Potential Games and Interactive Decisions with Multiple Criteria

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PROEFSCHRIFT

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Prologue

It was in my first year as an undergraduate student that I took the game theory course of Stef Tijs. According to the course description, this should not be a problem, since knowledge from other courses was not required. This was reassuring: as a first year undergraduate student, I still had an abundant lack of previous knowledge from other courses. The fact that I was surrounded by students in their third year and up, for whom this course was originally intended, did not temper my enthusiasm. Neither did the frequent use of correspondences, fixed point theorems, duality theory and other rather advanced mathematical tools, although I did learn that 'No previous knowledge required' *really* meant: 'No previous knowledge required, other than all this mathematical stuff we have been trying to teach you in the compulsory courses during your first three, four years as an undergraduate'. It was hard work, but I was succesful in the exam and have been fascinated by game theory ever since. I thank Stef Tijs for his stimulating way of teaching game theory and for his support over the years.

Peter Borm has been closely involved in my work. I am very grateful to him for his support and the discussions we had, many of them outside the scope of work.

I also want to thank the co-authors with whom I worked on chapters in this thesis, including Henk Norde, Dries Vermeulen, Maurice Koster, Hans Reijnierse, and Lina Mallozzi. I especially want to mention the collaboration with Edward Droste and Michael Kosfeld and the many discussions we had on bounded rationality.

Several colleagues have noticed my particular way of writing. One comment[1] on style, in particular on the use of personal pronouns. It is inconvenient, in referring to generic players, to continuously use 'he/she'. Moreover, formally, at the level of abstraction of this thesis, a player is neither a male nor a female, but an element i of a player set N. The gender, hair color, shoe size, and weight of this player $i \in N$ are of no concern in the models considered in this thesis. In cases where such matters are of concern, they should be modelled explicitly. A certain female player and I played a Battle of the Sexes about the use of gender labels. The outcome was that we both accepted the validity of the other person's arguments, but in no way saw our own arguments refuted. In the end, I decided to use male pronouns, which is equally incorrect as using female pronouns, but shorter.

[1] Well, two: footnotes should be avoided at all costs.

During the work on my thesis, I had the opportunity to do research at several foreign universities. I was a guest twice at the Department of Statistics and Operations Research at the University of Vigo, Spain. In addition to the work we did there, I am grateful for the hospitality I received, the friendship, and the time they took to show me around Galicia. Estela Sánchez Rodríguez, Gloria Fiestras Janeiro, Gustavo Bergantiños Cid, Ignacio García Jurado, thank you very much! I am particularly pleased that Ignacio also joined my Ph.D. committee.

I very much enjoyed the joint work I did with Anne van den Nouweland, another member of the thesis committee, part of which took place during my stay at the University of Oregon (USA). Anne and Ron Croonenberg made it a pleasant time.

The spring semester of 1999 I spent in Sweden as a guest at the Department of Economics of Stockholm University. My thanks to Martin Dufwenberg for this opportunity. It was a productive period. During my stay I finished three papers, including joint work with Sofia Grahn and Martin Dufwenberg. Moreover, the first part of my thesis was written there. Spring in Sweden in no way implies the absence of snow. I have particularly good memories of a barbecue where all participants, even though it took place in the middle of May, were wearing a full winter outfit, including gloves, scarf, and cap.

I also want to express my gratitude to the other members of the thesis committee, Dov Monderer and Peter Wakker, for the time and effort they spent on my thesis.

Finally, I want to thank my parents and Peter for their support and encouragement and Sofia for making it perfectly clear what is the most important result of my years as a Ph.D. student.

Mark Voorneveld
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Chapter 1

Introduction to Part I

1.1 Games

Game theory is a mathematical theory that designs and uses tools to study interaction between decision makers. This thesis mainly deals with noncooperative or strategic games, games in which the involved players cannot make binding agreements. Suppose that two classes in a school are working on a project and have to decide where to get their information. This can be either via the school library or their computer lab which gives access to the Internet.

If both classes decide to use the library, people are getting in each other's way and the required books will be hard to get hold of. In this case, each of the classes gets only half the information it needs. If both classes occupy the computer lab, network facilities will slow down due to crowding, but each class will be able to retrieve sixty percent of the necessary information. If one of the classes goes to the library and the other class goes to the computer lab, the library class will find only eighty percent of the information (due to some of the necessary books being lent), whereas the computer class will find all information it needs.

A schematic way to represent this situation is given in Figure 1.1: there are two players, namely class 1 and class 2. Each class has two strategies: go to the Library or

		Class 2	
		Library	Internet
Class 1	Library	$50, 50$	$80, 100$
	Internet	$100, 80$	$60, 60$

Figure 1.1: The information game

use the Internet. The numbers in the corresponding cells give the payoffs of the game, the percentage of information that the first, respectively the second class obtains. For instance, if class 1 goes to the library and class 2 to the computer lab, class 1 gets eighty percent of the necessary information and class 2 gets all necessary information.

The basic assumptions are that the two classes simultaneously and independently have to choose where to search for information and that each class tries to maximize its amount of information. But this is hard: the right thing to do depends on the choice of the opponent. The best thing to do is to go where the other class is not. Thus, the strategy profiles (Library, Internet) and (Internet, Library) where one class goes to the library and the other class goes to the computer lab are in some sense 'stable': each class chooses the best option given the choice of the other class. Such a strategy profile, where each player chooses a best strategy given the strategy choices of the other players, is a Nash equilibrium.

But not each game has a Nash equilibrium of this type. Consider the Matching Pennies game in Figure 1.2: Two players each have a penny. They have to decide to

		Player 2	
		H	T
Player 1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

Figure 1.2: Matching Pennies

show either heads or tails. If both show heads or both show tails, the penny of player 2 goes to player 1, otherwise the penny of player 1 goes to player 2. No matter what strategy pair is chosen, there will always be a player with an incentive to deviate.

1.2 Pure and mixed Nash equilibria

The absence of Nash equilibria as described above is usually solved by introducing a larger strategy space: instead of just choosing one of the (pure) strategies, a player may choose each of his pure strategies with a certain probability, a so-called mixed strategy. Assume that in the Matching Pennies game the first player shows heads with probability p and the second player with probability q. Then the expected payoff to player 1 is $pq - p(1-q) - (1-p)q + (1-p)(1-q) = (2p-1)(2q-1)$; the expected payoff to player 2 is $(1-2p)(2q-1)$. If player 2 chooses heads with probability smaller than $\frac{1}{2}$, then player 1's best response is to choose tails (with probability one). However, if player 2 chooses heads with probability equal to $\frac{1}{2}$ (assuming that it is a fair coin, one might say that he throws the penny and shows the side it comes down on), then player 1's expected payoff will be zero whatever he does: he is indifferent between all probability distributions over heads and tails. For higher probabilities ($q > \frac{1}{2}$), the unique best response is to show heads. A similar reasoning holds for the second player, so the unique Nash equilibrium of this game, i.e., the unique pair of mixed strategies in which each player chooses a best response against the strategy of his opponent, is the strategy pair in which $p = q = \frac{1}{2}$.

The game that arises if each player in a strategic game, with finitely many players and each player having finitely many pure strategies, is allowed to choose a probability distribution over his pure strategy set is called the mixed extension of the strategic game. Nash (1950a, 1951) established the existence of equilibria in mixed extensions of finite strategic games.

