since $1 = \mathcal{F}_0(e) = \tau\mathcal{F}_0(t_1) + (1-\tau)\mathcal{F}_0(t_2)$, we can write $e = \tau\mathcal{F}_0(t_1)\frac{t_1}{\mathcal{F}_0(t_1)} + (1-\tau)\mathcal{F}_0(t_2)\frac{t_2}{\mathcal{F}_0(t_2)}$ which is a convex combination of $\frac{t_1}{\mathcal{F}_0(t_1)}$ and $\frac{t_2}{\mathcal{F}_0(t_2)}$, both elements of K. Since e is an extreme point of K, this implies that $t_1/F_0(t_1) = t_2/F_0(t_2) = e$. We conclude that t_1 and t_2 (and therefore also s_1 and s_2) are elements of $ray(e)$.

Conversely, by Proposition 2.4.2, every extreme ray R of S corresponds with the extreme point $r \in \text{ext}(K)$, where $\{r\} = E \cap K$. By Corollary 2.4.8 the set $\text{ray}(\text{ext}(K))$ is saliently independent. Since $\text{sal}(K) = \text{sal}(\text{ext}(K)) = S$, we conclude that $\text{ext}(K)$ is a salient basis of S. □

Let S be a finite-dimensional, reflexive salient space and let $s_0 \in \text{int}(S)$. Then by Corollary 3.3.8.c, the set $L := \{\mathcal{F} \in S^* \mid \mathcal{F}(s_0) = 1\}$ is $\tau(S, S^*)$-compact. When we define $\mathcal{U}_{s_0} : S \to \mathbb{R}_+$ and $\mathcal{L}_{s_0} : S \to \mathbb{R}_+$ by

$$\begin{aligned}\mathcal{U}_{s_0}(s) &:= \max\{\mathcal{F}(s) \mid \mathcal{F} \in L\}\\ \mathcal{L}_{s_0}(s) &:= \min\{\mathcal{F}(s) \mid \mathcal{F} \in L\},\end{aligned}$$

then $\mathcal{L}_{s_0}(s) \leq \mathcal{F}(s) \leq \mathcal{U}_{s_0}(s)$ for all $\mathcal{F} \in L$ and $s \in S$. Clearly, $\mathcal{L}_{s_0}(s) > 0$ if $s \in \text{int}(S)$. Note, that this also proves that every $s_0 \in \text{int}(S)$ is an order unit.

3.3.12 Lemma. *Let S be a finite-dimensional, reflexive salient space, and let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\text{int}(S)$, with limit $s_0 \in \text{int}(S)$. Then, the functions $\mathcal{U}_{s_0} : S \to \mathbb{R}_+$ and $\mathcal{L}_{s_0} : S \to \mathbb{R}_+$ satisfy*

$$\forall n \in \mathbb{N} : \mathcal{L}_{s_0}(s_n)s_0 \leq_S s_n \leq_S \mathcal{U}_{s_0}(s_n)s_0 \text{ and } \lim_{n\to\infty} \mathcal{L}_{s_0}(s_n) = \lim_{n\to\infty} \mathcal{U}_{s_0}(s_n) = 1.$$

Proof.
Using the definition of L, $\mathcal{U}_{s_0}$ and $\mathcal{L}_{s_0}$, given above, let $\mathcal{F} \in L$ satisfy $\mathcal{F}(s_0) = \mathcal{U}_{s_0}(s_0) = 1$ and, similarly, for all $n \in \mathbb{N}$, let $\mathcal{F}_n \in L$ satisfy $\mathcal{F}_n(s_n) = \mathcal{U}_{s_0}(s_n)$. Since, for all $n \in \mathbb{N} : \mathcal{U}_{s_0}(s_n) \geq \mathcal{F}(s_n)$, we find that $\liminf_{n\to\infty} \mathcal{U}_{s_0}(s_n) \geq \mathcal{F}(s_0) = 1$. Let $(s_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(s_n)_{n\in\mathbb{N}}$, satisfying $\limsup_{n\to\infty} \mathcal{U}_{s_0}(s_n) = \lim_{k\to\infty}\mathcal{U}_{s_0}(s_{nk})$. The sequence $(\mathcal{F}_{nk})_{k\in\mathbb{N}}$ lies in the compact set L, so $(\mathcal{F}_{nk})_{k\in\mathbb{N}}$ can be assumed convergent with limit $\mathcal{G} \in L$. Now, we find

$$\limsup_{n\to\infty} \mathcal{U}_{s_0}(s_n) = \lim_{k\to\infty} \mathcal{U}_{s_0}(s_{nk}) = \lim_{k\to\infty} \mathcal{F}_{nk}(s_{nk}) = \mathcal{G}(s_0) = 1 \leq \liminf_{n\to\infty} \mathcal{U}_{s_0}(s_n).$$

A similar argument can be used to prove $\lim_{n\to\infty} \mathcal{L}_{s_0}(s_n) = 1$. □

3.3.13 Schauder-Tychonoff Fixed-Point Theorem *([11, Theorem V.10.5])* *Let K be a non-empty compact convex subset of a locally convex linear topological space and let $\mathcal{F} : K \to K$ be a continuous mapping. Then there exists $x \in K$ with $x = \mathcal{F}(x)$.*

This theorem is a generalisation of Brouwer's Fixed Point Theorem.

3.3.14 Brouwer's Fixed-Point Theorem *([7, Theorem V.9.1])* *Let K be a non-empty compact convex subset of a finite-dimensional normed vector space X and let $\mathcal{F} : K \to K$ be a continuous mapping, then there exists $x \in K$ such that $\mathcal{F}(x) = x$, i.e., $\mathcal{F}$ has a fixed point in K.*

Brouwer's Fixed Point Theorem has the following consequence for continuous functions on a finite-dimensional salient space.

3.3.15 Theorem. *Let S be a finite-dimensional, reflexive salient space. Let $\mathcal{H} : S \setminus \{0\} \to S$ be a continuous function with respect to $\tau(S, S^*)$. Then there exists an $s \in S \setminus \{0\}$ such that $\mathcal{H}(s) = \alpha s$ for some $\alpha \geq 0$. In fact, for all $\mathcal{F}_0 \in \mathrm{int}(S^*)$ there is $s \in S$ such that $\mathcal{H}(s) = \mathcal{F}_0(\mathcal{H}(s))s$.*

Proof.
Let $\mathcal{F}_0 \in \mathrm{int}(S^*)$. The set $L := \{s \in S \mid \mathcal{F}_0(s) = 1\}$ is non-empty, convex, and compact by Corollary 3.3.8.c. Define the function $\mathcal{H}_0 : L \to L$ by

$$\mathcal{H}_0(s) := \frac{s + \mathcal{H}(s)}{1 + \mathcal{F}_0(\mathcal{H}(s))}.$$

Then $\mathcal{H}_0$ is a continuous function. By the preceding theorem the function $\mathcal{H}_0$ has a fixed point s in L, so $s = \mathcal{H}_0(s) = \frac{s+\mathcal{H}(s)}{1+\mathcal{F}_0(\mathcal{H}(s))}$. □

Chapter 4

Models and Theorems

Introduction

In this chapter, we present models of pure exchange economies, with and without price rigidities, and of private ownership economies. These models are extensions of, or new approaches to the neoclassical models as introduced by Arrow and Debreu and the model concerning price rigidities and rationing (cf. Chapter 1 for a short description). We recall that these neoclassical models have the Euclidean structure of $\mathbb{R}^{k_0}$, $k_0 \in \mathbb{N}$, as a mathematical basis. In particular, the consumption set of each agent is modelled by (a subset of) the positive orthant $\mathbb{R}_+^{k_0}$ of $\mathbb{R}^{k_0}$, thus implying that every commodity can be considered separately.

We will show that the notion of salient space, as introduced in Chapter 2 is a more natural concept to model the consumption set of agents. The use of salient spaces instead of Euclidean spaces results in a more general model of a pure exchange economy; this approach allows for links between commodities, and induces a more natural way to order the set of all exchangeable objects in an economy. As a consequence, the mathematical assumption that preferences are monotonous, reflects the economic intuition that agents prefer more over less, is a better way. Furthermore, the equilibrium existence results, stated at the end of this chapter, are proved without the requirement that this ordering has a lattice structure. Recall that the set $\mathbb{R}^{k_0}$, which is used by Arrow and Debreu to model the consumption set of each agent, has a lattice structure.

We model situations in which the concept of commodity does not exist, or situations in which commodities do not occur separately. However, our model can be used also for the neoclassical situation of separate commodities, or for a combination of these situations.
Since, for our models, commodities do not need to occur separately, this approach can be used to model a situation in which links between commodities occur. For

example, apart from the neoclassical situation of a pure exchange economy in which separate commodities occur, our model of a pure exchange economy is able to capture the non-neoclassical situation in which there is trade in fixed combinations, or packages, of commodities without affecting the natural order relation of these packages (cf. Example 1.3.1 and Example 4.1.1). Moreover, this approach is able to capture a model of an exchange economy in which the agents value specific characteristics of commodities instead of the commodities themselves (cf. Example 4.1.2 and the work of Lancaster, [21]).

In case of price rigidities (cf. Section 1.2.3), the concept of equilibrium is redefined by imposing additional constraints on excess supply and excess demand for the agents. As we described shortly in Section 1.2.3, Drèze introduces these restrictions for every commodity, separately. This approach is not applicable to our case where the concept of commodity is not present. In Section 4.4, we will introduce a way of rationing supply and demand that does not depend on the existence of separate markets.

In Section 1.2.2, we have seen that in the Arrow-Debreu model of a private ownership economy, a production plan is modelled by an element of the vector space $\mathbb{R}^{k_0}$. As an example, we saw that the vector $(-1, 2, -3, 1) \in \mathbb{R}^4$ represents the production plan where two units of commodity 2 and one unit of commodity 1 are produced from one unit of commodity 1 and three units of commodity 3. In our models concerning production, we assume that the salient space C, which represents the set of all objects of trade, is a sum of two salient subspaces C_{prod} and C_{cons}. Each element $x \in C$ is a unique concatenation $(x^{\text{prod}}, x^{\text{cons}})$ of a production part $x^{\text{prod}} \in C_{\text{prod}}$ and a consumption part $x^{\text{cons}} \in C_{\text{cons}}$. Only elements of C_{prod} can be used as input for a production process whereas the output is always an element of C_{cons}. Thus, $x = (x^{\text{prod}}, x^{\text{cons}}) \in C$ represents not only an exchangeable object but also a production process which produces x^{cons} out of x^{prod}.

In the first part of this chapter, we introduce the primary concepts of our models. Each section deals with one of them. After the introduction of all the primary concepts needed for a model, that particular model is presented. Model A concerns a pure exchange economy, and Model B introduces price rigidities into this setting. Both Model C and Model D describe a private ownership economy with two types of bundles of exchange: production bundles that can be used as input of a production process and consumption bundles which can represent the output of a production process. In Model C it is assumed that agents consume both types of bundles, whereas in Model D the preferences of the agents are restricted to the set of consumption bundles. In the final section of this chapter we state several equilibrium existence theorems for these models and we discuss the extra mathematical assumptions which are made to guarantee the existence.

4.1 Objects of exchange

We represent the collection of all exchangeable objects, being it separate commodities, bundles of commodities or other objects, by a salient space C. Every element x of C represents "something that can be exchanged or traded". We refer to the elements of the salient space C as "exchangeable object", "bundles of exchange", or "bundles of trade".

In Section 1.3, we motivated the use of a salient space C to model the set of all bundles of exchange. There, we saw that the two salient operations, addition and (scalar) multiplication over $\mathbb{R}_+$, represent the construction of new bundles of exchange from other bundles; if x and y are bundles of exchange, then $x+y$ represents the bundle of exchange obtained by joining x and y. Furthermore, $3x \in C$, represents the bundle $x + x + x$, and so on. The adjective "salient", indicating that C does not contain a linear subset, reflects that two bundles of exchange cannot cancel each other out. Finally, the partial order relation $\leq_C$ which is induced by C (cf. Definition 2.2.5), represents a natural way to order bundles of exchange. By means of an example, we explained in Section 1.3 the difference between the nature of these exchangeable objects and the familiar term "commodity", as used in the neoclassical models. In this section, we give two more examples and elaborate on this difference. We want to emphasise that the conditions for Example 4.1.1 are essentially different from the conditions for Example 1.3.1.

4.1.1 Example. Consider a model of a pure exchange economy in which three separate commodities are available: commodities a, b and c. We denote the neoclassical commodity bundles by $x = (x_a, x_b, x_c) \in \mathbb{R}^3_+$. Assume that these commodities can only be purchased by trade in the following fixed commodity bundles:

- bundle 1: $(0, 1, 1)$,
- bundle 2: $(1, 2, 2)$.

The agents are allowed to break up the traded bundles into smaller pieces, as long as the trade is established in the above described proportions. For example, bundle 1 can represent a small box of breakfast cereal (commodity b denoting cereal and commodity c denoting cardboard), and bundle 2 a large box of cereal with a plastic toy (commodity a) in it.
The set of exchangeable objects consists of all linear combinations of these two bundles, such that the resulting bundle is greater than or equal to the zero bundle

$(0,0,0) \in \mathbb{R}^3_+$, i.e., every $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_1(0,1,1) + \alpha_2(1,2,2) \geq_E (0,0,0),$$

represents a bundle of trade. Here $\geq_E$ denote the Euclidean order relation on $\mathbb{R}^3$. Hence, every pair $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ represents a bundle of trade if and only if $\alpha_1 + 2\alpha_2 \geq 0$ and $\alpha_2 \geq 0$. Clearly, the set C of all exchangeable objects in this model is equal to the salient space

$$C = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 \mid \alpha_1 + 2\alpha_2 \geq 0 \text{ and } \alpha_2 \geq 0\}.$$

For example, the exchangeable object $h = (-2, 1)$ denotes the commodity bundle $(1, 0, 0)$. Hence, agents can exchange the separate plastic toy, since a large box of cereal can be split up in two smaller boxes of cereal which can be exchanged.

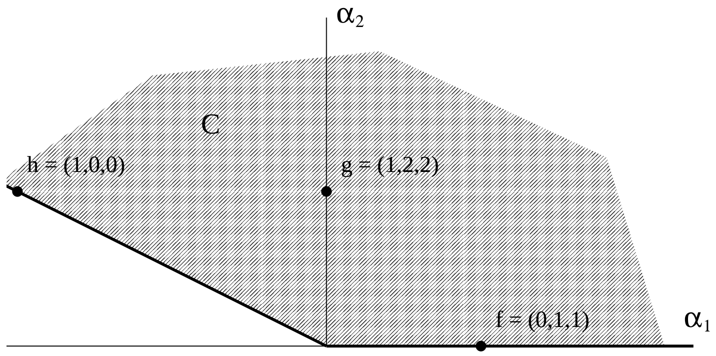

FIGURE 4.1.1: An example of the set of exchangeable objects in $\mathbb{R}^3$

Furthermore, an element $(\alpha_1, \alpha_2) \in C$ is greater than or equal to $(\beta_1, \beta_2) \in C$, with respect to the order relation $\leq_C$, if and only if $\alpha_1 + 2\alpha_2 \geq \beta_1 + 2\beta_2$ and $\alpha_2 \geq \beta_2$. Note that if bundle $\alpha_1(0,1,1) + \alpha_2(1,2,2)$ contains at least as much of all three commodities a, b and c as bundle $\beta_1(0,1,1) + \beta_2(1,2,2)$, then $(\beta_1, \beta_2) \leq_C (\alpha_1, \alpha_2)$. Hence, the Euclidean order relation is stronger than $\leq_C$. The two order relations are not equivalent. For example exchangeable object $g = (0, 1)$, which represents commodity bundle 2, is greater than $f = (1, 0)$, which represents commodity bundle 1 (see Figure 4.1.1).

The neoclassical approach to model this situation would be to consider the two fixed bundles as separate commodities and introduce $\mathbb{R}^2_+$ as the set of all commodity bundles. Note that in this case, the bundle $(1, 0, 0)$ is not a commodity bundle. Furthermore, the Euclidean order relation on $\mathbb{R}^2$, restricted to $\mathbb{R}^2_+$, does not represent the natural ordering of the exchangeable objects; commodity bundle 1, represented by $(1, 0)$ is not comparable to commodity bundle 2, represented by $(0, 1)$.

◇

The above example and Example 1.3.1, show two different situations in which a model based upon a salient space can incorporate fixed links between different commodities. For instance, an economy can be modelled in which only fixed, prescribed combinations of commodities can be exchanged. Examples are pre-packed offers, or special products received when purchasing a large amount of a commodity, examples which are frequently observed in e.g. supermarkets or drugstores.

Also, using a salient space-based model, we can describe a situation in which the preferences of the agents are in terms of characteristics of commodities rather than in terms of the commodities themselves. For instance, an economic agent may ask for a specific colour, the possibility of extensions, certain quality of service, etc.

4.1.2 Example. Consider a hotel where each day the guests can choose between breakfast combination A or B. In the evening the guests can choose between dinner menu C or D. Suppose a guest (a professional cyclist) is looking for a healthy meal for today. Consider Figure 4.1.2. Here x_1, x_2 and x_3 denote amounts of

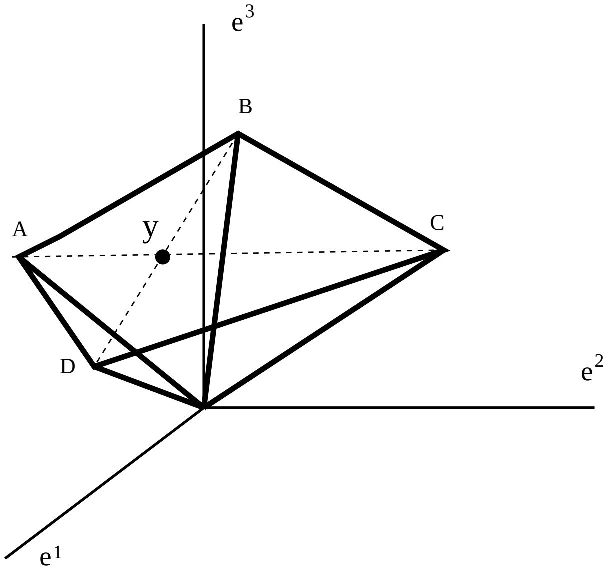

Figure 4.1.2: Preferences based on characteristics of commodities

carbohydrates, proteins and fibers, respectively. Suppose the bundle $y = (y_1, y_2, y_3)$ denotes the proper proportion of the nutritional values. The picture suggests that there are two different ways to obtain the bundle y: by combining breakfast A with menu C, or by combining breakfast B with menu D. ◇

In general, an "exchangeable object" can be considered to be a carrier of several attributes (cf. the work of Lancaster, [21]). Moreover, the same attribute may appear in more than one object of exchange. The models introduced in this chapter, allow for this mixture of attributes to be inextricable both in characteristics and in time.

In the labour market, for instance, a firm may ask for an employee with a certain education, intelligence and working experience. In this setting, one can consider an "object of exchange" to be a person with such (and perhaps other) specific attributes.

On the other hand, we want to emphasise, that although it is not assumed especially, separate commodities may be present. Note that $\mathbb{R}^k_+$ with the Euclidean order relation is a salient space. Also, a situation in which several commodities are linked and some of these and possibly other commodities are also available separately, fits into each of our models.

4.2 Pricing functions

In Section 1.3, we have seen that if the set C of bundles of exchange is a salient space then the set of all possible price systems also satisfies the definition of salient space. For the moment, we denote the salient space of all possible price systems by S, and by $\mathcal{B}(x, p)$, we denote the value that p assigns to bundle of exchange x. We assume that every price system $p \in S$ acts in a salient way on the elements of C: if $x, y, z \in C$ satisfy $x = \alpha y + \beta z$ for certain $\alpha, \beta \in \mathbb{R}_+$ then the value $\mathcal{B}(x, p)$ that p assigns to x has to be equal to α times $\mathcal{B}(y, p)$ plus β times $\mathcal{B}(z, p)$. Mathematically speaking, we find that $\mathcal{B}$ is a bi-salient form (cf. Definition 2.3.6).

The salient space C, which represents the set of exchangeable objects, the salient space S, which represents the set of all possible price systems, and the form $\mathcal{B}$ which represents the assigning of value to exchangeable objects given a price system, form a salient pairing $\{\, C\,,\ S\,; \mathcal{B}\,\}$. In Chapter 2, on page 57, we have seen that the salient space S is salient isomorphic to (a subset of) the adjoint C^* of C. Since S represents the set of all possible price systems that meet the requirements stated in Section 1.3, we find that, mathematically speaking, S is isomorphic with the adjoint C^* consisting of all salient functionals on C.

Henceforth, we take the adjoint C^* of C as the set of all price systems, i.e., we say that price systems are salient functions on C, and that for every $\mathcal{P} \in C^*$ the value of an element $x \in C$ is equal to $\mathcal{P}(x)$. Hence, we consider the canonical pairing $\{\, C\,,\ C^*\,; \mathcal{B}_{\text{can}}\,\}$ (cf. Corollary 2.3.16). Proposition 2.3.26 guarantees that in case the salient space C has an order unit with respect to the partial order relation $\leq_C$, then $C^* \neq \{0\}$, i.e., pricing functions unequal to the zero function exist. Due to price regulations or other reasons, it is possible that in a model some elements of C^* are not recognised as pricing functions. We denote the subset of C^* of all admissible pricing functions for a model, by P. In case P is a salient subspace of C^* the model is based on the salient pairing $\{\, C\,,\ P\,; \mathcal{B}_{\text{can}}\,\}$.

4.2.1 Example. Recall the model introduced in Example 4.1.1. We have seen that the set C of all exchangeable objects in this model is equal to the salient space

$$C = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 \mid \alpha_1 + 2\alpha_2 \geq 0 \text{ and } \alpha_2 \geq 0\}.$$

It is not difficult to derive (see Figure 4.2.1) that the set S consisting of all possible salient functions on C is represented by

$$S = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 \mid \alpha_2 \geq 2\alpha_1 \text{ and } \alpha_1 \geq 0\}.$$

Here, the bi-salient form $\mathcal{B}$ on $C \times S$ is equal to the Euclidean inner product.

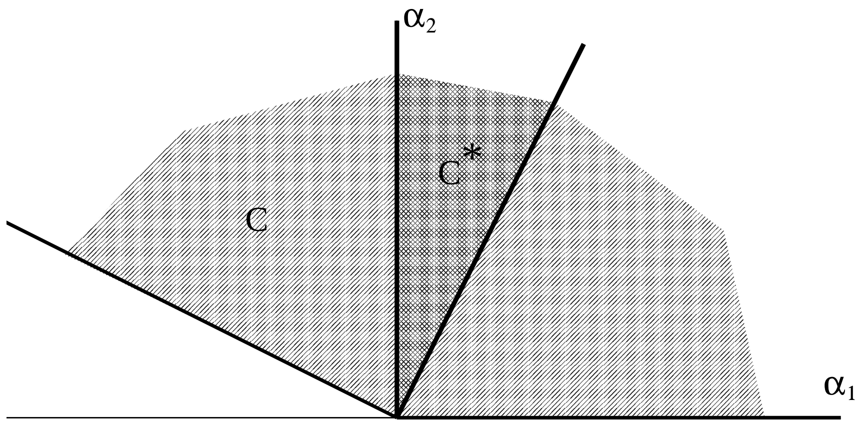

FIGURE 4.2.1: The set of exchangeable objects C and its adjoint C^*

◇

4.2.2 Example. Recall the model in Example 1.3.1, where the set of all exchangeable objects is represented by

$$\begin{aligned} C &= \left\{(x_a, x_b, x_c) \in \mathbb{R}^3_+ \mid \exists \alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0 : \left\{ \begin{array}{l} \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 = x_a \\ \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 = x_b \\ 2\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 = x_c, \end{array} \right\} \right\} \\ &= \operatorname{sal}((1,1,2),(1,2,1),(2,1,1),(1,2,2)). \end{aligned}$$

Also in this example, the bi-salient form is equal to the inner product.

The set of all possible pricing functions, i.e., the set of all salient functions on C, for this model is represented by

$$\begin{aligned} S &= \{(p_a, p_b, p_c) \in \mathbb{R}^3 \mid \forall (x_a, x_b, x_c) \in C : p_a x_a + p_b x_b + p_c x_c \geq 0\} \\ &= \operatorname{sal}((-1,3,-1),(-1,-1,3),(2,-1,0),(2,0,-1)). \end{aligned}$$

Note, that in the first neoclassical approach, as described in Example 1.3.1, where the consumption set was taken to be the restriction of $\mathbb{R}^3_+$ to the pointed convex cone

C, the set of all possible pricing functions is equal to $\mathbb{R}^3_+$, which is a strict subset of C^*. Hence, the neoclassical approach boils down to taking the set of all admissible price vectors equal to (a subset of) $\mathbb{R}^3_+$, whereas the salient space approach allows for a larger choice of P. ◇

4.2.3 Example. Consider a model for a pure exchange economy, where the set of all objects of trade is modelled by the salient space C of all positive 2×2-matrices (cf. Examples 2.1.20, 2.3.9 and 2.3.29). Then the set S of all possible price systems is also represented by C. The bi-salient form $\mathcal{B}$ is for each bundle of exchange $M \in C$ and each pricing function $P \in C$ given by $\mathcal{B}(M, P) = \mathrm{tr}(MP)$. ◇

Above, we have seen the first two primary concepts of a salient space based model of a pure exchange economy: the set of all objects of trade is represented by a salient space C and the set of all admissible pricing functions is represented by a subset P of C^*. In case P is a salient subspace of C^*, then the two primary concepts of a pure exchange economy form a salient pairing $\{\, C\,,\, P\,; \mathcal{B}_{\mathrm{can}}\,\}$, where, for every $x \in C$ and every $\mathcal{P} \in P$,

$$\mathcal{B}_{\mathrm{can}}(x, \mathcal{P}) = \mathcal{P}(x),$$

denotes the value of x at pricing function $\mathcal{P}$.

4.3 Agents

The third primary concept of a pure exchange economy concerns the agents. Similar to the neoclassical model of a pure exchange economy (cf. Section 1.2.1), the features of an economic agent are an element $w \in C$, called initial endowment, and a preference relation $\succeq$ defined on C, on the basis of which the agents makes choices. This preference relation $\succeq$ on C is assumed to satisfy (cf. page 10):

- reflexivity: $\forall x \in C : x \succeq x$,
- transitivity: $\forall x, y, z \in C : \;(\, x \succeq y \text{ and } y \succeq z\,) \implies x \succeq z$,
- completeness: $\forall x, y \in C : x \succeq y \text{ or } y \succeq x$.

This completes the introduction of all three primary concepts of a model of a pure exchange economy; we have defined the set of all exchangeable objects, the set of all pricing functions and the agents. Next, we briefly introduce the secondary concepts, of which the construction is similar to the construction in the neoclassical model of Section 1.2.1.

For a given pricing function $\mathcal{P} \in P$, the budget set of an agent with initial endowment $w \in C$ is given by

$$B(\mathcal{P}, w) := \{x \in C \mid \mathcal{P}(x) \leq \mathcal{P}(w)\},$$

and consists of all exchangeable objects that can be afforded at pricing function $\mathcal{P} \in P$. For an agent with an initial endowment $w \in C$ and a preference relation $\succeq$ on C, the set

$$D(\mathcal{P}, w, \succeq) := \{x \in B(\mathcal{P}, w) \mid \forall y \in B(\mathcal{P}, w) : x \succeq y\}$$

of all best (most preferable) elements of the budget set $B(\mathcal{P}, w)$, with respect to preference relation $\succeq$, is called the demand set.

4.3.1 Example. Reconsider the model of a pure exchange economy, presented in Example 4.1.1, where the set of all bundles of exchange is represented by the salient space

$$C = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 \mid \alpha_1 + 2\alpha_2 \geq 0 \text{ and } \alpha_2 \geq 0\}.$$

In Example 4.2.1, we have seen the the set of all possible pricing functions is represented by

$$S = \{(\pi_1, \pi_2) \in \mathbb{R}^2 \mid \pi_2 \geq 2\pi_1 \text{ and } \pi_1 \geq 0\},$$

where the value of exchangeable object (α_1, α_2) at pricing function (π_1, π_2) is equal to the inner product $\pi_1\alpha_1 + \pi_2\alpha_2$.

Consider an agent with initial endowment $w = (2, 2) \in C$ and preference relation $\succeq$ on C, for every $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in C$, given by

$$(\alpha_1, \alpha_2) \succeq (\beta_1, \beta_2) \quad :\Longleftrightarrow \quad \min\{\alpha_2, \alpha_1 + 2\alpha_2\} \geq \min\{\beta_2, \beta_1 + 2\beta_2\}.$$

It is not difficult to check that this is indeed a preference relation on C. At pricing function $(1, 3) \in S$, the budget set of this agent is equal to

$$B((1, 3), w) = \{(\alpha_1, \alpha_2) \in C \mid \alpha_1 + 3\alpha_2 \leq 8\}.$$

Furthermore, it is not difficult to derive that the demand set of this agent at pricing function $(1, 3)$ is equal to $D((1, 3), w, \succeq) = \{(-4, 4)\}$. ◇

With the concepts thus far introduced, we are ready to state the first model, with corresponding equilibrium concept. In Section 4.8, we give two existence theorems related to this model, Theorems A1 and A2, and we discuss the additional mathematical assumptions made in these theorems. Both of these theorems will explicitly assume that the set P of all admissible pricing functions is equal to the adjoint C^* of C.

Model A: pure exchange economy

Primary concepts:

- the set of all exchangeable objects is modelled by a salient space C;
- the set of price systems is modelled by a subset P of the adjoint C^* of C;
- there is a finite number, i_0, of agents, where agent i, $i \in \{1, \ldots, i_0\}$, is characterised by an initial endowment $w_i \in C$ and a preference relation $\succeq_i$ on C.

Secondary concepts:

- for every $i \in \{1, \ldots, i_0\}$ and for every $\mathcal{P} \in P$, the budget set of agent i is given by
$$B_i^A(\mathcal{P}) := B(\mathcal{P}, w_i) = \{x \in C \mid \mathcal{P}(x) \leq \mathcal{P}(w_i)\};$$
- for every $i \in \{1, \ldots, i_0\}$ and for every $\mathcal{P} \in P$, the demand set of agent i is given by
$$D_i^A(\mathcal{P}) := D(\mathcal{P}, w_i, \succeq_i) = \{x \in B_i^A(\mathcal{P}) \mid \forall y \in B_i^A(\mathcal{P}) : x \succeq_i y\}.$$

A Walrasian equilibrium for Model A is an (i_0+1)-tuple $(d_1, \ldots, d_{i_0}, \mathcal{P}_{\text{eq}}) \in C^{i_0} \times P$ such that

- $\mathcal{P}_{\text{eq}} \neq 0$,
- $d_i \in D_i^A(\mathcal{P}_{\text{eq}})$ for all $i \in \{1, \ldots, i_0\}$,
- $\sum_{i=1}^{i_0} d_i \leq_C \sum_{i=1}^{i_0} w_i$.

In words, an i_0-tuple of exchangeable objects for each agent, and a pricing function, form a Walrasian equilibrium if the pricing function is nonzero, each agent chooses a most preferable object in his budget set at the given pricing function, and the total demand is smaller than or equal to the total initial endowment.

A pricing function $\mathcal{P}_{\text{eq}} \in P \setminus \{0\}$, satisfying the conditions in the above definition, is called a (Walrasian) equilibrium pricing function.

4.4 Rationing schemes

In the neoclassical Arrow-Debreu model, where $C = \mathbb{R}_+^{k_0}$, Drèze introduced price rigidities by restricting prices to the set

$$P = \{p \in \mathbb{R}_+^{k_0} \mid f(p) = 1 \text{ and } \overline{p} \geq_E p \geq_E \underline{p}\}.$$

Then, a rationing scheme is introduced on the budget sets of the agents, ensuring the existence of a constrained equilibrium price vector $p \in P$. If p is on the boundary of P, then there is a set $K \subset \{1, \ldots, k_0\}$ such that $\forall k \in K : p_k = \overline{p}_k$ or $p_k = \underline{p}_k$. The number of elements of K determines the number of restrictions in the rationing scheme. If $p_k = \overline{p}_k$ then the net demand of each agent for commodity k is rationed, and if $p_k = \underline{p}_k$, the net supply is rationed.

We now introduce price regulations or price rigidities into the above described model, Model A, of a pure exchange economy. So, we reconsider the model of a pure exchange economy where the set of all bundles of exchange is represented by the salient space C, where the adjoint C^* represents the set of all pricing functions, and where the features of an agent are an exchangeable object $w \in C$, called his initial endowment, and a preference relation $\succeq$ defined on C. However, now, we assume that the set P of admissible pricing functions is a strict salient subspace of C^*. As a consequence, we cannot apply Theorem A1 or Theorem A2 of Section 4.8 which correspond to Model A, since they both explicitly assume $P = C^*$. In other words, the equilibrium pricing functions, of which the theorems corresponding to Model A prove the existence, may be elements of $C^* \setminus P$. When this is the case, we find that for every pricing function in P, the demand of the agents cannot be realised from the present initial endowment. In this situation, the description of a new equilibrium concept should include how commodities are allocated. This allocation is regulated by using a so-called rationing scheme for each agent.

A rationing scheme is a regulation which restricts the set of exchangeable objects that an agent is allowed to purchase or to offer for trade with regard to his initial endowment. The idea behind a rationing scheme is that if a pricing function lies on the boundary of the set P and cannot be used to equalise total demand and total supply, the rationing scheme takes over this role.

We let a rationing scheme be determined by three variables: $\mathcal{N}_1, \mathcal{N}_2 \in C^*$ and $\alpha \in \mathbb{R}_+$, in the following way. At a rationing scheme $(\mathcal{N}_1, \mathcal{N}_2, \alpha) \in C^* \times C^* \times \mathbb{R}_+$, the demand set of an agent with initial endowment w in C and preference relation $\succeq$ on C, is restricted to the set

$$R(\mathcal{N}_1, \mathcal{N}_2, \alpha, w) := \{x \in C \mid \mathcal{N}_1(x) - \mathcal{N}_1(w) - \mathcal{N}_2(x) + \mathcal{N}_2(w) \leq \alpha\}.$$

Note, that in $V[C]$-terminology this means that the demand is restricted to a set which is isomorphic with the intersection of $V_+[C]$ and a half-space. This half-space consists of all the elements of $V[C]$ that have a value smaller than or equal to α plus the value of w, both with respect to the linear function $[\mathcal{N}_1, \mathcal{N}_2] \in V[C]$. Furthermore, if $\mathcal{N}_1 = \mathcal{N}_2$, then restriction set $R(\mathcal{N}_1, \mathcal{N}_2, \alpha, w)$ is equal to C.

For every $\mathcal{P} \in P$, for every $\mathcal{N}_1, \mathcal{N}_2 \in C^*$ and for every $\alpha \in \mathbb{R}_+$, the constrained budget set of the agent is given by

$$B(\mathcal{P}, w; \mathcal{N}_1, \mathcal{N}_2, \alpha) := B(\mathcal{P}, w) \cap R(\mathcal{N}_1, \mathcal{N}_2, \alpha, w).$$

The constrained demand set for the agent, at $\mathcal{P} \in P$, at $\mathcal{N}_1, \mathcal{N}_2 \in C^*$ and at $\alpha \in \mathbb{R}_+$, is the set of all best elements of $B(\mathcal{P}, w; \mathcal{N}_1, \mathcal{N}_2, \alpha)$ with respect to the preference relation $\succeq$, i.e.,

$$D(\mathcal{P}, w, \succeq; \mathcal{N}_1, \mathcal{N}_2, \alpha) := \{x \in B(\mathcal{P}, w; \mathcal{N}_1, \mathcal{N}_2, \alpha) \mid \forall y \in B(\mathcal{P}, w; \mathcal{N}_1, \mathcal{N}_2, \alpha) : x \succeq y\}.$$

Note, that for all $\mathcal{P} \in P$, for all $\mathcal{N}_1, \mathcal{N}_2 \in C^*$ and for all $\alpha \in \mathbb{R}_+$, we find $w \in B(\mathcal{P}, w; \mathcal{N}_1, \mathcal{N}_2, \alpha)$. Secondly, the restriction of the demand to a half-space makes sure that the rationing scheme does not simultaneously affect supply and demand in any direction. Finally, we mention that for all $w \in C, \mathcal{P} \in P$, $\mathcal{N} \in C^*$ and $\alpha \in \mathbb{R}_+$, the constrained budget set $B(\mathcal{P}, w; \mathcal{N}, \mathcal{N}, \alpha)$ equals the unconstrained budget set $\{x \in C \mid \mathcal{P}(x) \leq \mathcal{P}(w)\}$ defined in Model A.

The choice of the rationing is one of the things to be specified in a model allowing for price rigidities. In the general set up as we have chosen for Model A, such a specification becomes highly untransparent. Therefore, we restrict ourselves to the following situation.

- V is a finite-dimensional inner product space, where the inner product is denoted by $\langle ., . \rangle$.
- C is a salient space, represented by the solid, pointed convex cone K in V (cf. Definition 2.1.21).
- C^* is represented by the solid, pointed convex cone K^* in V, given by

$$K^* := \{p \in V \mid \forall k \in K : \langle k, p \rangle \geq 0\}.$$

The above introduced restriction set $R(\mathcal{N}_1, \mathcal{N}_2, \alpha, w)$ is described by a half space in V intersected by K. Henceforth, we use a different notation for a rationing scheme and for the restriction set. For every $w \in K$, for every $n \in V$ and for every $\alpha \in \mathbb{R}_+$, we define

$$R(n, \alpha, w) := \{x \in K \mid \langle x - w, n \rangle \leq \alpha\}.$$

Note that $n \in V$ represents $[(\mathcal{N}_1, \mathcal{N}_2)] \in V[C^*]$, so a rationing scheme can be represented by a pair (n, α), with $n \in V$ and $\alpha \in \mathbb{R}_+$.