The use of mixed strategies can be motivated in several ways and is valid in many situations. Osborne and Rubinstein (1994, Section 3.2), for instance, give a detailed discussion of interpretations of mixed strategies, including critical comments. Equilibria in pure strategies, however, are particularly appealing. They are simple and allowing mixed strategies does not provide the players with opportunities for profitable deviation: a pure-strategy Nash equilibrium is also an equilibrium in the mixed extension of a finite strategic game. The simplicity argument applies to other games than mixed extensions of finite strategic games as well. If players have an infinite set of pure strategies it is relatively uncommon to consider the additional complication of allowing players to choose probability measures over these strategies. Moreover, it is natural to consider pure-strategy equilibria in one-shot games: only pure strategies can be observed as outcomes of these games. This motivates the search for games possessing pure-strategy Nash equilibria.

1.3 Potential games

The first part of this thesis, consisting of chapters 2 through 9, is concerned with a special tool for detecting games with pure-strategy Nash equilibria: so-called *potential functions*. Recall the game in Figure 1.1 and consider the real-valued function P on the strategy space of this game as given in Figure 1.3. Notice that the change in payoff to a

		Class 2	
		Library	Internet
Class 1	Library	130	180
	Internet	180	160

Figure 1.3: A potential function

unilaterally deviating player exactly matches the change in the function P. For instance, if player 2 deviates from (Library, Library) to (Library, Internet) his payoff increases by $100 - 50 = 50$ and the function P increases by $180 - 130 = 50$. The function P is therefore called an *exact potential* of the information game in Figure 1.1. Abstracting from irrelevant information — namely how unilateral deviations affect the payoffs to players other than the deviating player — this potential function provides the necessary information for the computation of the (pure) Nash equilibria: both (Library, Internet) and (Internet, Library) are pure Nash equilibria, since every unilateral deviation from these strategy profiles decreases the value of the potential function.

Thus, a potential function is an economical way to summarize the information concerning pure Nash equilibria into a single function. Moreover, every finite game with a potential function has an equilibrium in pure strategies: since the strategy space is finite, the potential achieves its maximum at a certain pure-strategy profile. This must be a Nash equilibrium. If not, a player could benefit from deviating; but by definition, the potential function would then increase as well, contradicting the assumption that it achieved its maximum.

Although potential functions already appeared implicitly in several earlier papers (Rosenthal, 1973, Slade, 1994), Monderer and Shapley (1996) were the first to formally define several classes of potential games. The first part of this thesis studies potential games in detail.

One of its main focuses is on the structure of several types of potential games: what are necessary and sufficient conditions for a certain type of potential function to exist? This topic is taken up in chapters 2, 5, 7, and 9. The relation between these chapters is that in all cases it turns out that a certain condition on cycles in the strategy space is of key importance.

Having derived the existence of pure-strategy Nash equilibria in finite potential games, the question arises whether infinite potential games have pure-strategy Nash equilibria as well. This matter is studied in chapter 8; it turns out that such games are less well-behaved than hoped for. In infinite games the existence of a potential function is of little help to establish existence results.

Applications of potential games are given in chapters 2, 3, 4, and 6. In chapters 2 and 3 the focus is on congestion games, games where players choose facilities from a common set and costs for using these facilities depend only on the number of simultaneous users. Chapter 4 studies a production process that takes place in different stages; costs of the production departments depend only on production techniques chosen by earlier departments and departments operating in the same stage. This chapter extends the notion of potential games to a specific class of extensive form games with incomplete information. Chapter 6 deals with the question whether or not players in a noncooperative game in which they contribute to the financing of a collection of public goods can be motivated to act in the interest of social welfare — measured through the utilitarian welfare function — rather than their own payoffs. A building rule is derived which specifies for each profile of contributions the set of public goods that is built. It is shown that this building rule makes the noncooperative game strategically equivalent to a potential game where the utilitarian welfare function is a potential.

1.4 Preliminaries

This section contains several definitions and matters of notation. Some mathematical maturity of the reader is assumed. Theorems and definitions that are of central concern will be stated where necessary in the chapters themselves; the reader is assumed to be familiar with other (mainly standard game theoretic and topological) notions. The books by Osborne and Rubinstein (1994) and Myerson (1991) provide a good background reading in game theory; Aliprantis and Border (1994) provide many of the mathematical notions that readers with interest in mathematics and economics will need.

A *(strategic) game* is a tuple $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where

- N is a nonempty, finite set of players;
- each player $i \in N$ has a nonempty set X_i of pure strategies;
- each player $i \in N$ has a payoff function $u_i : \prod_{j \in N} X_j \to \mathbb{R}$ specifying for each strategy profile $x = (x_j)_{j \in N} \in \prod_{j \in N} X_j$ player i's payoff $u_i(x) \in \mathbb{R}$.

The player set of a game is assumed to be finite throughout this thesis. A game is *finite* if, moreover, each player $i \in N$ has a finite set X_i of pure strategies.

Conventional game theoretic notation is used. For instance: $X = \prod_{j \in N} X_j$ denotes the set of strategy profiles. Let $i \in N$. $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$ denotes the set of strategy profiles of i's opponents. Let $S \subseteq N$. $X_S = \prod_{j \in S} X_j$ denotes the set of strategy profiles of players in S. With a slight abuse of notation strategy profiles $x = (x_j)_{j \in N} \in X$ will be denoted by (x_i, x_{-i}) or $(x_S, x_{N \setminus S})$ if the strategy choice of player i or of the set of players S needs stressing.

The set of probability distributions over a finite set A is denoted $\Delta(A)$:

$$\Delta(A) = \{\sigma : A \to [0,1] \mid \sum_{a \in A} \sigma(a) = 1\}.$$

The mixed extension of a finite game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ allows each player $i \in N$ to choose a mixed strategy $\sigma_i \in \Delta(X_i)$; payoffs are extended to mixed strategies as follows:

$$u_i((\sigma_j)_{j \in N}) = \sum_{x \in X} \left(\prod_{j \in N} \sigma_j(x_j) \right) u_i(x),$$

i.e., the payoff to a mixed strategy profile is simply the expected payoff. A pure strategy $x_i \in X_i$ can be identified with the mixed strategy that assigns probability one to x_i.

A pure-strategy profile $x \in X$ is a *pure Nash equilibrium* of the game G if players cannot benefit from unilateral deviation:

$$\forall i \in N, \forall y_i \in X_i : \quad u_i(x) \geqq u_i(y_i, x_{-i}).$$

Similarly, a mixed-strategy profile $\sigma = (\sigma_j)_{j \in N} \in \prod_{j \in N} \Delta(X_j)$ is a *(mixed-strategy) Nash equilibrium* of the game G if

$$\forall i \in N, \forall y_i \in X_i : \quad u_i(\sigma) \geqq u_i(y_i, \sigma_{-i}).$$

Notice that attention can be restricted to deviations to pure strategies due to the multilinearity of the payoff functions. The set of Nash equilibria of a game G is denoted $NE(G)$.

In Part I of this thesis, mixed strategies are not taken into account. In that case, $NE(G)$ stands for the set of *pure* Nash equilibria.

A *transferable utility game* or TU-game for ease of notation is a tuple (N, v) consisting of a finite, nonempty set N of players and a characteristic function $v : 2^N \to \mathbb{R}$ attaching to each coalition $S \subseteq N$ its value $v(S) \in \mathbb{R}$. By assumption $v(\emptyset) = 0$.

For a finite set A, the number of elements of A is denoted by $|A|$. Let A and B be two sets. $A \subseteq B$ denotes weak set inclusion, $A \subset B$ denotes proper set inclusion:

$$\begin{aligned} A \subseteq B &\Leftrightarrow \forall a \in A : a \in B \\ A \subset B &\Leftrightarrow (A \subseteq B \text{ and } A \neq B) \end{aligned}$$

The set of functions from A to B is denoted B^A. A binary relation on a set A is a subset of $A \times A$, i.e., a set of ordered pairs (a, b) with $a, b \in A$. The collection of all subsets of A is denoted 2^A. For instance, if M and N are sets, then $(2^M)^N$ is the set of functions that assign a subset of M to each element of N.