With the concepts thus far introduced, we are ready to state the second model, with corresponding equilibrium concept. In Section 4.8, we give an existence theorem that is related to this model, and we discuss the additional mathematical assumptions made in that theorem.

Model B: price rigidities and rationing

Primary concepts:

- the set of all exchangeable objects is modelled by a solid pointed convex cone K in a finite-dimensional inner product space V;
- the set of price systems is modelled by a strict subcone P of K^*, where K^* in V is given by

$$K^* = \{x \in V \mid \forall k \in K : \langle x, k \rangle \geq 0\};$$

- there is a finite number, i_0, of agents, where agent i, $i \in \{1, \ldots, i_0\}$, is characterised by an initial endowment $w_i \in K$ and a preference relation $\succeq_i$ on K,
- the set of all rationing schemes is modelled by $V \times \mathbb{R}_+$.

Secondary concepts:

- for every $i \in \{1, \ldots, i_0\}$, for every $p \in P$, for every $n \in V$ and for every $\alpha \in \mathbb{R}_+$, the constrained budget set of agent i is given by

$$B_i^B(p, n, \alpha) := \{x \in K \mid \langle x - w_i, p \rangle \leq 0 \text{ and } \langle x - w_i, n \rangle \leq \alpha\};$$

- for every $i \in \{1, \ldots, i_0\}$, for every $p \in P$, for every $n \in V$ and for every $\alpha \in \mathbb{R}_+$, the constrained demand set of agent i is given by

$$D_i^B(p, n, \alpha) := \{x \in B_i^B(p, n, \alpha) \mid \forall y \in B_i^B(p, n, \alpha) : x \succeq_i y\}.$$

A constrained equilibrium for Model B is an $(i_0 + 3)$-tuple

$$(d_1, \ldots, d_{i_0}, p_{\text{eq}}, n_{eq}, \alpha_{eq}) \in C^{i_0} \times P \times V \times \mathbb{R}_+,$$

Model B (continued)

such that

- $p_{\text{eq}} \neq 0$,
- $d_i \in D_i^B(\mathcal{P}_{\text{eq}}, n_{eq}, \alpha_{eq})$ for all $i \in \{1, \ldots, i_0\}$,
- $\sum_{i=1}^{i_0} d_i = \sum_{i=1}^{i_0} w_i$,
- if $p_{\text{eq}} \in \text{int}(P)$, then $n_{eq} = 0$,
- if $p_{\text{eq}} \in \text{bd}(P)$ then there is $x_0 \in \text{int}(K)$ such that $\langle x_0, n_{\text{eq}} \rangle = 0$ and the set

$$\{x \in V \mid \langle x, n_{\text{eq}} \rangle = \langle p_{\text{eq}}, n_{\text{eq}} \rangle\}$$

is a supporting hyperplane of $P_{x_0} = \{p \in P \mid \langle p, x_0 \rangle = 1\}$ at p_{eq}.

In words, an i_0-tuple of exchangeable objects for each agent, a price vector, and a rationing scheme form a constrained equilibrium if the price vector is nonzero, each agent chooses a most preferable object in his constrained budget set at the given price vector, and the total demand is equal to the total initial endowment. Furthermore, if the equilibrium price vector is not on the boundary of P, then there is no rationing. If the equilibrium price vector p_{eq} is an element of the boundary of P, then the rationing vector n_{eq} is orthogonal to P_{x_0} at p_{eq}, for some $x_0 \in \text{int}(K)$. So, if the pricing function cannot adapt in a certain direction, because it lies on the boundary of P, a rationing scheme is introduced which constrains every agent's net demand in the same direction.

We call $p_{\text{eq}} \in P \setminus \{0\}$, satisfying the conditions in the above definition, a (constrained) equilibrium price vector.

We note that the choice of the inner product is not specified, and the definition of orthogonality is related to this choice. Therewith the choice of the rationing scheme fixed by $n \in V$, is still arbitrary since it depends on the choice of the inner product. Of course, when $V = \mathbb{R}_+^{k_0}$, there is the natural tendency to choose the Euclidean inner product. A choice for the cone K in the Euclidean setting with $V = \mathbb{R}_+^3$ is depicted in Figure 2.1.3, page 36.

Finally, we remark that, contrary to the approach of Drèze, a rationing scheme in the above model, where the price rigidity set P is a convex subcone of K^*, is specified by one restriction only.

4.5 Firms

In a model of a private ownership economy, four primary concepts occur: the set of all exchangeable objects, the price set, the agents, and the firms. On the first of these primary concepts, we have to be a bit more specific. The last one has to be introduced.

We model a private ownership economy in which production and consumption will play a distinguished role. Each bundle of exchange, i.e., each element of C is a unique concatenation of a production bundle and a consumption bundle, where only production bundles can be used as input for a production process whereas the output of this process is a consumption bundle. However, bundles of both types are allowed to be consumed by economic agents and bundles of both types may be present in the initial endowment. Each of the two types of bundles is assumed to establish a salient space. The set consisting of all exchangeable objects is taken to be the direct sum of two salient spaces C_{prod} and C_{cons}. Here C_{prod} is the salient space consisting of all production bundles, and C_{cons} is the salient space containing all consumption bundles.

4.5.1 Definition (direct sum of two salient spaces). The direct sum of two salient spaces C_{prod} and C_{cons} is the salient space $C_{\text{prod}} \oplus C_{\text{cons}}$, consisting of all ordered pairs $x = (x^{\text{prod}}, x^{\text{cons}})$ with $x^{\text{prod}} \in C_{\text{prod}}$ and $x^{\text{cons}} \in C_{\text{cons}}$. The salient space operations are for all $x, y \in C_{\text{prod}} \oplus C_{\text{cons}}$ and for all $\alpha \geq 0$ given by:

$$\left\{ \begin{array}{lcl} (x+y)^{\text{prod}} & := & x^{\text{prod}} + y^{\text{prod}} \\ (\alpha x)^{\text{prod}} & := & \alpha x^{\text{prod}} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lcl} (x+y)^{\text{cons}} & := & x^{\text{cons}} + y^{\text{cons}} \\ (\alpha x)^{\text{cons}} & := & \alpha x^{\text{cons}}. \end{array} \right.$$

For every $x \in C_{\text{prod}} \oplus C_{\text{cons}}$, there are unique $x^{\text{prod}} \in C_{\text{prod}}$ and $x^{\text{cons}} \in C_{\text{cons}}$ such that $x = (x^{\text{prod}}, x^{\text{cons}})$. Since $C_{\text{prod}} \oplus C_{\text{cons}}$ is a salient space, every property derived for salient spaces (in Chapters 2 and 3) is also applicable to $C_{\text{prod}} \oplus C_{\text{cons}}$.

The partial order relation $\leq_C$ of the salient space $C := C_{\text{prod}} \oplus C_{\text{cons}}$ satisfies:

$$x \leq_C y \iff \left\{ \begin{array}{l} x^{\text{prod}} \leq_{\text{prod}} y^{\text{prod}} \\ x^{\text{cons}} \leq_{\text{cons}} y^{\text{cons}}, \end{array} \right.$$

and the adjoint C^* of C satisfies

$$C^* = C^*_{\text{prod}} \oplus C^*_{\text{cons}}.$$

Hence, if we model the set of all bundles of trade by a salient space $C = C_{\text{prod}} \oplus C_{\text{cons}}$, then for every pricing function $\mathcal{P} \in C^*$ there are $\mathcal{P}^{\text{prod}} \in C^*_{\text{prod}}$ and $\mathcal{P}^{\text{cons}} \in C^*_{\text{cons}}$ such that $\mathcal{P} = (\mathcal{P}^{\text{prod}}, \mathcal{P}^{\text{cons}})$. The value of an element $x = (x^{\text{prod}}, x^{\text{cons}}) \in C$ at pricing function $\mathcal{P} \in C^*_{\text{prod}} \oplus C^*_{\text{cons}}$ is defined by

$$\mathcal{P}(x) = \mathcal{P}^{\text{prod}}(x^{\text{prod}}) + \mathcal{P}^{\text{cons}}(x^{\text{cons}}).$$

Both C_{prod} and C_{cons} are assumed to be non-trivial, i.e., assumed to be not equal to $\{0^{\text{prod}}\}$ and $\{0^{\text{cons}}\}$, where 0^{prod} and 0^{cons} denote the vertex of C_{prod} and C_{cons}, respectively. As a consequence, $C := C_{\text{prod}} \oplus C_{\text{cons}}$ is also non-trivial.

Thus, we have presented the realisation of the set of all objects of trade, as being the direct sum $C = C_{\text{prod}} \oplus C_{\text{cons}}$, and the set of all admissible pricing functions as a subset of the direct sum $C^*_{\text{prod}} \oplus C^*_{\text{cons}}$. Now, we concentrate on the introduction of the primary concept of firm.

We start modelling so-called production processes, i.e., processes that incorporate the possibility of converting production bundles into consumption bundles. We present a production process by an element $x = (x^{\text{prod}}, x^{\text{cons}}) \in C$; a production process $(x^{\text{prod}}, x^{\text{cons}})$ converts production bundle $x^{\text{prod}} \in C_{\text{prod}}$ into consumption bundle $x^{\text{cons}} \in C_{\text{cons}}$. A collection of production processes being technologically feasible is said to be a production technology. So, a production technology is a subset T of C. One may think of a production technology as being the set of all production processes that can be executed due to the presence of a specific group of machinery. The primary concept of firm is completely characterised by a production technology.

From a feasibility point of view, each production technology T satisfies the following natural assumptions.

a) The production process "no production" belongs to T;

b) A production process in T with zero input has zero output;

c) Free disposal, both of input and of output.

If $x = (x^{\text{prod}}, x^{\text{cons}})$ is a feasible production process and $\tilde{x}^{\text{prod}} = x^{\text{prod}} + y^{\text{prod}}$ for some $y^{\text{prod}} \in C_{\text{prod}}$, then $(\tilde{x}^{\text{prod}}, x^{\text{cons}})$ is also a feasible production process since after disposal of y^{prod}, production process x can be executed. This is what we mean by free disposal of input. Put differently, if $x \in T$ and $\tilde{x}^{\text{prod}} \in C_{\text{prod}}$ with $x^{\text{prod}} \leq_{\text{prod}} \tilde{x}^{\text{prod}}$ then $(\tilde{x}^{\text{prod}}, x^{\text{cons}}) \in T$. Similarly, we say that there is free disposal of output if $x = (x^{\text{prod}}, x^{\text{cons}})$ is a feasible production process and $x^{\text{cons}} = y^{\text{cons}} + \tilde{x}^{\text{cons}}$ for some $y^{\text{cons}}, \tilde{x}^{\text{cons}} \in C_{\text{cons}}$, then $(x^{\text{prod}}, \tilde{x}^{\text{cons}})$ is also a feasible production process since after production of x^{cons} out of x^{prod}, y^{cons} can be disposed of, leaving $\tilde{x}^{\text{cons}}$ as output. So, if $x \in T$ and $\tilde{x}^{\text{cons}} \in C_{\text{cons}}$ with $\tilde{x}^{\text{cons}} \leq_{\text{cons}} x^{\text{cons}}$ then $(x^{\text{prod}}, \tilde{x}^{\text{cons}}) \in T$.

For the formal definition of production technology, we need the following notation.

For all $x \in C_{\text{prod}} \oplus C_{\text{cons}}$ the set F_x is given by

$$F_x := \{z \in C \mid x^{\text{prod}} \leq_{\text{prod}} z^{\text{prod}} \text{ and } z^{\text{cons}} \leq_{\text{cons}} x^{\text{cons}}\}.$$

Let A be a subset of C. For all $x \in A$ the set $R_x(A)$ is given by

$$R_x(A) := \{z \in A \mid x \in F_z \text{ and } F_z \subset A\}.$$

Furthermore, the set $E(A)$ is given by

$$E(A) := \{e \in A \mid R_e(A) = \{e\}\}.$$

The following three properties immediately follow.

4.5.2 Lemma. *Let $A \subset C$ and let $x \in C$. Then*

- $x \in R_x(A) \Longleftrightarrow F_x \subset A$,
- $\forall y \in F_x : F_y \subset F_x$,
- $y \in F_x$ and $x \in F_y \quad \Longleftrightarrow \quad x = y$.

In fact, if $T \subset C$ satisfies the afore mentioned properties a, b and c, then for every $x \in T$, the set F_x is a subset of T, since F_x consists of precisely all the production processes in C which are executable due to the two free disposal property c and the fact that $x \in T$. It turns out that $E(T)$ describes the set of all efficient elements of a set T of production processes.

So, translating the properties a, b and c with the help of the new notation, we come to the following definition of the concept of production technology.

4.5.3 Definition (production technology). A set $T \subset C$ is a production technology if

a) $(0^{\text{prod}}, 0^{\text{cons}}) \in T$,

b) If $(0^{\text{prod}}, x^{\text{cons}}) \in T$ then $x^{\text{cons}} = 0^{\text{cons}}$,

c) $T = \bigcup_{x \in T} F_x$.

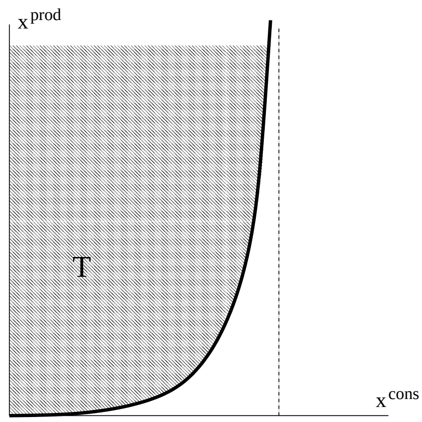

FIGURE 4.5.1: Production technology with bounded set of consumption bundles

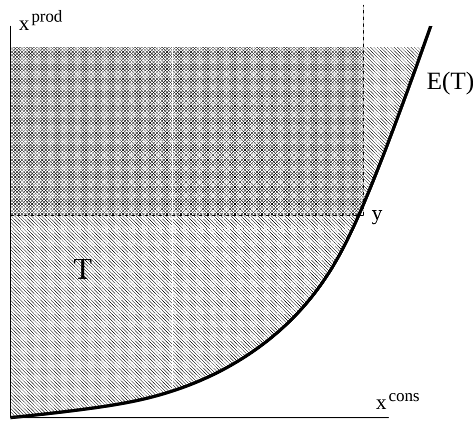

FIGURE 4.5.2: Production technology with unbounded set of consumption bundles

We remark that, in comparison with the neoclassical private ownership economy, presented in Section 1.2.2, it is not assumed that there is a "maximally possible output". Figure 4.5.1 and Figure 4.5.2 show two examples of a production technology, in case $C_{\text{prod}} = C_{\text{cons}} = \mathbb{R}_+$. Note, that in Figure 4.5.2 the non-smooth set $F_y \cup \{(x^{\text{prod}}, 0) \in C \mid x^{\text{prod}} \in C_{\text{prod}}\}$ fits the definition of a production technology also.

We call a production process $(x^{\text{prod}}, x^{\text{cons}})$ of a technology T efficient, if at least x^{prod} is needed to produce x^{cons}, and if it is not possible to produce more than x^{cons} out of x^{prod}. Mathematically speaking, this boils down to the following definition.

4.5.4 Definition (efficient production process). Let T be a production technology. A production process $e \in T$ is efficient if $\forall x \in T$:

$$\left.\begin{array}{l} x^{\text{prod}} \leq_{\text{prod}} e^{\text{prod}} \\ e^{\text{cons}} \leq_{\text{cons}} x^{\text{cons}} \end{array}\right\} \implies x = e.$$

Put differently, e is efficient if and only if $e \in E(T)$. Note that $(0^{\text{prod}}, 0^{\text{cons}}) \in E(T)$.

The following lemma gives conditions on an arbitrary set $T \subset C$ under which this set is convex. These conditions will be assumed in the equilibrium existence theorems concerning this model of a private ownership economy.

4.5.5 Lemma. *Let T be a subset of the salient space $C = C_{\text{prod}} \oplus C_{\text{cons}}$. Assume $T = \bigcup_{e \in E(T)} F_e$ and assume $\forall e, f \in E(T)\ \forall \tau \in [0,1] : \tau e + (1-\tau) f \in T$. Then the set T is convex.*

Proof.
Let $x, y \in T$ and $\tau \in [0,1]$. By the first property of T, there exist $e, f \in E(T)$ such that $x \in F_e$ and $y \in F_f$. Thus,

$$\exists \tilde{x} \in C : \begin{cases} x^{\text{prod}} = e^{\text{prod}} + \tilde{x}^{\text{prod}} \\ e^{\text{cons}} = x^{\text{cons}} + \tilde{x}^{\text{cons}} \end{cases} \quad \text{and} \quad \exists \tilde{y} \in C : \begin{cases} y^{\text{prod}} = f^{\text{prod}} + \tilde{y}^{\text{prod}} \\ f^{\text{cons}} = y^{\text{cons}} + \tilde{y}^{\text{cons}}. \end{cases}$$

To prove convexity of T we shall show that $\tau x + (1-\tau)y \in F_{(\tau e + (1-\tau)f)}$. Indeed, this proves the assertion since both properties of T, combined with the second property of Lemma 4.5.2, yield $F_{(\tau e + (1-\tau)f)} \subset T$.
Firstly, note that

$$\begin{aligned} \tau x^{\text{prod}} + (1-\tau)y^{\text{prod}} &= \tau(e^{\text{prod}} + \tilde{x}^{\text{prod}}) + (1-\tau)(f^{\text{prod}} + \tilde{y}^{\text{prod}}) \\ &= (\tau e^{\text{prod}} + (1-\tau)f^{\text{prod}}) + (\tau\tilde{x}^{\text{prod}} + (1-\tau)\tilde{y}^{\text{prod}}), \end{aligned}$$

and secondly,

$$(\tau x^{\text{cons}} + (1-\tau)y^{\text{cons}}) + (\tau\tilde{x}^{\text{cons}} + (1-\tau)\tilde{y}^{\text{cons}}) = \tau e^{\text{cons}} + (1-\tau)f^{\text{cons}}.$$

Since $\tau\tilde{x}^{\text{prod}} + (1-\tau)\tilde{y}^{\text{prod}} \in C_{\text{prod}}$ and $\tau\tilde{x}^{\text{cons}} + (1-\tau)\tilde{y}^{\text{cons}} \in C_{\text{cons}}$, we conclude that $\tau x + (1-\tau)y \in F_{(\tau e + (1-\tau)f)}$. □

As far as we know, in the neoclassical models, consumption bundles and production bundles are not distinguished explicitly. In Section 1.2.2, we have seen that the neoclassical models recognise a production technology (production set) as a subset Y of the Euclidean vector space $\mathbb{R}^{k_0}$. Although $\mathbb{R}^{k_0}$ can be regarded as the product of the positive cone $\mathbb{R}^{k_0}_+$ and the negative cone $-(\mathbb{R}^{k_0}_+)$ (cf. page 15), and thus Y can be seen as a subset $\tilde{Y}$ of $\mathbb{R}^{k_0}_+ \times \mathbb{R}^{k_0}_+$, the set $\tilde{Y}$ does not satisfy the conditions we impose on T, in general. In fact, it is the lattice structure of $\mathbb{R}^{k_0}$ which makes this "comparison" possible, and in our model, lattice structures are not involved at all. We shall not explicitly discuss whether the neoclassical notion of production technology (Y) is generalised by our notion of production technology (T). The following example indicates the similarity between these two concepts, when the set C has a lattice structure.

4.5.6 Example. We take $C_{\text{prod}} = C_{\text{cons}} = \mathbb{R}_+$. Let production technology T be given by $T = \{x \in \mathbb{R}^2_+ \mid x^{\text{prod}} = (x^{\text{cons}})^2\}$ (see Figure 4.5.3). The corresponding neoclassical production set Y which models the same technology is depicted in Figure 4.5.4.

◇

In order to come to the secondary concept of supply, we introduce the concept of gain of a production process. Given a pricing function $\mathcal{P} \in C^*$ and a production process $x \in T$, the profit or gain $\mathcal{G}(x, \mathcal{P})$ of the pair $(x, \mathcal{P})$ equals the value of the consumption bundle x^{cons}, produced as output, minus the value of the production bundle x^{prod}, used as input. So,

$$\mathcal{G}(x, \mathcal{P}) := \mathcal{P}^{\text{cons}}(x^{\text{cons}}) - \mathcal{P}^{\text{prod}}(x^{\text{prod}}).$$

Note that the following two properties are a direct consequence of the definition of $\mathcal{G}$ and F_x.

- Let $x \in C$, $\mathcal{P} \in C^*$ and $y \in F_x$, then $\mathcal{G}(x, \mathcal{P}) \geq \mathcal{G}(y, \mathcal{P})$.
- Let $x \in C$, $\mathcal{P} \in \text{int}(C^*)$ and let $y \in F_x$ satisfy $y \neq x$, then $\mathcal{G}(x, \mathcal{P}) > \mathcal{G}(y, \mathcal{P})$.

Given $\mathcal{P} \in C^*$, the (possibly empty) set of all gain maximizing production processes in T is called the supply set $S(\mathcal{P}, T)$ of T, i.e.,

$$S(\mathcal{P}, T) = \{x \in T \mid \forall y \in T : \mathcal{G}(x, \mathcal{P}) \geq \mathcal{G}(y, \mathcal{P})\}.$$

The assumptions on T and the definition of $E(T)$ imply that $\forall \mathcal{P} \in C^* : S(\mathcal{P}, T) \subseteq E(T)$. Note, that $(0^{\text{prod}}, 0^{\text{cons}}) \in T$ implies $\forall \mathcal{P} \in C^* \; \forall x \in S(\mathcal{P}, T) : \mathcal{G}(x, \mathcal{P}) \geq 0$.

Thus far, we have treated three of the four primary concepts of a private ownership economy: the set of exchangeable objects, the price set, and the firms. In the next section we redefine the primary concept of agent (cf. Section 4.3). In fact, all we have to do is to characterise an agent not only by initial endowment and preference relation, but also by shares in the profit of production.

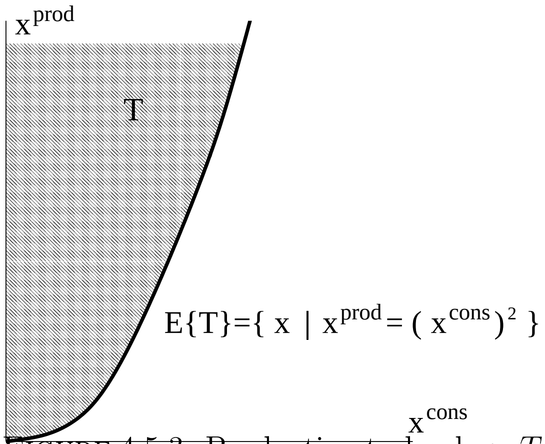

FIGURE 4.5.3: Production technology T in $\mathbb{R}_+ \oplus \mathbb{R}_+$

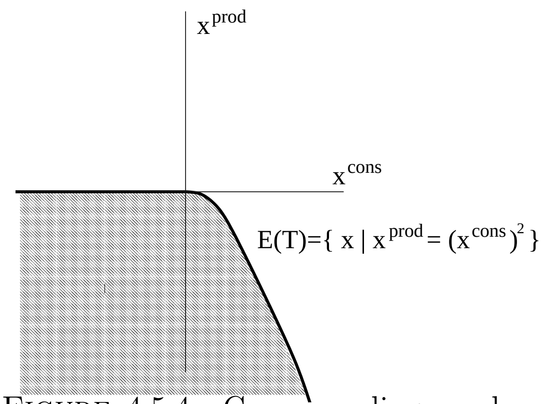

FIGURE 4.5.4: Corresponding production set Y in $\mathbb{R}^2$

4.6 Agents and shares

An economic agent as described in Section 4.3 is characterised by his initial endowment w which is an element of the set C of all bundles of trade, and a preference relation $\succeq$ defined on C. Let j_0 be the number of firms present. For every $j \in \{1, \dots, j_0\}$, firm j is characterised by production technology T_j. For each $j \in \{1, \dots, j_0\}$, the agent has a share θ_j, where $0 \le \theta_j \le 1$, in the gain of firm j, i.e., if firm j executes production process $x_j \in T_j$ at pricing function $\mathcal{P} \in C^*$, then the agent may add the amount of $\theta_j \mathcal{G}(x_j, \mathcal{P})$ to the value $\mathcal{P}(w)$ of his initial endowment at pricing function $\mathcal{P}$. By θ, we denote the vector of shares $(\theta_1, \dots, \theta_{j_0})$.

Thus, at pricing function $\mathcal{P} \in C^*$ and executed production processes $x_j \in T_j$, (one for each firm j, $j \in \{1, \dots, j_0\}$), the income or capital of the agent is defined by

$$\mathcal{K}(\mathcal{P}, w, \theta; x_1, \dots, x_{j_0}) := \mathcal{P}(w) + \sum_{j=1}^{j_0} \theta_j \mathcal{G}(x_j, \mathcal{P}),$$

where the first term denotes the value of the initial endowment and the second term denotes the total value received from shares in the gain that the firms obtain by executing the chosen production processes from their production technologies. This income represents the value which is available to him, so he is allowed to choose the most preferable element which has a value which is less than or equal to this income. Hence, at given pricing function $\mathcal{P} \in C^*$, and given production processes $x_1, \dots, x_{j_0}$, the budget set and the demand set of this agent are given by

$$B(\mathcal{P}, w, \theta; x_1, \dots, x_{j_0}) := \{x \in C \mid \mathcal{P}(x) \le \mathcal{K}(\mathcal{P}, w, \theta; x_1, \dots, x_{j_0})\},$$

and

$$D(\mathcal{P}, w, \theta, \succeq; x_1, .., x_{j_0}) :=$$

$$\{x \in B(\mathcal{P}, w, \theta; x_1, .., x_{j_0}) \mid \forall y \in B(\mathcal{P}, w, \theta; x_1, .., x_{j_0}) : x \succeq y\},$$

respectively.

With the concepts thus far introduced, we are ready to state the third model, concerning a private ownership economy. Furthermore, we give the corresponding equilibrium concept. In Section 4.8, we give two existence theorems related to this model, and we discuss the additional mathematical assumptions made in these theorems.

Model C: private ownership economy

Primary concepts:

- the set of all exchangeable objects is represented by the salient space $C := C_{\text{prod}} \oplus C_{\text{cons}}$, where the non-trivial salient space C_{prod} represents the set of all production bundles, and the non-trivial salient space C_{cons} represents the set of all consumption bundles;

- the set of admissible pricing functions is represented by a subset P of the adjoint $C^* = C^*_{\text{prod}} \oplus C^*_{\text{cons}}$ of the salient space C;

- there is a finite number, j_0, of firms, where for every $j \in \{1, \dots, j_0\}$, firm j is characterised by production technology $T_j \subset C$;

- there is a finite number, i_0, of agents, where for every $i \in \{1, \dots, i_0\}$, agent i is characterised by an initial endowment $w_i = (w_i^{\text{prod}}, w_i^{\text{cons}}) \in C$, a preference relation $\succeq_i$ defined on C, and share vector θ_i in the gains of each firm. These share rates satisfy $\forall j \in \{1, \dots, j_0\}$:

$$\sum_{i=1}^{i_0} \theta_{ij} = 1,$$

  and $\forall i \in \{1, \dots, i_0\}\ \forall j \in \{1, \dots, j_0\} : \theta_{ij} \geq 0$.

Secondary concepts:

- for every $j \in \{1, \dots, j_0\}$, and for every $\mathcal{P} \in P$, the supply set of firm j is given by

$$S_j^C(\mathcal{P}) := S(\mathcal{P}, T_j) = \{x \in T_j \mid \forall y \in T_j : \mathcal{G}(x, \mathcal{P}) \geq \mathcal{G}(y, \mathcal{P})\};$$

- for every $i \in \{1, \dots, i_0\}$, for every $\mathcal{P} \in P$, the budget set of agent i at given executed production processes $x_1, \dots x_{j_0} \in T_1 \times \cdots \times T_{j_0}$, is given by

$$B_i^C(\mathcal{P}, x_1, \dots, x_{j_0}) := B(\mathcal{P}, w_i, \theta_i; x_1, \dots, x_{j_0});$$

- for every $i \in \{1, \dots, i_0\}$, for every $\mathcal{P} \in P$, the demand set of agent i at given executed production processes $x_1, \dots x_{j_0} \in T_1 \times \cdots \times T_{j_0}$, is given by

$$D_i^C(\mathcal{P}, x_1, \dots, x_{j_0}) := D(\mathcal{P}, w_i, \theta_i, \succeq_i; x_1, \dots, x_{j_0}).$$

Model C (continued)

A Walrasian equilibrium for Model C is a $(j_0 + i_0 + 1)$-tuple

$$(s_1, \ldots, s_{j_0}, d_1, \ldots, d_{i_0}, \mathcal{P}_{\text{eq}}) \in C^{(j_0+i_0)} \times P$$

such that

- $\mathcal{P}_{\text{eq}} \neq 0$,
- $s_j \in S_j^C(\mathcal{P}_{\text{eq}})$ for all $j \in \{1, \ldots, j_0\}$,
- $d_i \in D_i^C(\mathcal{P}_{\text{eq}}, s_1, \ldots, s_{j_0})$ for all $i \in \{1, \ldots, i_0\}$,
- $\sum_{i=1}^{i_0} d_i + \sum_{j=1}^{j_0} (s_j^{\text{prod}}, 0^{\text{cons}}) \leq_C \sum_{i=1}^{i_0} w_i + \sum_{j=1}^{j_0} (0^{\text{prod}}, s_j^{\text{cons}})$.

In words: a j_0-tuple of production plans (to be executed) for each firm, an i_0-tuple of exchangeable objects for each agent, and a pricing function, form a Walrasian equilibrium if the pricing function is nonzero, if each firm maximises profit, and if each agent chooses a most preferable object in his budget set at the given pricing function. Furthermore, the total demand (including the production bundles needed for production) is smaller than or equal to the total supply (after production).

We call $\mathcal{P}_{\text{eq}} \in P \setminus \{0\}$, satisfying the conditions in the above definition, a (Walrasian) equilibrium pricing function.

4.6.1 Remark. The market clearance of a Walrasian equilibrium for Model C, is equivalent with

$$\begin{cases} \sum_{i=1}^{i_0} d_i^{\text{prod}} + \sum_{j=1}^{j_0} s_j^{\text{prod}} \leq_{\text{prod}} \sum_{i=1}^{i_0} w_i^{\text{prod}}, \\ \sum_{i=1}^{i_0} d_i^{\text{cons}} \leq_{\text{cons}} \sum_{i=1}^{i_0} w_i^{\text{cons}} + \sum_{j=1}^{j_0} s_j^{\text{cons}}, \end{cases}$$

i.e., the total demand of the agents for production bundles plus the production bundles needed for production must not exceed the available production bundles of the total initial endowment. Furthermore, the total demand of the agents for consumption bundles must not exceed the total initial endowment regarding consumption bundles plus the consumption bundles created by the executed production processes. ◇

4.7 Agents revisited

In order to obtain a model in which agents show a disinterest for production bundles, assuming that they are unable to consume production bundles (for instance due to the absence of a "second time period" in which the agents are able to exchange purchased production bundles), we adapt the definition of the primary concept of agents. We assume that the preference relations of the agents are not defined on the direct sum $C_{\text{prod}} \oplus C_{\text{cons}}$, but only on the set C_{cons} of consumption bundles.

So, regarding the primary concepts, we consider an economy in which two different types of bundles of trade occur: production bundles and consumption bundles. Bundles of both types may be present in the initial endowment of the agents, however only consumption bundles can be consumed by economic agents. Their preference relation is defined on the set of consumption bundles, only.

Let j_0 denote the number of production technologies, as defined in Definition 4.5.3, in the economy, and let for every $j \in \{1, \dots, j_0\}$ firm j be characterised by production technology $T_j \subset C$. As mentioned above, an agent chooses only a consumption bundle, for example because production bundles are of no use to him. In this setting, an economic agent is characterised by an element $w = (w^{\text{prod}}, w^{\text{cons}}) \in C$, representing his initial endowment, a preference relation $\succeq$ defined on C_{cons}, modelling his taste over the set of all consumption bundles, and share rates θ_j, $j \in \{1, \dots, j_0\}$, introduced similarly as in Section 4.6.

For a given pricing function $\mathcal{P} \in C^*$ and given executed production processes $x_j \in T_j$, for every $j \in \{1, \dots, j_0\}$, the income and the budget set of this agent are given by

$$\mathcal{K}(\mathcal{P}, w, \theta; x_1, \dots, x_{j_0}) := \mathcal{P}(w) + \sum_{j=1}^{j_0} \theta_j \mathcal{G}(x_j, \mathcal{P}), \text{ and}$$

$$B_{\text{cons}}(\mathcal{P}, w; x_1, \dots, x_{j_0}) := \{x^{\text{cons}} \in C_{\text{cons}} \mid \mathcal{P}^{\text{cons}}(x^{\text{cons}}) \leq \mathcal{K}(\mathcal{P}, w, \theta; x_1, \dots, x_{j_0})\}.$$

Consequently, the demand set $D_{\text{cons}}(\mathcal{P}, w, \theta, \succeq; x_1, \dots, x_{j_0})$ of this agent is equal to

$$\{x^{\text{cons}} \in B_{\text{cons}}(\mathcal{P}, w, \theta; x_1, \dots, x_{j_0}) \mid \forall y^{\text{cons}} \in B_{\text{cons}}(\mathcal{P}, w, \theta; x_1, \dots, x_{j_0}) : x^{\text{cons}} \succeq y^{\text{cons}}\}.$$

We emphasise that both the budget set and the demand set are subsets of C_{cons}.

Having introduced all four primary concepts for a model of a private ownership economy, we now come to the description of Model D.

In Section 4.8, we give an existence theorem related to this model, Theorem D, and we discuss the additional mathematical assumptions made in that theorem.

Model D: preferences on the consumption bundles

Primary concepts:

- the set of all exchangeable objects is represented by the salient space $C := C_{\text{prod}} \oplus C_{\text{cons}}$, where the non-trivial salient space C_{prod} represents the set of all production bundles, and the non-trivial salient space C_{cons} represents the set of all consumption bundles;
- the set of admissible pricing functions is represented by a subset P of the adjoint $C^* = C^*_{\text{prod}} \oplus C^*_{\text{cons}}$ of the salient space C;
- there is a finite number, j_0, of firms, where for every $j \in \{1, \ldots, j_0\}$, firm j is characterised by production technology $T_j \subset C$;
- there is a finite number, i_0, of agents, where for every $i \in \{1, \ldots, i_0\}$, agent i, is characterised by an initial endowment $w_i = (w_i^{\text{prod}}, w_i^{\text{cons}}) \in C$, a preference relation $\succeq_i$ defined on C_{cons} and share vector θ_i in the gains of each firm. These shares satisfy $\forall j \in \{1, \ldots, j_0\}$:
$$\sum_{i=1}^{i_0} \theta_{ij} = 1,$$
and $\forall i \in \{1, \ldots, i_0\}\ \forall j \in \{1, \ldots, j_0\} : \theta_{ij} \geq 0$.

Secondary concepts:

- for every $j \in \{1, \ldots, j_0\}$, and for every $\mathcal{P} \in P$, the supply set of firm j is given by
$$S_j^D(\mathcal{P}) := S(\mathcal{P}, T_j) = \{x \in T_j \mid \forall y \in T_j : \mathcal{G}(x, \mathcal{P}) \geq \mathcal{G}(y, \mathcal{P})\};$$
- for every $i \in \{1, \ldots, i_0\}$, for every $\mathcal{P} \in P$, the budget set of agent i at given executed production processes $x_1, \ldots x_{j_0} \in T_1 \times \cdots \times T_{j_0}$, is given by
$$B_i^D(\mathcal{P}, x_1, \ldots, x_{j_0}) := B_{\text{cons}}(\mathcal{P}, w_i, \theta_i; x_1, \ldots, x_{j_0});$$
- for every $i \in \{1, \ldots, i_0\}$, for every $\mathcal{P} \in P$, the demand set of agent i at given executed production processes $x_1, \ldots x_{j_0} \in T_1 \times \cdots \times T_{j_0}$, is given by
$$D_i^D(\mathcal{P}, x_1, \ldots, x_{j_0}) = D_{\text{cons}}(\mathcal{P}, w_i, \theta_i, \succeq_i; x_1, \ldots, x_{j_0}).$$

Model D (continued)

A Walrasian equilibrium for Model D, is an $(j_0 + i_0 + 1)$-tuple

$$(s_1, \ldots, s_{j_0}, d_1^{\text{cons}}, \ldots, d_{i_0}^{\text{cons}}, \mathcal{P}_{\text{eq}}) \in C^{j_0} \times (C_{\text{cons}})^{i_0} \times P$$

such that

- $\mathcal{P}_{\text{eq}} \neq 0$,
- $s_j \in S_j^D(\mathcal{P}_{\text{eq}})$ for all $j \in \{1, \ldots, j_0\}$,
- $d_i^{\text{cons}} \in D_i^D(\mathcal{P}_{\text{eq}}, s_1, \ldots, s_{j_0})$ for all $i \in \{1, \ldots, i_0\}$,
- $\sum_{i=1}^{i_0}(0^{\text{prod}}, d_i^{\text{cons}}) + \sum_{j=1}^{j_0}(s_j^{\text{prod}}, 0^{\text{cons}}) \leq_C \sum_{i=1}^{i_0} w_i + \sum_{j=1}^{j_0}(0^{\text{prod}}, s_j^{\text{cons}})$.