Summation over the empty set yields zero. The infimum of the empty set equals infinity: $\inf(\emptyset) = \infty$.

Some specific sets:

$\mathbb{N} = \{1, 2, 3, \ldots\}$	set of positive integers
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	set of nonnegative integers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rationals
$\mathbb{R}$	set of reals
$\mathbb{R}_+ = [0, \infty)$	set of nonnegative reals
$\mathbb{R}_{++} = (0, \infty)$	set of positive reals

The symbol $\triangleleft$ indicates the end of definitions, remarks, and examples. The symbol $\Box$ indicates the end of a proof.

Chapter 2

Exact Potential Games

2.1 Introduction

Monderer and Shapley (1996) introduced several classes of potential games. A common feature of these classes is the existence of a real-valued function on the strategy space that incorporates information about the strategic possibilities of all players simultaneously.

This chapter reviews results concerning exact potential games. Exact potential games are defined in Section 2.2. Two characterizations of exact potential games are provided. The purpose of Section 2.3 is to describe the congestion model of Rosenthal (1973) and to establish an isomorphism between the class of exact potential games and the class of Rosenthal's congestion games.

2.2 Exact potential games

This section defines exact potential games, surveys some simple results, and provides two characterizations of exact potential games.

Definition 2.1 A strategic game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is an *exact potential game* if there exists a function $P : X \to \mathbb{R}$ such that for all $i \in N$, for all $x_{-i} \in X_{-i}$, and all $x_i, y_i \in X_i$:

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}).$$

The function P is called an *(exact) potential (function)* for G. $\triangleleft$

In other words, a strategic game is an exact potential game if there exists a real-valued function on the strategy space which exactly measures the difference in the payoff that accrues to a player if he unilaterally deviates.

Example 2.2 In the Prisoner's Dilemma game of Figure 2.1, two suspects of a crime are put into separate cells. If both confess (strategy c), each will be sentenced to 3 years in prison. If exactly one of them confesses, he will be freed and used as a witness against

	c	d
c	1,1	4,0
d	0,4	3,3

Figure 2.1: Prisoner's Dilemma

the other person, who will be sentenced to 4 years in prison. If both do not confess (strategy d), they will both be punished for a minor offense and spend 1 year in jail. Payoffs are represented by 4, minus the number of years spent in prison. This is an exact potential game. An exact potential function is given by $P(c,c) = 5, P(c,d) = P(d,c) = 4, P(d,d) = 3$. ◁

The definition of an exact potential game immediately implies

Proposition 2.3 *If $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ has an exact potential P, then the Nash equilibria of G and $\langle N, (X_i)_{i\in N}, (P)_{i\in N}\rangle$, i.e., the game obtained by replacing each payoff function by the potential P, coincide.*

An important implication is the following result.

Proposition 2.4 *Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a finite exact potential game. Then G has at least one (pure-strategy) Nash equilibrium.*

Proof. Let P be an exact potential for G. Since X is finite, $\arg\max_{x\in X} P(x)$ is a nonempty set. Clearly, all elements in this set are pure-strategy Nash equilibria. □

Facchini *et al.* (1997) provide a characterization of exact potential games by splitting them up into coordination games and dummy games.

Definition 2.5 A game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a

- *coordination game* if there exists a function $u : X \to \mathbb{R}$ such that $u_i = u$ for all $i \in N$;
- *dummy game* if for all $i \in N$ and all $x_{-i} \in X_{-i}$ there exists a $k \in \mathbb{R}$ such that $u_i(x_i, x_{-i}) = k$ for all $x_i \in X_i$.

◁

In a coordination game, players pursue the same goal, reflected by the identical payoff functions. In a dummy game, a player's payoff does not depend on his own strategy.

Theorem 2.6 *Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a strategic game. G is an exact potential game if and only if there exist functions $(c_i)_{i\in N}$ and $(d_i)_{i\in N}$ such that*

- *$u_i = c_i + d_i$ for all $i \in N$,*

- $\langle N, (X_i)_{i \in N}, (c_i)_{i \in N} \rangle$ *is a coordination game, and*
- $\langle N, (X_i)_{i \in N}, (d_i)_{i \in N} \rangle$ *is a dummy game.*

Proof. The 'if'-part is obvious: the payoff function of the coordination game is an exact potential function of G. To prove the 'only if'-part, let P be an exact potential for G. For all $i \in N$, $u_i = P + (u_i - P)$. Clearly, $\langle N, (X_i)_{i \in N}, (P)_{i \in N} \rangle$ is a coordination game. Let $i \in N, x_{-i} \in X_{-i}$, and $x_i, y_i \in X_i$. Then $u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i})$ implies $u_i(x_i, x_{-i}) - P(x_i, x_{-i}) = u_i(y_i, x_{-i}) - P(y_i, x_{-i})$. Consequently, $\langle N, (X_i)_{i \in N}, (u_i - P)_{i \in N} \rangle$ is a dummy game. $\square$

Facchini *et al.* (1997) proceed to derive from this theorem the dimension of the linear space of finite exact potential games. Let $N = \{1, \ldots, n\}$, $n \in \mathbb{N}$, be a fixed player set and $X = \prod_{i \in N} X_i$ a fixed strategy space. Let $m_i = |X_i| \in \mathbb{N}$ be the cardinality of player i's strategy set. The set of strategic games with player set N and strategy space X is denoted $\Gamma^{N,X}$ and is clearly isomorphic to the linear space $(\mathbb{R}^N)^X$ of functions from X to $\mathbb{R}^N$. A game in $\Gamma^{N,X}$ can be identified with an n-dimensional vector of payoff functions $u = (u_1, \ldots, u_n)$. Addition and scalar multiplication on $\Gamma^{N,X}$ is then defined by using the standard addition and scalar multiplication for functions. By $\Gamma_C^{N,X}, \Gamma_D^{N,X}, \Gamma_P^{N,X}$ we denote the subset of coordination, dummy, and exact potential games, respectively. These are all linear subspaces of $\Gamma^{N,X}$. From Theorem 2.6 we know that $\Gamma_P^{N,X} = \Gamma_C^{N,X} + \Gamma_D^{N,X}$. Hence $\dim(\Gamma_P^{N,X}) = \dim(\Gamma_C^{N,X}) + \dim(\Gamma_D^{N,X}) - \dim(\Gamma_C^{N,X} \cap \Gamma_D^{N,X})$. By isomorphism, $\dim(\Gamma_C^{N,X}) = \dim(\mathbb{R}^X) = \prod_{i \in N} m_i$, since it suffices to specify one real number for every $x \in X$ in a coordination game, and $\dim(\Gamma_D^{N,X}) = \dim(\mathbb{R}^{X_{-1}} \times \cdots \times \mathbb{R}^{X_{-n}}) = \sum_{i \in N} \prod_{j \in N \setminus \{i\}} m_j$, since it suffices, in dummy games, to specify for each player $i \in N$ a payoff for each element in X_{-i}. It remains to determine $\dim(\Gamma_C^{N,X} \cap \Gamma_D^{N,X})$. Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma^{N,X}$ be both a dummy game and a coordination game. Then there exists a function $u : X \to \mathbb{R}$ such that $u_i = u$ for each player $i \in N$. The dummy property implies that for each $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X$: $u(x) = u(y_1, x_2, x_3, \ldots, x_n) = u(y_1, y_2, x_3, \ldots, x_n) = \cdots = u(y)$. Hence $\dim(\Gamma_C^{N,X} \cap \Gamma_D^{N,X}) = \dim(\mathbb{R}) = 1$. This finishes the proof of

Proposition 2.7 *The dimension of the linear space of exact potential games* $\Gamma_P^{N,X}$ *equals*

$$\prod_{i \in N} m_i + \sum_{i \in N} \prod_{j \in N \setminus \{i\}} m_j - 1.$$

The following result shows that the difference between two exact potential functions of a game is a constant function.