In words, a j_0-tuple of production plans (to be executed) for each firm, an i_0-tuple of exchangeable objects for each agent, and a pricing function, form a Walrasian equilibrium if the pricing function is nonzero, if each firm maximises profit and if each agent chooses a most preferable consumption bundle in his budget set at the given pricing function. Furthermore, the total demand (i.e., the consumption bundles demanded by the agents and the production bundles needed for production) is smaller than or equal to the total supply (after production).

We call $\mathcal{P}_{\text{eq}} \in P \setminus \{0\}$, satisfying the conditions in the above definition, a (Walrasian) equilibrium pricing function.

4.7.1 Remark. The market clearance of Walrasian equilibrium for Model D is similar to the corresponding item of Model C, with the exception that here $d_i^{\text{prod}} = 0$ for all $i \in \{1, \ldots, i_0\}$:

$$\begin{cases} \sum_{j=1}^{j_0} s_j^{\text{prod}} & \leq_{\text{prod}} & \sum_{i=1}^{i_0} w_i^{\text{prod}}, \\ \sum_{i=1}^{i_0} d_i^{\text{cons}} & \leq_{\text{cons}} & \sum_{i=1}^{i_0} w_i^{\text{cons}} + \sum_{j=1}^{j_0} s_j^{\text{cons}}, \end{cases}$$

i.e., the total input needed to execute the production processes must not exceed the total initial endowment regarding production bundles. Furthermore, the total demand of the agents must not exceed the available consumption bundles of the total initial endowment plus the consumption bundles created by the production processes. ◇

4.8 Equilibrium existence theorems

In this section, we state equilibrium existence theorems for the models introduced in this chapter. Although all these models are presented in the general terms of salient spaces, existence of a corresponding equilibrium situation will be guaranteed only if some assumptions are made, of which the assumption that the salient space C, representing the set of all exchangeable objects, is finite-dimensional, is the strongest. Among other things, it guarantees that $\text{int}(C^*)$ is nonempty. Furthermore, the assumption that C is also reflexive implies that C^* separates the elements of C. An important conclusion is that every closed bounded set in C is $\tau(C, C^*)$-compact. In Chapter 5, we will use this to prove that every budget set corresponding with an interior pricing function is compact.

Despite this strong assumption regarding the dimension of C, we feel that the essential idea of our models is the use of the concept of salient space and concepts related to it. Forcing ourselves to cope with this general model structure, we have to apply an analysis and techniques which may be of use when tackling models for economies where the finite-dimensionality restriction is not satisfied.

Theorem A1

Model A of a pure exchange economy admits a corresponding Walrasian equilibrium under the following assumptions:

Assumption A1.1 C is a finite-dimensional, reflexive salient space.

Assumption A1.2 $P = C^*$.

Assumption A1.3 For every $i \in \{1, \ldots, i_0\}$, preference relation $\succeq_i$ on C is

- **a)** monotonous: $\forall\, x, y \in C : x \geq_C y$ implies $x \succeq_i y$,
- **b)** strictly convex: $\forall\, x, y \in C, \tau \in (0, 1) : x \succeq_i y$ and $x \neq y$ imply $\tau x + (1 - \tau) y \succ_i y$,
- **c)** continuous: $\forall\, y \in C$ the sets $\{x \in C \mid x \succeq_i y\}$ and $\{x \in C \mid y \succeq_i x\}$ are $\tau(C, C^*)$-closed in C.

Assumption A1.4 $\forall\, \mathcal{P} \in C^* \setminus \{0\} : \mathcal{P}(w_{\text{total}}) > 0$.

As mentioned at the beginning of this section, Assumption A1.1 guarantees that every $\tau(C, C^*)$-bounded set (and also every $\tau(C^*, C)$-bounded set) is pre-compact.

Reflexivity of C implies that $V_+[C]$ is $\tau(V[C], C^*)$-closed in $V[C]$.

Assumption A1.2 states that a pricing function can be any element of the adjoint C^*, thus implying that "prices are flexible".

Assumption A1.3 is the salient space equivalence of the assumptions made on the preference relations of the agents in the neoclassical model of a pure exchange economy (cf. Section 1.2.1). It yields that every demand set is a singleton, thus implying the use of demand functions instead of demand sets. Note that continuity, stated in Assumption A1.3.c, is with respect to topology $\tau(C, C^*)$ and that monotony of the preference relations (Assumption A1.3.a) is with respect to the partial order relation $\leq_C$.

Finally, Assumption A1.4 is the salient space equivalence of the neoclassical assumption, related to the minimum income hypothesis, that the total initial endowment is strictly positive.

In case $C = \mathbb{R}^n_+$, the assumptions of Theorem A1 coincide with the assumptions Arrow and Debreu made (cf. Section 1.2.1). So, Theorem A1 can be seen as a true generalisation of the Arrow and Debreu equilibrium existence theorem.

The proof of Theorem A1 can be found in Section 5.4. We will see that the equilibrium pricing function of which the existence is proved is an element of $\text{int}(C^*)$, by construction. Also, we will see that Assumption A1.3 allows for the use of demand functions.

Similar to the neoclassical situation, we have made the assumption that in each model the consumption set of each agent is equal to the set C of all bundles of trade. However, we remark that all the equilibrium existence theorems, presented in this section, still hold when every agent is assumed to have a consumption set that is a non-empty, closed, convex subset of C, of which the initial endowment of the agent is a saliently internal point.

The following equilibrium theorem, Theorem A2, also concerns Model A, but differs from the previous theorem in the following two aspects. Firstly, the monotony assumption (A1.3.a) is weakened to the non-saturation assumption (A2.3.a). Secondly, the assumption that the total initial endowment w_{total} is strictly positive (A1.4) is strengthened to the assumption that every initial endowment is strictly positive (A2.4).

Theorem A2

Model A of a pure exchange economy admits a corresponding Walrasian equilibrium under the following assumptions:

Assumption A2.1 C is a finite-dimensional, reflexive salient space.

Assumption A2.2 $P = C^*$.

Assumption A2.3 For every $i \in \{1, \dots, i_0\}$, preference relation $\succeq_i$ on C is

- **a)** non-saturated: $\forall\, x \in C\ \exists y \in C, y \neq x : y \succeq_i x$,
- **b)** strictly convex: $\forall\, x, y \in C, \tau \in (0,1) : x \succeq_i y$ and $x \neq y$ imply $\tau x + (1-\tau)y \succ_i y$,
- **c)** continuous: $\forall\, y \in C$ the sets $\{x \in C \mid x \succeq_i y\}$ and $\{x \in C \mid y \succeq_i x\}$ are $\tau(C, C^*)$-closed in C.

Assumption A2.4 $\forall i \in \{1, \dots, i_0\}\ \forall \mathcal{P} \in C^* \setminus \{0\} : \mathcal{P}(w_i) > 0$.

The proof of Theorem A2 can be found in Section 5.6. This proof allows for an equilibrium pricing function in the set $\mathrm{bd}(C^*)$. We explain by means of an example, how this can be achieved.

4.8.1 Example. Take $C = P = \mathbb{R}^2_+$. Consider an agent with initial endowment $w = (6, 1)$ and preference relation $\succeq$ on $\mathbb{R}^2_+$, for every $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2_+$, given by

$$(x_1, x_2) \succeq (y_1, y_2) \quad :\Longleftrightarrow \quad x_2 - (x_1 - 4)^2 \geq y_2 - (y_1 - 4)^2.$$

It is not difficult to check that this preference relation satisfies Assumption A2.3. The demand for this agent, at price vector $p_n = (\frac{1}{n}, 1)$ is equal to the bundle

$$(4 - \frac{1}{2n}, 1 + \frac{2}{n} + \frac{1}{2n^2}).$$

Furthermore, the demand of the agent at $p = (0, 1)$ is equal to the bundle $(4, 1)$. $\diamond$

The next theorem concerns Model B, the model of a pure exchange economy with price rigidities and rationing.

Theorem B

Model B of a pure exchange economy with price restrictions and rationing admits a constrained equilibrium under the following assumptions:

Assumption B.1: K is a closed solid pointed convex cone in a finite-dimensional inner product space V.

Assumption B.2: P is a closed convex subcone of $\text{int}(K^*) \cup \{0\}$ and satisfies $P \cap \text{int}(K) \neq \emptyset$, where

$$K^* = \{x \in V \mid \forall k \in K : \langle x, k \rangle \geq 0\}.$$

Assumption B.3: For every $i \in \{1, \ldots, i_0\}$, preference relation $\succeq_i$ is

a) monotonous: $\forall\, x, y \in K : x \leq_K y$ implies $y \succeq_i x$,

b) strictly convex: $\forall x, y \in K$, $\tau \in (0, 1) : x \succeq_i y$ and $x \neq y$ imply $\tau x + (1 - \tau) y \succ_i y$,

c) continuous: $\forall y \in K$ the sets $\{x \in K \mid x \succeq_i y\}$ and $\{x \in K \mid y \succeq_i x\}$ are closed in K.

Assumption B.4: For every $i \in \{1, \ldots, i_0\}$ initial endowment w_i satisfies $w_i \in \text{int}(K)$.

Assumption B.1 is the regular assumption, already discussed below Theorem A1. Assumption B.2 is a technical condition on the restricted price set P. Assumption B.3 is similar to Assumption A1.3, and Assumption B.4 is equivalent with Assumption A2.4. Theorem B is proved in Section 5.13.

The next theorem is related to Model C, the model of a private ownership economy.

Theorem C1

Model C of a private ownership economy admits a corresponding Walrasian equilibrium under the following assumptions:

Assumption C1.1 C is a finite-dimensional, reflexive salient space.

Theorem C1 (continued)

Assumption C1.2 $P = C^*$.

Assumption C1.3 For every $j \in \{1, \ldots, j_0\}$, production technology T_j satisfies

a) $T_j = \bigcup_{e \in E(T_j)} F_e$.

b) T_j is closed with respect to topology $\tau(C, C^*)$,

c) if $e_1, e_2 \in E(T_j)$, $e_1 \neq e_2$, $\tau \in (0,1)$ then $\tau e_1 + (1-\tau)e_2 \in T_j$ and $\tau e_1 + (1-\tau)e_2 \notin E(T_j)$.

Assumption C1.4 For every $i \in \{1, \ldots, i_0\}$, preference relation $\succeq_i$ is

a) monotonous: $\forall\, x, y \in C : x \leq_C y$ implies $y \succeq_i x$,

b) strictly convex: $\forall x, y \in C$, $\tau \in (0,1) : x \succeq_i y$ and $x \neq y$ imply $\tau x + (1-\tau)y \succ_i y$,

c) continuous: $\forall y \in C$ the sets $\{x \in C \mid x \succeq_i y\}$ and $\{x \in C \mid y \succeq_i x\}$ are $\tau(C, C^*)$-closed in C.

Assumption C1.5 Furthermore,

a) $\exists \mathcal{P} \in \mathrm{int}(C^*)\ \forall j \in \{1, \ldots, j_0\} : S_j^C(\mathcal{P}) \neq \emptyset$,

b) for all $\mathcal{P}^{\mathrm{cons}} \in C^*_{\mathrm{cons}} \setminus \{0^{\mathrm{cons}}\}$ satisfying $\forall i \in \{1, \ldots, i_0\} : \mathcal{P}^{\mathrm{cons}}(w_i^{\mathrm{cons}}) = 0$, there is $j \in \{1, \ldots, j_0\}$ and $x \in T_j$ such that $\mathcal{P}^{\mathrm{cons}}(x^{\mathrm{cons}}) > 0$,

c) $\forall \mathcal{P}^{\mathrm{prod}} \in C^*_{\mathrm{prod}} \setminus \{0^{\mathrm{prod}}\} : \mathcal{P}^{\mathrm{prod}}(\sum_{i=1}^{i_0} w_i^{\mathrm{prod}}) > 0$.

Assumption C1.1 and Assumption C1.2 are the regular assumptions made in an economy without price rigidities, and already discussed with regard to Theorem A1. The interpretation of Assumption C1.3.a is that for every production process $x \in T_j$, there is an efficient production process $e \in E(T_j)$ such that $x \in F_e$, i.e., x is the result of e being an efficient production process and the possibility of free disposal. Assumption C1.3.b, replaces the "strict convexity condition" made by Arrow and Debreu (cf. Section 1.2.2), and guarantees uniqueness of the supply. Assumptions C1.3.b and c guarantee the continuity of the supply functions.
Similar to Assumption A1.4, Assumption C1.4 implies that we deal with continuous demand functions.
Assumption C1.5.a yields that the total supply function has a non-empty domain. Assumption C1.5.b states that if $\mathcal{P}^{\mathrm{cons}} \neq 0$ is such that $\mathcal{P}^{\mathrm{cons}}(w_i^{\mathrm{cons}}) = 0$ for every $i \in$

$\{1,\ldots,i_0\}$, there is a firm $j \in \{1,\ldots,j_0\}$ which can produce something with positive value at $\mathcal{P}^{\text{cons}}$. If this were not the case, every agent would have zero income at pricing function $(0^{\text{prod}}, \mathcal{P}^{\text{cons}})$. Assumption C1.5.c requires only that the production part $\sum_{i=1}^{i_0} w_i^{\text{prod}}$ of the total initial endowment is strictly positive. This is an assumption more natural than the one which is usually made (cf. [9], Section 1.2.2), stating that the total initial endowment w_{total} is strictly positive. Hence, in this model, the existence of a Walrasian equilibrium is guaranteed even if $\sum_{i=1}^{i_0} w_i^{\text{cons}} = 0^{\text{cons}}$. In this situation, all consumption bundles have to be produced from the available production bundles. Assumption C1.5.b guarantees that production actually takes place. Moreover, Assumptions C1.5.b and C1.5.c can be replaced by the weaker, but rather technical Assumption C1.5.b'.

Assumption C1.5.b') For every sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ in the domain of the total supply function with nonzero limit, there is $i \in \{1,\ldots,i_0\}$ such that

$$\limsup_{n\to\infty} \mathcal{K}_i^C(\mathcal{P}_n) = \limsup_{n\to\infty} \mathcal{P}_n(w_i) + \sum_{j=1}^{j_0} \theta_{ij}\mathcal{G}(\mathcal{S}_j(\mathcal{P}_n), \mathcal{P}_n) > 0.$$

Here $\mathcal{S}_j$ denotes the supply function of firm j, $j \in \{1,\ldots,j_0\}$.

4.8.2 Lemma. *Assumptions C1.5.b and C1.5.c imply Assumption C1.5.b'.*

Proof.
Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be a sequence in the domain of the total supply function, with limit $\mathcal{P} \in C^* \setminus \{0\}$. We have to prove

$$\exists \hat{\imath} \in \{1,\ldots,i_0\} : \limsup_{n\to\infty} \Big(\underbrace{\mathcal{P}_n^{\text{prod}}(w_{\hat{\imath}}^{\text{prod}})}_{\geq 0} + \underbrace{\mathcal{P}_n^{\text{cons}}(w_{\hat{\imath}}^{\text{cons}})}_{\geq 0} + \sum_{j=1}^{j_0} \theta_{\hat{\imath}j} \underbrace{\mathcal{G}(\mathcal{S}_j(\mathcal{P}_n), \mathcal{P}_n)}_{\geq 0} \Big) > 0.$$

Since, by Assumption C1.5.c, $\sum_{i=1}^{i_0} w_i^{\text{prod}} \in \text{int}(C_{\text{prod}})$, we may as well assume $\mathcal{P}^{\text{prod}} = 0^{\text{prod}}$. Furthermore, we may as well assume that $\forall i \in \{1,\ldots,i_0\} : \mathcal{P}^{\text{cons}}(w_i^{\text{cons}}) = 0$. By Assumption C1.5.b, $\exists \hat{\jmath} \in \{1,\ldots,j_0\}\ \exists x \in T_{\hat{\jmath}} : \mathcal{P}^{\text{cons}}(x^{\text{cons}}) > 0$. The continuity of the function $\mathcal{G}$ yields $\exists n_0 \in \mathbb{N}\ \forall n > n_0 : \mathcal{G}(\mathcal{S}_{\hat{\jmath}}(\mathcal{P}_n), \mathcal{P}_n) \geq \mathcal{G}(x, \mathcal{P}_n) > \frac{1}{2}\mathcal{G}(x, \mathcal{P}) > 0$. Take $\hat{\imath} \in \{1,\ldots,i_0\}$ such that $\theta_{\hat{\imath}\hat{\jmath}} \neq 0$ and the proof is done. □

We remark that Assumption C1.5.b' is implied by Assumption A1.4:

$$\forall \mathcal{P} \in C^* \setminus \{0\} : \mathcal{P}(\sum_{i=1}^{i_0} w_i) > 0.$$

The proof of Theorem C1 can be found in Section 5.8. There, we will see that, by construction, the equilibrium pricing function of which the existence is proved, is an element of $\text{int}(C^*)$.

The difference between the next equilibrium theorem and the previous theorem lies in the assumptions concerning the production technologies, only.

Theorem C2

Model C of a private ownership economy admits a corresponding Walrasian equilibrium under the following assumptions:

Assumption C2.1 C is a finite-dimensional, reflexive salient space.

Assumption C2.2 $P = C^*$.

Assumption C2.3 For every $j \in \{1, \ldots, j_0\}$, production technology T_j satisfies

a) $T_j = \bigcup_{e \in E(T_j)} F_e$,

b) $E(T_j)$ is closed with respect to topology $\tau(C, C^*)$,

c) if $e_1, e_2 \in E(T_j)$, $e_1 \neq e_2$, $\tau \in (0,1)$ then $\tau e_1 + (1-\tau)e_2 \in \operatorname{int}(T_j)$.

Assumption C2.4 For every $i \in \{1, \ldots, i_0\}$, preference relation $\succeq_i$ is

a) monotonous: $\forall\, x, y \in C : x \leq_C y$ implies $y \succeq_i x$,

b) strictly convex: $\forall x, y \in C$, $\tau \in (0,1) : x \succeq_i y$ and $x \neq y$ imply $\tau x + (1-\tau)y \succ_i y$,

c) continuous: $\forall y \in C$ the sets $\{x \in C \mid x \succeq_i y\}$ and $\{x \in C \mid y \succeq_i x\}$ are $\tau(C, C^*)$-closed in C.

Assumption C2.5 Furthermore,

a) $\exists \mathcal{P} \in \operatorname{int}(C^*)\ \forall j \in \{1, \ldots, j_0\} : S_j^C(\mathcal{P}) \neq \emptyset$,

b) for all $\mathcal{P}^{\text{cons}} \in C^*_{\text{cons}} \setminus \{0^{\text{cons}}\}$ satisfying $\forall i \in \{1, \ldots, i_0\} : \mathcal{P}^{\text{cons}}(w_i^{\text{cons}}) = 0$, there is $j \in \{1, \ldots, j_0\}$ and $x \in T_j$ such that $\mathcal{P}^{\text{cons}}(x^{\text{cons}}) > 0$,

c) $\forall \mathcal{P}^{\text{prod}} \in C^*_{\text{prod}} \setminus \{0^{\text{prod}}\} : \mathcal{P}^{\text{prod}}(\sum_{i=1}^{i_0} w_i^{\text{prod}}) > 0$.

Similar to the previous situation, Assumptions C2.5.b and C2.5.c can be replaced by a weaker assumption.

Assumption C2.5.b') For every sequence $(\mathcal{P}_n)_{n \in \mathbb{N}}$ in the domain of the total supply function with nonzero limit, there is $i \in \{1, \ldots, i_0\}$ such that

$$\limsup_{n \to \infty} \mathcal{K}_i^C(\mathcal{P}_n) > 0.$$

As mentioned above Theorem C2, the assumptions on the production technologies of this theorem, are different from the corresponding ones concerning production technologies in Theorem C1. Firstly, Assumption C2.3.c is related to the interior of T_j, introduced in Definition 2.1.28, where the former version was not. Secondly, Assumption C2.3.b only requires closedness of $E(T_j)$ instead of closedness of T_j.

The connection between closedness of T_j and $E(T_j)$ is investigated in Lemmas 4.8.3, 4.8.4 and 4.8.5.

The following observations (Lemma 4.8.3, 4.8.4 and 4.8.5), concerning the difference between Assumption C1.3.b and C2.3.b, are made under the assumption that the salient space C is finite-dimensional and reflexive.

4.8.3 Lemma. *Let $T \subset C$ be a production technology satisfying*

$$T = \bigcup_{e \in E(T)} F_e.$$

Let $a^{\text{prod}} \in C_{\text{prod}}$. If T is $\tau(C, C^*)$-closed, then the set $\{x^{\text{cons}} \in C_{\text{cons}} \mid (a^{\text{prod}}, x^{\text{cons}}) \in T\}$ is $\tau(C, C^*)$-bounded.

Proof.
Since C_{cons} is a finite-dimensional salient space, $\text{int}(C_{\text{cons}}) \neq \emptyset$ (Corollary 2.2.11). Let $b_0^{\text{cons}} \in \text{int}(C_{\text{cons}})$. Suppose the set $\{x^{\text{cons}} \in C_{\text{cons}} \mid (a^{\text{prod}}, x^{\text{cons}}) \in T\}$ is unbounded, then, by Lemma 3.3.9, for every $n \in \mathbb{N}$ there exists $x_n^{\text{cons}} \in C_{\text{cons}}$ such that

$$\begin{cases} (a^{\text{prod}}, x_n^{\text{cons}}) \in T \\ x_n^{\text{cons}} \geq_{\text{cons}} n b_0^{\text{cons}}. \end{cases}$$

By Definition 4.5.3.c we find $(a^{\text{prod}}, n b_0^{\text{cons}}) \in T$ for all $n \in \mathbb{N}$. Since T is convex (Lemma 4.5.5) and contains $(0^{\text{prod}}, 0^{\text{cons}})$ (Definition 4.5.3.a), we find $\forall n \in \mathbb{N}$: $(\frac{1}{n} a^{\text{prod}}, b_0^{\text{cons}}) \in T$. Taking the limit for $n \to \infty$, the $\tau(C, C^*)$-closedness of T implies $(0^{\text{prod}}, b_0^{\text{cons}}) \in T$, which is in contradiction with Definition 4.5.3.b. □

4.8.4 Lemma. *Let T be a production technology satisfying*

$$T = \bigcup_{e \in E(T)} F_e.$$

Let the set $S \subset T$ satisfy $\exists a^{\text{prod}} \in C_{\text{prod}}\ \forall s \in S : s^{\text{prod}} \leq_{\text{prod}} a^{\text{prod}}$. If T is $\tau(C, C^*)$-closed, then S is $\tau(C, C^*)$-bounded.

Proof.
Let $s \in S$. By Definition 4.5.3.c, we find that $s^{\text{prod}} \leq_{\text{prod}} a^{\text{prod}}$ implies $(a^{\text{prod}}, s^{\text{cons}}) \in F_s \subset T$, so $S \subset \{x \in C \mid (a^{\text{prod}}, x^{\text{cons}}) \in T\}$. By the previous lemma we find that S is $\tau(C, C^*)$-bounded. □

4.8.5 Lemma. *Let T be a production technology, satisfying*

$$T = \bigcup_{e \in E(T)} F_e.$$

Let $E(T)$ be $\tau(C, C^*)$-closed, and assume every sequence $(e_n)_{n\in\mathbb{N}}$ in $E(T)$ satisfies

$$(e_n^{\text{prod}})_{n\in\mathbb{N}} \text{ is } \tau(C, C^*)\text{-bounded} \quad \Longrightarrow \quad (e_n)_{n\in\mathbb{N}} \text{ is } \tau(C, C^*)\text{-bounded.}$$

Then T is $\tau(C, C^*)$-closed.

Proof.
Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in T with limit $x \in C$. By assumption, we find a sequence $(e_n)_{n\in\mathbb{N}}$ in $E(T)$ satisfying $\forall n \in \mathbb{N} : x_n \in F_{e_n}$. Hence, $\forall n \in \mathbb{N} : x_n^{\text{prod}} \geq_{\text{prod}} e_n^{\text{prod}}$. Since the sequence $(e_n^{\text{prod}})_{n\in\mathbb{N}}$ is bounded, the assumption implies that $(e_n)_{n\in\mathbb{N}}$ is bounded. Without loss of generality, we may assume that $(e_n)_{n\in\mathbb{N}}$ is convergent with limit $e \in E(T)$. By the continuity of the order relations $\geq_{\text{prod}}$ and $\geq_{\text{cons}}$, and by Definition 4.5.3.c we find $x \in F_e \subset T$. □

In case $\text{int}(C_{\text{cons}}) \neq \emptyset$, the previous three lemmas imply that for a production technology T satisfying Assumptions C2.3.a and C2.3.b, the following two statements are equivalent:

- T is closed,
- "bounded input yields bounded output".

The proof of Theorem C2 can be found in Section 5.10. There, we will see that, by construction, the equilibrium pricing function of which the existence is proved, is an element of $\text{int}(C^*)$.

We end this section with the equilibrium existence theorem concerning Model D.

Theorem D

Model D of a private ownership economy admits a corresponding Walrasian equilibrium under the following assumptions:

Assumption D.1 $C = C_{\text{prod}} \oplus C_{\text{cons}}$ is a finite-dimensional, reflexive salient space.

Assumption D.2 $P = C^*$.

Assumption D.3 For every $j \in \{1, \ldots, j_0\}$, production technology T_j satisfies

- **a)** $T_j = \bigcup_{e \in E(T_j)} F_e$,
- **b)** $E(T_j)$ is closed with respect to topology $\tau(C, C^*)$,
- **c)** if $e_1, e_2 \in E(T_j)$, $e_1 \neq e_2$, $\tau \in (0,1)$ then $\tau e_1 + (1-\tau)e_2 \in \text{int}(T_j)$.

Assumption D.4 For every $i \in \{1, \ldots, i_0\}$, preference relation $\succeq_i$ is

- **a)** monotonous: $\forall\, x^{\text{cons}}, y^{\text{cons}} \in C_{\text{cons}} : x^{\text{cons}} \leq_{\text{cons}} y^{\text{cons}}$ implies $y^{\text{cons}} \succeq_i x^{\text{cons}}$,
- **b)** strictly convex: $\forall x^{\text{cons}}, y^{\text{cons}} \in C_{\text{cons}}$, $\tau \in (0,1) : x^{\text{cons}} \succeq_i y^{\text{cons}}$ and $x^{\text{cons}} \neq y^{\text{cons}}$ imply $\tau x^{\text{cons}} + (1-\tau) y^{\text{cons}} \succ_i y^{\text{cons}}$,
- **c)** continuous: $\forall y^{\text{cons}} \in C_{\text{cons}}$ the sets $\{x^{\text{cons}} \in C \mid x^{\text{cons}} \succeq_i y^{\text{cons}}\}$ and $\{x^{\text{cons}} \in C \mid y^{\text{cons}} \succeq_i x^{\text{cons}}\}$ are closed in C_{cons}.

Assumption D.5 Furthermore,

- **a)** $\exists \mathcal{P} \in \text{int}(C^*)\ \forall j \in \{1, \ldots, j_0\} : S_j^D(\mathcal{P}) \neq \emptyset$,
- **b)** for all $\mathcal{P}^{\text{cons}} \in C^*_{\text{cons}} \setminus \{0^{\text{cons}}\}$ satisfying $\forall i \in \{1, \ldots, i_0\} : \mathcal{P}^{\text{cons}}(w_i^{\text{cons}}) = 0$, there is $j \in \{1, \ldots, j_0\}$ and $x \in T_j$ such that $\mathcal{P}^{\text{cons}}(x^{\text{cons}}) > 0$,
- **c)** $\forall \mathcal{P}^{\text{prod}} \in C^*_{\text{prod}} \setminus \{0^{\text{prod}}\} : \mathcal{P}^{\text{prod}}(\sum_{i=1}^{i_0} w_i^{\text{prod}}) > 0$.

Similar to the previous two situations, Assumption D.5.b and D.5.c can be replaced by the following weaker assumption.

Assumption D.5.b') For every sequence $(\mathcal{P}_n)_{n \in \mathbb{N}}$ in the domain of the total supply function with nonzero limit, there is $i \in \{1, \ldots, i_0\}$ such that

$$\limsup_{n \to \infty} \mathcal{K}_i^D(\mathcal{P}_n) = \limsup_{n \to \infty} \mathcal{P}_n(w_i) + \sum_{j=1}^{j_0} \theta_{ij} \mathcal{G}(\mathcal{S}_j(\mathcal{P}_n), \mathcal{P}_n) > 0.$$

Here $\mathcal{S}_j$ denotes the supply function of production firm j that corresponds to

Model D.

Assumptions D.1 and D.2 are the regular assumptions on the salient space C and the price set P, discussed below Theorem A1. Similar to the assumptions regarding Theorem C2, Assumptions D.3 and D.4 imply that we can deal with supply and demand functions. However, it is possible that the supply functions are continuous on a domain which is larger than the domain defined in the proofs concerning the previous model. More specifically, the assumptions of Theorem D allow for zero value for certain production bundles with respect to an equilibrium pricing function. Typically, production bundles which can only be used to produce certain consumption bundles for which there is a cheaper way of producing them, will have zero-prices. As a consequence, several proofs of the stated lemmas and propositions differ from the corresponding ones concerning Theorem C2.

We remark that since the domain of the total supply function may contain pricing functions $\mathcal{P}$ satisfying $\mathcal{P}^{\text{prod}} \notin \text{int}(C^*)$, Assumption D.5.b' is slightly stronger than Assumption C2.5.b'.

The proof of Theorem D can be found in Section 5.11. There, we will see that the equilibrium pricing function of which the existence is proved, possibly is an element of $\text{bd}(C^*_{\text{prod}}) \oplus \text{int}(C^*)$.

Chapter 5

Proofs

Introduction

The goal of this chapter is to prove the equilibrium existence theorems, Theorems A1, A2, B, C1, C2 and D, stated in Section 4.8. We recall that in each of these theorems it is assumed that the salient space C, representing the set of all exchangeable objects, is finite-dimensional and reflexive. Hence, the salient topology $\tau(C, C^*)$ on C is generated by any element of $\text{int}(C^*)$, and topology $\tau(C^*, C)$ on C^* is generated by any element of $\text{int}(C)$. Throughout this chapter, we identify the salient space C and C^{**}, i.e., we identify $x \in C$ with its action $\mathcal{P}(x)$ on every $\mathcal{P} \in C^*$. To show this duality to full advantage, we use lower case letters, e.g. p, q, to denote elements of C^*. Furthermore, instead of $p(x)$, we write $[x, p]$ for every $p \in C^*$ and $x \in C$.

In Section 5.1, we explain the general structure of the proofs of the theorems that are stated in Section 4.8. We will describe the successive steps we use in these proofs, and explain in which section of this chapter the precise description of each step can be found. As indicated in the previous chapters, the adaption of Brouwer's Fixed Point Theorem for salient spaces (cf. Theorem 3.3.15) plays an important role in the proofs of this section. In Section 5.2, we will show that applying Theorem 3.3.15 to a specific function, which is built around the excess demand value function as defined in Section 5.1, results in a proof of the existence of an equilibrium pricing function.

5.1 Structure of the proofs

Roughly speaking, each proof of the existence theorems of Chapter 4 has the same structure. In this section, we explain this general structure in an informal way; for the exact definitions and constructions we refer to the other sections of this chapter.

The main path we follow, when proving one of the equilibrium existence theorems of Chapter 4, is the following.

Step a) If there is production (cf. Models C and D), then we **show that each firm has a continuous supply function**. In Sections 5.7 and 5.9, we give general conditions on the salient space representing all exchangeable objects and on the production technologies T_j, $j \in \{1, \dots, j_0\}$, under which, for every $j \in \{1, \dots, j_0\}$ and for every $p \in C^*$, the supply set $S_j(p)$ contains at most one element in $E(T_j)$. As a consequence, we can introduce supply functions on a suitable domain in C^* as follows. For every $j \in \{1, \dots, j_0\}$, the supply function $\mathcal{S}_j$ is defined such that $\mathcal{S}_j(p)$ denotes the unique element of the supply set $S_j(p)$, where p is an element of the domain $\{q \in C^* \mid S_j(q) \neq \emptyset\}$ of $\mathcal{S}_j$. Furthermore, we show that the conditions presented in Sections 5.7 and 5.9 imply that each supply function is continuous on its domain and we show certain limit behaviour of these functions. The imposed conditions on the salient space of all bundles of trade are implied by Assumptions C1.1, C2.1 and D.1. The conditions on the production technologies are implied by Assumptions C1.3, C2.3 and D.3. Hence, the conditions needed to take step a, are met in theorems C1, C2 and D.

Step b) We **show that each agent has a continuous demand function**. In Sections 5.3 and 5.5 we give general conditions on the salient space of all bundles of trade, on the preference relations $\succeq_i$, $i \in \{1, \dots, i_0\}$, and on the income functions $\mathcal{K}_i$, $i \in \{1, \dots, i_0\}$. For each model A, B, C and D, introduced in Chapter 4, and for every agent i, $i \in \{1, \dots, i_0\}$, the income function is given by

$$\begin{aligned} \mathcal{K}_i^A(p) &= \mathcal{K}_i^B(p) = [w_i, p], \\ \mathcal{K}_i^C(p) &= \mathcal{K}_i^D(p) = [w_i, p] + \sum_{j=1}^{j_0} \theta_{ij} \mathcal{G}(\mathcal{S}_j(p), p). \end{aligned}$$

Here, $\mathcal{S}_j$ denotes the supply function of firm j (This is why, in case of production, Step a has to be taken before Step b). We show that for every $i \in \{1, \dots, i_0\}$, and for every $p \in C^*$ for which the income function is defined, the demand set $D_i(p)$ contains at most one element. Hence, we can introduce demand functions on a suitable domain in C^*, by defining the demand function $\mathcal{D}_i$ of agent i, for every $i \in \{1, \dots, i_0\}$, as follows. For every $p \in \{q \in C^* \mid D_i(q) \neq \emptyset\}$, we let $\mathcal{D}_i(p)$ denote the unique element of the demand set $D_i(p)$. Also, we show that these demand functions are continuous, we show that Walras' law holds (i.e. the value of the total demand of the agents is equal to the total income of the agents), and we show some limit behaviour concerning these functions. The imposed conditions on the salient space of all bundles of trade are implied by Assumptions A1.1, A2.1, B.1, C1.1, C2.1 and D.1. The conditions concerning the agents are implied by Assumptions A1.3, A2.3, B.3, C1.4, C2.4 and D.4. Hence, the conditions needed to take step b, are met in each theorem of Section 4.8.

Step c) We **construct the excess demand value function** $\mathcal{Z}$. If there is production, then we define the total supply function $\mathcal{S}$ to be the sum of all individual supply functions. We define the total demand function $\mathcal{D}$ to be the sum of all individual demand functions. The function $\mathcal{Z}$ denotes the value, with respect to an arbitrary $q \in C^*$, of the total demand at pricing system p minus the value of the total supply at p. This function is for every model of Chapter 4 given by

$$\begin{aligned} \mathcal{Z}^A(p,q) = \mathcal{Z}^B(p,q) &= [\mathcal{D}(p), q] - [\sum_{i=1}^{i_0} w_i, q], \\ \mathcal{Z}^C(p,q) &= [\mathcal{D}(p), q] - \mathcal{G}(\mathcal{S}(p), q) - [\sum_{i=1}^{i_0} w_i, q], \\ \mathcal{Z}^D(p,q) &= [(0^{\text{prod}}, \mathcal{D}^{\text{cons}}(p)), q] - \mathcal{G}(\mathcal{S}(p), q) - [\sum_{i=1}^{i_0} w_i, q], \end{aligned}$$

where $\mathcal{G}(\mathcal{S}(p), q)$ denotes the profit of executing production process $\mathcal{S}(p) \in C$ at pricing function $q \in C^*$ (cf. page 118). Now, Lemma 2.3.23 implies that $p^{\text{eq}} \in P \setminus \{0\}$ is an equilibrium pricing function if and only if $\mathcal{Z}(p^{\text{eq}}, q) \leq 0$ for all $q \in C^*$.

Step d) We **construct an equilibrium function**. (We do this only for Models A, C and D.) We call a function $\mathcal{F} : C^* \to C^*$ an "equilibrium function" if $\mathcal{F}$ is continuous, and satisfies that precisely those $p \in C^* \setminus \{0\}$ for which there is $\alpha \geq 0$ such that $\mathcal{F}(p) = \alpha p$, are equilibrium pricing functions. In Section 5.2 we construct an equilibrium function from an excess demand value function: we define $\mathcal{F}$ by

$$\mathcal{F}(p) := \int_{L(x_0)} \max\{0, \mathcal{Z}(p,q)\} q \, d\mu(q),$$

where $L(x_0) := \{q \in C^* \mid [x_0, q] = 1\}$, where μ is the Lebesgue measure on $L(x_0)$ and where $x_0 \in \text{int}(C)$ can be taken arbitrarily.

Step e) We **use a fixed point argument to prove existence of an equilibrium**. To this end we prove a general theorem in Section 5.2. In case of Models A, C and D, Theorem 3.3.15 proves that if an equilibrium function exists, then there exists an equilibrium pricing function. For Model B, we use a stationary point argument to prove existence of an equilibrium pricing function.

In each of the following sections of this chapter, we concentrate on one of the steps of the above described structure at a time. As soon as we are ready to give the complete proof of one of the equilibrium existence theorems of Section 4.8, we give this proof in a separate section.

More precisely, in Section 5.3, we concentrate on Step b, i.e., the construction of demand functions, as needed in the proof of Theorems A1, B, C1, C2 and D. In Section 5.5, Step b is established for the proof of Theorem A2. This separate treatment is needed due to the different assumptions of Theorem A2, concerning the preference relations of the agents in Model A of a pure exchange economy.