Proposition 2.8 *Let* $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ *be a game with exact potential functions* P *and* Q. *Then* $P - Q$ *is a constant function.*

Proof. Let $i \in N$. By Theorem 2.6, $u_i - Q$ and $u_i - P$ do not depend on the strategy choice of player i. Hence $(P - Q) = (u_i - Q) - (u_i - P)$ does not depend on the strategy choice of player i. This holds for every player $i \in N$: $(P - Q)$ is a constant function. $\square$

Proposition 2.8 implies that the set of strategy profiles maximizing a potential function of an exact potential game does not depend on the particular potential function that is chosen. Potential-maximizing strategies were used in the proof of Proposition 2.4 to show that finite exact potential games have pure-strategy Nash equilibria. The potential maximizer, formally defined for an exact potential game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ as

$$PM(G) = \{x \in X \mid x \in \arg\max_{y \in X} P(y) \text{ for some potential function } P \text{ of } G\}$$

can therefore act as an equilibrium refinement tool. See Monderer and Shapley (1996). Peleg, Potters, and Tijs (1996) provide an axiomatic approach to potential-maximizing strategies.

Remark 2.9 In a more general setting, Balder (1997) considers games with additively coupled payoffs. A game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ has additively coupled payoffs if for each pair $(i, j) \in N \times N$ there exists a function $v_{i,j} : X_j \to \mathbb{R}$ such that each player i's payoff function decomposes as

$$u_i : x \mapsto \sum_{j \in N} v_{i,j}(x_j).$$

Define $P : X \to \mathbb{R}$ by $P : x \mapsto \sum_{i \in N} v_{i,i}(x_i)$. Then $(u_i - P) : x \mapsto \sum_{j \in N \setminus \{i\}} (v_{i,j}(x_j) - v_{j,j}(x_j))$ does not depend on the strategy choice of player i, proving that a game with additively coupled payoffs is an exact potential game with potential P. $\triangleleft$

Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game. A *path* in the strategy space X is a sequence $\gamma = (x^1, x^2, \ldots)$ of elements $x^k \in X$ such that for all $k = 1, 2, \ldots$ the strategy combinations x^k and x^{k+1} differ in exactly one, say the $i(k)$-th, coordinate. A finite path $\gamma = (x^1, \ldots, x^k)$ is called *closed* or a *cycle* if $x^1 = x^k$. It is a *simple closed path* if it is closed and apart from the initial and the terminal point of the path all strategy combinations are different: for all $l, m \in \{1, \ldots, k-1\}, l \neq m : x^l \neq x^m$. The number of distinct strategy combinations in a simple closed path is called the *length* of the path.

Let $u = (u_i)_{i \in N}$ be the vector of payoff functions and $\gamma = (x^1, \ldots, x^k)$ be a finite path. Define $I(\gamma, u) = \sum_{m=1}^{k-1} [u_{i(m)}(x^{m+1}) - u_{i(m)}(x^m)]$, where $i(m)$ is the unique deviating player at step m, i.e., $x^{m+1}_{i(m)} \neq x^m_{i(m)}$.

These concepts will be illustrated in an example.

Example 2.10 Consider the two-player game given below. An exact potential for this game is given by $P(T, L) = 0, P(T, R) = 1, P(B, L) = 2, P(B, R) = 3$. The sequence

	L	R
T	0,2	2,3
B	2,5	4,6

$((T, L), (B, R))$ is not a path, since the consecutive elements differ in both coordinates. $\gamma = ((T, L), (T, R), (B, R), (B, L), (T, L))$ is a path, which is also closed and simple. Its length is 4. Notice that $I(\gamma, u) = (3 - 2) + (4 - 2) + (5 - 6) + (0 - 2) = 0$. $\triangleleft$

The following characterization of exact potential games was given by Monderer and Shapley (1996). It shows that it is no coincidence that the game in the example above is an exact potential game and that the payoff differences over the closed path sum to zero.

Theorem 2.11 *Let* $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ *be a game. The following claims are equivalent:*

(a) *G is an exact potential game;*

(b) *$I(\gamma, u) = 0$ for all closed paths γ;*

(c) *$I(\gamma, u) = 0$ for all simple closed paths γ;*

(d) *$I(\gamma, u) = 0$ for all simple closed paths γ of length 4.*

Proof.
(a) $\Rightarrow$ (b) Let P be an exact potential of G and $\gamma = (x^1, \ldots, x^k)$ a closed path. Then $I(\gamma, u) = I(\gamma, P) = P(x^k) - P(x^1) = 0$.
(b) $\Rightarrow$ (a) Fix $x \in X$ and take $P(x) = 0$. Let $y \in X, y \neq x$, and let $\gamma = (x^1, \ldots, x^k)$ be a path from x to y: $x^1 = x, x^k = y$. Define $P(y) = I(\gamma, u)$. To show that this yields an exact potential for G, one needs to show that P is well-defined and that the conditions of Definition 2.1 hold.

Let $\gamma' = (y^1, \ldots, y^m)$ be any other path from x to y. For P to be well-defined, $P(y)$ should equal $I(\gamma', u)$. This follows from the fact that $\gamma'' = (x^1, \ldots, x^k, y^{m-1}, \ldots, y^1)$ is a closed path (from x to y via γ and back by reversing γ') and $I(\gamma'', u) = I(\gamma, u) - I(\gamma', u) = 0$.

To check that P is indeed an exact potential, let $i \in N, x_{-i} \in X_{-i}$, and $y_i, z_i \in X_i, y_i \neq z_i$. Let $\gamma = (y^1, \ldots, y^k)$ be a path from x to $y = (y_i, x_{-i})$ and $\gamma' = (z^1, \ldots, z^m)$ a path from x to $z = (z_i, x_{-i})$. Remains to show that $P(y) - P(z) = u_i(y) - u_i(z)$. Consider the closed path $\gamma'' = (y^1, \ldots, y^{k-1}, y^k, z^m, z^{m-1}, \ldots, z^1)$. By assumption, $0 = I(\gamma'', u) = I(\gamma, u) + u_i(z) - u_i(y) - I(\gamma', u) = P(y) + u_i(z) - u_i(y) - P(z)$.
(b) $\Rightarrow$ (c) $\Rightarrow$ (d) Trivial.
(d) $\Rightarrow$ (b) Assume $I(\gamma, u) = 0$ for all simple closed paths of length 4. Suppose there is a closed path $\gamma = (x^1, \ldots, x^k)$ such that $I(\gamma, u) \neq 0$. W.l.o.g. γ has minimal length ($\geqq 5$) among all closed paths with this property. Since $i(1)$ deviates at the first step and $x^1 = x^k$, there must be another step m with $i(m) = i(1)$. By minimality, $i(1)$ does not make two consecutive deviations: $m \in \{3, \ldots, k-1\}$. Define the closed path $\mu = (x^1, \ldots, x^{m-1}, y^m, x^{m+1}, \ldots, x^k)$ in such a way that the deviations of players $i(m-1)$ and $i(m) = i(1)$ are reversed, i.e.,

$$y_i^m = \begin{cases} x_i^{m+1} & \text{if } i = i(m) = i(1), \\ \\ x_i^{m-1} & \text{otherwise.} \end{cases}$$