In Section 5.7, we concentrate on Step a, i.e., the construction of supply functions, needed in the proof of Theorem C1. As explained in Section 4.8, there is a difference between assumptions concerning the firms in Theorem C1 and the assumptions concerning the firms in Theorems C2 and D. As a result, the supply functions derived for the proof of Theorem C1 are only defined for strictly positive pricing functions, whereas the supply functions used in the proofs of Theorems C2 and D can also be defined for pricing functions which assign zero value to some bundles of exchange. Section 5.9 deals with Step b, the construction of supply functions, as needed in the proof of Theorems C2 and D. The structure of this section is essentially the same as the structure of Section 5.7. However, some of the proofs presented in Section 5.9 are more elaborate than their counterparts in Section 5.7 due to the possibility of bundles of exchange with zero value.

The definition of the excess demand value function $\mathcal{Z}$, i.e., Step c, will be given for each theorem separately. Hence, Step c can be found in each of the Sections 5.4, 5.6, 5.8, 5.10 and 5.11.

The heart of each equilibrium existence proof concerns Step d and Step e. In the following section we prove a general theorem, Theorem 5.2.1, that realises these two final steps. Theorem 5.2.1 will be used in the proofs of Theorem A1, A2, C1, C2 and D.

The proof of Theorem B has a different structure, since it makes use of an inner product structure on the set of all bundles of exchange. In Section 5.12 we give a mathematical introduction to some notions concerning inner product spaces and hyperplanes. Thereafter, we give the proof of Theorem B, in Section 5.13.

5.2 Equilibrium functions

This section is devoted completely to the proof of the central theorem of this chapter. As mentioned in the previous section, Theorem 5.2.1 represents the heart of the proofs of Theorems A1, A2, C1, C2 and D. In order to be applicable to all the different environments of these theorems, we state Theorem 5.2.1 for general salient spaces R and Q. When this theorem is applied, R will be represented by the set of all bundles of trade, R^* will be represented by the set of all possible pricing functions and $Q \setminus \{0\}$ will be represented by the domain of the total demand function. Finally, $\mathcal{W}$ will be replaced by the excess demand value function $\mathcal{Z}$ as described in Step c of the previous section.

5.2.1 Theorem. *Let R be a finite-dimensional, reflexive salient space. Let Q be a salient subspace of R^* such that $Q \cap \operatorname{int}(R^*) \neq \emptyset$. Let $p_0 \in Q \cap \operatorname{int}(R^*)$ and let $\mathcal{W} : Q \setminus \{0\} \times R^* \to \mathbb{R}$ be a function for which*

- **I)** *$\mathcal{W}(p,p) = 0$ for all $p \in Q \setminus \{0\}$, and $\mathcal{W}(p, \alpha q) = \alpha \mathcal{W}(p,q)$ for all $\alpha \in \mathbb{R}_+$, for all $p \in Q \setminus \{0\}$ and for all $q \in R^*$.*
- **II)** *For every $p \in Q \setminus \{0\}$, the function $q \mapsto \mathcal{W}(p,q)$ is continuous on R^*.*
- **III)** *There is $x_0 \in \operatorname{int}(R)$ such that for every $p_1 \in Q \setminus \{0\}$ and for every $\varepsilon > 0$, there is $\delta > 0$ such that for all $p_2 \in Q \setminus \{0\}$:*

  $$d(p_1,p_2) < \delta \implies \max\{|\mathcal{W}(p_1,q) - \mathcal{W}(p_2,q)| \mid q \in \mathbb{R}^* \text{ with } [x_0,q] = 1\} < \varepsilon,$$

  where, $d : R^* \times R^* \to \mathbb{R}_+$ is a salient metric on R^*.
- **IV)** *There is $\xi_0 > 0$ such that for any sequence $(p_n)_{n\in\mathbb{N}}$ in $Q \setminus \{0\}$ with limit $p \in R^* \setminus Q$, there is $n \in \mathbb{N}$ such that $\mathcal{W}(p_n, p_0) > \xi_0$.*

Then there is $p_* \in Q$ with $[x_0, p_*] = 1$ such that for all $q \in R^* : \mathcal{W}(p_*, q) \leq 0$.

5.2.2 Remark. In fact, for the proof of Theorem 5.2.1, it is sufficient that Q is a subset of R^*, that is closed under scalar multiplication over $\mathbb{R}^+$. ◇

The remaining part of this section is devoted tot he proof of Theorem 5.2.1.

Choose $x_0 \in \operatorname{int}(R)$ as indicated in Assumption 5.2.1.III. (Observe that by Corollary 2.2.11, the finite-dimensionality of R implies that $\operatorname{int}(R) \neq \emptyset$.) Then Corollary 3.3.8.c and the reflexivity of R^* imply that the set $L(x_0) := \{q \in R^* \mid [x_0,q] = 1\}$ is compact. Because of Assumption 5.2.1.II, we can define the function $\mathcal{F}_0 : Q \setminus \{0\} \to R^*$ by

$$\mathcal{F}_0(p) := \int\limits_{L(x_0)} \max\{0, \mathcal{W}(p,q)\} q \, d\mu(q). \tag{5.1}$$

Here, μ is the Lebesgue measure on $L(x_0)$. Note that

$$\forall p \in Q \setminus \{0\} : \mathcal{W}(p, \mathcal{F}_0(p)) \geq 0. \tag{5.2}$$

In order to establish Step d and e as described in the previous section, we want to obtain a continuous function $\mathcal{F} : R^* \to R^*$ that extends $\mathcal{F}_0$ (that is, in case $Q \neq R^*$). Then Theorem 3.3.15 states that there is a $p \in R^*$ satisfying $\mathcal{F}(p) = \alpha p$ for some $\alpha \in \mathbb{R}_+$. This will yield the p_* required in Theorem 5.2.1.

Choose ξ_0 as indicated in Assumption 5.2.1.IV, and define the sigma-oidal function $\eta : \mathbb{R} \to [0, 1]$ by

$$\eta(\xi) := \begin{cases} 0 & \text{if } \xi \leq 0 \\ \frac{\xi}{\xi_0} & \text{if } 0 < \xi < \xi_0 \\ 1 & \text{if } \xi_0 \leq \xi. \end{cases} \tag{5.3}$$

Note that

$$\forall \xi \in \mathbb{R} : \xi\eta(\xi) \geq 0, \text{ and} \tag{5.4}$$

$$\xi\eta(\xi) = 0 \text{ if and only if } \xi \leq 0. \tag{5.5}$$

By assumption, $p_0 \in Q \cap \operatorname{int}(R^*)$. The function $\mathcal{F} : R^* \to R^*$ is defined by

$$\mathcal{F}(p) := \begin{cases} (1 - \eta(\mathcal{W}(p, p_0)))\mathcal{F}_0(p) + \eta(\mathcal{W}(p, p_0))p_0 & p \in Q \setminus \{0\} \\ p_0 & p \in (R^* \setminus Q) \cup \{0\}. \end{cases} \tag{5.6}$$

Next, we prove that the function $\mathcal{F} : R^* \to R^*$ is an equilibrium function as defined in Step d of Section 5.1.

5.2.3 Lemma. *Let $p \in R^*$. Then*

$$\mathcal{F}(p) = 0 \iff \exists\, \alpha \geq 0 : \mathcal{F}(p) = \alpha p \iff \begin{cases} p \in Q \setminus \{0\} \text{ and} \\ \forall q \in R^* : \mathcal{W}(p, q) \leq 0. \end{cases}$$

Proof.
Suppose $p \in Q \setminus \{0\}$ and $\forall q \in R^* : \mathcal{W}(p, q) \leq 0$. Then, by (5.1), $\mathcal{F}_0(p) = 0$, and by (5.3), $\eta(\mathcal{W}(p, p_0)) = 0$. By (5.6), we conclude that $\mathcal{F}(p) = 0$.
For the converse, suppose $\mathcal{F}(p) = \alpha p$ for some $\alpha \geq 0$. From (5.6) and the fact that Q is closed under scalar multiplication over $\mathbb{R}_+$, it follows that $p \in Q \setminus \{0\}$. Assumption I of Theorem 5.2.1 yields

$$\mathcal{W}(p, \mathcal{F}(p)) = \mathcal{W}(p, \alpha p) = \alpha\mathcal{W}(p, p) = 0.$$

By (5.6), (5.2) and (5.4) , we find

$$0 = \mathcal{W}(p, \mathcal{F}(p)) = \underbrace{(1 - \eta(\mathcal{W}(p, p_0)))\mathcal{W}(p, \mathcal{F}_0(p))}_{\geq 0} + \underbrace{\eta(\mathcal{W}(p, p_0))\mathcal{W}(p, p_0)}_{\geq 0}.$$

Clearly,

$$(1-\eta(\mathcal{W}(p,p_0)))\mathcal{W}(p,\mathcal{F}_0(p))=0 \tag{5.7}$$

and

$$\eta(\mathcal{W}(p,p_0))\mathcal{W}(p,p_0)=0. \tag{5.8}$$

By (5.8) and (5.5) we find $\mathcal{W}(p,p_0)\leq 0$, hence, using the definition of η, (5.7) implies

$$0=\mathcal{W}(p,\mathcal{F}_0(p))=\int\limits_{L(x_0)}\max\{0,\mathcal{W}(p,q)\}\mathcal{W}(p,q)d\mu(q).$$

So, we conclude that for all $q\in L(x_0):\mathcal{W}(p,q)\leq 0$. Assumption A implies $\forall q\in R^*:\mathcal{W}(p,q)\leq 0$. □

We want to use Theorem 3.3.15 to prove that $\exists p\in R^*\ \exists\alpha\geq 0:\mathcal{F}(p)=\alpha p$. Hence, we need to prove that the function $\mathcal{F}$ is continuous on $R^*\setminus\{0\}$, and for this, we need the following lemma.

5.2.4 Lemma. *The function $\mathcal{F}_0$ is continuous on $Q\setminus\{0\}$ with respect to $\tau(Q,R)$.*

Proof.
Recall the definition of x_0 and $L(x_0)$ in the definition of the function $\mathcal{F}_0$. Impose on R^* the norm $\|\ .\ \|_{x_0}$, for every $p\in R^*$, given by $\|\ p\ \|_{x_0}:=[x_0,p]$. Then, by definition, for all $q\in L(x_0)$, we have $\|\ q\ \|_{x_0}=1$. Let d be a salient metric on R^*. Using Assumption III of Theorem 5.2.1 and the fact that for all $\alpha,\beta\in\mathbb{R}$: $|\max\{0,\alpha\}-\max\{0,\beta\}|\leq|\alpha-\beta|$, we find that for all $p_1\in Q\setminus\{0\}$ and for all $\varepsilon>0$, there is $\delta>0$ such that for all $p_2\in Q\setminus\{0\}$ satisfying $d\,(p_1,p_2)<\delta$ we have

$$\begin{aligned}
d_{x_0}\left(\mathcal{F}_0(p_1),\mathcal{F}_0(p_2)\right) &= \|\ \mathcal{F}_0(p_1)-\mathcal{F}_0(p_2)\ \|_{x_0}\\
&\leq \int\limits_{L(x_0)}|\max\{0,\mathcal{W}(p_1,q)\}-\max\{0,\mathcal{W}(p_2,q)\}|d\mu(q)\\
&\leq \int\limits_{L(x_0)}|\mathcal{W}(p_1,q)-\mathcal{W}(p_2,q)|d\mu(q)\\
&\leq \varepsilon\mu(L(x_0)).
\end{aligned}$$

□

5.2.5 Proposition. *The function $\mathcal{F}:R^*\setminus\{0\}\to R^*$ is continuous.*

Proof.
Define $\tilde{p}_0:=\frac{p_0}{[x_0,p_0]}\in L(x_0)$. By Assumption III of Theorem 5.2.1, the mapping is continuous, so by Assumption I of Theorem 5.2.1 the mapping $r\mapsto\eta(\mathcal{W}(r,p_0))=\eta([x_0,p_0]\mathcal{W}(r,\tilde{p}_0))$ is continuous on $Q\setminus\{0\}$. We have seen that $\mathcal{F}_0$ is continuous

on $Q \setminus \{0\}$, so the function $\mathcal{F}$ is continuous on $Q \setminus \{0\}$. Remains to prove the continuity of $\mathcal{F}$ on $R^* \setminus Q$. By definition, $\mathcal{F}(p) = p_0$ for all $p \in R^* \setminus Q$, so we only have to consider a sequence $(p_n)_{n\in\mathbb{N}}$ in $Q \setminus \{0\}$ with limit $p \notin Q$. Now, suppose the sequence $(\mathcal{F}(p_n))_{n\in\mathbb{N}}$ does not converge to p_0. Taking a subsequence if necessary, we may assume $\mathcal{F}(p_n) \neq p_0$, for all $n \in \mathbb{N}$. By Assumption IV of Theorem 5.2.1, $\exists n_0 \in \mathbb{N} : \mathcal{W}(p_{n_0}, p_0) \geq \xi_0$. So, by (5.6) and (5.3), $\mathcal{F}(p_{n_0}) = p_0$. This is in contradiction with the assumption that $\mathcal{F}(p_n) \neq p_0$ for all $n \in \mathbb{N}$. □

Applying Theorem 3.3.15 and Lemma 5.2.3 to the previous proposition, proves Theorem 5.2.1.

5.2.6 Example. Let R be a finite-dimensional reflexive salient space and let Q be a salient subspace of R^*. Let $\mathcal{E} : Q \setminus \{0\} \to R$ be a continuous function. Then the mapping $\mathcal{W} : Q \setminus \{0\} \times R^* \to \mathbb{R}$, for every $p \in Q \setminus \{0\}$ and every $q \in R^*$ given by

$$\mathcal{W}(p, q) := [\mathcal{E}(p), q],$$

satisfies Assumption III of Theorem 5.2.1. Indeed, let $x_0 \in \operatorname{int}(R)$. Since the set $L(x_0)$ is compact (Corollary 3.3.8.c), there is $q_0 \in \operatorname{int}(R^*)$ such that $\forall q \in L(x_0) : q \leq_{R^*} q_0$. For every $p_1, p_2 \in Q \setminus \{0\}$ we find (cf. Corollary 3.2.13)

$$\begin{aligned}
&\max\{|\mathcal{W}(p_1, q) - \mathcal{W}(p_2, q)| \mid q \in L(x_0)\} \\
&= \max\{|[\mathcal{E}(p_1), q] - [\mathcal{E}(p_2), q]| \mid q \in L(x_0)\} \\
&\leq \max\{|d_q(\mathcal{E}(p_1), \mathcal{E}(p_2))| \mid q \in L(x_0)\} \\
&\leq d_{q_0}(\mathcal{E}(p_1), \mathcal{E}(p_2)).
\end{aligned}$$

◇

5.2.7 Example. Let R be a finite-dimensional reflexive salient space and let Q be a salient subspace of R^*. Let $\mathcal{E}_1 : Q \setminus \{0\} \to R$ and $\mathcal{E}_2 : Q \setminus \{0\} \to R$ be continuous functions. Then, for every $\alpha, \beta \in \mathbb{R}$, the mapping $\mathcal{W} : Q \setminus \{0\} \times R^* \to \mathbb{R}$, for every $p \in Q \setminus \{0\}$ and every $q \in R^*$ given by

$$\mathcal{W}(p, q) := \alpha[\mathcal{E}_1(p), q] + \beta[\mathcal{E}_2(p), q]$$

satisfies Assumption III of Theorem 5.2.1. Indeed, this is implied by

$$|\mathcal{W}(p_1, q) - \mathcal{W}(p_2, q)| \leq |\alpha||[\mathcal{E}_1(p_1), q] - [\mathcal{E}_1(p_2), q]| + |\beta||[\mathcal{E}_2(p_1), q] - [\mathcal{E}_2(p_2), q]|.$$

◇

5.3 Demand functions 1

The goal of this section is to prove Lemma 5.3.2, stated below. This lemma will be used in the proofs of Theorem A1, B, C1, C2 and D, to realise Step b of Section 5.1: the step from demand sets to demand functions. Also, several consequences of Lemma 5.3.2 will be needed in order to apply Theorem 5.2.1 in the above mentioned proofs.

For the same reason as in the previous section, the central lemma of this section is stated in terms of general salient spaces Q, R and S. When this lemma is applied, Q will be represented by the domain of the total demand function, R will be represented by the set of all bundles of trade, R^* by the set of all possible pricing functions, and S will be represented by the subset of the set of all bundles of trade on which the preference relations of the agents are defined.

Let R be a salient space. In this section, we consider a finite number i_0 of economic agents. Each agent i, $i \in \{1, \ldots, i_0\}$, has a preference relation $\succeq_i$ defined on a salient subspace S of R. We assume that R and the preference relations on S satisfy the following assumption.

5.3.1 Assumption.

I) *The salient space R is finite-dimensional and reflexive, and S is a reflexive salient subspace of R.*

II) *For all $i \in \{1, \ldots, i_0\}$, preference relation $\succeq_i$ on S is*

- **a)** *monotonous: $\forall\, s_1, s_2 \in S : s_1 \leq_S s_2$ implies $s_2 \succeq_i s_1$,*
- **b)** *strictly convex: $\forall s_1, s_2 \in S$, $\tau \in (0,1) : s_1 \succeq_i s_2$ and $s_1 \neq s_2$ imply $\tau s_1 + (1-\tau)s_2 \succ_i s_2$,*
- **c)** *continuous: $\forall s_1 \in S$ the sets $\{s \in S \mid s \succeq_i s_1\}$ and $\{s \in S \mid s_1 \succeq_i s\}$ are closed in S, with respect to the relative topology of $\tau(R, R^*)$ on $S \subset R$.*

Assumption 5.3.1.I corresponds to the first assumption of every theorem stated in Section 4.8. Furthermore, Assumption 5.3.1.II corresponds to Assumptions A1.3, B.3, C1.3, C2.3 and D.3.

Throughout this section we assume that the above assumption holds.

Since R is assumed to be a finite-dimensional salient space, topology $\tau(R, R^*)$ is equal to the relative topology of the unique norm topology of the finite-dimensional vector space $V[R]$. Furthermore, in Chapter 3 (on page 89), we have seen that

$\tau(R, R^*)$ is generated by any element $p_0 \in \mathrm{int}(R^*)$, and $\tau(R^*, R)$ on R^* is the relative topology of the unique norm topology on the vector space $V[R^*]$, generated by any element of $\mathrm{int}(R)$. We denote the relative topology of $\tau(R, R^*)$ on S by $\tau(S, R^*)$.

Next to a preference relation, we assume that each agent i, $i \in \{1, \ldots, i_0\}$, has an income or capital function $\mathcal{K}_i : Q \to \mathbb{R}_+$. Here, Q is a salient subspace of $\mathrm{int}(R^*) \cup \{0\}$, not dependent on the choice of $i \in \{1, \ldots, i_0\}$. At given $p \in Q$, the value $\mathcal{K}_i(p)$ denotes the maximum value of each element in the budget set of agent i. Hence, for every $i \in \{1, \ldots, i_0\}$, budget set $B_i(p)$ at pricing function $p \in Q$ is given by

$$B_i(p) := \{s \in S \mid [s, p] \leq \mathcal{K}_i(p)\}.$$

The demand set consists of the most preferable elements of the budget set, i.e.,

$$D_i(p) := \{b \in B_i(p) \mid \forall c \in B_i(p) : b \succeq_i c\}.$$

The purpose of this section is to prove the following lemma.

5.3.2 Lemma. *Let R be salient space, let S be salient subspace of R and let Q be salient subspace of* $\mathrm{int}(R^*) \cup \{0\}$. *Because of Assumption 5.3.1 the following statements are valid.*

a) *For all $i \in \{1, \ldots, i_0\}$ and for all $p \in Q$, the demand set $D_i(p)$ consists of exactly one element.*

b) *Define the demand function $\mathcal{D}_i : Q \to S$, such that $D_i(p) = \{\mathcal{D}_i(p)\}$, for all $p \in Q$. If the function $\mathcal{K}_i$ is continuous on Q, then $\mathcal{D}_i$ is continuous on Q, with respect to $\tau(R^*, R)$ and $\tau(S, R^*)$.*

c) *Walras' law: $[\sum_{i=1}^{i_0} \mathcal{D}_i(p), p] = \sum_{i=1}^{i_0} \mathcal{K}_i(p)$, for every $p \in Q$.*

d) *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Q, let $(p_n)_{n\in\mathbb{N}}$ be $\tau(R^*, R)$-convergent with limit $p \in$* $\mathrm{bd}(R^*) \setminus \{0\}$ *such that $[s, p] = 0$ for some $s \in S$, and let $\limsup_{n\to\infty} \mathcal{K}_i(p_n) > 0$ for certain $i \in \{1, \ldots, i_0\}$. Then the sequence $(\mathcal{D}_i(p_n))_{n\in\mathbb{N}}$ is $\tau(S, R^*)$-unbounded.*

The proof of each item of Lemma 5.3.2 will be a direct result of several of the following lemmas. More precisely, the first part of Lemma 5.3.2 is a direct result of Lemma 5.3.3 and Lemma 5.3.4. The continuity of the demand function is proved in Lemma 5.3.10. Lemma 5.3.2.c is a direct result of Lemma 5.3.6, and, finally, Lemma 5.3.7 yields the last part of Lemma 5.3.2.

In the remainder of this section we consider agent i, $i \in \{1, \ldots, i_0\}$, with preference relation $\succeq_i$ defined on the salient space $S \subset R$, and a capital function $\mathcal{K}_i : Q \to \mathbb{R}_+$.

5.3.3 Lemma. *Let $p \in Q$. Then the demand set $D_i(p)$ contains at most one element.*

Proof.
Suppose both d_1 and d_2 belong to $D_i(p)$ and $d_1 \neq d_2$. On the one hand, using II.b of Assumption 5.3.1, we find $\tau d_1 + (1-\tau)d_2 \succ_i d_1$ for all $\tau \in (0,1)$. And, on the other hand, using convexity of the budget set, we find $\tau d_1 + (1-\tau)d_2 \in B_i(p)$ for all $\tau \in (0,1)$. □

5.3.4 Lemma. *Let $p \in Q$. Then the demand set $D_i(p)$ at pricing function p is non-empty.*

Proof.
Since $Q \subset \text{int}(R^*)$, Corollary 3.3.8.(c) and Assumption 5.3.1.I imply that the budget set $B_i(p)$ is compact in S. For every $b \in B_i(p)$, define the set $G(b) := \{c \in B_i(p) \mid b \succ_i c\}$. The preference relation $\succeq_i$ is continuous (II.c of Assumption 5.3.1), so every set $G(b)$ is $\tau(S, R^*)$-open. Suppose the demand set were empty, then every $b_0 \in B_i(p)$ is an element of at least one $G(b)$. The collection $\{G(b) \mid b \in B_i(p)\}$ is an open cover of the compact set $B_i(p)$, so there is a finite subset $F \subset B_i(p)$ such that $B_i(p) = \bigcup_{f \in F} G(f)$. The preference relation $\succeq_i$ being transitive, F has a maximal element $f_1 \in F$. Since, $f_1 \in G(f_2)$ for some $f_2 \in F$, $f_2 \neq f_1$, we arrive at a contradiction. □

Lemma 5.3.2.a is a direct result from the above two lemmas. As a consequence, we can define the demand function $\mathcal{D}_i : Q \to S$, where for every $p \in Q$, $\mathcal{D}_i(p)$ is the unique element of demand set $D_i(p)$.

Before we prove the continuity of this demand function, let us state some preliminary lemmas concerning the budget set and the demand set of this agent. These lemmas imply Lemmas 5.3.2.c and d.

5.3.5 Lemma. *Let $p \in R^*$, let $\kappa > 0$, let $s_0 \in S$, and suppose $s_0 \succeq_i b$ for all $b \in \{s \in S \mid [s,p] < \kappa\}$. Then $s_0 \succeq_i b$ for all $b \in \{s \in S \mid [s,p] \leq \kappa\}$.*

Proof.
Let $b \in S$ satisfy $[b,p] = \kappa$. We shall prove that $s_0 \succeq_i b$. Clearly, $b \neq 0$. So, for all $\tau \in [0,1)$ we have $[\tau b, p] < \kappa$ and thus $s_0 \succeq_i \tau b$. By II.c of Assumption 5.3.1, the preference relation $\succeq_i$ is continuous, so $s_0 \succeq_i b$. □

By the following lemma, Lemma 5.3.2.c is proved.

5.3.6 Lemma. *Let $p \in Q$. Then $[\mathcal{D}_i(p), p] = \mathcal{K}_i(p)$.*

Proof.
In case $\mathcal{K}_i(p) = 0$, the budget set $B_i(p)$ equals $\{0\}$, and thus $[\mathcal{D}_i(p), p] = [0, p] = 0$. Now, suppose $\mathcal{K}_i(p) > 0$ and $[\mathcal{D}_i(p), p] < \mathcal{K}_i(p)$. Since $\mathrm{int}(S) \subset \mathrm{int}(R)$, there is $s_0 \in \mathrm{int}(S)$ such that $s_0 >_S \mathcal{D}_i(p)$ and $[s_0, p] > \mathcal{K}_i(p)$ (cf. Lemma 2.2.12 and Lemma 3.3.7). Consider the convex combination $\tau s_0 + (1-\tau)\mathcal{D}_i(p)$ with $\tau \in (0,1)$ so small that $[\tau s_0 + (1-\tau)\mathcal{D}_i(p), p] \leq \mathcal{K}_i(p)$. Then $\tau s_0 + (1-\tau)\mathcal{D}_i(p) \in B_i(p)$ and $\tau s_0 + (1-\tau)\mathcal{D}_i(p) >_S \mathcal{D}_i(p)$. By the monotony of preference relation $\succeq_i$ (II.a of Assumption 5.3.1), $\tau s_0 + (1-\tau)\mathcal{D}_i(p) \succeq_i \mathcal{D}_i(p)$. Since $s_0 \neq \mathcal{D}_i(p)$, we come to a contradiction with Lemma 5.3.3. □

5.3.7 Lemma. *Let $(p_n)_{n\in\mathbb{N}}$ be a convergent sequence in Q with limit $p \in R^*$, and assume the sequence $(\mathcal{K}_i(p_n))_{n\in\mathbb{N}}$ is convergent with limit κ. If $\kappa > 0$ and the sequence $(\mathcal{D}_i(p_n))_{n\in\mathbb{N}}$ is bounded, then $\forall s \in S \setminus \{0\} : [s, p] > 0$.*

Proof.
Let $\kappa > 0$ and let the sequence $(\mathcal{D}_i(p_n))_{n\in\mathbb{N}}$ be bounded. We may as well assume that the sequence $(\mathcal{D}_i(p_n))_{n\in\mathbb{N}}$ is convergent. Define $B_i(p,\kappa) := \{s \in S \mid [s,p] \leq \kappa\}$. Suppose there is an element $s \in S \setminus \{0\}$, such that $[s,p] = 0$. Let $b \in B_i(p,\kappa)$, then by the monotony of $\succeq_i$ (II.a of Assumption 5.3.1), $b + s \succeq_i b + \frac{1}{2}s \succeq_i b$. By the strict convexity of $\succeq_i$ (II.b of Assumption 5.3.1), we find $b + s \succ_i b$. Since $b+s \in B_i(p,\kappa)$, we conclude that $B_i(p,\kappa)$ contains no maximal element with respect to preference relation $\succeq_i$. In order to arrive at a contradiction, we prove that the limit d of the sequence $(\mathcal{D}_i(p_n))_{n\in\mathbb{N}}$ is maximal in $B_i(p,\kappa)$. Indeed, let $b \in B_i(p,\kappa)$ satisfy $[b,p] < \kappa$. Then there is $N \in \mathbb{N}$ such that $\forall\, n > N : [b, p_n] < \mathcal{K}_i(p_n)$, i.e., $b \in B_i(p_n)$. So, $\mathcal{D}_i(p_n) \succeq_i b$ for all $n > N$. Continuity of the preference relation (II.c of Assumption 5.3.1) yields $d \succeq_i b$, and by Lemma 5.3.5 we conclude that d is maximal in $B_i(p,\kappa)$. □

5.3.8 Corollary. *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Q. Let $(p_n)_{n\in\mathbb{N}}$ be $\tau(R^*, R)$-convergent to $p \in \mathrm{bd}(R^*) \setminus \{0\}$ with $[s,p] = 0$ for some $s \in S \setminus \{0\}$, and let*

$$\limsup_{n\to\infty} \mathcal{K}_i(p_n) > 0.$$

Then the sequence $\mathcal{D}_i(p_n)$ is $\tau(S, R^*)$-unbounded.

Proof.
Suppose $\limsup_{n\to\infty} \mathcal{K}_i(p_n) = \infty$. Then Lemma 5.3.6 implies that $([\mathcal{D}_i(p_n), p_0])_{n\in\mathbb{N}}$

is an unbounded sequence. Since $(p_n)_{n\in\mathbb{N}}$ is convergent, the sequence $(\mathcal{D}_i(p_n))_{n\in\mathbb{N}}$ cannot contain a convergent subsequence, and so $(\mathcal{D}_i(p_n))_{n\in\mathbb{N}}$ is unbounded. Hence, we may as well assume $\lim_{n\to\infty}\mathcal{K}_i(p_n) = \kappa > 0$. If the sequence $\mathcal{D}_i(p_n)$ is bounded, then Lemma 5.3.7 implies $\forall s \in S \setminus \{0\} : [s,p] > 0$. □

To conclude this section, we prove Lemma 5.3.2.b, namely that continuity of the function $\mathcal{K}_i$ implies the continuity of the demand function. For this we need the following lemma.

5.3.9 Lemma. *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Q convergent to $p \in Q$, and let $\lim_{n\to\infty}\mathcal{K}_i(p_n) = \mathcal{K}_i(p)$. Then the following two properties hold.*

1) *If $b_n \in B_i(p_n)$ for each $n \in \mathbb{N}$, then there is a subsequence $(b_{nk})_{k\in\mathbb{N}}$ that converges to some $b \in B_i(p)$.*

2) *For each $b \in B_i(p)$ there exists a convergent sequence $(b_n)_{n\in\mathbb{N}}$ with limit b, such that $b_n \in B_i(p_n)$ for all $n \in \mathbb{N}$.*

Proof.

1) Since $p \in Q \subset \mathrm{int}(R^*)$ is an order unit for R^*, Lemma 3.3.12 implies that the function $\mathcal{L}_p : R^* \to \mathbb{R}_+$ satisfies

$$\lim_{n\to\infty} \mathcal{L}_p(p_n) = 1 \text{ and } \forall n \in \mathbb{N} : \mathcal{L}_p(p_n)p \leq_{R^*} p_n.$$

Because $b_n \in B_i(p_n)$ for all $n \in \mathbb{N}$, we find

$$\mathcal{L}_p(p_n)[b_n, p] \leq [b_n, p_n] \leq \mathcal{K}_i(p_n).$$

And since $p \in \mathrm{int}(R^*)$, by Corollary 3.3.8.b, boundedness of $[b_n, p]$ implies that the sequence $(b_n)_{n\in\mathbb{N}}$ is bounded in $S \subseteq R$. So, $(b_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(b_{nk})_{k\in\mathbb{N}}$ with limit $b \in S$ (Assumption 5.3.1.I). Since $\forall k \in \mathbb{N}$: $[b_{nk}, p_{nk}] \leq \mathcal{K}_i(p_{nk})$, the limit b belongs to $B_i(p)$.

2) Let $b \in B_i(p)$. Since $0 \in B_i(p)$, for every $p \in Q$, we may as well assume $b \neq 0$. If $[b,p] < \mathcal{K}_i(p)$ then $\exists N \in \mathbb{N}\ \forall n > N : [b, p_n] < \mathcal{K}_i(p_n)$, and so, if we choose $b_n := b$ for all $n > N$, we are done. Therefore, we may as well assume $[b,p] = \mathcal{K}_i(p)$. For every $n \in \mathbb{N}$, define $\tau_n := \frac{\mathcal{K}_i(p_n)}{[b,p_n]}$. Note that $\lim\limits_{n\to\infty} \tau_n = 1$. Now put $b_n := \tau_n b$, then $\forall n \in \mathbb{N} : [b_n, p_n] = \mathcal{K}_i(p_n)$ and $\lim\limits_{n\to\infty} b_n = b$.

□

Lemma 5.3.9 expresses the type of continuity that we need in order to prove the continuity of the demand function $\mathcal{D}_i$ (Lemma 5.3.2.b).

5.3.10 Lemma. *If $\mathcal{K}_i : Q \to \mathbb{R}_+$ is continuous on Q, then demand function $\mathcal{D}_i$ is continuous on Q.*

Proof.
Suppose $\mathcal{D}_i$ is not continuous in $p \in Q$, then there is a sequence $(p_n)_{n\in\mathbb{N}}$ in Q, converging to p, such that any subsequence of $(\mathcal{D}_i(p_n))_{n\in\mathbb{N}}$ does not converge to $\mathcal{D}_i(p)$. By 1) of the preceding lemma, the sequence $(\mathcal{D}_i(p_n))_{n\in\mathbb{N}}$ has a subsequence $(\mathcal{D}_i(p_{n_k}))_{k\in\mathbb{N}}$ that converges to some $b \in B_i(p)$. Now, the proof is done if we can show that $b = \mathcal{D}_i(p)$. Let $c \in B_i(p)$. By 2) of the preceding lemma, for all $n \in \mathbb{N}$ there is $c_n \in B_i(p_n)$ satisfying $\lim_{n\to\infty} c_n = c$. Since the preference relation $\succeq_i$ is continuous (II.c of Assumption 5.3.1), we find that if $\forall n \in \mathbb{N} : \mathcal{D}_i(p_n) \succeq_i c_n$, then $b \succeq_i c$. So, $b = \mathcal{D}_i(p)$. □

5.4 Proof of Theorem A1

Consider Model A, introduced on page 108, and assume that all the assumptions of Theorem A1 are satisfied.

For every $i \in \{1, \ldots, i_0\}$, the income function $\mathcal{K}_i^{A1} : C^* \to \mathbb{R}_+$, for every $p \in C^*$ is defined by

$$\mathcal{K}_i^{A1}(p) := [w_i, p].$$

Since $\mathcal{K}_i^{A1}$ is continuous on C^*, Lemma 5.3.2.b (with $Q = \text{int}(C^*) \cup \{0\}$ and $S = R = C$) implies that every agent i, $i \in \{1, \ldots, i_0\}$, has a demand function $\mathcal{D}_i^{A1} : \text{int}(C^*) \to C$, which is continuous with respect to $\tau(C^*, C)$ and $\tau(C, C^*)$. This completes Step b, described in Section 5.1.

We define the total demand function $\mathcal{D}^{A1} : \text{int}(C^*) \to C$ by

$$\mathcal{D}^{A1} := \sum_{i=1}^{i_0} \mathcal{D}_i^{A1}.$$

The mapping $\mathcal{Z}^{A1} : \text{int}(C^*) \times C^* \to \mathbb{R}$ is, for every $p \in \text{int}(C^*)$ and every $q \in C^*$, defined by

$$\mathcal{Z}^{A1}(p, q) := [\mathcal{D}^{A1}(p), q] - [w_{\text{total}}, q].$$

This completes Step c, as described in Section 5.1.

In order to realise Steps d and e of Section 5.1, we want to apply Theorem 5.2.1 to this mapping, with $\mathcal{W} = \mathcal{Z}^{A1}, R = C$ and $Q = \text{int}(C^*) \cup \{0\}$. So, let $p_0 \in \text{int}(C^*)$. We have to check whether $\mathcal{Z}^{A1}$ satisfies the requirements for this theorem.

I: By Lemma 5.3.2.c (with $Q = \mathrm{int}(C^*) \cup \{0\}$ and $S = R = C$) we find $\mathcal{Z}^{A1}(p,p) = 0$ for all $p \in \mathrm{int}(C^*)$. Clearly, $\mathcal{Z}^{A1}(\alpha p, q) = \alpha \mathcal{Z}^{A1}(p,q)$ for every $\alpha \in \mathbb{R}_+$, $p \in \mathrm{int}(C^*)$ and $q \in C^*$.

II: For every $p \in \mathrm{int}(C^*)$, the mapping $q \mapsto \mathcal{Z}^{A1}(p,q)$ is continuous.

III: Let $x_0 \in \mathrm{int}(C)$. Since the function $\mathcal{D}^{A1}$ is continuous on $\mathrm{int}(C^*)$, Example 5.2.6 implies that the mapping $\mathcal{Z}^{A1}$ satisfies Condition III of Theorem 5.2.1.

IV: Let $\xi_0 = 1$ and let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{int}(C^*) \cup \{0\}$ with limit $p \in \mathrm{bd}(C^*)$. Since $\forall r \in C^* \setminus \{0\} : [w_{\text{total}}, r] > 0$ (Assumption A1.4), there is $i \in \{1, \ldots, i_0\}$ such that $\mathcal{K}_i^{A1}(p) = [w_i, p] > 0$. Lemma 5.3.2.d (with $Q = \mathrm{int}(C^*) \cup \{0\}$ and $S = R = C$) implies that the sequence $\left(\mathcal{D}_i^{A1}(p_n)\right)_{n\in\mathbb{N}}$ is unbounded, and so the sequence $\left(\mathcal{D}^{A1}(p_n)\right)_{n\in\mathbb{N}}$ is unbounded. Since $p_0 \in \mathrm{int}(C^*)$, there is $n \in \mathbb{N}$ such that

$$[\mathcal{D}^{A1}(p_n), p_0] - [w_{\text{total}}, p_0] \geq 1.$$

We conclude that the mapping $\mathcal{Z}^{A1} : \mathrm{int}(C^*) \times C^* \to \mathbb{R}$ meets all the requirements for Theorem 5.2.1 (with $R = C$ and $Q = \mathrm{int}(C^*) \cup \{0\}$), hence

$$\exists p_* \in \mathrm{int}(C^*) \forall q \in C^* : \mathcal{Z}^{A1}(p_*, q) \leq 0.$$

Lemma 2.3.23 implies $\mathcal{D}^{A1}(p_*) \leq_C w_{\text{total}}$, so we conclude that

$$(\mathcal{D}_1^{A1}(p_*), \ldots, \mathcal{D}_{i_0}^{A1}(p_*), p_*)$$

is a Walrasian equilibrium for Model A.

5.5 Demand functions 2

This section is the A2-counterpart of Section 5.3. Hence, the goal of this section is to prove Lemma 5.5.2, which will be used in the proof of Theorem A2 to realise Step b of Section 5.1. The need for this separate treatment lies in the fact that the assumptions concerning the agents in Theorem A2 are different from the assumptions concerning the agents in the other theorems.

Also in this section, the central lemma is stated in terms of general salient spaces Q, R and S. When this lemma is applied, Q will be represented by the domain of the total demand function, R will be represented by the set of all bundles of trade, R^* by the set of all possible pricing functions, and S will be represented by the set on which the preference relations of the agents are defined.

Let R be a salient space, and let S be a salient subspace of R. Similarly to Section 5.3, we consider a finite number i_0 of economic agents. Each agent i, $i \in \{1, \ldots, i_0\}$, has a preference relation $\succeq_i$ defined on S, and an income or capital function $\mathcal{K}_i : Q \to \mathbb{R}_+$, where Q is a salient subspace of $\mathrm{int}(R^*)$. Again, for every $i \in \{1, \ldots, i_0\}$ and every $p \in Q \setminus \{0\}$, the budget set and demand set of agent i are given by

$$B_i(p) := \{s \in S \mid [s, p] \leq \mathcal{K}_i(p)\}.$$

and

$$D_i(p) := \{b \in B_i(p) \mid \forall c \in B_i(p) : b \succeq_i c\}.$$

We assume that R and the preference relations on S satisfy the following assumption. The only difference between Assumptions 5.3.1 and 5.5.1 lies in Assumption II.a. Furthermore, Assumptions 5.3.1.I and II correspond with Assumptions A2.1 and A2.3.

5.5.1 Assumption.

I) *The salient space R is finite-dimensional and reflexive, and S is a reflexive salient subspace of R.*

II) *For all $i \in \{1, \ldots, i_0\}$, preference relation $\succeq_i$ on S is*

- **a)** *non-saturated: $\forall s_1 \in S\ \exists s_2 \in S \setminus \{s_1\} : s_2 \succeq_i s_1$,*
- **b)** *strictly convex: $\forall s_1, s_2 \in S$, $\tau \in (0, 1)$: $s_1 \succeq_i s_2$ and $s_1 \neq s_2$ imply $\tau s_1 + (1 - \tau)s_2 \succ_i s_2$,*
- **c)** *continuous: $\forall s_1 \in S$ the sets $\{s \in S \mid s \succeq_i s_1\}$ and $\{s \in S \mid s_1 \succeq_i s\}$ are closed in S, with respect to the relative topology of $\tau(R, R^*)$ on $S \subset R$.*

Throughout this section we assume that the above assumption holds.

Similar to the situation in Section 5.3, the first statement of the assumption implies that the topology $\tau(R, R^*)$ is equal to the relative topology of the unique norm topology of $V[R]$. We denote the relative topology of $\tau(R, R^*)$ on S by $\tau(S, R^*)$.

The purpose of this section is to prove the following lemma.

5.5.2 Lemma. *Let R be a salient space, let S be a salient subspace of R and let Q be salient subspace of* $\mathrm{int}(R^*) \cup \{0\}$*. Then, because of Assumption 5.3.1, the following statements are valid.*

a) *For all $i \in \{1, \ldots, i_0\}$ and for all $p \in Q$, the demand set $D_i(p)$ consists of exactly one element.*

b) *Define the demand function $\mathcal{D}_i : Q \to S$, such that $D_i(p) = \{\mathcal{D}_i(p)\}$, for all $p \in Q$. If the function $\mathcal{K}_i$ is continuous on Q, then $\mathcal{D}_i$ is continuous on Q, with respect to $\tau(R^*, R)$ and $\tau(S, R^*)$.*

c) *Walras' law: $[\sum_{i=1}^{i_0} \mathcal{D}_i(p), p] = \sum_{i=1}^{i_0} \mathcal{K}_i(p)$, for every $p \in Q$.*

We remark that due to the altered Assumption II.a, it is not possible in this setting, to prove a statement like Lemma 5.3.2.d. Example 4.8.1 shows a situation in which the demand function is continuous in $p \in \{q \in \mathrm{bd}(R^*) | \exists s \in S : [s, q] = 0\}$.

The proof of each item of Lemma 5.5.2 will be a direct result of several of the following lemmas. More precisely, the first part of Lemma 5.5.2 is a direct result of Lemma 5.5.3. The continuity of the demand function is proved in Lemma 5.5.5. Finally, Lemma 5.5.2.c is a direct result of Lemma 5.5.4.

In the remainder of this section we consider agent i, $i \in \{1, \ldots, i_0\}$, with preference relation $\succeq_i$ defined on the salient space $S \subset R$.

The proof of the following lemma is a combination of the proofs of Lemmas 5.3.3 and 5.3.4.

5.5.3 Lemma. *Let $p \in Q$. Then the demand set $D_i(p)$ contains precisely one element.*

Lemma 5.5.2.a is a direct result of Lemma 5.5.3. As a consequence, we can define the demand function $\mathcal{D}_i : Q \to S$ where for every $p \in Q$, $\mathcal{D}_i(p)$ is the unique element of demand set $D_i(p)$.

The following lemma proves Lemma 5.3.2.c.

5.5.4 Lemma. *Let $p \in Q$. Then $[\mathcal{D}_i(p), p] = \mathcal{K}_i(p)$.*

Proof.
In case $\mathcal{K}_i(p) = 0$, the budget set $B_i(p)$ equals $\{0\}$, and thus $[\mathcal{D}_i(p), p] = [0, p] = 0$. Now, suppose $\mathcal{K}_i(p) > 0$ and $[\mathcal{D}_i(p), p] < \mathcal{K}_i(p)$. By the non-saturation of $\succeq_i$ (II.a of Assumption 5.5.1), there is $x \in C$ such that $x \succeq_i \mathcal{D}_i(p)$. On the one hand, we find (II.b of Assumption 5.5.1) that $\forall \tau \in (0, 1) : \tau x + (1-\tau)\mathcal{D}_i(p) \succ_i \mathcal{D}_i(p)$. On the other hand, $\exists \tau > 0 : [\tau x + (1-\tau)\mathcal{D}_i(p), p] \leq \mathcal{K}_i(p)$. This is in contradiction with the optimality of $\mathcal{D}_i(p)$. □

The proof of the following lemma is a combination of the proofs of Lemma 5.3.9 and 5.3.10 of Section 5.3. Note that herewith Lemma 5.5.2 is proved.

5.5.5 Lemma. *If $\mathcal{K}_i : Q \to \mathbb{R}_+$ is continuous on Q, then demand function $\mathcal{D}_i$ is continuous on Q.*

5.6 Proof of Theorem A2

Consider Model A, introduced on page 108, and assume that the assumptions of Theorem A2 are satisfied.

For every $i \in \{1, \dots, i_0\}$, the income function $\mathcal{K}_i^{A2} : C^* \to \mathbb{R}_+$, for every $p \in C^*$ is defined by

$$\mathcal{K}_i^{A2}(p) := [w_i, p].$$

Let $p_0 \in \text{int}(C^*)$ and let $x_0 \in \text{int}(C)$. For every $n \in N$, we define

$$C_n^* := \{(1 - \frac{1}{n})p + \frac{1}{n}[x_0, p]p_0 \mid p \in C^*\}.$$

Then

- $\forall n \in \mathbb{N} : C_n^*$ is a $\tau(C^*, C)$-closed salient subspace of $\text{int}(C^*) \cup \{0\}$,
- $\forall n \in \mathbb{N} : C_n^* \subset C_{n+1}^*$,
- $C^* = \bigcup_{n \in \mathbb{N}} C_n^*$ is $\tau(C^*, C)$-dense in C^*.

Let $n \in \mathbb{N}$. Since $\mathcal{K}_i^{A2}$ is continuous on C^*, Lemma 5.5.2.b (with $Q = C_n^* \cup \{0\}$ and $S = R = C$) implies that every agent i, $i \in \{1, \dots, i_0\}$, has a demand function $\mathcal{D}_i^n : C_n^* \to C$, which is continuous with respect to topology $\tau(C_n^*, C)$ and with respect to $\tau(C, C^*)$. This completes Step b, as described in Section 5.1.

We define the total demand function $\mathcal{D}^n : C_n^* \to C$ by

$$\mathcal{D}^n := \sum_{i=1}^{i_0} \mathcal{D}_i^n.$$

Note, that for every $n \in \mathbb{N}$, the function $\mathcal{D}^{n-1}$ is the restriction of $\mathcal{D}^n$ to the set C_{n-1}^*.

The mapping $\mathcal{Z}^n : C_n^* \times C_n^* \to \mathbb{R}$ is, for every $p, q \in C_n^*$, defined by

$$\mathcal{Z}^n(p, q) := [\mathcal{D}^n(p), q] - [w_{\text{total}}, q].$$

This completes Step c, as described in Section 5.1.

In order to realise Steps d and e of Section 5.1, we want to apply Theorem 5.2.1 to this mapping, with $\mathcal{W} = \mathcal{Z}^n$, $R = (C_n^*)^*$ and $Q = C_n^*$. We have to check whether $\mathcal{Z}^n$ satisfies the requirements for this theorem.

I: By Lemma 5.5.2.c (with $Q = C_n^*$ and $S = R = C$) we find $\forall p \in C_n^* : \mathcal{Z}^n(p,p) = 0$. Clearly, $\mathcal{Z}^{A1}(\alpha p, q) = \alpha \mathcal{Z}^{A1}(p,q)$ for every $\alpha \in \mathbb{R}_+$ and all $p, q \in C_n^*$.

II: The total demand function $\mathcal{D}^n$ is continuous on C_n^*, so for every $q_0 \in C^*$ the function $p \mapsto \mathcal{Z}^n(p, q_0)$ is continuous on C_n^*.

III: Let $x_0 \in \mathrm{int}(C)$. Since the function $\mathcal{D}^n$ is continuous on C_n^*, Example 5.2.6 implies that the mapping $\mathcal{Z}^n$ satisfies Condition III of Theorem 5.2.1.

IV: Since $(C_n^*)^* = C_n^*$, Condition IV of Theorem 5.2.1 is satisfied by default.

We conclude that the mapping $\mathcal{Z}^n : C_n^* \times C^* \to \mathbb{R}$ meets all the requirements for Theorem 5.2.1 (with $R = (C_n^*)^*$ and $Q = C_n^*$), hence

$$\exists p_{n,*} \in C_n^* \text{ such that } [x_0, p_{n,*}] = 1 \text{ and } \mathcal{Z}^n(p_{n,*}, q) \leq 0 \text{ for all } q \in C_n^*.$$

Lemma 2.3.23 implies $\mathcal{D}^n(p_{n,*}) \leq_{(C_n^*)^*} w_{\text{total}}$. Since, $C^* \setminus C_n^* \neq \emptyset$, this does not imply $\mathcal{D}^n(p_{n,*}) \leq_C w_{\text{total}}$, i.e, it may not be that $p_{n,*}$ is an equilibrium pricing function.

We prove that the sequence $(p_{n,*})_{n\in\mathbb{N}}$ has a convergent subsequence with limit p_* and that p_* is an equilibrium pricing function for model A. For this, we need the following two lemmas.

Consider agent i, $i \in \{1, \ldots, i_0\}$.

5.6.1 Lemma. *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{int}(C^*)$ convergent to $p \in C^*$. If $[w_i, p] > 0$, then for each $b \in B_i(p)$ there exists a convergent sequence $(b_n)_{n\in\mathbb{N}}$ with limit b, such that $b_n \in B_i(p_n)$ for all $n \in \mathbb{N}$.*

Proof.
Let $b \in B_i(p)$. Since $\forall n \in \mathbb{N} : 0 \in B_i(p_n)$, this lemma is proved in case $b = 0$. Now, assume $b \neq 0$. If $[b,p] < [w_i, p]$ then $\exists N \in \mathbb{N}\ \forall n > N : [b, p_n] < [w_i, p_n]$, and so, if we choose $b_n := b$ for all $n > N$, we are done. Therefore, we may as well assume $[b,p] = [w_i,p]$. For every $n \in \mathbb{N}$, define $\tau_n := \frac{\mathcal{K}_i(p_n)}{[b,p_n]}$. Note that $\lim_{n\to\infty} \tau_n = 1$. Now put $b_n := \tau_n b$, then $\forall n \in \mathbb{N} : [b_n, p_n] = [w_i, p_n]$ and $\lim_{n\to\infty} b_n = b$. □

5.6.2 Lemma. *Let* $(p_n)_{n\in\mathbb{N}}$ *be a sequence in* $\operatorname{int}(C^*)$ *convergent to* $p \in C^*$ *with* $[w_i, p] > 0$. *If the sequence* $(\mathcal{D}_i(p_n))_{n\in\mathbb{N}}$ *is convergent with limit* $b \in C$, *then* b *is the unique best element of* $B_i(p)$ *with respect to preference relation* $\succeq_i$.

Proof.
Let $x \in B_i(p)$. We prove that $b \succeq_i x$. By the previous lemma, there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N} : x_n \in B_i(p_n)$ and $\lim_{n\to\infty} x_n = x$. Since the preference relation is continuous (II.c of Assumption 5.5.1), we find that if $\forall n \in \mathbb{N} : \mathcal{D}_i(p_n) \succeq_i x_n$, then $b \succeq_i x$. The fact that b is the unique best element follows from II.a and II.b of Assumption 5.5.1, using the same proof as in Lemma 5.3.3. □

Since for every $n \in \mathbb{N}$, we have $p_{n,*} \in \{q \in C^* \mid [x_0, q] = 1\}$, we may as well assume that the sequence $(p_{n,*})_{n\in\mathbb{N}}$ is convergent with limit $p_* \in \{q \in C^* \mid [x_0, q] = 1\}$.

We distinguish two cases.

- If $p_* \in \operatorname{int}(C^*)$, then there is $n_0 \in \mathbb{N}$ such that $p_* \in C^*_{n_0}$ The total demand function $\mathcal{D}^{n_0} : C^*_{n_0} \to C$ is continuous. Hence

$$\lim_{n\to\infty} \mathcal{D}^{n_0}(p_{n,*}) = \mathcal{D}^{n_0}(p_*).$$

  In this case, define $b := \mathcal{D}^{n_0}(p_*)$ and for every $i \in \{1, \ldots, i_0\}$, define $b_i := \mathcal{D}_i^{n_0}(p_*)$.

- If $p_* \in \operatorname{bd}(C^*)$, then

$$\forall n \in \mathbb{N}\ \forall i \in \{1, \ldots, i_0\}\ \forall q \in C^*_n : [\mathcal{D}_i^n(p_{n,*}), q] \leq [\mathcal{D}^n(p_{n,*}), q] \leq [w_{\text{total}}, q].$$

  Let $p_1 \in C^*_1$, then $p_1 \in \operatorname{int}(C^*)$ and $\forall n \in \mathbb{N} : p_1 \in C^*_n$. By Corollary 3.3.8.b, the sequence $\mathcal{D}_i^n(p_{n,*})$ is bounded for every $i \in \{1, \ldots, i_0\}$ since $\forall i \in \{1, \ldots, i_0\} : [\mathcal{D}_i^n(p_{n,*}), p_1] \leq [w_i, p_1]$. Without loss of generality, we may assume $\forall i \in \{1, \ldots, i_0\} : \lim\limits_{n\to\infty} \mathcal{D}_i^n(p_{n,*}) = b_i$, where Lemma 5.6.2 implies that for every $i \in \{1, \ldots, i_0\}$, b_i is the unique best element of $B_i(p_*)$ (Assumption A2.4). Define $b := \sum_{i=1}^{i_0} b_i$.

We prove that $(b_1, \ldots, b_{i_0}, p_*)$ is a Walrasian equilibrium for Model A, i.e., we prove

$$\forall q \in C^* : [b, q] \leq [w_{\text{total}}, q].$$

Let $q \in C^*$, then there is a sequence $(q_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N} : q_n \in C^*_n$ and $\lim_{n\to\infty} q_n = q$. Since $\forall n \in \mathbb{N}\ \forall q_n \in C^*_n : [\mathcal{D}_i^n(p_{n,*}), q_n] \leq [w_{\text{total}}, q_n]$, we find $[b, q] \leq [w_{\text{total}}, q]$.

5.7 Supply functions 1

The goal of this section is to prove Lemma 5.7.2, stated below. This lemma will be used in the proof of Theorem C1, to realise Step a of Section 5.1: the step from supply sets to supply functions. Also, several consequences of Lemma 5.7.2 will be needed in order to apply Theorem 5.2.1 in the above mentioned proof.

For the same reason as in Section 5.3, the central lemma of this section is stated in terms of a general salient space R. When this lemma will be applied, R will be replaced by the set of all bundles of trade. Consequently, R^* will represent the set of all possible pricing functions.

In this section, we consider a finite number j_0 of firms. For every $j \in \{1, \ldots, j_0\}$, production technology T_j is a subset of a salient space $R = R_{\text{prod}} \oplus R_{\text{cons}}$. We assume that R and every T_j, $j \in \{1, \ldots, j_0\}$, satisfy the following assumption.

5.7.1 Assumption.

I) *The salient space R is finite-dimensional and reflexive.*

II) *For all $j \in \{1, \ldots, j_0\}$, production technology T_j satisfies*

- **a)** $T_j = \bigcup_{e \in E(T_j)} F_e$,
- **b)** *T_j is closed with respect to topology $\tau(R, R^*)$,*
- **c)** *if $e_1, e_2 \in E(T_j)$, $e_1 \neq e_2$, $\tau \in (0,1)$ then $\tau e_1 + (1-\tau)e_2 \in T_j$ and $\tau e_1 + (1-\tau)e_2 \notin E(T_j)$.*

Assumption 5.7.1.I corresponds with the first assumption of every theorem of Section 4.8. Furthermore, Assumption 5.7.1.II corresponds with Assumption C1.3.

Throughout this section we assume that the above assumption holds.

Recall that topology $\tau(R, R^*)$ is equal to the relative topology of the norm topology of $V[R]$.

For every $j \in \{1, \ldots, j_0\}$, the supply set $\mathcal{S}_j(p)$ at pricing function $p \in R^*$ is given by

$$\mathcal{S}_j(p) := \{t \in T_j \mid \forall s \in T_j : \mathcal{G}(s,p) \leq \mathcal{G}(t,p)\}.$$

For every $j \in \{1, \ldots, j_0\}$, we define

$$\text{Domain}[j] := \{p \in \text{int}(R^*) \mid S_j(p) \neq \emptyset\},$$

hence, by definition every Domain$[j]$, $j \in \{1, \ldots, j_0\}$, is a subset of $\text{int}(R^*)$.

The purpose of this section is to prove the following lemma.

5.7.2 Lemma. *Assumption 5.7.1 implies the following.*

a) *For every $j \in \{1, \ldots, j_0\}$, the set* Domain$[j] \cup \{0\}$ *is a salient subspace of R^*.*

b) *For all $j \in \{1, \ldots, j_0\}$ and for all $p \in$ Domain$[j]$, the supply set $S_j(p)$ consists of exactly one element.*

c) *For every $j \in \{1, \ldots, j_0\}$, define the supply function $\mathcal{S}_j :$ Domain$[j] \to R$, such that $S_j(p) = \{\mathcal{S}_j(p)\}$, for all $p \in$ Domain$[j]$. Then the function $\mathcal{S}_j$ is continuous on its domain with respect to the relative topology of $\tau(R^*, R)$.*

d) *Let $j \in \{1, \ldots, j_0\}$ and let $p_0 \in \text{int}(R^*) \cap$ Domain$[j]$. If $(p_n)_{n\in\mathbb{N}}$ is a sequence in* Domain$[j]$, *convergent to $p \in \text{int}(R^*) \backslash$*Domain$[j]$, *then* $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$.

The proof of each item of Lemma 5.7.2 will be a direct result of several of the following lemmas. More precisely, Lemma 5.7.3 and Proposition 5.7.5 correspond to Lemma 5.7.2.b and c, respectively. Lemma 5.7.2.d is proved in Corollary 5.7.13. Finally, Proposition 5.7.14 proves 5.7.2.a.

In the remainder of this section we consider a fixed production technology T_j, with $j \in \{1, \ldots, j_0\}$. We assume that Domain$[j] \neq \emptyset$. Recall, that Lemma 4.5.5 implies that T_j is a convex subset of R.

5.7.3 Lemma. *Let $p \in \text{int}(R^*)$. Then the supply set $S_j(p)$ contains at most one element.*

Proof.
Suppose both s_1 and $s_2 \in S_j(p)$ and $s_1 \neq s_2$. By II.c of Assumption 5.7.1, $s := \frac{1}{2}(s_1 + s_2) \in T_j \setminus E(T_j)$. Recall that for all $y \in R$ the set F_y is given by $\{x \in R \mid y^{\text{prod}} \leq_{\text{prod}} x^{\text{prod}} \text{ and } x^{\text{cons}} \leq_{\text{cons}} y^{\text{cons}}\}$. Since $T_j \setminus E(T_j) = \{x \in T_j \mid \exists y \in E(T_j), y \neq x : x \in F_y\}$, there exists $y \in E(T_j) : s \in F_y$. Now, since $p \in \text{int}(R^*)$, $\mathcal{G}(y,p) > \mathcal{G}(s,p) = \mathcal{G}(s_1,p)$, which is in contradiction with s_1 being an element of the supply set $S_j(p)$. □

Lemma 5.7.2.b is a direct result of the above lemma, and the fact that, by definition, Domain$[j] \subset \text{int}(R^*)$. As a consequence, we can define the supply function $\mathcal{S}_j : \text{int}(R^*) \to E(T_j)$ where, for every $p \in$ Domain$[j]$, $\mathcal{S}_j(p)$ is the unique element of $S_j(p)$.

5.7.4 Lemma. *Let* $(p_n)_{n\in\mathbb{N}}$ *be a sequence in* Domain$[j]$*, with limit* $p \in \text{int}(R^*)$*. If the sequence* $(\mathcal{S}_j(p_n))_{n\in\mathbb{N}}$ *is convergent with limit* $s \in R$*, then* $p \in$ Domain$[j]$ *and* $s = \mathcal{S}_j(p)$*.*

Proof.
Since $\forall n \in \mathbb{N}\ \forall x \in T_j : \mathcal{G}(\mathcal{S}_j(p_n), p_n) \geq \mathcal{G}(x, p_n)$, continuity of the function $\mathcal{G} : R \times R^* \to \mathbb{R}$ guarantees that $\forall x \in T_j : \mathcal{G}(s, p) \geq \mathcal{G}(x, p)$. Since T_j is closed (II.2 of Assumption 5.7.1), we find $s \in T_j$, and so $p \in$ Domain$[j]$. Furthermore, Lemma 5.7.3 implies $s = \mathcal{S}_j(p)$. □

5.7.5 Proposition. *The supply function* $\mathcal{S}_j$: Domain$[j] \to E(T_j)$ *is continuous with respect to the relative topology on* Domain$[j]$*.*

Proof.
Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Domain$[j]$ with limit $p \in$ Domain$[j]$. Let $d : R \times R \to \mathbb{R}_+$ be a salient metric which generates the norm topology $\tau(R, R^*)$. Suppose the sequence $(\mathcal{S}_j(p_n))_{n\in\mathbb{N}}$ does not converge to $\mathcal{S}_j(p)$. Taking a subsequence if necessary, we may assume that

$$\exists \varepsilon > 0\ \forall n \in \mathbb{N} : d\left(\mathcal{S}_j(p_n), \mathcal{S}_j(p)\right) \geq \varepsilon.$$

Define $x_n := \lambda_n \mathcal{S}_j(p_n) + (1 - \lambda_n)\mathcal{S}_j(p)$ with $\lambda_n := \frac{\varepsilon}{d(\mathcal{S}_j(p_n), \mathcal{S}_j(p))} \in (0, 1]$, then, by II.c of Assumption 5.7.1, $x_n \in T_j \setminus E(T_j)$ and $d\left(x_n, \mathcal{S}_j(p)\right) = \varepsilon$. The sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, so there is a convergent subsequence $(x_{nk})_{k\in\mathbb{N}}$ with limit $x \in T_j$ (II.b of Assumption 5.7.1), satisfying $d\left(x, \mathcal{S}_j(p)\right) = \varepsilon$. Since $x_n = \lambda_n \mathcal{S}_j(p_n) + (1 - \lambda_n)\mathcal{S}_j(p)$ with $\lambda \in (0, 1]$, we find $\mathcal{G}(x_n, p_n) \geq \min\{\mathcal{G}(\mathcal{S}_j(p_n), p_n), \mathcal{G}(\mathcal{S}_j(p), p_n)\} = \mathcal{G}(\mathcal{S}_j(p), p_n)$. The function $\mathcal{G} : R \times R^* \to \mathbb{R}$ is continuous, so $\mathcal{G}(x, p) \geq \mathcal{G}(\mathcal{S}_j(p), p)$. Since $x \in T_j$, $x \neq \mathcal{S}_j(p)$, this is in contradiction with Lemma 5.7.3. □

For the proof of the final part of Lemma 5.7.2, we need the following definition and lemmas.

5.7.6 Definition (extended real valued function $\boldsymbol{\chi_j}$). For every production technology T_j, $j \in \{1, \ldots, j_0\}$, the extended real valued function $\chi_j : R^* \to [0, \infty) \cup \{\infty\}$ is given by

$$\chi_j(p) := \sup_{x \in T_j} \mathcal{G}(x, p) = \sup_{e \in E(T_j)} \mathcal{G}(e, p).$$

Note that for every $j \in \{1, \ldots, j_0\}$, the function χ_j is convex, i.e.,

$$\forall p_1, p_2 \in R^*\ \forall \tau \in [0, 1] : \chi_j(\tau p_1 + (1 - \tau)p_2) \leq \tau \chi_j(p_1) + (1 - \tau)\chi_j(p_2).$$

5.7.7 Lemma. *Let $p_0 \in \mathrm{int}(R^*)$, let $\alpha \in \mathbb{R}$, and let $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ be an unbounded subset of R. Then the set $\{x \in T_j \mid \mathcal{G}(x, p_0) = \alpha\}$ is unbounded.*

Proof.
Since R_{prod} is a finite-dimensional salient space, $\mathrm{int}(R_{\text{prod}}) \neq \emptyset$. Let $u_0^{\text{prod}} \in \mathrm{int}(R_{\text{prod}})$. Then, by the free-disposal property of T_j, for every $y \in \{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ there is $\lambda > 0$ such that $(y^{\text{prod}} + \lambda u_0^{\text{prod}}, y^{\text{cons}}) \in \{x \in T_j \mid \mathcal{G}(x, p_0) = \alpha\}$. □

5.7.8 Lemma. *Let $p_0 \in \mathrm{int}(R^*)$, let $\alpha \in \mathbb{R}$ satisfy $\alpha < \chi_j(p_0)$ and let $\{x \in T_j \mid \mathcal{G}(x, p_0) = \alpha\}$ be a bounded set. Then $\chi_j(p_0) < \infty$.*

Proof.
Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in T_j, satisfying $\sup\{\mathcal{G}(e_n, p_0) \mid n \in \mathbb{N}\} = \chi_j(p_0)$. Lemma 5.7.7 implies that the set $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ is bounded, so $(e_n)_{n\in\mathbb{N}}$ has a convergent subsequence with limit $e \in T_j$ (II.b of Assumption 5.7.1). □

5.7.9 Corollary. *Let $p_0 \in \mathrm{int}(R^*)$ and let $\alpha \in \mathbb{R}$. If $\chi_j(p_0) = \infty$ then the set $\{x \in T_j \mid \mathcal{G}(x, p_0) = \alpha\}$ is unbounded.*

5.7.10 Lemma. *Let $p_0 \in \mathrm{Domain}[j] \cap \mathrm{int}(R^*)$. Then there is a $\tau(R^*, R)$-open neighbourhood O of p_0 such that every $q \in O$ satisfies $\chi_j(q) < \infty$.*

Proof.
The proof of this lemma is by contradiction. So, let $(q_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{int}(R^*)$, converging to p_0, such that $\forall n \in \mathbb{N} : \chi_j(q_n) = \infty$. By the previous corollary, for all $n \in \mathbb{N}$, the set $L_n := \{z \in T_j \mid \mathcal{G}(z, q_n) = \mathcal{G}(\mathcal{S}_j(p_0), q_n)\}$ is unbounded, so $\forall n \in \mathbb{N}\ \exists y_n \in L_n : [y_n, p_0] > 1 + [\mathcal{S}_j(p_0), p_0]$. Since L_n is convex, and contains $\mathcal{S}_j(p_0)$, for all $\tau \in [0, 1]$ we find $\tau y_n + (1 - \tau)\mathcal{S}_j(p_0) \in L_n$. Now choose $\tau_n := \frac{1}{[y_n,p_0]-[\mathcal{S}_j(p_0),p_0]} \in (0, 1)$ then $x_n := \tau_n y_n + (1 - \tau_n)\mathcal{S}_j(p_0) \in L_n \cap U$ where $U := \{z \in R \mid [z, p_0] = 1 + [\mathcal{S}_j(p_0), p_0]\}$. Since U is compact (Corollary 3.3.8.c), we may as well assume that $(x_n)_{n\in\mathbb{N}}$ is convergent, with limit $x \in R$. Note that the continuity of $\mathcal{G}$ implies $\mathcal{G}(x, p_0) = \chi_j(p_0)$. However, since by construction $x \neq \mathcal{S}_j(p)$ this is in contradiction with Lemma 5.7.3. □

5.7.11 Lemma. *Let $p_0 \in \mathrm{int}(R^*)$, let $\alpha \in \mathbb{R}$, and let $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ be an unbounded set. Then $p_0 \notin \mathrm{Domain}[j]$.*

Proof.
Let $(x_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$. Let $0 < \varepsilon \leq 1$ and define $p_\varepsilon := ((1-\varepsilon)p_0^{\text{prod}}, (1+\varepsilon)p_0^{\text{cons}})$. Since for all $n \in \mathbb{N}$ the gain $\mathcal{G}(x_n, p_\varepsilon)$ equals $\mathcal{G}(x_n, p_0) + \varepsilon[x_n^{\text{prod}}, p_0^{\text{prod}}]_{\text{prod}} + \varepsilon[x_n^{\text{cons}}, p_0^{\text{cons}}]_{\text{cons}}$, the sequence $(\mathcal{G}(x_n, p_\varepsilon))_{n\in\mathbb{N}}$ is unbounded. Hence, $\forall \varepsilon \in (0,1] : \chi_j(p_\varepsilon) = \infty$. Using Lemma 5.7.10, we conclude $p_0 \notin \text{Domain}[j]$. □

5.7.12 Corollary. *Let $p_0 \in \text{int}(R^*)$. If $p_0 \in \text{Domain}[j]$, then for all $\alpha \in \mathbb{R}$, the set $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ is compact.*

The following corollary implies Lemma 5.7.2.d.

5.7.13 Corollary. *Let $(p_n)_{n\in\mathbb{N}}$ be a convergent sequence in* Domain$[j]$, *with limit in* $\text{int}(R^*) \setminus \text{Domain}[j]$. *Then*

$$\forall p_0 \in \text{Domain}[j] \cap \text{int}(R^*) : \limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty.$$

Proof.
The sequence $(\mathcal{S}_j(p_n))_{n\in\mathbb{N}}$ does not have a point of accumulation, since existence of such a point would lead to a contradiction with Lemma 5.7.4. Let $p_0 \in \text{Domain}[j] \cap \text{int}(R^*)$. For all $\alpha \in \mathbb{R}$, the set $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ is compact (Corollary 5.7.12), and so we find that $\forall \alpha \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n > N : \mathcal{G}(\mathcal{S}_j(p_n), p_0) < \alpha$. We conclude $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$. □

We end this section with the following Proposition, which proves Lemma 5.7.2.a.

5.7.14 Proposition. *The set* $\text{Domain}[j] \cup \{0\}$ *is a salient subspace of* $\text{int}(R^*) \cup \{0\}$.

Proof.
Since the function $\mathcal{G} : R \times R^* \to \mathbb{R}$ is homogeneous of degree one, $\text{Domain}[j] \cup \{0\}$ is closed under scalar multiplication over $\mathbb{R}^+$. Let $p_1, p_2 \in \text{Domain}[j]$ and let $\tau \in (0,1)$. We prove that $q := \tau p_1 + (1-\tau)p_2 \in \text{Domain}[j]$. We first note that $p_1, p_2 \in \text{Domain}[j]$ implies $\chi_j(q) \leq \tau\chi_j(p_1) + (1-\tau)\chi_j(p_2)$. Since there is nothing to prove in case $\mathcal{G}(\mathcal{S}_j(p_1), q) = \chi_j(q)$, we may as well assume that $\exists \varepsilon > 0$ such that $\mathcal{G}(\mathcal{S}_j(p_1), q) < \chi_j(q) - \varepsilon$. Define $U := \{x \in T_j \mid \mathcal{G}(x, p_2) \geq \mathcal{G}(\mathcal{S}_j(p_1), p_2)\}$, then U is non-empty and compact (Lemma 5.7.12). By Definition 5.7.6, there is a sequence $(e_n)_{n\in\mathbb{N}}$ in $E(T_j)$ satisfying $\sup\{\mathcal{G}(e_n, q) \mid n \in \mathbb{N}\} = \chi_j(q)$.
Let $n \in \mathbb{N}$. If $e_n \notin U$, i.e., if $\mathcal{G}(e_n, p_2) < \mathcal{G}(\mathcal{S}_j(p_1), p_2)$ then $\mathcal{G}(e_n, q) = \tau\mathcal{G}(e_n, p_1) + (1-\tau)\mathcal{G}(e_n, p_2) < \tau\mathcal{G}(\mathcal{S}_j(p_1), p_1) + (1-\tau)\mathcal{G}(\mathcal{S}_j(p_1), p_2) = \mathcal{G}(\mathcal{S}_j(p_1), q) < \chi_j(q) - \varepsilon$. We conclude that $\exists N \in \mathbb{N}\ \forall n > N : e_n \in U$. Since U is compact, Lemma 5.7.4 implies that $q \in \text{Domain}[j]$. □

5.8 Proof of Theorem C1

Consider Model C, introduced on page 120, and assume that the assumptions of Theorem C1 are satisfied.

For every $j \in \{1, \ldots, j_0\}$, Lemma 5.7.2 implies that every firm j, $j \in \{1, \ldots, j_0\}$, has a supply function $\mathcal{S}_j^{C1} : \text{Domain}[j] \to C$, which is continuous with respect to $\tau(C^*, C)$ and $\tau(C, C^*)$. This completes Step a, as described in Section 5.1.

Define

$$\text{Domain} := \bigcap_{j=1}^{j_0} \text{Domain}[j].$$

By Assumption C1.5.a, the set Domain is non-empty. Define the total supply function $\mathcal{S}^{C1} : \text{Domain} \to C$ by

$$\mathcal{S}^{C1} := \sum_{j=1}^{j_0} \mathcal{S}_j^{C1}.$$

For every $i \in \{1, \ldots, i_0\}$, the income function $\mathcal{K}_i^{C1} : \text{Domain} \to \mathbb{R}_+$ is, for every $p \in \text{Domain}$, defined by

$$\mathcal{K}_i^{C1}(p) := [w_i, p] + \sum_{j=1}^{j_0} \theta_{ij} \mathcal{G}(\mathcal{S}_j^{C1}(p), p),$$

where $\mathcal{G}(x, p)$ denotes the gain (or profit) from executing production process x at pricing functional p (cf. page 118). Since $\mathcal{K}_i^{C1}$ is continuous on Domain, Lemma 5.3.2.b (with $Q = \text{Domain} \cup \{0\}$ and $R = S = C$) implies that every agent i, $i \in \{1, \ldots, i_0\}$, has a demand function $\mathcal{D}_i^{C1} : \text{Domain} \to C$, which is continuous with respect to $\tau(C^*, C)$ and $\tau(C, C^*)$. This completes Step b, as described in Section 5.1.

We define the total demand function $\mathcal{D}^{C1} : \text{Domain} \to C$ by

$$\mathcal{D}^{C1} := \sum_{i=1}^{i_0} \mathcal{D}_i^{C1}.$$

The mapping $\mathcal{Z}^{C1} : \text{Domain} \times C^* \to \mathbb{R}$ is, for every $p \in \text{Domain}$ and every $q \in C^*$, defined by

$$\mathcal{Z}^{C1}(p, q) := [\mathcal{D}^{C1}(p), q] - \mathcal{G}(\mathcal{S}^{C1}(p), q) - [w_{\text{total}}, q].$$

This completes Step c, as described in Section 5.1.

In order to realise Steps d and e of Section 5.1, we want to apply Theorem 5.2.1 to this mapping, with $\mathcal{W} = \mathcal{Z}^{C1}$, $R = C$ and $Q = \text{Domain} \cup \{0\}$. So, let $p_0 \in \text{int}(C^*)$. We have to check whether $\mathcal{Z}^{C1}$ satisfies the requirements for this theorem.