The simple closed path $\nu = (x^{m-1}, x^m, x^{m+1}, y^m, x^{m-1})$ of length 4 satisfies $I(\nu, u) = 0$, so $I(\gamma, u) = I(\mu, u)$, but in the closed path μ player $i(1)$ deviates one step earlier than in γ. Continuing in this way one finds a closed path τ of the same length as γ with $I(\gamma, u) = I(\tau, u) \neq 0$ in which $i(1)$ deviates in two consecutive steps, contradicting the minimality assumption on γ. □

2.3 Rosenthal's congestion model

In a congestion model, players use several facilities — also called machines or (primary) factors — from a common pool. The costs or benefits that a player derives from the use of a facility are, possibly among other factors, determined by the number of users of a facility. The purpose of this section is to describe the congestion model of Rosenthal (1973). In his model, each player chooses a subset of facilities. The benefit associated with each facility is a function only of the number of players using it. The payoff to a player is the sum of the benefits associated with each facility in his strategy choice, given the choices of the other players. By constructing an exact potential function for such congestion games, the existence of pure-strategy Nash equilibria can be established. Moreover, Monderer and Shapley (1996) showed that every finite exact potential game is isomorphic to a congestion game. Their proof is rather complex. In this section we present a different proof which is shorter and in our opinion more intuitive. In fact, we use the decomposition of exact potential games into dummy games and coordination games stated in Theorem 2.6 to decompose the problem into two subproblems. It is shown that each coordination game and each dummy game is isomorphic to a congestion game.

Rosenthal (1973) defines a *congestion model* as a tuple $\langle N, M, (X_i)_{i \in N}, (c_j)_{j \in M} \rangle$, where

- $N = \{1, \dots, n\}$ is the set of players;
- M is the finite set of facilities, machines, or factors;
- For each player $i \in N$, his collection of pure strategies X_i is a finite family of subsets of M;
- For each facility $j \in M$, $c_j : \{1, \dots, n\} \to \mathbb{R}$ is the cost function of facility j, with $c_j(r), r \in \{1, \dots, n\}$, the costs to each of the users of machine j if there is a total of r users.

This gives rise to a *congestion game* $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where N and $(X_i)_{i \in N}$ are as above and for $i \in N$, $u_i : X \to \mathbb{R}$ is defined thus: for each $x = (x_1, \dots, x_n) \in X$, and each $j \in M$, let $n_j(x) = |\{i \in N : j \in x_i\}|$ be the number of users of machine j if the players choose x. Then $u_i(x) = -\sum_{j \in x_i} c_j(n_j(x))$. This definition implies that each player pays for the facilities he uses, with costs depending only on the number of users of the facility. It is usually assumed that costs are an increasing function of the number of users. This, however, is not necessary to prove the existence of an equilibrium. Notice that cost functions can achieve negative values, representing benefits of using a facility.

The main result from Rosenthal's paper, formulated in terms of exact potentials, is given in the next proposition. Its proof is straightforward and therefore omitted.

Proposition 2.12 *Let $\langle N, M, (X_i)_{i\in N}, (c_j)_{j\in M}\rangle$ be a congestion model and G its congestion game. Then G is an exact potential game. A potential function is given by $P : X \to \mathbb{R}$ defined for all $x = (x_i)_{i\in N} \in X$ as*

$$P(x) = -\sum_{j\in\cup_{i\in N} x_i} \sum_{l=1}^{n_j(x)} c_j(l).$$

Since the game is finite, it has a Nash equilibrium in pure strategies.

Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ and $H = \langle N, (Y_i)_{i\in N}, (v_i)_{i\in N}\rangle$ be two strategic games with identical player set N. G and H are *isomorphic* if for all $i \in N$ there exists a bijection $\varphi_i : X_i \to Y_i$ such that

$$u_i(x_1, \dots, x_n) = v_i(\varphi_1(x_1), \dots, \varphi_n(x_n)) \text{ for all } (x_1, \dots, x_n) \in X.$$

A congestion game where the machines have non-zero costs only if all players use it as part of their strategy choice is clearly a coordination game. Also, each coordination game can be expressed in this form, as shown in the proof of the next theorem.

Theorem 2.13 *Each finite coordination game is isomorphic to a congestion game.*

Proof. Let $G = \langle N, (X_i)_{i\in N}, (u)_{i\in N}\rangle$ be a finite n-player coordination game in which each player has payoff function u. Introduce for each $x \in X$ a different machine $m(x)$. Define the congestion model $\langle N, M, (Y_i)_{i\in N}, (c_j)_{j\in M}\rangle$ with $M = \cup_{x\in X} \{m(x)\}$, for each $i \in N$: $Y_i = \{f_i(x_i) \mid x_i \in X_i\}$ where $f_i(x_i) = \cup_{x_{-i}\in X_{-i}} \{m(x_i, x_{-i})\}$, and for each $m(x) \in M$:

$$c_{m(x)}(r) = \begin{cases} -u(x) & \text{if } r = n \\ \\ 0 & \text{otherwise} \end{cases}$$

For each $x \in X$: $\cap_{i\in N} f_i(x_i) = \{m(x)\}$, so the game corresponding to this congestion model is isomorphic to G (where the isomorphisms map x_i to $f_i(x_i)$). □

The proof is illustrated with a simple example.

Example 2.14 Consider the coordination game in Figure 2.2a. For each strategy profile we introduce a machine as in Figure 2.2b. These are the machines that we want to be used by both players if they play the corresponding strategy profile. To do this, give

0,0	1,1
2,2	3,3

a

A	B
C	D

b

	$\{A, C\}$	$\{B, D\}$
$\{A, B\}$	0,0	1,1
$\{C, D\}$	2,2	3,3

c

Figure 2.2: A coordination game

each player in a certain row (column) all machines mentioned in this row (column). For instance, the second strategy of the row player will correspond with choosing machine set $\{C, D\}$. Now indeed, if both players play their second strategy, machine D is used by both players and all other machines have one or zero users. Defining the costs of D in case of two simultaneous users to be -3 and in case of less users zero, we obtain the payoff $(3, 3)$ in the lower righthand corner of Figure 2.2c. Similar reasoning applies to the other cells. ◁

Consider a congestion game in which costs for a facility are non-zero only if it is used by a single player. If for each player, given the strategy choices of the other players, it holds that his costs arise from using one and the same facility, irrespective of his own strategy choice, we have a dummy game. Also, as shown in the next theorem, each dummy game is isomorphic to a congestion game with this property.

Theorem 2.15 *Each finite dummy game is isomorphic to a congestion game.*

Proof. Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a finite n-player dummy game. Introduce for each $i \in N$ and each $x_{-i} \in X_{-i}$ a different machine $m(x_{-i})$. Define the congestion model $\langle N, M, (Y_i)_{i\in N}, (c_j)_{j\in M}\rangle$ with $M = \cup_{i\in N} \cup_{x_{-i}\in X_{-i}} \{m(x_{-i})\}$, for each $i \in N$: $Y_i = \{g_i(x_i) \mid x_i \in X_i\}$ where

$$\begin{aligned} g_i(x_i) &= \{m(x_{-i}) \mid x_{-i} \in X_{-i}\} \\ &\cup \{m(y_{-j}) \mid j \in N \setminus \{i\} \text{ and } y_{-j} \in X_{-j} \text{ is such that } y_i \neq x_i\}, \end{aligned}$$

and for each $m(x_{-i}) \in M$:

$$c_{m(x_{-i})}(r) = \begin{cases} -u_i(x_i, x_{-i}) & \text{if } r = 1 \text{ (with } x_i \in X_i \text{ arbitrary)} \\ 0 & \text{otherwise} \end{cases}$$

For each $i \in N$, $\overline{x}_{-i} \in X_{-i}$, and $\overline{x}_i \in X_i$: i is the unique user of $m(\overline{x}_{-i})$ in $(g_j(\overline{x}_j))_{j\in N}$ and all other machines in $g_i(\overline{x}_i)$ have more than one user. Why? Let $i \in N$, $\overline{x}_{-i} \in X_{-i}$, and $\overline{x}_i \in X_i$. Then $m(\overline{x}_{-i}) \in g_i(\overline{x}_i)$ and for each $j \in N \setminus \{i\}$: $m(\overline{x}_{-i}) \notin g_j(\overline{x}_j)$, so i is indeed the unique user of $m(\overline{x}_{-i})$ in $(g_j(\overline{x}_j))_{j\in N}$. Let $m \in g_i(\overline{x}_i), m \neq m(\overline{x}_{-i})$.