I: By Lemma 5.3.2.c (with $Q = \text{Domain} \cup \{0\}$ and $S = R = C$) we find that $\mathcal{Z}^{C1}(p,p) = 0$, for every $p \in \text{Domain}$. Clearly, $\mathcal{Z}^{C1}(\alpha p, q) = \alpha \mathcal{Z}^{C1}(p,q)$ for every $\alpha \in \mathbb{R}_+$, $p \in \text{Domain}$ and $q \in C^*$.

II: For every $p \in \text{Domain}$, the mapping $q \mapsto \mathcal{Z}^{C1}(p,q)$ is continuous.

III: Let $x_0 \in \text{int}(C)$. Since the functions $\mathcal{S}_j^{C1}$, $j \in \{1, \dots, j_0\}$, and $\mathcal{D}^{C1}$ are continuous on Domain, Example 5.2.7 implies that the mapping $\mathcal{Z}^{C1}$ satisfies Condition III of Theorem 5.2.1.

IV: Let $\xi_0 = 1$ and let $(p_n)_{n \in \mathbb{N}}$ be a sequence in Domain, with limit $p \notin \text{Domain} \cup \{0\}$. Note that $p \notin \text{Domain}$ means either $p \in \text{bd}(C^*)$ or $p \in \text{int}(C^*) \backslash \text{Domain}$. In the first situation, Assumption C1.5.b' states that there is $i \in \{1, \dots, i_0\}$ such that

$$\limsup_{n \to \infty} \ \mathcal{K}_i^{C1}(p_n) > 0.$$

Taking a subsequence if necessary, Lemma 5.3.2.d (with $Q = \text{Domain} \cup \{0\}$ and $S = R = C$), implies that the sequence $\left(\mathcal{D}_i^{C1}(p_n)\right)_{n \in \mathbb{N}}$, and therefore also the sequence $\left(\mathcal{D}^{C1}(p_n)\right)_{n \in \mathbb{N}}$, is unbounded.
In the second situation, there is $j \in \{1, \dots, j_0\}$ such that $p \notin \text{Domain}[j]$. By Lemma 5.7.2.d, we find

$$\limsup_{n \to \infty} \ \mathcal{G}(\mathcal{S}_j^{C1}(p_n), p_0) = -\infty,$$

and thus

$$\limsup_{n \to \infty} \ \mathcal{G}(\mathcal{S}^{C1}(p_n), p_0) = -\infty.$$

Either way, since $\mathcal{G}(\mathcal{S}^{C1}(p), p_0) \leq \sum_{j=1}^{j_0} \chi_j(p_0) < \infty$ for every $p \in \text{Domain}$, we find

$$\begin{aligned} &\limsup_{n \to \infty} \ \mathcal{Z}^{C1}(p_n, p_0) \\ &\quad = \limsup_{n \to \infty} \ \left([\mathcal{D}^{C1}(p_n), p_0] - \mathcal{G}(\mathcal{S}^{C1}(p_n), p_0) - [w_{\text{total}}, p_0]\right) = \infty. \end{aligned}$$

We conclude that the mapping $\mathcal{Z}^{C1} : \text{Domain} \times C^* \to \mathbb{R}$ meets all the requirements for Theorem 5.2.1 (with $R = C$ and $Q = \text{Domain} \cup \{0\}$), hence

$$\exists p_* \in \text{int}(C^*) \forall q \in C^* : \mathcal{Z}^{C1}(p_*, q) \leq 0.$$

Lemma 2.3.23 implies $\mathcal{D}^{C1}(p_*) + (\mathcal{S}^{C1,\text{prod}}(p_*), 0) \leq_C w_{\text{total}} + (0, \mathcal{S}^{C1,\text{cons}}(p_*))$, so we conclude that

$$(\mathcal{S}_1^{C1}(p_*), \dots, \mathcal{S}_{j_0}^{C1}(p_*), \mathcal{D}_1^{C1}(p_*), \dots, \mathcal{D}_{i_0}^{C1}(p_*), p_*)$$

is a Walrasian equilibrium for Model C.

5.9 Supply functions 2

This section is the C2 and D-counterpart of Section 5.7. The goal of this section is to prove Lemma 5.9.3, which will be used in the proofs of Theorems C2 and D, to realise Step a of Section 5.1: the step from supply sets to supply functions.
Similar to Section 5.7, the central lemma of this section is stated in terms of a general salient space R. When this lemma will be applied, R will be replaced by the set of all bundles of trade. Consequently, R^* will represent the set of all possible pricing functions.

Similar to the situation in Section 5.7, we consider a finite number j_0 of production technologies where for every $j \in \{1, \ldots, j_0\}$, production technology T_j is a subset of a salient space $R = R_{\text{prod}} \oplus R_{\text{cons}}$.

In this section, we assume that R and the production technologies satisfy the following assumption, which differs from Assumption 5.7.1 with respect to II.b and II.c, only. Assumption 5.9.1.II.b is weaker than 5.7.1.II.b, Assumption 5.9.1.II.c is stronger than 5.7.1.II.c. In Section 4.8, we discussed the interpretation of these assumptions and compared the two.

5.9.1 Assumption.

I) *The salient space R is finite-dimensional and reflexive.*

II) *For all $j \in \{1, \ldots, j_0\}$, production technology T_j satisfies*

- **a)** $T_j = \bigcup_{e \in E(T_j)} F_e$,
- **b)** *$E(T_j)$ is closed with respect to topology $\tau(R, R^*)$,*
- **c)** *if $e_1, e_2 \in E(T_j)$, $e_1 \neq e_2$, $\tau \in (0, 1)$ then $\tau e_1 + (1 - \tau)e_2 \in \text{int}(T_j)$.*

Throughout this section we assume that the above assumption holds.

Basically, the structure of this section is the same as the structure of Section 5.7; almost every lemma in this section has a counterpart in Section 5.7. Where possible, we refer to Section 5.7 for lemmas and proofs.

Similar to Definition 5.7.6, the extended real valued function $\chi_j : R^* \to [0, \infty) \cup \{\infty\}$ is for every $j \in \{1, \ldots, j_0\}$, given by

$$\chi_j(p) := \sup_{x \in T_j} \mathcal{G}(x, p) = \sup_{e \in E(T_j)} \mathcal{G}(e, p).$$

Recall that for every $j \in \{1, \ldots, j_0\}$, the function χ_j is convex.

5.9.2 Definition (Domain). For every $j \in \{1, \dots, j_0\}$ the set Domain$[j]$ is given by

$$\text{Domain}[j] := \{q \in R^* \setminus \{0\} \mid \exists x_q \in T_j : \mathcal{G}(x_q, q) = \chi_j(q)\}.$$

Contrary to the situation as discussed in Section 5.7, where for every $j \in \{1, \dots, j_0\}$ the set Domain$[j]$ was a subset of $\text{int}(R^*)$ by definition, here we allow that also elements of $\text{bd}(R^*)$ are in Domain$[j]$. In comparison with Section 5.7, this extension of the definition of Domain$[j]$, will alter the proofs of several lemmas.

For every $j \in \{1, \dots, j_0\}$ and for every $p \in$ Domain, the supply set $S_j(p)$ is given by

$$S_j(p) = \{x \in T_j \mid \mathcal{G}(x, p) = \chi_j(p)\}.$$

For every $p \notin$ Domain$[j]$ we find $S_j(p) = \emptyset$.

The main purpose of this section is to prove the following lemma.

5.9.3 Lemma. *Assumption 5.9.1 implies the following.*

a) *For every $j \in \{1, \dots, j_0\}$, the set $(\text{Domain}[j] \cap \text{int}(R^*)) \cup \{0\}$ is a salient subspace of R^*.*

b) *For all $j \in \{1, \dots, j_0\}$ and for all $p \in$ Domain$[j]$, the supply set $S_j(p)$ contains exactly one element of $E(T_j)$.*

c) *For every $j \in \{1, \dots, j_0\}$, define the supply function $\mathcal{S}_j : \text{Domain}[j] \to R$, such that $S_j(p) \cap E(T_j) = \{\mathcal{S}_j(p)\}$, for all $p \in$ Domain$[j]$. Then the function $\mathcal{S}_j$ is continuous on its domain with respect to the relative topology of $\tau(R^*, R)$.*

d) *Let $j \in \{1, \dots, j_0\}$ and let $p_0 \in \text{int}(R^*) \cap \text{Domain}[j]$. If $(p_n)_{n \in \mathbb{N}}$ is a sequence in Domain$[j]$, convergent to $p \in R^* \setminus \text{Domain}[j]$, then $\limsup_{n \to \infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$.*

The proof of each item of Lemma 5.9.3 will be a direct result of several of the following lemmas. More precisely, Lemma 5.9.4 and Lemma 5.9.6 correspond to Lemma 5.9.3.b and c, respectively. Lemma 5.9.3.d is proved in Corollary 5.9.10. Finally, Proposition 5.9.12 proves Lemma 5.9.3.a.

Next, we show that Assumption 5.9.1 indeed implies that we can deal with continuous supply functions $\mathcal{S}_j$, defined on the set Domain$[j]$, $j \in \{1, \dots, j_0\}$. For the remainder of this section, let j be any fixed element of $\{1, \dots, j_0\}$, and assume Domain$[j] \neq \emptyset$. Before we are able to define the supply function $\mathcal{S}_j$, we need uniqueness of the supply, for every $p \in$ Domain$[j]$. It turns out that allowing elements of

$\mathrm{bd}(R^*)$ in the set Domain$[j]$ results in the loss of the uniqueness of the supply. Indeed, if for certain $p \in \mathrm{bd}(C^*)$, there is $x^{\mathrm{prod}} \in C_{\mathrm{prod}}$ such that $[x^{\mathrm{prod}}, p^{\mathrm{prod}}]_{\mathrm{prod}} = 0$, then $y \in S_j(p)$ implies $y + x \in S_j(p)$. However, Assumption 5.9.1.II.c implies that if the supply set at p is non-empty, then it contains exactly one element of $E(T_j)$. We will use this unique element to define the supply function of firm j. Combined with some properties of the unique efficient element of the supply set, this is proved in the following lemma.

5.9.4 Lemma. *Let $p \in$ Domain$[j]$. Then there is a unique $e_p \in E(T_j)$ such that $\mathcal{G}(e_p, p) = \chi_j(p)$. This element e_p satisfies $\forall x \in S_j(p) : x \in F_{e_p}$. Moreover, if $p \in$ Domain$[j] \cap \mathrm{int}(R^*)$, then e_p is the unique element of the supply set $S_j(p)$.*

Proof.
Since $p \in$ Domain$[j]$, the set $S_j(p) \cap E(T_j)$ is non-empty. Suppose $e_1, e_2 \in S_j(p) \cap E(T_j)$ and $e_1 \neq e_2$. Then by Assumption 5.9.1.II.c, $x := \tau e_1 + (1-\tau)e_2$ is a (linearly) internal point of T_j. Hence, for a fixed order unit u_0 of R there exists $\varepsilon > 0$ such that $(x^{\mathrm{prod}}, x^{\mathrm{cons}} + \varepsilon u_0^{\mathrm{cons}}) \in (\varepsilon u_0^{\mathrm{prod}}, 0^{\mathrm{cons}}) + T_j$. Let $y \in T_j$ satisfy $(\varepsilon u_0^{\mathrm{prod}}, 0^{\mathrm{cons}}) + y = (x^{\mathrm{prod}}, x^{\mathrm{cons}} + \varepsilon u_0^{\mathrm{cons}})$. Since $p \neq 0$, we find $\mathcal{G}(y,p) > \mathcal{G}(x,p)$ which is in contradiction with the optimality of e_1 and e_2. We conclude that there is a unique $e_p \in S_j(p) \cap E(T_j)$, maximising $\mathcal{G}(e,p)$, $e \in E(T_j)$.
Let $x \in S_j(p)$, then (Assumption 5.9.1.II.a) there is $e \in E(T_j)$ such that $x \in F_e$. Since $e \in S_j(p)$, we conclude $e = e_p$.
Let $p \in$ Domain$[j] \cap \mathrm{int}(R^*)$ and let $x \in T_j \setminus E(T_j)$. Then $\exists e_x \in E(T_j) : x \in F_{e_x}$. Since $p \in \mathrm{int}(R^*)$ and $x \neq e_x$, we find $\mathcal{G}(x,p) < \mathcal{G}(e_x,p) \leq \mathcal{G}(e_p,p)$. □

Lemma 5.9.3.b is a direct result from the above lemma. As a consequence, we can define the supply function $\mathcal{S}_j : \mathrm{int}(R^*) \to E(T_j)$ where for every $p \in$ Domain$[j]$, $\mathcal{S}_j(p)$ is the unique element of $S_j(p) \cap E(T_j)$.

The following lemma is the C2-counterpart of Lemma 5.7.4. However, in this section, the limit p of the stated sequence is allowed to be an element of $\mathrm{bd}(R^*) \setminus \{0\}$.

5.9.5 Lemma. *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Domain$[j]$, with limit $p \neq 0$. If the sequence $(\mathcal{S}_j(p_n))_{n\in\mathbb{N}}$ is convergent with limit $s \in R$, then $p \in$ Domain$[j]$ and $s = \mathcal{S}_j(p)$.*

Proof.
Since $\forall n \in \mathbb{N}\ \forall x \in T_j : \mathcal{G}(\mathcal{S}_j(p_n), p_n) \geq \mathcal{G}(x, p_n)$, continuity of $\mathcal{G}$ guarantees $\forall x \in T_j : \mathcal{G}(s,p) \geq \mathcal{G}(x,p)$. Since $E(T_j)$ is closed (Assumption 5.9.1.II.b), $s \in E(T_j)$, so $p \in$ Domain$[j]$. Furthermore, Lemma 5.9.4 implies $s = \mathcal{S}_j(p)$. □

The following lemma proves Lemma 5.9.3.c. Compared to the proof of Lemma 5.7.5 (the counterpart of the following lemma), the proof of Lemma 5.9.6 is longer since it is possible that the set $S_j(p)$ is not a singleton.

5.9.6 Lemma. *The supply function* $\mathcal{S}_j : \text{Domain}[j] \to E(T_j)$ *is continuous with respect to the relative topology on* $\text{Domain}[j]$.

Proof.
Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $\text{Domain}[j]$, with limit $p \in \text{Domain}[j]$. Let $d : R \times R \to \mathbb{R}_+$ be a salient metric generating the salient topology $\tau(R, R^*)$. Suppose $(\mathcal{S}_j(p_n))_{n\in\mathbb{N}}$ does not converge to $\mathcal{S}_j(p)$. Without loss of generality, we may assume $\exists \varepsilon > 0 \ \ \forall n \in \mathbb{N} : d(\mathcal{S}_j(p_n), \mathcal{S}_j(p)) > \varepsilon$. Define $x_n := \tau_n \mathcal{S}_j(p_n) + (1 - \tau_n)\mathcal{S}_j(p)$, with $\tau_n := \frac{\varepsilon}{d(\mathcal{S}_j(p_n),\mathcal{S}_j(p))} \in (0, 1)$. Then $d(x_n, \mathcal{S}_j(p)) = \varepsilon$ and by Assumption 5.9.1.II.c we find that x_n is an internal point of T_j. Both the sequences $(\tau_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are bounded. Without loss of generality assume $\lim_{n\to\infty} \tau_n = \tau$ and $\lim_{n\to\infty} x_n = x \in R$. Note that $x \neq \mathcal{S}_j(p)$ implies $\tau > 0$. Since $\mathcal{G}(x_n, p_n) \geq \min\{\mathcal{G}(\mathcal{S}_j(p_n), p_n), \mathcal{G}(\mathcal{S}_j(p), p_n)\} = \mathcal{G}(\mathcal{S}_j(p), p_n)$, the continuity of $\mathcal{G}$ implies $\mathcal{G}(x, p) \geq \mathcal{G}(\mathcal{S}_j(p), p) = \chi_j(p)$. Since $\forall n \in \mathbb{N} : \mathcal{G}(x_n, p) \leq \chi_j(p)$, we find $\mathcal{G}(x, p) = \chi_j(p)$.
Since for all $n \in \mathbb{N} : x_n = \tau_n \mathcal{S}_j(p_n) + (1 - \tau_n)\mathcal{S}_j(p)$, the sequence $(\mathcal{S}_j(p_n))_{n\in\mathbb{N}}$ in $E(T_j)$ is convergent with limit $e \in E(T_j)$ (II.b of Assumption 5.9.1) satisfying $x = \tau e + (1 - \tau)\mathcal{S}_j(p))$. Note that $x \neq \mathcal{S}_j(p)$ implies $e \neq \mathcal{S}_j(p)$. However, $\mathcal{G}(e, p) = \frac{1}{\tau}(\mathcal{G}(x, p) - (1 - \tau)\mathcal{G}(\mathcal{S}_j(p), p)) = \chi_j(p)$. This is in contradiction with $\mathcal{S}_j(p)$ being the unique element of the set $S_j(p) \cap E(T_j)$. □

The continuity of the supply function is proved, so we can now concentrate on some other properties of this function. First, we derive some limit behaviour, especially regarding a sequence $(p_n)_{n\in\mathbb{N}} \in \text{Domain}[j]$, with limit $p \notin \text{Domain}[j]$. Also, we will investigate the set $\text{Domain}[j]$ in more detail.

The following lemma is equal to Corollary 5.7.9. Lemma 5.9.8 is the counterpart of Lemma 5.7.10. However, since in this section it is not assumed that the set T_j is closed, the proof of Lemma 5.7.10 needs a supplement.

5.9.7 Lemma. *Let* $p_0 \in \text{int}(R^*)$ *and let* $\alpha \in \mathbb{R}$. *If* $\chi_j(p_0) = \infty$ *then the set* $\{x \in T_j \mid \mathcal{G}(x, p_0) = \alpha\}$ *is unbounded.*

5.9.8 Lemma. *Let* $p_0 \in \text{Domain}[j] \cap \text{int}(R^*)$. *Then there is a* $\tau(R^*, R)$*-open neighbourhood* O *of* p_0 *such that every* $q \in O$ *satisfies* $\chi_j(q) < \infty$.

Proof.
The proof of this lemma is by contradiction. So, let $(q_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{int}(R^*)$, converging to p_0, such that $\forall n \in \mathbb{N} : \chi_j(q_n) = \infty$. By the previous lemma, for all $n \in \mathbb{N}$, the set $L_n := \{z \in T_j \mid \mathcal{G}(z, q_n) = \mathcal{G}(\mathcal{S}_j(p_0), q_n)\}$ is unbounded, so $\forall n \in \mathbb{N}\ \exists y_n \in L_n : [y_n, p_0] > 1 + [\mathcal{S}_j(p_0), p_0]$. Since L_n is convex and contains $\mathcal{S}_j(p_0)$, for all $\tau \in [0, 1]$ we find $\tau y_n + (1 - \tau)\mathcal{S}_j(p_0) \in L_n$. Now choose $\tau_n := \frac{1}{[y_n,p_0]-[\mathcal{S}_j(p_0),p_0]} \in (0, 1)$ then $x_n := \tau_n y_n + (1 - \tau_n)\mathcal{S}_j(p_0) \in L_n \cap U$ where $U := \{z \in R \mid [z, p_0] = 1 + [\mathcal{S}_j(p_0), p_0]\}$. Since U is compact (Corollary 3.3.8.c), we may as well assume that $(x_n)_{n\in\mathbb{N}}$ is convergent, with limit $x \in R$. Note that the continuity of $\mathcal{G}$ implies $\mathcal{G}(x, p_0) = \chi_j(p_0)$. If we can prove $x \in T_j$, we are done.
By II.a of Assumption 5.9.1, there is a sequence $(e_n)_{n\in\mathbb{N}}$ in $E(T_j)$ satisfying $\forall n \in \mathbb{N} : x_n \in F_{e_n}$. Hence, $\mathcal{G}(x_n, p_0) \leq \mathcal{G}(e_n, p_0) \leq \chi_j(p_0)$ and $x_n^{\mathrm{prod}} \geq_{\mathrm{prod}} e_n^{\mathrm{prod}}$. So, the sequence $(\mathcal{G}(e_n, p_0))_{n\in\mathbb{N}}$ is convergent with limit $\chi_j(p_0)$, and the sequence $(e_n^{\mathrm{prod}})_{n\in\mathbb{N}}$ is bounded. Moreover, $[e_n^{\mathrm{cons}}, p_0^{\mathrm{cons}}]_{\mathrm{cons}} \leq \chi_j(p_0) + [e_n^{\mathrm{prod}}, p_0^{\mathrm{prod}}]_{\mathrm{prod}}$, so the sequence $(e_n^{\mathrm{cons}})_{n\in\mathbb{N}}$ is bounded. Without loss of generality, we assume that $(e_n)_{n\in\mathbb{N}}$ is convergent. Let $e \in E(T_j)$ be its limit, so $\mathcal{G}(e, p_0) = \chi_j(p_0)$. By Lemma 5.9.4 we find $e = \mathcal{S}_j(p_0)$. Continuity of $\geq_{\mathrm{prod}}$ and $\geq_{\mathrm{cons}}$ implies $x \in F_e \subset T_j$. Now, $x \in T_j$ and $\mathcal{G}(x, p_0) = \chi_j(p_0)$ imply that x is an element of the supply set $S_j(p_0)$. Since $x \in U$ implies $x \neq \mathcal{S}_j(p_0)$, we arrive at a contradiction since $p_0 \in \mathrm{Domain}[j] \cap \mathrm{int}(R^*)$ combined with Lemma 5.9.4 implies that $\mathcal{S}_j(p_0)$ is the unique element of the supply set $S_j(p_0)$. □

For the proof of Corollary 5.9.9, we refer to the proof of Corollary 5.7.12.

5.9.9 Corollary. *Let* $p_0 \in \mathrm{int}(R^*)$. *If* $p_0 \in \mathrm{Domain}[j]$, *then for all* $\alpha \in \mathbb{R}$, *the set* $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ *is compact.*

The following corollary proves Lemma 5.9.3.d.

5.9.10 Corollary. *Let* $(p_n)_{n\in\mathbb{N}}$ *be a convergent sequence in* $\mathrm{Domain}[j]$, *with limit* $p \in R^* \setminus \mathrm{Domain}[j]$. *Then* $\forall p_0 \in \mathrm{Domain}[j] \cap \mathrm{int}(R^*) : \limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$.

Proof.
The sequence $(\mathcal{S}_j(p_n))_{n\in\mathbb{N}}$ does not have a point of accumulation, since existence of such a point would lead to a contradiction with Lemma 5.9.5. Let $p_0 \in \mathrm{Domain}[j] \cap \mathrm{int}(R^*)$. For all $\alpha \in \mathbb{R}$, the set $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ is compact (Corollary 5.9.9) and so we find that $\forall \alpha \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n > N : \mathcal{G}(\mathcal{S}_j(p_n), p_0) \leq \alpha$. We conclude that $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$. □

5.9.11 Proposition. $\mathrm{Domain}[j] \cap \mathrm{int}(R^*) = \mathrm{int}(\mathrm{Domain}[j])$.

Proof.
We only have to prove $\mathrm{Domain}[j] \cap \mathrm{int}(R^*) \subset \mathrm{int}(\mathrm{Domain}[j])$. Let $p_0 \in \mathrm{Domain}[j] \cap \mathrm{int}(R^*)$. By Lemma 5.9.8, there is a $\tau(R^*, R)$-open neighbourhood O of p_0 such that every $q \in O$ satisfies $\chi_j(q) < \infty$. Let $q \in O$. We shall prove that $\exists e \in E(T_j) : \mathcal{G}(e, q) = \chi_j(q)$.
Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in $E(T_j)$ satisfying $\lim_{n\to\infty} \mathcal{G}(e_n, q) = \chi_j(q) < \infty$. Then, for $\alpha \in \mathbb{R}$ chosen sufficiently small, $(e_n)_{n\in\mathbb{N}}$ is a sequence in $\{x \in T_j \mid \mathcal{G}(x, q) \geq \alpha\}$. So, by Corollary 5.9.9, without loss of generality, we may assume $(e_n)_{n\in\mathbb{N}}$ is convergent with limit $e \in E(T_j) \subset T_j$ (II.b of Assumption 5.9.1). Since $\mathcal{G}(e, q) = \chi_j(q)$, we conclude that $q \in \mathrm{Domain}[j]$. □

Except for an occasional replacement of Domain[j] with int(Domain[j]), the proof of Proposition 5.9.12 is similar to the proof of Proposition 5.7.14.

5.9.12 Proposition. *The set* $\mathrm{int}(\mathrm{Domain}[j]) \cup \{0\}$ *is a salient subspace of* R^*.

Proof.
Since the function $\mathcal{G} : R \times R^* \to \mathbb{R}$ is homogeneous of degree one, $\mathrm{Domain}[j] \cup \{0\}$ is closed under scalar multiplication over $\mathbb{R}^+$. Let $p_1, p_2 \in \mathrm{int}(\mathrm{Domain}[j])$ and let $\tau \in (0, 1)$. Define $q \in \mathrm{int}(R^*)$ by $q := \tau p_1 + (1 - \tau)p_2$. We have to prove that $q \in \mathrm{Domain}[j]$. Since $p_1, p_2 \in \mathrm{Domain}[j]$, we find $\chi_j(q) \leq \tau\chi_j(p_1) + (1 - \tau)\chi_j(p_2)$. There is nothing to prove in case $\mathcal{G}(\mathcal{S}_j(p_1), q) = \chi_j(q)$, so we may as well assume that $\exists\varepsilon > 0$ such that $\mathcal{G}(\mathcal{S}_j(p_1), q) < \chi_j(q) - \varepsilon$. Define $U := \{x \in T_j \mid \mathcal{G}(x, p_2) \geq \mathcal{G}(\mathcal{S}_j(p_1), p_2)\}$, then U is compact (Lemma 5.9.9). Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in $E(T_j)$ satisfying $\sup\{\mathcal{G}(e_n, q) \mid n \in \mathbb{N}\} = \chi_j(q)$.
Let $n \in \mathbb{N}$. If $e_n \notin U$, i.e., if $\mathcal{G}(e_n, p_2) < \mathcal{G}(\mathcal{S}_j(p_1), p_2)$ then $\mathcal{G}(e_n, q) = \tau\mathcal{G}(e_n, p_1) + (1 - \tau)\mathcal{G}(e_n, p_2) < \tau\mathcal{G}(\mathcal{S}_j(p_1), p_1) + (1 - \tau)\mathcal{G}(\mathcal{S}_j(p_1), p_2) = \mathcal{G}(\mathcal{S}_j(p_1), q) < \chi_j(q) - \varepsilon$. We conclude that $\exists N \in \mathbb{N}\ \forall n > N : e_n \in U$. Since U is compact, $q \in \mathrm{Domain}[j]$. □

5.10 Proof of Theorem C2

Consider Model C, introduced on page 120, and assume that the assumptions of Theorem C2 are satisfied.

For every $j \in \{1, \ldots, j_0\}$, Lemma 5.9.3 implies that every firm j, $j \in \{1, \ldots, j_0\}$, has a supply function $\mathcal{S}_j^{C2} : \mathrm{Domain}[j] \to C$, which is continuous with respect to $\tau(C^*, C)$ and $\tau(C, C^*)$. This completes Step a, as described in Section 5.1.

Define

$$\text{Domain} := \left(\bigcap_{j=1}^{j_0} \text{Domain}[j] \right) \cap \operatorname{int}(C^*).$$

By Assumption C2.5.a, the set Domain is non-empty. Define the total supply function $\mathcal{S}^{C2} : \text{Domain} \to C$ by

$$\mathcal{S}^{C2} := \sum_{j=1}^{j_0} \mathcal{S}_j^{C2}.$$

Note that Lemma 5.9.3.a implies that Domain$\cup\{0\}$ is a salient subspace of $\operatorname{int}(C^*)\cup\{0\}$.

For every $i \in \{1, \dots, i_0\}$, the income function $\mathcal{K}_i^{C2}$: Domain $\to \mathbb{R}_+$, is for every $p \in$ Domain defined by

$$\mathcal{K}_i^{C2}(p) := [w_i, p] + \sum_{j=1}^{j_0} \theta_{ij} \mathcal{G}(\mathcal{S}_j^{C2}(p), p).$$

Since $\mathcal{K}_i^{C2}$ is continuous on Domain, Lemma 5.3.2.b (with $Q = \text{Domain} \cup \{0\}$ and $R = S = C$) implies that every agent i, $i \in \{1, \dots, i_0\}$, has a demand function $\mathcal{D}_i^{C2}$: Domain $\to C$, which is continuous with respect to $\tau(C^*, C)$ and $\tau(C, C^*)$. This completes Step b, as described in Section 5.1.

We define the total demand function $\mathcal{D}^{C2}$: Domain $\to C$ by

$$\mathcal{D}^{C2} := \sum_{i=1}^{i_0} \mathcal{D}_i^{C2}.$$

The mapping $\mathcal{Z}^{C2}$: Domain $\times\, C \to \mathbb{R}$ is, for every $p \in$ Domain and every $q \in C^*$, defined by

$$\mathcal{Z}^{C2}(p, q) := [\mathcal{D}^{C2}(p), q] - \mathcal{G}(\mathcal{S}^{C2}(p), q) - [w_{\text{total}}, q].$$

This completes Step c, as described in Section 5.1.

In order to realise Steps d and e of Section 5.1, we want to apply Theorem 5.2.1 to this mapping, with $\mathcal{W} = \mathcal{Z}^{C2}$, $R = C$ and $Q = \text{Domain} \cup \{0\}$. So, let $p_0 \in \operatorname{int}(C^*)$. We have to check whether $\mathcal{Z}^{C2}$ satisfies the requirements for this theorem.

I: By Lemma 5.3.2.c (with $Q = \text{Domain} \cup \{0\}$ and $S = R = C$) we find that $\mathcal{Z}^{C2}(p, p) = 0$ for every $p \in$ Domain. Clearly, $\mathcal{Z}^{C2}(\alpha p, q) = \alpha \mathcal{Z}^{C2}(p, q)$ for every $\alpha \in \mathbb{R}_+$, $p \in$ Domain and $q \in C^*$.

II: For every $p \in$ Domain, the mapping $q \mapsto \mathcal{Z}^{C2}(p, q)$ is continuous.

III: Let $x_0 \in \mathrm{int}(C)$. Since the functions $\mathcal{S}_j^{C2}$, $j \in \{1, \dots, j_0\}$, and $\mathcal{D}^{C2}$ are continuous on Domain, Example 5.2.7 implies that the mapping $\mathcal{Z}^{C2}$ satisfies Condition III of Theorem 5.2.1.

IV: Let $\xi_0 = 1$ and let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Domain, with limit $p \notin \mathrm{Domain} \cup \{0\}$. Note that $p \notin \mathrm{Domain}$ means either $p \in \mathrm{bd}(C^*)$ or $p \in \mathrm{int}(C^*)\backslash\mathrm{Domain}$. In the first situation, Assumption C2.5.b' states that there is $i \in \{1, \dots, i_0\}$ such that

$$\limsup_{n\to\infty} \mathcal{K}_i^{C2}(p_n) > 0.$$

Taking a subsequence if necessary, Lemma 5.3.2.d (with $Q = \mathrm{Domain} \cup \{0\}$ and $S = R = C$) implies that the sequence $\left(\mathcal{D}_i^{C2}(p_n)\right)_{n\in\mathbb{N}}$, and therefore also the sequence $\left(\mathcal{D}^{C2}(p_n)\right)_{n\in\mathbb{N}}$, is unbounded.
In the second situation, there is $j \in \{1, \dots, j_0\}$ such that $p \notin \mathrm{Domain}[j]$. By Lemma 5.9.3.d, we find

$$\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_j^{C2}(p_n), p_0) = -\infty,$$

and thus

$$\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}^{C2}(p_n), p_0) = -\infty.$$

Either way, since $\mathcal{G}(\mathcal{S}^{C2}(p), p_0) \leq \sum_{j=1}^{j_0} \chi_j(p_0) < \infty$ for every $p \in \mathrm{Domain}$, we find

$$\limsup_{n\to\infty} \mathcal{Z}^{C2}(p_n, p_0) = \limsup_{n\to\infty} \left([\mathcal{D}^{C2}(p_n), p_0] - \mathcal{G}(\mathcal{S}^{C2}(p_n), p_0) - [w_{\mathrm{total}}, p_0]\right) = \infty.$$

We conclude that the mapping $\mathcal{Z}^{C2} : \mathrm{Domain} \times C^* \to \mathbb{R}$ meets all the requirements for Theorem 5.2.1 (with $R = C$ and $Q = \mathrm{Domain} \cup \{0\}$), hence

$$\exists p_* \in \mathrm{int}(C^*) \forall q \in C^* : \mathcal{Z}^{C2}(p_*, q) \leq 0.$$

Lemma 2.3.23 implies $\mathcal{D}^{C2}(p_*) + (\mathcal{S}^{C2,\mathrm{prod}}(p_*), 0) \leq_C w_{\mathrm{total}} + (0, \mathcal{S}^{C2,\mathrm{cons}}(p_*))$, so we conclude that

$$(\mathcal{S}_1^{C2}(p_*), \dots, \mathcal{S}_{j_0}^{C2}(p_*), \mathcal{D}_1^{C2}(p_*), \dots, \mathcal{D}_{i_0}^{C2}(p_*), p_*)$$

is a Walrasian equilibrium for Model C.

5.11 Proof of Theorem D

Consider Model D, introduced on page 123, and assume that the assumptions of Theorem D are satisfied.

For every $j \in \{1, \ldots, j_0\}$, Lemma 5.9.3 implies that every firm j, $j \in \{1, \ldots, j_0\}$, has a supply function $\mathcal{S}_j^D$: Domain$[j] \to C$, which is continuous with respect to $\tau(C^*, C)$ and $\tau(C, C^*)$. This completes Step a, as described in Section 5.1.

Define

$$\text{Domain} := \left(\bigcap_{j=1}^{j_0} \text{Domain}[j] \right) \cap \{p \in C^* \mid p^{\text{cons}} \in \text{int}(C^*_{\text{cons}})\}.$$

By Assumption D.5.a, the set Domain is non-empty. Furthermore, the set Domain is closed under scalar multiplication over $\mathbb{R}_+$. Define the total supply function $\mathcal{S}^D$: Domain $\to C$ by

$$\mathcal{S}^D := \sum_{j=1}^{j_0} \mathcal{S}_j^D.$$

For every $i \in \{1, \ldots, i_0\}$, the income function $\mathcal{K}_i^D$: Domain $\to \mathbb{R}_+$, is for every $p \in$ Domain defined by

$$\mathcal{K}_i^D(p) := [w_i, p] + \sum_{j=1}^{j_0} \theta_{ij} \mathcal{G}(\mathcal{S}_j^D(p), p).$$

We remark that C_{cons}, when identified with $\{(0, c^{\text{cons}}) \in C \mid c^{\text{cons}} \in C_{\text{cons}}\}$ is a salient subspace of C. Since $\mathcal{K}_i^D$ is continuous on Domain, Lemma 5.3.2.b (with $Q =$ Domain $\cup \{0\}$ and $R = C$ and $S = C_{\text{cons}}$) implies that every agent i, $i \in \{1, \ldots, i_0\}$, has a demand function $\mathcal{D}_i^D$: Domain $\to C_{\text{cons}}$, which is continuous with respect to the relative topology of $\tau(C^*, C)$ on C_{cons}. This completes Step b, as described in Section 5.1.

We define the total demand function $\mathcal{D}^D$: Domain $\to C_{\text{cons}}$ by

$$\mathcal{D}^D(p) := \sum_{i=1}^{i_0} \mathcal{D}_i^D(p).$$

The mapping $\mathcal{Z}^D$: Domain $\times\, C \to \mathbb{R}$ is, for every $p \in$ Domain and every $q \in C^*$, defined by

$$\mathcal{Z}^D(p, q) := [(\mathcal{D}^D(p), q^{\text{cons}}]_{\text{cons}} - \mathcal{G}(\mathcal{S}^D(p), q) - [w_{\text{total}}, q].$$

This completes Step c, as described in Section 5.1.

In order to realise Steps d and e of Section 5.1, we want to apply Theorem 5.2.1 to this mapping, with $\mathcal{W} = \mathcal{Z}^D$, $R = C$ and $Q =$ Domain $\cup \{0\}$. Remark 5.2.2 states that this is possible even though Domain may not be a salient space. Let $p_0 \in$ Domain $\cap\, \text{int}(C^*)$ (Assumption D.5.a). We have to check whether $\mathcal{Z}^D$ satisfies the requirements for this theorem.

I: By Lemma 5.3.2.c (with $Q = \text{Domain} \cup \{0\}$, $R = C$ and $S = C_{\text{cons}}$) we find that $\mathcal{Z}^D(p,p) = 0$ for every $p \in \text{Domain}$. Clearly, $\mathcal{Z}^D(\alpha p, q) = \alpha \mathcal{Z}^D(p,q)$ for every $\alpha \in \mathbb{R}_+$, $p \in \text{Domain}$ and $q \in C^*$.

II: For every $p \in \text{Domain}$, the mapping $q \mapsto \mathcal{Z}^D(p,q)$ is continuous.

III: Let $x_0 \in \text{int}(C)$. Since the functions $\mathcal{S}_j^D, j \in \{1,\ldots,j_0\}$, and $\mathcal{D}^D$ are continuous on Domain, Example 5.2.7 implies that the mapping $\mathcal{Z}^D$ satisfies Condition III of Theorem 5.2.1.

IV: Let $\xi_0 = 1$ and let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Domain, with limit $p \notin \text{Domain} \cup \{0\}$. Note that $p \notin \text{Domain}$ means either $p^{\text{cons}} \in \text{bd}(C^*_{\text{cons}})$ or $p^{\text{cons}} \in \text{int}(C^*_{\text{cons}}) \setminus \text{Domain}$. In the first situation, Assumption D.5.b' states that there is $i \in \{1,\ldots,i_0\}$ such that

$$\limsup_{n\to\infty} \mathcal{K}_i^D(p_n) > 0.$$

Taking a subsequence if necessary, Lemma 5.3.2.d (with $Q = \text{Domain} \cup \{0\}$, $R = C$ and $S = C_{\text{cons}}$), implies that the sequence $\big(\mathcal{D}_i^D(p_n)\big)_{n\in\mathbb{N}}$, and therefore also the sequence $\big(\mathcal{D}^D(p_n)\big)_{n\in\mathbb{N}}$, is unbounded.
In the second situation, there is $j \in \{1,\ldots,j_0\}$ such that $p \notin \text{Domain}[j]$. By Lemma 5.9.3.d, we find

$$\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_j^D(p_n), p_0) = -\infty,$$

and thus

$$\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}^D(p_n), p_0) = -\infty.$$

Either way, we find

$$\begin{aligned}&\limsup_{n\to\infty} \mathcal{Z}^D(p_n, p_0)\\ &\qquad = \limsup_{n\to\infty} \Big([\mathcal{D}^D(p_n), p_0^{\text{cons}}]_{\text{cons}} - \mathcal{G}(\mathcal{S}^D(p_n), p_0) - [w_{\text{total}}, p_0]\Big) = \infty.\end{aligned}$$

We conclude that the mapping $\mathcal{Z}^D : \text{Domain} \times C^* \to \mathbb{R}$ meets all the requirements for Theorem 5.2.1 (with $R = C$ and $Q = \text{Domain} \cup \{0\}$), hence

$$\exists p_* \in \text{int}(C^*) \forall q \in C^* : \mathcal{Z}^D(p_*, q) \leq 0.$$

Lemma 2.3.23 implies $(0, \mathcal{D}^D(p_*)) + (\mathcal{S}^{D,\text{prod}}(p_*), 0) \leq_C w_{\text{total}} + (0, \mathcal{S}^{D,\text{cons}}(p_*))$, so we conclude that

$$(\mathcal{S}_1^D(p_*), \ldots, \mathcal{S}_{j_0}^D(p_*), \mathcal{D}_1^D(p_*), \ldots, \mathcal{D}_{i_0}^D(p_*), p_*)$$

is a Walrasian equilibrium for Model D.

5.12 Hyperplanes

Consider a finite-dimensional inner product space X, with inner product denoted by $\langle .,. \rangle$, and with corresponding norm $\| \, . \, \|$. For every $n \in X \setminus \{0\}$ and every $\lambda \in \mathbb{R}$, the set $H = \{x \in X \mid \langle x, n \rangle = \lambda\}$ is called a hyperplane in X. We call n the normal of H. In the remainder of this chapter, we use the following notation concerning (subsets of) hyperplanes: for $n, a \in X$, $n \neq 0$, let

$$\begin{array}{lclcl} H(n) & = & \{x \in X \mid \langle x, n \rangle = 0\}, & & \\ H(n,a) & = & \{a\} + H(n) & = & \{x \in X \mid \langle x, n \rangle = \langle a, n \rangle\}. \end{array}$$

Hence, $H(n)$ is the subspace of X with normal n, and $H(n,a)$ is the unique hyperplane of X with normal n which contains a. Note that $\forall \alpha, \beta \in \mathbb{R} \setminus \{0\} : H(n,a) = H(\alpha n, a + \beta n^{\perp})$, where $n^{\perp} \in H(n)$. Hence, there are many different ways to describe a hyperplane. Below, we will state a property of hyperplanes which will depend on the choice of the normal and the choice of the vector a.

Further, for every subset A of X, we introduce the following notation: for $n, a \in X$, $n \neq 0$, let

$$\begin{array}{lcl} H_A(n) & = & H(n) \cap A, \\ H_A(n,a) & = & H(n,a) \cap A. \end{array}$$

5.12.1 Remark. Let $x, y \in X$. Then $\langle x, y \rangle = \| x \| \, \| y \|$ if and only if the set $\{x, y\}$ is linearly dependent. For a linearly independent subset $\{x, y\} \in X$, where $\| x \| = \| y \| = 1$, the Cauchy-Schwartz inequality implies that $\langle y, x \rangle > -\langle x, x \rangle = -1$ and therefore $\langle x + y, x \rangle > 0$. $\diamond$

For the proof of the following theorem, we refer to the proofs of Theorems 1.4.1 and 1.4.2 of [4].

5.12.2 Theorem. *Let K be a non-empty, closed convex set in X. For any $x \in X$ there is a unique element in K closest to x; that is, there is a unique element $k_x \in K$ such that*

$$\| x - k_x \| = \inf\{ \, \| x - k \| \mid k \in K\}.$$

The element k_x is uniquely determined by

$$\langle x - k_x, k - k_x \rangle \leq 0 \text{ for every } k \in K.$$

Let K be a non-empty compact convex set in X, let $k_0 \in K$ and let $n \in X \setminus \{0\}$. We say that the hyperplane $H(n, k_0)$ is a *supporting hyperplane of K at k_0* if

$$\langle k_0, n \rangle = \max\{\langle k, n \rangle \mid k \in K\}.$$

So K is on one side of the hyperplane $H(n, k_0)$ and the hyperplane $H(n, k_0)$ supports K at k_0.
Note that herewith we choose an orientation for the normal at $k_0 \in K$. We say that the vector n is perpendicular to the surface of K at k_0 (cf. Figure 5.12.1 for two examples of supporting hyperplanes of a polytope at $\mathbb{R}^2$).

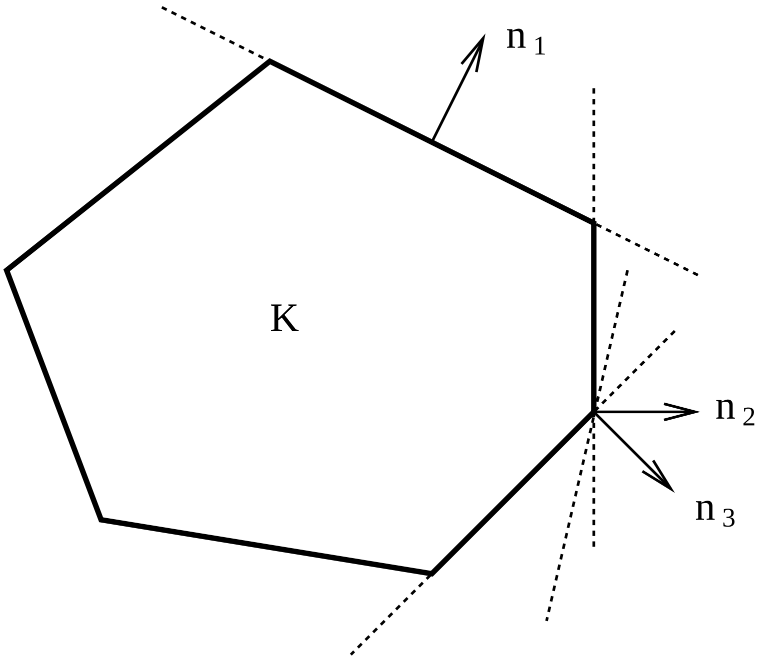

FIGURE 5.12.1: Supporting hyperplanes

The following corollary is a direct result of the definition of supporting hyperplane and Theorem 5.12.2.

5.12.3 Corollary. *Let K be a non-empty, closed convex set in X, let $x \in X \setminus K$ and let k_x be the unique element in* $\mathrm{bd}(K)$ *closest to x. Then $H(x - k_x, k_x)$ is a, not necessarily unique, supporting hyperplane of K at k_x.*

5.12.4 Definition (projection). Let K be a non-empty, closed convex set in X. The projection $\mathcal{P}_K : X \to K$, assigns to each $x \in X$, the element closest to x in K.

5.12.5 Theorem. *Let K be a non-empty, closed convex set in X and let $\mathcal{P}_K$ be the projection onto K. Then the mapping $\mathcal{P}_K : X \to K$ is continuous.*

Proof.
Let $x, y \in X$ and consider the following sequence of (in)equalities, where the last inequality is obtained by applying Theorem 5.12.2 twice.

$$\begin{aligned}
&\| \mathcal{P}_K(x) - \mathcal{P}_K(y) \|^2 \\
&= \langle \mathcal{P}_K(x) - \mathcal{P}_K(y), \mathcal{P}_K(x) - \mathcal{P}_K(y) \rangle \\
&= \langle \mathcal{P}_K(x) - \mathcal{P}_K(y), \mathcal{P}_K(x) - x \rangle + \langle \mathcal{P}_K(x) - \mathcal{P}_K(y), x - y \rangle \\
&+ \langle \mathcal{P}_K(x) - \mathcal{P}_K(y), y - \mathcal{P}_K(y) \rangle \\
&\leq \langle \mathcal{P}_K(x) - \mathcal{P}_K(y), x - y \rangle .
\end{aligned}$$

We conclude that $\| \mathcal{P}_K(x) - \mathcal{P}_K(y) \| \leq \| x - y \|$. □

Let K be a closed, convex set in X with non-empty interior, and let the mapping $\mathcal{P}_K : X \to K$ denote the projection onto the set K, as defined above. Since K has an interior point, $\forall x \in X : \mathcal{P}_K(x) \neq x \implies \mathcal{P}_K(x) \in \text{bd}(K)$. The mapping $\mathcal{N}_K : X \to X$ is for every $x \in X$ defined by $\mathcal{N}_K(x) = x - \mathcal{P}_K(x)$. So $\forall x \in X\ \forall k \in K : \langle \mathcal{N}_K(x), k - \mathcal{P}_K(x) \rangle \leq 0$. Clearly, the projection $\mathcal{P}_K$ being continuous on X, the mapping $\mathcal{N}_K$ is continuous on X.

5.12.6 Lemma. *Let K be a non-empty, closed convex set in X. Then the mapping $\mathcal{N}_K : X \to X$ satisfies $\forall x, y \in X\ \forall \tau \in [0, 1]$:*

$$\| \mathcal{N}_K(\tau x + (1 - \tau) y) \| \leq \tau \| \mathcal{N}_K(x) \| + (1 - \tau) \| \mathcal{N}_K(y) \| .$$

Proof.
Let $x, y \in X$ and $\tau \in [0, 1]$. Then

$$\begin{aligned}
&\| \mathcal{N}_K(\tau x + (1 - \tau) y) \| \\
&= \| \tau x + (1 - \tau) y - \mathcal{P}_K(\tau x + (1 - \tau) y) \| \\
&= \min\{ \| (\tau x + (1 - \tau) y) - k \| \mid k \in K \} \\
&= \min\{ \| (\tau x + (1 - \tau) y) - (\tau k_1 + (1 - \tau) k_2) \| \mid k_1, k_2 \in K \} \\
&\leq \min\{ \tau \| x - k_1 \| + (1 - \tau) \| y - k_2 \| \mid k_1, k_2 \in K \} \\
&= \tau \min\{ \| x - k_1 \| \mid k_1 \in K \} + (1 - \tau) \min\{ \| y - k_2 \| \mid k_2 \in K \} \\
&= \tau \| \mathcal{N}_K(x) \| + (1 - \tau) \| \mathcal{N}_K(y) \| .
\end{aligned}$$

□

Let $n \in X$, with $\| n \| = 1$, and let L be a non-empty, closed convex subset of $H(n)$. We introduce the cylinder L_C in X, generated by L, by defining

$$L_C := \{ L + \lambda n \mid \lambda \in \mathbb{R} \}.$$

Let $\mathcal{P}_{L_C} : X \to L_C$ denote the projection on the cylinder L_C.

Regarding $H(n)$ as an inner product space (thus replacing X with the subspace $H(n)$), we define the projection $\mathcal{P}_L : H(n) \to L$.

5.12.7 Proposition. *Let* $x = x_0 + \lambda_0 n \in X$*, with* $x_0 \in H(n)$*. Then* $\mathcal{P}_{L_C}(x) = \mathcal{P}_L(x_0) + \lambda_0 n$*.*

Proof.
Theorem 5.12.2 implies

$$\forall l \in L\ \forall \lambda \in \mathbb{R} : \langle x - \mathcal{P}_{L_C}(x), l + \lambda n - \mathcal{P}_{L_C}(x) \rangle \leq 0.$$

Let $l_1 \in L$ and $\lambda_1 \in \mathbb{R}$ satisfy $\mathcal{P}_{L_C}(x) = l_1 + \lambda_1 n$. Using $\langle x, n \rangle = \lambda_0$ and $\langle \mathcal{P}_{L_C}(x), n \rangle = \lambda_1$, we find

$$\begin{aligned}&\langle x - \mathcal{P}_{L_C}(x), l + \lambda n - \mathcal{P}_{L_C}(x) \rangle \\ &\quad = \langle x - l_1 - \lambda_1 n, l + \lambda n - l_1 - \lambda_1 n \rangle \\ &\quad = \langle (x - l_1) - \lambda_1 n, l - l_1 + (\lambda - \lambda_1) n \rangle \\ &\quad = \langle x - l_1, l - l_1 \rangle + (\lambda - \lambda_1)(\lambda_0 - \lambda_1).\end{aligned}$$

Hence, the above inequality implies

$$\forall l \in L\ \forall \lambda \in \mathbb{R} : \langle x - l_1, l - l_1 \rangle \leq -(\lambda - \lambda_1)(\lambda_0 - \lambda_1).$$

This is only possible if $\lambda = \lambda_1$. Furthermore, substituting $\lambda = \lambda_1$, we find

$$\forall l \in L : 0 \geq \langle x - l_1, l - l_1 \rangle = \langle x_1 - l_1, l - l_1 \rangle,$$

i.e., $\mathcal{P}_L(x_1) = l_1$. □

5.12.8 Corollary. *The projection* $\mathcal{P}_L : H(n) \to L$ *is the restriction of* $\mathcal{P}_{L_C} : X \to L_C$ *to* $H(n)$*.*

Next, we derive a stationary point theorem for finite-dimensional inner product spaces. This theorem is a consequence of the well known Brouwer Fixed Point Theorem (3.3.14).

5.12.9 Stationary Point Theorem
Let K *be a non-empty, convex and compact subset of a finite-dimensional inner product space* X *and let* $\mathcal{G} : K \to X$ *be a continuous mapping. Then there exists* $x^* \in K$ *such that* $\forall k \in K : \langle \mathcal{G}(x^*), k - x^* \rangle \leq 0$*, i.e.,* $\mathcal{G}$ *has a stationary point in* K*.*

Proof.
Since the function $\mathcal{G} : K \to X$ is continuous, the mapping $\mathcal{F} : K \to K$ defined by $\forall x \in K : \mathcal{F}(x) := \mathcal{P}_K(x + \mathcal{G}(x))$, is continuous. Brouwer's Fixed Point Theorem implies the existence of $x^* \in K$ such that $\mathcal{F}(x^*) = x^*$. By Theorem 5.12.2, we find $\forall x \in X\ \forall k \in K : \langle x - \mathcal{P}_K(x), k - \mathcal{P}_K(x) \rangle \leq 0$. Since $x^* + \mathcal{G}(x^*) \in X$, we find that $\forall k \in K : 0 \geq \langle x^* + \mathcal{G}(x^*) - \mathcal{P}_K(x^* + \mathcal{G}(x^*)), k - \mathcal{P}_K(x^* + \mathcal{G}(x^*)) \rangle = \langle x^* + \mathcal{G}(x^*) - x^*, k - x^* \rangle = \langle \mathcal{G}(x^*), k - x^* \rangle$. Hence, x^* is a stationary point of $\mathcal{G}$. □

5.13 Proof of Theorem B

Consider Model B, introduced on page 111, and assume that the assumptions of Theorem B are satisfied.

Let us restate Model B. We consider a model of a pure exchange economy, with the following primary concepts:

- the set of all exchangeable objects is modelled by solid pointed convex cone K in a finite-dimensional inner product space V;
- the set of price systems is modelled by a strict subcone P of K^*, where K^* in V is given by
$$K^* = \{x \in V \mid \forall k \in K : \langle x, k\rangle \geq 0\};$$
- there is a finite number, i_0, of agents, where agent i, $i \in \{1, \ldots, i_0\}$, is characterised by an initial endowment $w_i \in K$ and a preference relation $\succeq_i$ on K;
- the set of all rationing schemes is modelled by $V \times \mathbb{R}_+$.

Concerning the secondary concepts: the constrained budget set of agent i is for every $p \in P$, for every $n \in V$ and for every $\alpha \in \mathbb{R}_+$ given by

$$B_i^B(p, n, \alpha) := B_i(p, w_i) \cap R(n, \alpha),$$

where

$$B_i(p, w_i) = \{x \in K \mid \langle x, p\rangle \leq \langle w_i, p\rangle\} \text{ and } R(n, \alpha) := \{x \in K \mid \langle x - w_i, n\rangle \leq \alpha\}.$$

The constrained demand set $D_i^B(p, n)$ of agent i contains all most preferable elements of the constrained budget set, with respect to $\succeq_i$.

Assumption B.2 of Theorem B states that P is a closed convex subcone of $\text{int}(K^*) \cup \{0\}$, satisfying $P \cap \text{int}(K) \neq \emptyset$. Let $p_0 \in P \cap \text{int}(K)$, then p_0 satisfies

$$\forall p \in P : \langle p_0, p\rangle > 0.$$

Without loss of generality, we assume $\| p_0 \| = 1$. Next, consider $S = \{p \in P \mid \langle p, p_0\rangle = 1\}$. Then, by Lemma 3.3.8.c, the set S is a compact convex subset of the compact set $H_{K^*}(p_0, p_0)$. Define the set L in $H(p_0)$ by $L := S - \{p_0\}$, and define the cylinder $L_C := \{L + \lambda p_0 \mid \lambda \in \mathbb{R}\}$. Let $\mathcal{P}$ and $\mathcal{N}$ denote the restrictions of the mappings $\mathcal{P}_{L_C}$ and $\mathcal{N}_{L_C}$ to the hyperplane $H(p_0, p_0)$. Then $\mathcal{P} : H(p_0, p_0) \to S$ and $\mathcal{N} : H(p_0, p_0) \to H(p_0)$.

Define

$$\lambda_{\max} := \min\{d(a,p) \mid a \in H(p_0,p_0) \cap \mathrm{bd}(K^*) \text{ and } p \in S\} > 0,$$

and define

$$Q := \{h \in H(p_0,p_0) \mid \| \mathcal{N}(h) \| \le \lambda_{\max}\}.$$

Note that by Lemma 5.12.6, the set Q is a closed and convex subset of the compact set $H_{K^*}(p_0,p_0)$ and that the relative interior of Q, with respect to the hyperplane $H(p_0,p_0)$, is non-empty, even when the relative interior P is empty. Also note that the boundary of Q, with respect to $H(p_0,p_0)$, is given by

$$\mathrm{bd}(Q) := \{q \in Q \mid \| \mathcal{N}(q) \| = \lambda_{\max}\}.$$

In the remainder of this section, we let the vector $\mathcal{N}(q)$, with $q \in Q$, represent a rationing scheme. More precisely, for every $i \in \{1,\ldots,i_0\}$ and $q \in Q$ the constrained budget set $B_i^B(q)$ of agent i, at q, is given by

$$B_i^B(q) := B_i(\mathcal{P}(q),w_i) \cap R(q),$$

where

$$B_i(\mathcal{P}(q),w_i) = \{x \in K \mid \langle x,\mathcal{P}(q)\rangle \le \langle w_i,\mathcal{P}(q)\rangle\}$$

and

$$R(q) = \{x \in K \mid \langle x-w_i,\mathcal{N}(q)\rangle \le \lambda_{\max} - \| \mathcal{N}(q) \| \}.$$

The constrained demand set $D_i^B(q)$ for agent i, at $q \in Q$, is the set of all best elements of $B_i^B(q)$ with respect to the preference relation $\succeq_i$, i.e.,

$$D_i^B(q) := \{x \in B_i^B(q) \mid \forall y \in B_i^B(q) : x \succeq_i y\}.$$

Note, that for all $q \in Q$ and for all $i \in \{1,\ldots,i_0\}$, we find that $w_i \in B_i^B(q)$. Moreover, $\forall i \in \{1,\ldots,i_0\}\ \forall q \in Q \cap P : R(q) = K$, i.e., $B_i^B(q) = \{x \in K \mid \langle x,q\rangle \le \langle w_i,q\rangle\}$. And if $\| \mathcal{N}(q) \| = \lambda_{\max}$ then $R(q) = \{x \in K \mid \langle x-w_i,\mathcal{N}(q)\rangle \le 0\}$.

In the following, we consider agent i, $i \in \{1,\ldots,i_0\}$, with the following characteristics: initial endowment $w_i \in K$ and preference relation $\succeq_i$ defined on K. Under Assumptions B.3 and B.4 of Theorem B, we derive the demand function of this agent, and we will show that this demand function is continuous.

5.13.1 Lemma. *Let $q_0 \in Q$. Then the constrained demand set $D_i^B(q_0)$ is non-empty.*

Proof.
Since $\mathcal{P}(q_0) \in \mathrm{int}(K^*)$ and since the set $R(q_0)$ is closed, Lemma 3.3.8.c implies that the constrained budget set $B_i^B(q_0)$ is compact in K. For every $b \in B_i^B(q_0)$, define

the set $G(b) := \{x \in B_i^B(q_0) \mid b \succ_i x\}$. The preference relation $\succeq_i$ is continuous (Assumption B.3.c), so every set $G(b)$ is open. Suppose the constrained demand set were empty, then every $b_0 \in B_i^B(q_0)$ is an element of at least one $G(b)$. The collection $\{G(b) \mid b \in B_i^B(q_0)\}$ is an open cover of the compact set $B_i^B(q_0)$, so there is a finite subset $F \subset B_i^B(q_0)$ such that $B_i^B(q_0) = \bigcup_{f \in F} G(f)$. The preference relation $\succeq_i$ being transitive, F has a maximal element $f_1 \in F$. Since $f_1 \in G(f_2)$ for some $f_2 \in F$, $f_2 \neq f_1$, we arrive at a contradiction. □

Note that by Assumption B.3.b, for every $q \in Q$, the constrained demand set $D_i^B(q)$ contains at most one element. So, as a direct result of the above lemma and Assumption B.3.b, we can define the constrained demand function $\mathcal{D}_i^B : Q \to K$, where $\forall q \in Q : \{\mathcal{D}_i^B(q)\} = D_i^B(q)$, i.e., $\mathcal{D}_i^B(q)$ is the unique element of the constrained demand set $D_i^B(q)$.

Using Theorem 5.12.2, (taking $x_0 = p_0$) the proof of the following lemma is straightforward.

5.13.2 Lemma. *If $q \in Q$ satisfies $\sum_{i=1}^{i_0} \mathcal{D}_i^B(q) = \sum_{i=1}^{i_0} w_i$, then*

$$(\mathcal{D}_1^B(q), \ldots, \mathcal{D}_{i_0}^B(q), \mathcal{P}(q), \mathcal{N}(q), \lambda_{\max} - \parallel \mathcal{N}(q) \parallel)$$

is a constrained equilibrium.

Recalling the notation regarding (subsets of) hyperplanes from the previous section, we find that Lemma 3.3.8.c implies that $H_{K^*}(p_0, a)$ is compact if $a \in \mathrm{int}(K)$ and $p_0 \in \mathrm{int}(K^*)$.

5.13.3 Lemma. *For every $h \in H_{K^*}(p_0, p_0)$, satisfying $\mathcal{N}(h) \neq 0$, we find*

$$\langle \frac{\mathcal{P}(h)}{\parallel \mathcal{P}(h) \parallel} + \frac{\mathcal{N}(h)}{\parallel \mathcal{N}(h) \parallel}, \mathcal{P}(h) \rangle > 0 \text{ and } \langle \frac{\mathcal{P}(h)}{\parallel \mathcal{P}(h) \parallel} + \frac{\mathcal{N}(h)}{\parallel \mathcal{N}(h) \parallel}, \mathcal{N}(h) \rangle > 0.$$

Proof.
Since $\forall h \in H_{K^*}(p_0, p_0) : \langle \mathcal{P}(h), p_0 \rangle > 0$ and $\langle \mathcal{N}(h), n \rangle = 0$, we conclude that $\forall h \in H_{K^*}(p_0, p_0)\ \forall \beta \in \mathbb{R} : \mathcal{P}(h) \neq \beta \mathcal{N}(h)$. The remainder of the proof is a direct consequence of Remark 5.12.1. □

In the following lemmas, we again consider an arbitrary agent i, $i \in \{1, \ldots, i_0\}$.

5.13.4 Lemma. *Let $q_0 \in Q$. Then $\langle \mathcal{D}_i^B(q_0), \mathcal{P}(q_0) \rangle = \langle w_i, \mathcal{P}(q_0) \rangle$.*

Proof.
By Assumption B.4, we find $\langle w_i, \mathcal{P}(q_0)\rangle > 0$. Suppose $\langle \mathcal{D}_i^B(q_0), \mathcal{P}(q_0)\rangle < \langle w_i, \mathcal{P}(q_0)\rangle$. Since $p_0 \in \text{int}(K)$, Proposition 2.2.12 implies that $\exists \mu_0 > 0 : \langle \mathcal{D}_i^B(q_0)+\mu_0 p_0, \mathcal{P}(q_0)\rangle = \langle w_i, \mathcal{P}(q_0)\rangle$. Note that, since $\forall q \in Q : \langle p_0, \mathcal{N}(q)\rangle = 0$, we find $\mathcal{D}_i^B(q_0) + \mu_0 p_0 \in B_i^B(q_0)$. By the monotony of preference relation $\succeq_i$ (Assumption B.3.a), we find that $\mathcal{D}_i^B(q_0) + \mu_0 p_0 \geq_K \mathcal{D}_i^B(q_0)$ implies $\mathcal{D}_i^B(q_0) + \mu_0 p_0 \succeq_i \mathcal{D}_i^B(q_0)$. Since $\mathcal{D}_i^B(q_0)$ is the unique best element of the constrained budget set, and since $\mu_0 p_0 \neq 0$, we arrive at a contradiction. □

To conclude this part concerning individual demand functions, we prove that the constrained demand function $\mathcal{D}_i^B : Q \to K$, is continuous on Q. Similar to the approach in the proofs of the theorems of the other models, we need the following lemma.

5.13.5 Lemma. *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence in Q convergent to $q_0 \in Q$. Then the following two properties hold.*

a) *If $b_n \in B_i^B(q_n)$ for each $n \in \mathbb{N}$, then there is a subsequence $(b_{nk})_{k\in\mathbb{N}}$ that converges to some $b \in B_i^B(q_0)$.*

b) *For each $b \in B_i^B(q_0)$ there exists a convergent sequence $(b_n)_{n\in\mathbb{N}}$ with limit b, such that $b_n \in B_i^B(q_n)$ for all $n \in \mathbb{N}$.*

Proof.

a) Since the sequence $(\mathcal{P}(q_n))_{n\in\mathbb{N}}$ is convergent with limit $\mathcal{P}(q_0)$, and since $\mathcal{P}(q_0) \in \text{int}(K^*)$ is an order unit for K^*, Lemma 3.3.12 implies that the function $\mathcal{L}_{\mathcal{P}(q_0)} : Q \to \mathbb{R}^+$ satisfies

$$\lim_{n\to\infty} \mathcal{L}_{\mathcal{P}(q_0)}(\mathcal{P}(q_n)) = 1 \text{ and } \forall\, n \in \mathbb{N} : \ \mathcal{L}_{\mathcal{P}(q_0)}(\mathcal{P}(q_n))\mathcal{P}(q_0) \leq_{K^*} \mathcal{P}(q_n).$$

Because $\forall n \in \mathbb{N} : b_n \in B_i^B(q_n)$, we find

$$\begin{cases} \mathcal{L}_{\mathcal{P}(q_0)}(\mathcal{P}(q_n))\,\langle b_n, \mathcal{P}(q_0)\rangle \ \leq\ \langle b_n, \mathcal{P}(q_n)\rangle \ \leq\ \langle w_i, \mathcal{P}(q_n)\rangle, \\ \qquad\qquad\qquad \langle b_n - w_i, \mathcal{N}(q_n)\rangle \ \leq\ \lambda_{\max}- \ \| \mathcal{N}(q_n) \| \ . \end{cases}$$

By Lemma 3.3.8.b, boundedness of $\langle b_n, \mathcal{P}(q_0)\rangle$ implies that the sequence $(b_n)_{n\in\mathbb{N}}$ is bounded. So, $(b_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(b_{nk})_{k\in\mathbb{N}}$ with limit $b \in K$. Since $\forall k \in \mathbb{N} : \langle b_{nk}, \mathcal{P}(q_{nk})\rangle \leq \langle w_i, \mathcal{P}(q_{nk})\rangle$ and $\forall k \in \mathbb{N} : \langle b_{n_k} - w_i, \mathcal{N}(q_{n_k})\rangle \leq \lambda_{\max}- \ \| \mathcal{N}(q_{n_k}) \|$, the limit b belongs to $B_i^B(q_0)$.

b) Suppose $\mathcal{N}(q_0) = 0$. Assumption B.4 implies that there is $\mu > 0$ such that $b_0 := w_i - \mu\mathcal{P}(q_0) \in \text{int}(K)$. In this situation we find $\langle b_0, \mathcal{P}(q_0)\rangle < \langle w_i, \mathcal{P}(q_0)\rangle$, i.e., $b_0 \in \text{int}(B_i^B(q_0))$.

Suppose $\mathcal{N}(q_0) \neq 0$. Assumption B.4 implies that there is $\mu > 0$ such that $b_0 := w_i - \mu(\frac{\mathcal{P}(q_0)}{\|\mathcal{P}(q_0)\|} + \frac{\mathcal{N}(q_0)}{\|\mathcal{N}(q_0)\|}) \in \text{int}(K)$. In this situation, Lemma 5.13.3 implies

$$\langle b_0, \mathcal{P}(q_0)\rangle < \langle w_i, \mathcal{P}(q_0)\rangle \text{ and } \langle b_0, \mathcal{N}(q_0)\rangle < \langle w_i, \mathcal{N}(q_0)\rangle,$$

i.e., $b_0 \in \text{int}(B_i^B(q_0))$.

Either way, since $\mathcal{P}$ and $\mathcal{N}$ are continuous on Q, we find that $\exists N_0 \in \mathbb{N}\ \forall n > N_0$:

$$\langle b_0, \mathcal{P}(q_n)\rangle < \langle w_i, \mathcal{P}(q_n)\rangle, \tag{5.9}$$

$$\text{if } \mathcal{N}(q_0) \neq 0 \text{ then } \langle b_0, \mathcal{N}(q_n)\rangle < \langle w_i, \mathcal{N}(q_n)\rangle. \tag{5.10}$$

Let $b \in B_i^B(q_0)$. For all $n \leq N_0$ we define $b_n := w_i$. Hence, $\forall n \leq N_0 : b_n \in B_i^B(q_n)$. We will construct a sequence $(\tau_n)_{n>N_0}$ in $[0,1]$ such that $\lim\limits_{n\to\infty} \tau_n = 1$ and $\forall n > N_0 : \tau_n b + (1-\tau_n)b_0 \in B_i^B(q_n)$.

We distinguish four cases.

- $\langle \boldsymbol{b}, \boldsymbol{\mathcal{P}(q_0)}\rangle < \langle \boldsymbol{w_i}, \boldsymbol{\mathcal{P}(q_0)}\rangle$ **and** $\langle \boldsymbol{b - w_i}, \boldsymbol{\mathcal{N}(q_0)}\rangle < \boldsymbol{\lambda_{\max}} - \| \boldsymbol{\mathcal{N}(q_0)} \|$

  In this situation

  $$\exists N_1 \geq N_0\ \forall n > N_1 : \begin{cases} \langle b, \mathcal{P}(q_n)\rangle < \langle w_i, \mathcal{P}(q_n)\rangle, \\ \langle b - w_i, \mathcal{N}(q_n)\rangle < \lambda_{\max} - \| \mathcal{N}(q_n) \| \end{cases}.$$

  Define $\tau_n := 0$ if $N_0 < n \leq N_1$. In case $n > N_1$, define $\tau_n := 1$. Then $\forall n > N_0 : \tau_n b + (1-\tau_n)b_0 \in \text{int}(B_i^B(q_n))$.

- $\langle \boldsymbol{b}, \boldsymbol{\mathcal{P}(q_0)}\rangle < \langle \boldsymbol{w_i}, \boldsymbol{\mathcal{P}(q_0)}\rangle$ **and** $\langle \boldsymbol{b - w_i}, \boldsymbol{\mathcal{N}(q_0)}\rangle = \boldsymbol{\lambda_{\max}} - \| \boldsymbol{\mathcal{N}(q_0)} \|$

  Note that this implies that $\mathcal{N}(q_0) \neq 0$. In this situation, (5.10) implies

  $$\exists N_1 \geq N_0\ \forall n > N_1 : \begin{cases} \langle b_0, \mathcal{N}(q_n)\rangle < \langle w_i, \mathcal{N}(q_n)\rangle, \\ \langle b_0, \mathcal{N}(q_n)\rangle < \langle b, \mathcal{N}(q_n)\rangle, \\ \langle b, \mathcal{P}(q_n)\rangle < \langle w_i, \mathcal{P}(q_n)\rangle. \end{cases}$$

  Define $\sigma_n := 0$ if $N_0 < n \leq N_1$. In case $n > N_1$, define

  $$\sigma_n := \frac{\langle (w_i - b_0), \mathcal{N}(q_n)\rangle + (\lambda_{\max} - \| \mathcal{N}(q_n) \|)}{\langle (b - b_0), \mathcal{N}(q_n)\rangle}.$$

  Then $\forall n > N_1 : \sigma_n > 0$ and $\lim\limits_{n\to\infty} \sigma_n = 1$.
  Since, $\forall n > N_1 : \sigma_n \langle (b-b_0), \mathcal{N}(q_n)\rangle = \langle (w_i - b_0), \mathcal{N}(q_n)\rangle + (\lambda_{\max} - \| \mathcal{N}(q_n) \|)$ and $b_0 \in \text{int}(B_i^B(q_n))$, we find that $\forall n > N_1 : \tau_n b + (1-\tau_n)b_0 \in B_i^B(q_n)$, where $\tau_n := \min\{\sigma_n, 1\}$.

- $\boldsymbol{\langle b, \mathcal{P}(q_0)\rangle = \langle w_i, \mathcal{P}(q_0)\rangle}$ **and** $\boldsymbol{\langle b - w_i, \mathcal{N}(q_0)\rangle < \lambda_{\max} - \parallel \mathcal{N}(q_0) \parallel}$

  In this situation, (5.9) implies

$$\exists N_1 \geq N_0 \; \forall n > N_1 : \begin{cases} \langle b_0, \mathcal{P}(q_n)\rangle < \langle w_i, \mathcal{P}(q_n)\rangle, \\ \langle b_0, \mathcal{P}(q_n)\rangle < \langle b, \mathcal{P}(q_n)\rangle, \\ \langle b - w_i, \mathcal{N}(q_n)\rangle < \lambda_{\max} - \parallel \mathcal{N}(q_n) \parallel \; . \end{cases}$$

  Define $\sigma_n := 0$ if $N_0 < n \leq N_1$. In case $n > N_1$, define

$$\sigma_n := \frac{\langle (w_i - b_0), \mathcal{P}(q_n)\rangle}{\langle (b - b_0), \mathcal{P}(q_n)\rangle}.$$

  Then $\forall n > N_1 : \sigma_n > 0$ and $\lim_{n\to\infty} \sigma_n = 1$.
  Since $\forall n > N_1 : \sigma_n \langle (b - b_0), \mathcal{P}(q_n)\rangle = \langle (w_i - b_0), \mathcal{P}(q_n)\rangle$ and $b_0 \in \operatorname{int}(B_i^B(q_n))$ we find that $\forall n > N_1 : \tau_n b + (1 - \tau_n) b_0 \in B_i^B(q_n)$, where $\tau_n := \min\{\sigma_n, 1\}$.

- $\boldsymbol{\langle b, \mathcal{P}(q_0)\rangle = \langle w_i, \mathcal{P}(q_0)\rangle}$ **and** $\boldsymbol{\langle b - w_i, \mathcal{N}(q_0)\rangle = \lambda_{\max} - \parallel \mathcal{N}(q_0) \parallel}$

  Note that this implies that $\mathcal{N}(q_0) \neq 0$. In this situation (5.9) and (5.10) imply

$$\exists N_1 \geq N_0 \; \forall n > N_1 : \begin{cases} \langle b_0, \mathcal{P}(q_n)\rangle < \langle w_i, \mathcal{P}(q_n)\rangle, \\ \langle b_0, \mathcal{P}(q_n)\rangle < \langle b, \mathcal{P}(q_n)\rangle, \\ \langle b_0, \mathcal{N}(q_n)\rangle < \langle w_i, \mathcal{N}(q_n)\rangle, \\ \langle b_0, \mathcal{N}(q_n)\rangle < \langle b, \mathcal{N}(q_n)\rangle. \end{cases}$$

  Define $\tau_n := 0$ if $N_0 < n \leq N_1$. In case $n > N_1$, define

$$\tau_n := \min\{\frac{\langle (w_i - b_0), \mathcal{P}(q_n)\rangle}{\langle (b - b_0), \mathcal{P}(q_n)\rangle}, \frac{\langle (w_i - b_0), \mathcal{N}(q_n)\rangle + (\lambda_{\max} - \parallel \mathcal{N}(q_n) \parallel)}{\langle (b - b_0), \mathcal{N}(q_n)\rangle}, 1\}.$$

  Then $\forall n > N_1 : \tau_n > 0$ and $\lim_{n\to\infty} \tau_n = 1$.
  Since $\forall n > N_1 : \tau_n \langle (b - b_0), \mathcal{P}(q_n)\rangle \leq \langle (w_i - b_0), \mathcal{P}(q_n)\rangle$ and $\forall n > N_1 : \tau_n \langle (b - b_0), \mathcal{N}(q_n)\rangle \leq \langle (w_i - b_0), \mathcal{N}(q_n)\rangle + (\lambda_{\max} - \parallel \mathcal{N}(q_n) \parallel)$, we find that $\forall n > N_1 : \tau_n b + (1 - \tau_n) b_0 \in B_i^B(q_n)$.