- If $m = m(y_{-i})$ for some $y_{-i} \in X_{-i}$, then $y_{-i} \neq \overline{x}_{-i}$ implies that $y_j \neq \overline{x}_j$ for some $j \in N \setminus \{i\}$, so $m = m(y_{-i}) \in g_j(\overline{x}_j)$.

- If $m = m(y_{-j})$ for some $j \in N \setminus \{i\}$ and $y_{-j} \in X_{-j}$ with $y_i \neq \overline{x}_i$, then $m = m(y_{-j}) \in g_j(\overline{x}_j)$.

In both cases m has more than one user. So the game corresponding to this congestion model is isomorphic to G (where the isomorphisms map x_i to $g_i(x_i)$). □

Once again, this argumentation is illustrated by an example.

0,2	1,2
0,3	1,3

a

α, γ	β, γ
α, δ	β, δ

b

	$\{\beta, \gamma, \delta\}$	$\{\alpha, \gamma, \delta\}$
$\{\alpha, \beta, \delta\}$	0,2	1,2
$\{\alpha, \beta, \gamma\}$	0,3	1,3

c

Figure 2.3: A dummy game

Example 2.16 Consider the dummy game in Figure 2.3a. Introduce a different machine for each profile of opponent strategies as in Figure 2.3b. Include a machine $m(x_{-i})$ in each player's strategy, except for those strategies of players $j \neq i$ playing according to the profile x_{-i} for which this machine was introduced. For instance, facility α was introduced for the first column of player 2; then α is part of every strategy, except for the first column of player 2. This yields the strategies as in Figure 2.3c. Define costs for multiple users equal to zero. No matter what player 1 does, if his opponent chooses his second strategy, the costs to player 1 can be attributed to machine β. Assign costs -1 to a single user of this facility. Similar reasoning for the other payoffs yields the isomorphic congestion game in Figure 2.3c. ◁

In the previous two theorems it was shown that coordination and dummy games are isomorphic to congestion games. Using the decomposition of Theorem 2.6 we obtain that every exact potential game is isomorphic to a congestion game.

Theorem 2.17 *Every finite exact potential game is isomorphic to a congestion game.*

Proof. Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite exact potential game. Split it into a coordination game and a dummy game as in Theorem 2.6 and take their isomorphic congestion games as in Theorems 2.13 and 2.15. W.l.o.g., take their machine sets disjoint. Construct a congestion game isomorphic to G by taking the union of the two machine sets, cost functions as in Theorems 2.13 and 2.15, and strategy sets $Y_i = \{f_i(x_i) \cup g_i(x_i) \mid x_i \in X_i\}$. □

Example 2.18 The exact potential game in Example 2.10 is the sum of the coordination game from Example 2.14 and the dummy game from Example 2.16. Combining the two isomorphic congestion games from these examples yields a congestion game isomorphic to the exact potential game. See Figure 2.4. ◁

0,2	2,3
2,5	4,6

a

	$\{A, C\} \cup \{\beta, \gamma, \delta\}$	$\{B, D\} \cup \{\alpha, \gamma, \delta\}$
$\{A, B\} \cup \{\alpha, \beta, \delta\}$	0+0,0+2	1+1,1+2
$\{C, D\} \cup \{\alpha, \beta, \gamma\}$	2+0,2+3	3+1,3+3

b

Figure 2.4: Exact potential game and isomorphic congestion game

Chapter 3

Strong Nash Equilibria and the Potential Maximizer

3.1 Introduction

In the previous chapter it was shown that the existence of pure Nash equilibria in Rosenthal's congestion games could be established through the construction of a potential function. Milchtaich (1996) and Quint and Shubik (1994) considered different classes of congestion games which in general do not admit a potential function, but were still able to prove the existence of pure Nash equilibria. Konishi, Le Breton, and Weber (1997), considering the same model as Milchtaich, have even shown the existence of a strong Nash equilibrium.

Combining features from the congestion models mentioned above, this chapter, which is based on Borm *et al.* (1997), introduces a class of congestion games with several interesting properties. In particular, it will be shown that for each game in this class the set of strong Nash equilibria is nonempty and coincides with the set of Nash equilibria and the set of potential-maximizing strategies. Similar results can be found in Holzman and Law-Yone (1997).

The situation considered in this chapter can be used to model, for example, the foraging behavior of a population of identical bees in a field of flowers. In deciding which flower to visit, each insect will take into account the quantity of nectar available and the number of bees already on the flower, because, as is intuitively clear, the more crowded the source of nectar, the less food is available per capita. In economics this kind of problems is studied in the literature on local public goods, where it is common to speak about "anonymous crowding" (cf. Wooders, 1989) to describe the negative externality arising from the presence of more than one user of the same facility. Another example is the problem faced by a set of unemployed workers who have to decide where to emigrate to get a job. The attraction of different countries depends on the conditions of the local labor market and, on the other hand, a crowding out effect reduces the appeal of emigrating.

This chapter is structured as follows. In Section 3.2 we investigate the various models mentioned above, clarifying the similarities and differences among them. After that we define a class of games which possess a strong Nash equilibrium and at the same time admit an exact potential function. In Section 3.3 we analyze the geometric properties of this class of games, showing that it can be represented by a finitely generated cone. In Section 3.4 we state our main theorems concerning the coincidence of equilibrium sets, where the representation of each game as an element of a cone is used. Attention is focused on the computation of the potential. The section is concluded with comments on strictly strong equilibria. Implications of relaxing some of the assumptions underlying the congestion effect are discussed in Section 3.5.

3.2 Congestion games

The games introduced by Milchtaich (1996), Konishi, Le Breton, and Weber (1997) and Quint and Shubik (1994) are rather similar, in the sense that the utility functions of the players are characterized by a "congestion effect". The various classes of games we are going to discuss are identified by means of different sets of properties concerning the structure of the strategic interaction. In particular, Konishi *et al.* (1997) impose the following assumptions (P1)–(P4) on a game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$.

(P1) There exists a finite set F such that $X_i = F$ for all players $i \in N$.

The set F is called the "facility set" and a strategy for player i is choosing an element of F.

(P2) For each strategy profile $x \in X$ and all players $i, j \in N$: if $x_i \neq x_j$ and $x'_j \in X_j$ is such that $x_i \neq x'_j$, then $u_i(x_j, x_{-j}) = u_i(x'_j, x_{-j})$.

Konishi *et al.* (1997) call this assumption *independence of irrelevant choices* and the meaning is that for each player $i \in N$ and each strategy profile x the utility of i will not be altered if the set of players that choose the same facility as player i is not modified.

Let $x \in X, f \in F$. Denote by $n_f(x)$ the number of users of facility f in the strategy profile x. Then the third property can be stated as follows:

(P3) For each player $i \in N$ and all strategy profiles $x, y \in X$ with $x_i = y_i$: if $n_f(x) = n_f(y)$ for all $f \in F$, then $u_i(x) = u_i(y)$.

This *anonymity* condition reflects the idea that the payoff of player i depends on the number of players choosing the facilities, rather than on their identity. The fourth assumption, called *partial rivalry*, states that each player i would not regret that other players, choosing the same facility, would select another one. Formally:

(P4) For each player $i \in N$, each strategy profile $x \in X$, each player $j \neq i$ such that $x_j = x_i$ and each $x'_j \neq x_i$: $u_i(x_j, x_{-j}) \leqq u_i(x'_j, x_{-j})$.

Although Milchtaich (1996) introduces his model in a slightly different way, the resulting class of games is the same. More specifically Milchtaich (1996) introduces the conditions (P1), (P4), and the following assumption:

(P2') For each player $i \in N$ and all strategy profiles x, y with $x_i = y_i = f$: if $n_f(x) = n_f(y)$, then $u_i(x) = u_i(y)$.

In other words the utility of player i depends only on the number of users of the facility that i has chosen. Assuming (P1), it is straightforward to prove that (P2') implies both (P2) and (P3). The converse implication is also true.

Lemma 3.1 *Any game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ satisfying (P1), (P2), and (P3) satisfies (P2').*

Proof. Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ satisfy (P1), (P2), and (P3). Let $i \in N$, $x, y \in X$ such that $x_i = y_i = f$ and assume that $n_f(x) = n_f(y)$. If $|F| = 1$, (P2') follows directly. Otherwise, from repeated use of (P2), we know that for a fixed $g \neq x_i$, $u_i(x_i, x_{-i}) = u_i(x_i, x'_{-i})$ where for each $j \in N \setminus \{i\}$:

$$x'_j = \begin{cases} x_i & \text{if } x_j = x_i, \\ \\ g & \text{otherwise,} \end{cases}$$

and that $u_i(x_i, y_{-i}) = u_i(x_i, y'_{-i})$, where for each $j \in N \setminus \{i\}$:

$$y'_j = \begin{cases} x_i & \text{if } y_j = x_i, \\ \\ g & \text{otherwise.} \end{cases}$$

Notice that for each $h \in F$, $n_h(x_i, x'_{-i}) = n_h(x_i, y'_{-i})$. So (P3) implies $u_i(x_i, x'_{-i}) = u_i(x_i, y'_{-i})$. Therefore, $u_i(x_i, x_{-i}) = u_i(x_i, x'_{-i}) = u_i(x_i, y'_{-i}) = u_i(y_i, y_{-i})$. □

Konishi *et al.* (1997) and Milchtaich (1996) independently proved the following

Theorem 3.2 *Each game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ satisfying (P1), (P2), (P3) and (P4), possesses a pure-strategy Nash equilibrium.*

Recall that, given a game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a strategy profile x is called a strong Nash equilibrium if for every $S \subseteq N$ and all strategy profiles $y_S \in \Pi_{i \in S} X_i$, there is at least one player $i \in S$ such that $u_i(y_S, x_{-S}) \leqq u_i(x)$. The set of strong Nash equilibria of a game G is denoted by $SNE(G)$. In general, the existence of a strong Nash equilibrium is not guaranteed, but Konishi *et al.* (1997) show

Theorem 3.3 *For each game satisfying (P1), (P2) (P3) and (P4), the set of strong Nash equilibria is nonempty.*

Finally, we mention the model introduced by Quint and Shubik (1994), where the assumption that all players have the same set of facilities (as stated by (P1)) is relaxed.

(P1') There exists a finite set F such that $X_i \subseteq F$ for all players $i \in N$.

Assuming that (P1') holds, it is still easy to see that (P2') implies (P2) and (P3). But the analogon of Lemma 3.1 does not hold.

Example 3.4 Take $N = \{1, 2, 3\}$, $F = \{a, b, c\}$ and strategy sets $X_1 = \{a, b\}$, $X_2 = \{a\}$, $X_3 = \{a, c\}$. This game satisfies (P1'). Assumption (P3) imposes no additional requirements and (P2) requires that $u_1(b, a, a) = u_1(b, a, c)$ and $u_3(a, a, c) = u_3(b, a, c)$. This does not imply $u_2(a, a, c) = u_2(b, a, a)$, which is required by (P2'). $\triangleleft$

Quint and Shubik (1994) are able to show

Theorem 3.5 *All strategic games satisfying (P1'), (P2') and (P4) possess a pure Nash equilibrium.*

Games in the classes considered so far not necessarily admit a potential function. Consider now the following *cross-symmetry* condition, which states that the payoffs on a certain facility are player-independent, provided that the number of users is the same.

(P5) For all strategy profiles $x, y \in X$ and all players $i, j \in N$: if $x_i = y_j = f$ and $n_f(x) = n_f(y)$, then $u_i(x) = u_j(y)$.

Notice that (P5) together with (P1) implies (P2'), and thus (P2) and (P3). Moreover, (P1) and (P5) guarantee the existence of a potential.

Theorem 3.6 *Each game satisfying (P1) and (P5) is an exact potential game.*

Proof. Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ satisfy (P1) and (P5). Let $i, j \in N, i \neq j, x_{-\{i,j\}} \in X_{-\{i,j\}}, x_i, y_i \in X_i, x_i \neq y_i, x_j, y_j \in X_j, x_j \neq y_j$. For notational convenience, define for $k \in \{i, j\}$ the function $v_k : X_i \times X_j \to \mathbb{R}$ with $v_k(a_i, b_j) = u_k(a_i, b_j, x_{-\{i,j\}})$ for all $(a_i, b_j) \in X_i \times X_j$. According to the cycle characterization of exact potential games in Theorem 2.11, it suffices to show that

$$\begin{array}{lcll} [v_i(y_i, x_j) - v_i(x_i, x_j)] & + & [v_j(y_i, y_j) - v_j(y_i, x_j)] & + \\ {[v_i(x_i, y_j) - v_i(y_i, y_j)]} & + & [v_j(x_i, x_j) - v_j(x_i, y_j)] & \end{array} \tag{3.1}$$

equals zero. We consider three cases:

- If there are two different machines in $\{x_i, y_i, x_j, y_j\}$, then without loss of generality $x_i = x_j = f$ and $y_i = y_j = g$. By axiom (P5):

$$\begin{array}{lllllll} v_i(x_i, x_j) & = & v_i(f, f) & = & v_j(f, f) & = & v_j(x_i, x_j), \\ v_i(y_i, y_j) & = & v_i(g, g) & = & v_j(g, g) & = & v_j(y_i, y_j), \\ v_i(y_i, x_j) & = & v_i(g, f) & = & v_j(f, g) & = & v_j(x_i, y_j), \\ v_i(x_i, y_j) & = & v_i(f, g) & = & v_j(g, f) & = & v_j(y_i, x_j). \end{array}$$

  Substituting this in (3.1) indeed yields 0.

- If there are three different machines in $\{x_i, y_i, x_j, y_j\}$, then without loss of generality $x_i = x_j = f$ and $y_i \neq y_j, y_i \neq f, y_j \neq f$. By axiom (P5):

$$\begin{array}{lllllll} v_i(x_i, x_j) & = & v_i(f, f) & = & v_j(f, f) & = & v_j(x_i, x_j), \\ v_i(x_i, y_j) & = & v_i(f, y_j) & = & v_j(y_i, f) & = & v_j(y_i, x_j), \\ v_i(y_i, x_j) & & & = & & & v_i(y_i, y_j), \\ v_j(y_i, y_j) & & & = & & & v_j(x_i, y_j). \end{array}$$

  Substituting this in (3.1) indeed yields 0.