□

Similar to the proofs of the theorems of the other models, the continuity of the constrained demand function $\mathcal{D}_i^B$, follows from the above lemma.

5.13.6 Lemma. *The constrained demand function* $\mathcal{D}_i^B$ *is continuous on* Q.

Proof.
Suppose $\mathcal{D}_i^B$ is not continuous in $q_0 \in Q$, then there is a sequence $(q_n)_{n\in\mathbb{N}}$ in Q, converging to q_0, such that $\mathcal{D}_i^B(q_0)$ is not a point of accumulation of the sequence $\left(\mathcal{D}_i^B(q_n)\right)_{n\in\mathbb{N}}$. By Lemma 5.13.5.a, the sequence $\left(\mathcal{D}_i^B(q_n)\right)_{n\in\mathbb{N}}$ has a subsequence $\left(\mathcal{D}_i^B(q_{nk})\right)_{k\in\mathbb{N}}$ that converges to some $b \in B_i^B(q_0)$. Now, the proof is done if we can show that $b = \mathcal{D}_i^B(q_0)$. Let $x \in B_i^B(q_0)$. By Lemma 5.13.5.b, for all $n \in \mathbb{N}$ there is $x_n \in B_i^B(q_n)$ satisfying $x_n \to x$. Since the preference relation $\succeq_i$ is continuous (Assumption B.3.c), we find that if $\forall n \in \mathbb{N} : \mathcal{D}_i^B(q_n) \succeq_i x_n$, then $b \succeq_i x$. So, $b = \mathcal{D}_i^B(q_0)$. □

Here, we end our exploration of the properties of the constrained demand for an individual agent i, $i \in \{1, \dots, i_0\}$. The total constrained demand function $\mathcal{D}^B : Q \to K$, is for every $q \in Q$, defined by

$$\mathcal{D}^B(q) := \sum_{i=1}^{i_0} \mathcal{D}_i^B(q).$$

Note that by Lemma 5.13.6, the total constrained demand function is continuous.

The constrained excess demand function $\mathcal{E}^B : Q \to V$ is, for every $q \in Q$, defined by

$$\mathcal{E}^B(q) := \mathcal{D}^B(q) - w_{\text{total}}.$$

Lemma 5.13.2 yields that in order to prove the existence of a constrained equilibrium in our model, we only have to prove that $\exists q \in Q : \mathcal{E}^B(q) = 0$.

Lemma 5.13.4 implies the following version of Walras' Law.

5.13.7 Walras' law
Let $q_0 \in Q$. Then $\langle \mathcal{E}^B(q_0), \mathcal{P}(q_0)\rangle = 0$.

The following lemma is a consequence of the definition of the rationing scheme.

5.13.8 Lemma. *Let $q_0 \in \text{bd}(Q)$. Then $\langle \mathcal{E}^B(q_0), \mathcal{N}(q_0)\rangle \leq 0$.*

Proof.
Since $\parallel \mathcal{N}(q_0) \parallel = \lambda_{\max}$, we find $R(q_0) = \{x \in K \mid \langle x - w_i, \mathcal{N}(q_0)\rangle \leq 0\}$. Hence, $\forall i \in \{1, \dots, i_0\} : \langle \mathcal{D}_i^B(q_0) - w_i, \mathcal{N}(q_0)\rangle \leq 0$, which implies $\langle \mathcal{D}^B(q_0) - w_{\text{total}}, \mathcal{N}(q_0)\rangle \leq 0$. □

In the previous section, we have seen that the supporting hyperplane in a specific point of a convex set does not have to be unique. Lemma 5.13.10 shows that for every element q_0 of $\text{bd}(Q)$, there is a unique supporting hyperplane $H(n, q_0)$ of Q for which $n \in H(p_0)$. The following lemma is needed in the proof of Lemma 5.13.10.

5.13.9 Lemma. *Let $q_0 \in \mathrm{bd}(Q)$. Then $H(\mathcal{N}(q_0), q_0)$ is a supporting hyperplane for the cylinder Q_C generated by Q, at q_0.*

Proof.
Since $q_0 \in \mathrm{bd}(Q)$, we find $\| \mathcal{N}(q_0) \| = \lambda_{\max}$. In Section 5.12, we have seen that $H(\mathcal{N}(q_0), \mathcal{P}(q_0))$ is a supporting hyperplane for S at $\mathcal{P}(q_0)$. Recall that this implies $\forall p \in H_P(p_0, p_0) : \langle p, \mathcal{N}(q_0)\rangle \leq \langle \mathcal{P}(q_0), \mathcal{N}(q_0)\rangle$. Since $\| \mathcal{N}(q) \| \leq \| \mathcal{N}(q_0) \| = \lambda_{\max}$, the Cauchy-Schwartz inequality implies $\langle \mathcal{N}(q), \mathcal{N}(q_0)\rangle \leq \| \mathcal{N}(q_0) \|^2$. So for all $q \in Q$:

$$\begin{aligned} \langle q, \mathcal{N}(q_0)\rangle &= \langle \mathcal{P}(q), \mathcal{N}(q_0)\rangle + \langle \mathcal{N}(q), \mathcal{N}(q_0)\rangle \\ &\leq \langle \mathcal{P}(q), \mathcal{N}(q_0)\rangle + \langle \mathcal{N}(q_0), \mathcal{N}(q_0)\rangle \\ &\leq \langle \mathcal{P}(q_0), \mathcal{N}(q_0)\rangle + \langle \mathcal{N}(q_0), \mathcal{N}(q_0)\rangle \\ &= \langle q_0, \mathcal{N}(q_0)\rangle. \end{aligned}$$

□

5.13.10 Lemma. *Let $q_0 \in \mathrm{bd}(Q)$ and let $n \in H(p_0)$ satisfy $n \neq 0$. If $H(n, q_0)$ is a supporting hyperplane of the cylinder Q_C at q_0, then $\exists \mu > 0 : n = \mu \mathcal{N}(q_0)$.*

Proof.
Define $\hat{q} \in H(q_0, q_0)$ by $\hat{q} := \mathcal{P}(q_0) + \frac{\lambda_{\max}}{\|n\|} n$. Clearly, $\| \mathcal{N}(\hat{q}) \| \leq \lambda_{\max}$, so $\hat{q} \in Q$. Since $\forall q \in Q : \langle q, n\rangle \leq \langle q_0, n\rangle$, we find that $\langle \mathcal{P}(q_0), n\rangle + \lambda_{\max} \; \| n \| = \langle \hat{q}, n\rangle \leq \langle q_0, n\rangle = \langle \mathcal{P}(q_0), n\rangle + \langle \mathcal{N}(q_0), n\rangle$. So, we find $\lambda_{\max} \; \| n \| \leq \langle \mathcal{N}(q_0), n\rangle$. On the other hand, the Cauchy-Schwartz inequality implies $\langle \mathcal{N}(q_0), n\rangle \leq \lambda_{\max} \; \| n \|$. Hence, we find $\langle \mathcal{N}(q_0), n\rangle = \| \mathcal{N}(q_0) \| \; \| n \|$, so $\exists \mu \in \mathbb{R} \setminus \{0\} : n = \mu \mathcal{N}(q_0)$. Lemma 5.13.9 implies that, like n, $\mathcal{N}(q_0)$ satisfies $\forall q \in Q : \langle q, \mathcal{N}(q_0)\rangle \leq \langle q_0, \mathcal{N}(q_0)\rangle$, so we conclude $\mu > 0$. □

Now, we are ready to give the proof of Theorem B.

Proof of Theorem B
We shall prove that $\exists \tilde{q} \in Q : \mathcal{E}^B(\tilde{q}) = 0$. In this situation, Theorem 5.13.2 implies that

$$(\mathcal{D}_1(\tilde{q}), \ldots, \mathcal{D}_{i_0}(\tilde{q}), \mathcal{P}(\tilde{q}), \mathcal{N}(\tilde{q}), \lambda_{\max} - \| \mathcal{N}(\tilde{q}) \|)$$

is a constrained equilibrium.
The projected constrained excess demand function $\mathcal{E}_0 : Q \to H(p_0, p_0)$ is for every $q \in Q$ defined by

$$\mathcal{E}_0(q) := \mathcal{H}(\mathcal{E}^B(q)),$$

where $\mathcal{H}$ denotes the projection from V onto the hyperplane $H(p_0, p_0)$. Then, $\forall q \in Q \; \exists \beta \in \mathbb{R} : \mathcal{E}^B(q) = \mathcal{E}_0(q) + \beta p_0$. Since the projected constrained excess demand function $\mathcal{E}_0$ is continuous on Q, and since Q is a non-empty, convex and compact set of the affine subspace $H(p_0, p_0)$, the function $\mathcal{E}_0$ has a stationary point:

$$\exists \tilde{q} \in Q \; \forall q \in Q : \langle q, \mathcal{E}_0(\tilde{q}) \rangle \leq \langle \tilde{q}, \mathcal{E}_0(\tilde{q}) \rangle.$$

So, $\tilde{q}$ maximises $\langle q, \mathcal{E}_0(\tilde{q}) \rangle$ over Q. Since $\langle q, \mathcal{E}_0(\tilde{q}) \rangle$ is linear in q, and since Q, considered as a subset of $H(p_0, p_0)$, has an interior point, this means that $\tilde{q} \in \text{bd}(Q)$ or $\mathcal{E}_0(\tilde{q}) = 0$. In case $\tilde{q} \in \text{bd}(Q)$, Lemma 5.13.10 implies that there is $\mu \geq 0$ such that $\mathcal{E}_0(\tilde{q}) = \mu \mathcal{N}(\tilde{q})$. Now, $\langle p_0, \mathcal{N}(\tilde{q}) \rangle = 0$ implies that

$$0 \leq \langle \mathcal{E}_0(\tilde{q}), \mathcal{E}_0(\tilde{q}) \rangle = \mu \langle \mathcal{E}_0(\tilde{q}), \mathcal{N}(\tilde{q}) \rangle = \mu \langle \mathcal{E}^B(\tilde{q}), \mathcal{N}(\tilde{q}) \rangle.$$

Using Lemma 5.13.8, we conclude $\mathcal{E}_0(\tilde{q}) = 0$, i.e., $\exists \beta \in \mathbb{R} : \mathcal{E}^B(\tilde{q}) = \beta p_0$. Either way, we conclude $\mathcal{E}_0(\tilde{q}) = 0$. Walras' Law implies $0 = \langle \mathcal{E}^B(\tilde{q}), \mathcal{P}(\tilde{q}) \rangle = \beta \langle p_0, \mathcal{P}(\tilde{q}) \rangle$, and since $\forall p \in Q \cap P : \langle p, p_0 \rangle > 0$, we conclude that $\beta = 0$. Hence $\mathcal{E}^B(\tilde{q}) = 0$. □

This concludes the proof of the existence of a constrained equilibrium price vector for Model B.

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Notation

general

$\mathbb{N}$	set of natural numbers, 0 excluded
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of nonnegative real numbers, 0 included, p. 28
$\mathbb{R}^n$	n-dimensional Euclidean vector space
$\mathbb{R}^n_+$	positive orthant of $\mathbb{R}^n$, p. 9
$x \cdot y$	Euclidean inner product of $x, y \in \mathbb{R}^n$, p. 9
$e^1, \ldots, e^n$	standard basis of $\mathbb{R}^n$, p. 8
$\mathrm{co}(A)$	convex hull of a set A, p. 95
$\mathrm{ext}(A)$	set of extreme points of a set A, p. 95
$A + B$	the sum of two sets, p. 32
$a + B$	$\{a\} + B$, p. 32

lattice

$\leq$	any partial order relation, p. 42
$\leq_E$	Euclidean partial order relation of $\mathbb{R}^n$, p. 9
$x \vee y$	least upper bound of x and y, p. 47
$x \wedge y$	greatest lower bound of x and y, p. 47
$x^+ = x \vee 0$	positive part of x, p. 52

$x^- = (-x) \vee 0$	negative part of x, p. 52
$\lvert x \rvert = x^+ + x^-$	absolute value of x, p. 52

vector space and salient space

V, W	real vector spaces
$(V, \leq)$	partially ordered vector space, p. 42
V_+	positive cone of a partially ordered vector space V, p. 43
S, T	salient spaces, p. 28
$v, 0$	vertex of a salient space, p. 23, 28, 36
$V[S]$	vector space reproduced by a salient space S, p. 35
$[(s_1, s_2)]$	element of $V[S]$, equivalence class of the pairing (s_1, s_2), where $s_1, s_2 \in S$, p. 35
$\mathrm{sal}(A)$	salient span of a set $A \subset S$, p. 31
$\mathrm{ray}(s)$	ray generated by an element $s \in S$, p. 31
$\mathrm{ray}(A)$	set of all rays generated by elements of $A \subset S$, p. 31
$\mathrm{lin\,dim}(S)$	linear dimension of a salient space S, p. 40
$\mathrm{int}(S)$	set of internal elements of a salient space S, p. 37
$\mathrm{bd}(S)$	$S \setminus \mathrm{int}(S)$, boundary of S, p. 37
$\mathrm{span}_V(A)$	linear span of a set $A \subset V$ in the vector space V, p. 35
$A_{V[S]}$	$\{[(a_1, a_2)] \in V[S] \mid a_1, a_2 \in A\}$, where $A \subset S$, p. 58
$\mathcal{L} : S \to T$	salient mapping, p. 30
$\mathcal{L}^{\mathrm{ext}} : V[S] \to V[T]$	extension of the salient mapping $\mathcal{L} : S \to T$, p. 37
$\mathcal{J}_S : S \to V_+[S]$	salient isomorphism between S and $V_+[S]$, p. 35
$\mathcal{F} : S \to \mathbb{R}_+$	salient function, p. 56
$\mathcal{F}^{\mathrm{ext}} : V[S] \to \mathbb{R}$	extension of the salient function $\mathcal{F} : S \to \mathbb{R}_+$, p. 56

S^*	adjoint of a salient space S, p. 57
V^*	adjoint of a vector space V, p. 54
$\mathcal{B} : S \times T \to \mathbb{R}_+$	bi-salient form, p. 55
$\mathcal{B} : V \times W \to \mathbb{R}$	bi-linear form, p. 53
$\mathcal{B}^{\text{ext}} : V[S] \times V[T] \to \mathbb{R}$	extension of the bi-salient form $\mathcal{B} : S \times T \to \mathbb{R}_+$, p. 56
$\mathcal{B}_{\text{can}} : S \times S^* \to \mathbb{R}_+$	canonical bi-salient form on $S \times S^*$, p. 57
$\mathcal{B}_{\text{can}} : V \times V^* \to \mathbb{R}$	canonical bi-linear form on $V \times V^*$, p. 54
$\mathcal{M}_t : S \to \mathbb{R}_+$	salient mapping induced by a bi-salient form on $S \times T$ and an element $t \in T$, p. 57
$\mathcal{M} : T \to S^*$	salient mapping induced by a bi-salient form on $S \times T$, p. 57
$\mathcal{M}(T)$	salient subspace of S^*, induced by a bi-salient form on $S \times T$, p. 57
$\mathcal{M}_w : V \to \mathbb{R}$	linear mapping induced by a bi-linear form on $V \times W$ and an element $w \in W$, p. 54
$\mathcal{M} : W \to V^*$	linear mapping induced by a bi-linear form on $V \times W$, p. 54
$\mathcal{M}(W)$	linear subspace of V^*, induced by a bi-linear form on $V \times W$, p. 54
$\{ S , T ; \mathcal{B} \}$	salient pairing; ordered triple of two salient spaces S and T and a bi-salient form $\mathcal{B}$, p. 55
$\{ V , W ; \mathcal{B} \}$	linear pairing; ordered triple of two vector spaces V and W and a bi-linear form $\mathcal{B}$, p. 53
$\leq_S$	partial order relation induced by a salient space S on S or on $V[S]$, p. 43, 113
$\leq_{S^*}, \leq_{S^{**}}$	partial order relation on S^* and on S^{**} respectively, p. 43, 57, 59
$\leq_{\mathcal{B}}$	partial order relation induced by a bi-salient form $\mathcal{B} : S \times T \to \mathbb{R}_+$, on S or on T, p. 59

$\leq_*$	partial order relation on $(V[S])^*$, induced by S^*, p. 57
$V_+[S]$	$\{[(s_1,s_2)] \in V[S] \mid \exists s \in S : [(s_1,s_2)] = [(s,0)]\}$, p. 35
$(V[S])_+$	$\{[(s_1,s_2)] \in V[S] \mid [(0,0)] \leq_S [(s_1,s_2)]\}$, p. 43
$d : S \times S \to \mathbb{R}_+$	(salient) (semi-)metric on a salient space, p. 69
$\varphi : S \to \mathbb{R}_+$	(semi-)norm on a salient space, p. 74
$\pi : V \to \mathbb{R}_+$	(semi-)norm on a vector space, p. 69
$d_\pi : S \times S \to \mathbb{R}_+$	salient semi-metric on S generated by a semi-norm π on $V[S]$, p. 69
$d_\varphi : S \times S \to \mathbb{R}_+$	salient semi-metric on S generated by a semi-norm φ on S, p. 76
$d_\mathcal{F} : S \times S \to \mathbb{R}_+$	salient semi-metric on S generated by an element $\mathcal{F} \in S^*$, p. 86
$d_{\mathcal{M}_t}$	salient semi-metric on S generated by a salient pairing $\{\, S\,,\, T\,;\mathcal{B}\,\}$ and a salient function $\mathcal{M}_t$, $t \in T$, p. 78
$\psi_d : S \to \mathbb{R}_+$	semi-norm on S generated by a salient semi-metric d on S, p. 79
$\pi_d : V[S] \to \mathbb{R}_+$	semi-norm on $V[S]$, generated by a salient semi-metric d on S, p. 72
D	collection of salient semi-metrics, p. 71
P	collection of semi-norms, p. 68, 70
D_P	$\{d_\pi \mid \pi \in P\}$, p. 73
P_D	$\{\pi_d \mid d \in D\}$, p. 70
$\leq_D$	partial order relation on a set of salient semi-metrics, p. 71
τ_D	salient topology generated by a collection D of salient semi-metrics, p. 71
τ_P	locally convex topology generated by a collection P of semi-norms, p. 70

$B_d(s,\varepsilon)$	$\{t \in S \mid d\,(s,t) < \varepsilon\}$, p. 71
$B_\pi([(s_1,s_2)],\varepsilon)$	$\{[(t_1,t_2)] \in V[S] \mid \pi([(s_1+t_2, s_2+t_1)]) < \varepsilon\}$, p. 73
$B_{d_\pi}(s,\varepsilon)$	$\{t \in S \mid d_\pi\,(s,t) < \varepsilon\}$, p. 69
$\rho(\mathcal{L})$	$\sup\{\varphi_T\,(\mathcal{L}(s)) \mid s \in S$ and $\varphi_S\,(s) = 1\}$, where $\mathcal{L} : S \to T$ and $\varphi_S : S \to \mathbb{R}_+$ and $\varphi_T : T \to \mathbb{R}_+$, p. 87
T_φ	$\{t \in T \mid \sup\{\mathcal{B}(s,t) \mid s \in S$ and $\varphi\,(s) \leq 1\} < \infty\}$, p. 88
$\varphi' : T_\varphi \to \mathbb{R}_+$	$\varphi'(t) = \inf\{\kappa \geq 0 \mid \forall s \in S : \mathcal{B}(s,t) \leq \kappa\varphi\,(s)\}$, p. 88
S^*_φ	$\{\mathcal{F} \in S^* \mid \sup\{\mathcal{F}(s) \mid s \in S$ and $\varphi\,(s) \leq 1\} < \infty\}$, p. 89
$\varphi^* : S^*_\varphi \to \mathbb{R}_+$	$\varphi^*(\mathcal{F}) = \sup\{\mathcal{F}(s) \mid s \in S$ and $\varphi\,(s) \leq 1\}$, p. 89
$\tau(S,T)$	locally convex topology on S induced by the salient pairing $\{\,S\,,\,T\,;\mathcal{B}\,\}$ and the collection $\{d_t \mid t \in T\}$, p. 78
$\tau(S,S^*)$	locally convex topology on S induced by S^*, p. 87, 89
$\mathcal{U}_{s_0} : S \to \mathbb{R}_+$	$\mathcal{U}_{s_0}(s) := \max\{\mathcal{F}(s) \mid \mathcal{F} \in L\}$, where $s_0 \in \mathrm{int}(S)$ and $L = \{\mathcal{F} \in S^* \mid \mathcal{F}(s_0) = 1\}$, p. 96
$\mathcal{L}_{s_0} : S \to \mathbb{R}_+$	$\mathcal{U}_{s_0}(s) := \min\{\mathcal{F}(s) \mid \mathcal{F} \in L\}$, where $s_0 \in \mathrm{int}(S)$ and $L = \{\mathcal{F} \in S^* \mid \mathcal{F}(s_0) = 1\}$, p. 96

neoclassical models

k_0	neoclassical number of commodities, indexed by $k \in \{1,\ldots,k_0\}$, p. 8
$\mathbb{R}^{k_0}_+$	neoclassical consumption set and neoclassical price set, p. 8, 9
x, y, z	elements of $\mathbb{R}^{k_0}_+$, neoclassical commodity bundles, p. 8
p, q, r	elements of $\mathbb{R}^{k_0}_+$, neoclassical price vectors, p. 9
$p \cdot x$	value of commodity bundle x and price vector p, p. 9
i_0	number of agents, indexed by $i \in \{1,\ldots,i_0\}$, p. 11, 15, 18
j_0	number of firms, indexed by $j \in \{1,\ldots,j_0\}$, p. 15
w, w_i	initial endowment (of agent i); element of $\mathbb{R}^{k_0}_+$, p. 11
w_{total}	total initial endowment, $\sum_{i=1}^{i_0} w_i$

$\succeq, \succeq_i$	preference relation (of agent i); defined on $\mathbb{R}_+^{k_0}$, p. 10, 11
θ_{ij}	share of agent i in the profit of firm j; element of $[0,1]$, p. 15
θ_i	shares of agent i; element of $[0,1]^{j_0}$, p. 16
Y, Y_j	neoclassical production set (of firm j), p. 14, 15
L, l	rationing scheme, $L \in \mathbb{R}_+^{k_0}$, $l \in -(\mathbb{R}_+^{k_0})$, p. 17
$S_j(.)$	supply set of firm j, p. 16
$\mathcal{K}_i(.)$	income function of agent i, p. 16
$B_i(.)$	budget set of agent i, p. 11, 16, 18
$D_i(.)$	demand set of agent i, p. 11, 16, 19

salient models

C	salient space representing the set of all bundles of trade, p. 21, 101
x, y, z	elements of C; bundles of trade, p. 21, 101
C_{prod}	salient space of all production bundles in a model with production, p. 113
C_{cons}	salient space of all consumption bundles in a model with production, p. 113
$C_{\text{prod}} \oplus C_{\text{cons}}$	the set C of all bundles of trade in a model with production, p. 113
$(x^{\text{prod}}, x^{\text{cons}})$	element of $C_{\text{prod}} \oplus C_{\text{cons}}$; bundle of exchange and/or production process, p. 100, 113, 114
$\leq_C$	natural ordering on C, p. 22, 101, 113
$\leq_{\text{prod}}$	natural ordering on the set C_{prod} of production bundles, p. 113
$\leq_{\text{cons}}$	natural ordering on the set C_{cons} of consumption bundles, p. 113
C^*	adjoint of C; set of all pricing functions, p. 104

P	set of all admissible pricing functions for a specific model; subset of C^*, p. 104
$\mathcal{P}$	element of C^*; pricing function, p. 104
$\mathcal{P}(x)$	value of exchangeable object $x \in C$ at pricing function $\mathcal{P} \in C^*$, p. 104, 114
$\mathcal{G}(x, \mathcal{P})$	profit of executing production process $x \in C$ at pricing function $\mathcal{P} \in C^*$, p. 118
$\tau(C, C^*)$	salient topology on C, induced by C^*, p. 125, 137
i_0	number of agents; indexed by $i \in \{1, \ldots, i_0\}$, p. 108, 111, 120, 123
j_0	number of firms; indexed by $j \in \{1, \ldots, j_0\}$, p. 120, 123
w, w_i	initial endowment (of agent i); element of C, p. 106, 108, 111, 120, 122, 123
w_{total}	total initial endowment, $\sum_{i=1}^{i_0} w_i$
$\succeq, \succeq_i$	preference relation (of agent i); defined on (a subset of) C, p. 106, 108, 111, 120, 122, 123
θ_j, θ_{ij}	share (of agent i) in the profit of firm j; element of $[0, 1]$, p. 119, 120, 123
θ, θ_i	share vector (of agent i); element of $[0, 1]^{j_0}$, p. 119, 120, 123
T, T_j	production technology (of firm j), p. 114, 115, 120, 123
$S(.)$	supply set, p. 118
$S_j(.)$	supply set of firm j, p. 120, 123, 157, 165
$\mathcal{S}_j(.)$	supply function of firm j, p. 138, 158, 165
$\mathcal{S}(.)$	total supply function, p. 139
$\mathcal{K}(.)$	income function, p. 119, 122, 146, 152
$B(.)$	budget set, p. 107, 110, 119
$B_i(.)$	budget set of agent i, p. 108, 111, 120, 123, 146, 152
$D(.)$	demand set, p. 107, 110, 119
$D_i(.)$	demand set of agent i, p. 108, 111, 120, 123, 146, 152

$\mathcal{D}_i(.)$	demand function of agent i, p. 138, 146, 153
$\mathcal{D}(.)$	total demand function, p. 139
$\mathcal{Z}(.,.)$	excess demand value function, p. 139

hyperplane and projection

$(X, \langle .,. \rangle)$	finite-dimensional inner product space, p. 174
$H(n)$	subspace of X with normal $n \in X$, p. 174
$H(n, a)$	hyperplane of X with normal $n \in X$ which contains $a \in X$, p. 174
$H_A(n)$	$H(n) \cap A$, p. 174
$H_A(n, a)$	$H(n, a) \cap A$, p. 174
K	non-empty convex compact set in X, p. 174
$\mathcal{P}_K : X \to K$	projection on K, p. 175
$\mathcal{N}_K : X \to X$	$\mathcal{N}_K(x) := x - \mathcal{P}_K(x)$, p. 176
K_C	cylinder generated by K, p. 176

Model B (specifically)

$(\mathcal{N}_1, \mathcal{N}_2, \alpha)$	rationing scheme; element of $C^* \times C^* \times \mathbb{R}_+$, p. 109
$R(\mathcal{N}_1, \mathcal{N}_2, \alpha, w)$	$\{x \in C \mid \mathcal{N}_1(x) - \mathcal{N}_1(w) - \mathcal{N}_2(x) + \mathcal{N}_2(w) \leq \alpha\}$, p. 109
$(V, \langle .,. \rangle)$	finite-dimensional inner product space, p. 110, 178
K, K^*	solid pointed convex cone in V representing the set of all bundles of trade and the set of all pricing functions, respectively, p. 110, 110, 178
x, y	elements of K; bundles of trade, p. 110
p, q	elements of K^*; pricing vectors, p. 110
(n, α)	rationing scheme for Model B; element of $V \times \mathbb{R}_+$, p. 111
$R(n, \alpha, w)$	$\{x \in K \mid \langle x - w, n \rangle \leq \alpha\}$, p. 110, 178
$\lambda_{\max}$	$\min\{d(a, p) \mid a \in H(p_0, p_0) \cap \mathrm{bd}(K^*) \text{ and } p \in S\}$, p. 179
Q	$\{h \in H(p_0, p_0) \mid \parallel \mathcal{N}(h) \parallel \leq \lambda_{\max}\}$, p. 179
$R(q)$	$\{x \in K \mid \langle x - w_i, \mathcal{N}(q) \rangle \leq \lambda_{\max} - \parallel \mathcal{N}(q) \parallel \}$, p. 179

Model C and D (specifically)

F_x	$\{z \in C \mid x^{\text{prod}} \leq_{\text{prod}} z^{\text{prod}}$ and $z^{\text{cons}} \leq_{\text{cons}} x^{\text{cons}}\}$, p. 115
$R_x(A)$	$\{z \in A \mid x \in F_z$ and $F_z \subset A\}$, p. 115
$E(A)$	$\{e \in A \mid R_e(A) = \{e\}\}$, p. 115
Domain[j]	domain of the supply function of firm j, p. 157, 165
Domain	domain of the total supply function, p. 162, 170, 172
χ_j	extended real valued function, p. 159

Index

Evenwichtstheorie

een saillante benadering

Samenvatting

Consumptie, productie en ruil van goederen door agenten en bedrijven zijn de voornaamste kenmerken van een economie. De onderliggende drijfveren hierbij zijn de persoonlijke voorkeur van de agenten en winstbejag van de bedrijven. De praktijk leert dat deze individuele acties niet leiden tot sociale chaos, maar dat er sprake is van een zeker evenwicht tussen vraag en aanbod. De centrale vraag in de economische wetenschap vloeit hieruit voort: "Hoe komt het dat in een economie de door de bedrijven en agenten gewenste productie mogelijk is en dat precies alle producten die zij wensen aanwezig zijn?".

Aan het eind van de negentiende eeuw heeft Leon Walras als eerste een formulering van dit probleem in wiskundige termen geponeerd. Hij wordt daarmee beschouwd als de grondlegger van de wiskundige economie, meer in het bijzonder van de algemene evenwichtstheorie. In een wiskundig model van een economie wordt een sterk vereenvoudigde voorstelling van de werkelijkheid gemaakt, gebaseerd op wiskundige concepten. In de modellen, die geïnspireerd zijn door het werk van Walras, wordt verondersteld dat er een eindig aantal goederen in de economie aanwezig is, waarbij voor elk goed een aparte markt bestaat. Walras veronderstelt dat de coördinatie van vraag naar en aanbod van een goed door prijsvorming op de desbetreffende markt tot stand komt. Aangezien in zulke modellen de prijs van een goed als het leidend mechanisme wordt gezien bij de totstandkoming van een evenwicht, kan de eerder gestelde centrale vraag in deze nieuwe termen geformuleerd worden als "Hoe komen evenwichtsprijzen tot stand op de verschillende goederenmarkten?".

In wiskundig-economische modellen worden economische ingrediënten beschreven door een beperkt aantal karakteristieken, die een wiskundige analyse toestaan. Zo

worden in het meest eenvoudige model van een ruileconomie zonder productie, economische agenten volledig beschreven door hun voorkeur en door hun beginvoorraad. Elke agent gaat met zijn beginvoorraad naar de 'markt' en keert na de ruil terug met een nieuwe voorraad goederen die naar de mening van de agent niet meer door middel van ruil te verbeteren valt. De ruil op zich wordt als volgt gemodelleerd: er wordt verondersteld dat elke economische agent de heersende prijzen op de verschillende goederenmarkten aanneemt als zijnde gegeven, en dat hij vervolgens de meest geprefereerde goederenbundel bepaalt die binnen zijn budget valt. Zijn budget is in dit geval de waarde van zijn beginvoorraad, bepaald aan de hand van de heersende prijzen. Dit model van een ruileconomie is in evenwicht indien elke agent, gegeven zijn beginvoorraad, zijn preferentie en de heersende prijzen, daadwerkelijk met zijn (binnen zijn budget) meest geprefereerde goederenbundel naar huis kan keren.

Bij wiskundige modelvorming in een economische context, zoals in de modellen geïnspireerd door het werk van Walras, zijn een aantal (mogelijk conflicterende) randvoorwaarden cruciaal. Enerzijds is een vereenvoudiging noodzakelijk om een wiskundige analyse mogelijk te maken, anderzijds dient het model zodanig dicht bij de werkelijkheid te liggen dat de centrale vraag nog steeds gesteld kan worden. Concreet impliceert deze laatste restrictie dat de Walrasiaanse modellen alleen dan bruikbaar zijn als de analyse ervan minimaal resulteert in de existentie van een evenwichtsprijs en daarmee van het bijbehorende evenwicht.

Een basis voor het bewijs van het bestaan van een evenwichtssituatie werd begin deze eeuw geleverd door de Nederlandse wiskundige L.E.J. Brouwer, die in 1912 zijn beroemde vaste punt stelling publiceerde. In de jaren vijftig waren het onder andere K.J. Arrow en G. Debreu die zich bewust werden van de mogelijkheden die de vaste punt stelling van Brouwer bood bij het oplossen van het probleem van de existentie van evenwichtsprijzen. In 1954 bewezen Arrow en Debreu dat er, onder bepaalde voorwaarden, een evenwichtssituatie bestaat in het model dat Walras geformuleerd had. Hun condities waaronder existentie van een evenwichtsprijs gegarandeerd is, zijn hoofdzakelijk wiskundig van aard, maar worden desondanks in het algemeen wel als hanteerbaar in een economische context beschouwd. Het voorafgaande heeft er mede toe geleid dat de modellen van een ruileconomie zowel zonder als met productie, inclusief de bijbehorende condities zoals Arrow en Debreu ze formuleerden, standaard zijn geworden. Ze worden ook wel met de term *'neo-klassiek'* aangeduid.

De oorsprong van dit promotie-onderzoek ligt in de herziening van de uitgangspunten waarop de modellen van Arrow en Debreu zijn gebaseerd. Hierbij wordt voor de wiskundige onderbouwing gebruik gemaakt van een nieuw wiskundig concept. In de neo-klassieke modellering wordt expliciet de veronderstelling gemaakt dat elk goed separaat verhandeld wordt, hetgeen betekent dat er voor elk goed een aparte markt bestaat. Hierdoor is er géén ruimte voor modellering van een economie waarin goederen gekoppeld voorkomen of waarin een goed uitsluitend beschouwd wordt als een samenstelling van karakteristieken of eigenschappen. Een eenvoudig voorbeeld uit de dagelijkse praktijk is een fruitmand als een 'gekoppeld' goed of een 'gezonde maaltijd' als een goed dat samengesteld is uit bepaalde karakteristieken zoals voedingswaarden en vitaminen.

In dit proefschrift worden modellen geconstrueerd waarin het begrip 'goederenbundel', zoals gehanteerd door Arrow en Debreu, in een ander perspectief wordt geplaatst. Als uitgangspunt wordt niet langer aangenomen dat goederen slechts separaat kunnen optreden, maar dat evenzeer combinaties van goederen kunnen voorkomen. In de nieuwe modellen is namelijk ook ruimte voor goederenpakketten waarvan de elementen niet los van elkaar ruilbaar zijn. Daarnaast kunnen ook abstracte economische begrippen gemodelleerd worden, zoals de eerder genoemde samenstelling van karakteristieken. In de geïntroduceerde modellen wordt het begrip '*ruilbaar object*' gehanteerd als verzamelnaam voor enerzijds deze nieuw ingevoerde combinaties van goederen en karakteristieken, en anderzijds voor de individuele goederen én voor de bundels bestaande uit los verhandelbare goederen, beide in neo-klassieke zin. Het begrip 'ruilbaar object' neemt daarmee in de modellen van dit proefschrift de rol over van de neo-klassieke begippen 'goed' en 'goederenbundel'.

Teneinde in staat te zijn deze nieuwe ideeën in wiskundige terminologie onder te brengen, wordt in dit proefschrift speciaal voor de modellering van de verzameling van alle ruilbare objecten een nieuw wiskundig formalisme opgebouwd. Het eerste deel bevat een axiomatische introductie van het wiskundige begrip saillante (i.e. gepunte) ruimte binnen dit formalisme. Vervolgens wordt systematisch onderzoek verricht naar de eigenschappen van deze ruimten. Eén van de belangrijke resultaten is een versie van de vaste punt stelling van Brouwer, speciaal geënt op saillante ruimten.

In het tweede deel van dit proefschrift wordt het concept saillante ruimte aangewend om de verzameling van alle ruilbare objecten, zoals hierboven beschreven, te representeren in verscheidene modellen van een ruileconomie. In Model A wordt op deze wijze een generalisatie verkregen van het Arrow-Debreu model van een ruileconomie zonder productie.

In Model B wordt een uitbreiding van Model A bereikt door middel van de introductie van prijsstarheden en rantsoeneringen. Deze aanpak is geïnspireerd door het werk van Drèze, met de kanttekening dat zijn benadering om per markt een (eventuele) restrictie op te leggen in Model B niet toegepast kan worden. Dit is een gevolg van het feit dat in een model gebaseerd op ruilbare objecten de verschillende markten niet noodzakelijk aanwezig zijn. Het blijkt dat op basis van maximaal één restrictie de rantsoenering in Model B vormgegeven kan worden.

In de Modellen C en D wordt de mogelijkheid tot productie toegevoegd aan Model A. Hiertoe wordt verondersteld dat een ruilbaar object uniek opgesplitst kan worden in twee delen: een productie- en een consumptiedeel. Hierbij wordt een productiedeel als input gebruikt voor een productieproces dat als output het consumptiedeel van een ruilbaar object heeft. In Model C wordt verondersteld dat de agenten geïnteresseerd zijn in ruilbare objecten als geheel, terwijl in Model D expliciet wordt aangenomen dat agenten slechts het consumptiedeel van een ruilbaar object prefereren.

Gebruik makend van de resultaten betreffende saillante ruimten uit het eerste deel van dit proefschrift, wordt er in het tweede deel, naast de beschrijving van de vier modellen ook voor elk model minimaal één stelling met betrekking tot het bestaan van een evenwichtssituatie geformuleerd en bewezen.