- If all elements in $\{x_i, y_i, x_j, y_j\}$ are different, then axiom (P5) implies

$$\begin{array}{lll} v_i(x_i, x_j) & = & v_i(x_i, y_j), \\ v_i(y_i, x_j) & = & v_i(y_i, y_j), \\ v_j(x_i, x_j) & = & v_j(y_i, x_j), \\ v_j(x_i, y_j) & = & v_j(y_i, y_j). \end{array}$$

  Substituting this in (3.1) indeed yields 0. $\square$

The proof does not change if (P1') is substituted for (P1).

As can be seen in the Prisoner's Dilemma in Example 2.2, exact potential games do not in general possess a strong Nash equilibrium. This chapter's focus is on games that admit an exact potential and have strong Nash equilibria. Therefore, attention is restricted to the class $\mathcal{C}$ of congestion games satisfying not only (P1) and (P5), but also (P4). So

$$\mathcal{C} = \{G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \mid G \text{ satisfies (P1), (P4), and (P5)}\}.$$

3.3 On the structure of the class $\mathcal{C}$

In the previous section we have defined the class $\mathcal{C}$. Now we will analyze its structure. For $n \in \mathbb{N}$, let $\mathcal{C}(n)$ denote the class of games $G \in \mathcal{C}$ with n players. It will be shown that each game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \mathcal{C}(n)$ can be identified with a finite set of vectors in $\mathbb{R}^n_+$, and that the subclass $\mathcal{C}(F,n)$, consisting of all games in $\mathcal{C}(n)$ with fixed facility set F, is a finitely generated cone in $(\mathbb{R}^n_+)^F$. The vector notation of the games simplifies the proofs of the theorems on strong equilibria and the potential maximizer presented in Sections 3.4 and 3.5.

Fix a number $n \in \mathbb{N}$, a finite facility set F and let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \mathcal{C}(F,n)$. For any $f \in F$ and $x, y \in X$ such that $n_f(x) = n_f(y)$, we have by (P5):

if there are $i, j \in N$ such that $x_i = y_j = f$, then $u_i(x) = u_j(y)$.

This shows that for all $f \in F$ there exists a function $w_f : \{1,\dots,n\} \to \mathbb{R}$ such that for all $x \in X$, if $x_i = f$, then $u_i(x) = w_f(n_f(x))$. This function is to be interpreted as the utility assigned to each player using this facility, given a certain number of users of this same facility. From (P4), we have for each $f \in F$ and $t \in \{1,\dots,n-1\}$ that $w_f(t) \geqq w_f(t+1)$. For convenience and without loss of generality we assume that $w_f(t) \geqq 0$ for all $f \in F, t \in \{1,\dots,n\}$. This means that the game $G \in \mathcal{C}(F,n)$ is described by $|F|$ vectors of the form $(w_f(1),\dots,w_f(n)), f \in F$, each in the set $V = \{v = (v_1,\dots,v_n) \in \mathbb{R}^n_+ \mid v_t \geqq v_{t+1}$ for all $t \in \{1,\dots,n-1\}\}$.

Proposition 3.7 *The set V is a finitely generated cone in $\mathbb{R}^n_+$. The extreme directions of V are the vectors $\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^n$ with $\mathbf{b}^i = (\underbrace{1,1,1,1}_{i \text{ times}}, 0, ..., 0)$. Furthermore, dim$(V) = n$.*

Proof. The vectors $\mathbf{b}^1 = (1,0,0,...,0)$, $\mathbf{b}^i = (\underbrace{1,1,1,1}_{i \text{ times}}, 0,...,0)$,..., $\mathbf{b}^n = (1,1,1,1,...,1)$ are elements of V and each vector $\mathbf{v} \in V$ can be uniquely written as a nonnegative combination of $\mathbf{b}^1, \mathbf{b}^2, ..., \mathbf{b}^n$. To show this, let $\mathbf{v} \in V$ and define

$$\mathbf{B}_n = \begin{bmatrix} \mathbf{b}^1 \\ \mathbf{b}^2 \\ \vdots \\ \mathbf{b}^n \end{bmatrix}$$

So $\mathbf{B}_n$ is the $n \times n$ matrix whose i-th row is $\mathbf{b}^i$. Since $\det(\mathbf{B}_n) = 1$, the equation $\alpha\mathbf{B}_n = \mathbf{v}$ has exactly one solution. Clearly, α is nonnegative because of the decreasingness property of $\mathbf{v}$. The set V is therefore the cone $C(\mathbf{B}_n)$ where $C(\mathbf{B}_n) := \{\alpha\mathbf{B}_n \mid \alpha \in \mathbb{R}^n_+\}$.

The extreme directions of the cone $C(\mathbf{B}_n)$ are the vectors $\mathbf{b}^i$, $i \in \{1,...,n\}$. This cone has furthermore the property that its dimension is the number of extreme directions. In other words we have that dim $C(\mathbf{B}_n) = rank(\mathbf{B}_n) = n$. □

Essentially we proved

Corollary 3.8 *The class of games $\mathcal{C}(F,n)$ can be identified with a cone in $(\mathbb{R}^n_+)^F$ and* $\dim(\mathcal{C}(F,n)) = |F| \times n$.

In the next example we consider an extreme game of $\mathcal{C}(F, n)$, i.e., a game with facility set F such that w_f is an extreme direction in the cone V for each $f \in F$.

Example 3.9 Let G be a game in $\mathcal{C}(\{f, g\}, 4)$ such that $w_f = (1, 0, 0, 0)$ and $w_g = (1, 1, 0, 0)$. Nash equilibria are either those strategy profiles in which one of the players chooses f and the other three g, or those in which both facilities are chosen by two players. These situations will be depicted

$$\begin{array}{c}(\boxed{1}, 0, 0, 0)\\(1, 1, \boxed{0}, 0)\end{array}$$

for the first case and

$$\begin{array}{c}(1, \boxed{0}, 0, 0)\\(1, \boxed{1}, 0, 0)\end{array}$$

for the second one, where the numbers in the square boxes indicate the payoff received by each player choosing this facility. Notice furthermore that the players are interchangeable as suggested by the cross-symmetry condition (P5). One easily checks that all Nash equilibria are strong. $\triangleleft$

3.4 Strong Nash equilibria and the potential maximizer

In this section it is shown that on the class $\mathcal{C}$, the set of Nash equilibria, strong Nash equilibria, and potential maximizers coincide:

Theorem 3.10 *On the class $\mathcal{C}$ of games, $SNE = NE = PM$.*

A proof of this result is given in parts. Recall that for any strategic game G, $SNE(G) \subseteq NE(G)$ and that for any exact potential game G, $PM(G) \subseteq NE(G)$. It therefore suffices to prove the following propositions.

Proposition 3.11 *For each game $G \in \mathcal{C}$, $NE(G) \subseteq SNE(G)$.*

Proposition 3.12 *For each game $G \in \mathcal{C}$, $NE(G) \subseteq PM(G)$.*

The proofs are based on the structure of the class of games described in the previous section. We assume $n \in \mathbb{N}$ and a finite facility set F to be fixed. Each game $G \in \mathcal{C}(F, n)$ is given by a collection of vectors

$$((w_f(1), ..., w_f(n)))_{f \in F}.$$

Proof. **[Proposition 3.11]**. Let $G \in \mathcal{C}\ (F, n)$ be given by $((w_f(1), \ldots, w_f(n)))_{f \in F}$ and let $x \in NE(G)$. Suppose $S \subseteq N$ can strictly improve the payoff for all its members by switching to a strategy combination $y_S \in F^S$. Call the resulting strategy combination $y = (y_S, x_{N \setminus S})$. If $n_f(y) > n_f(x)$ for some $f \in F$, a player $i \in S$ exists such that $y_i = f$ and $x_i = g, g \neq f$. This implies $w_f(n_f(x)+1) \geqq w_f(n_f(y)) > w_g(n_g(x))$, which contradicts the fact that x is a Nash equilibrium. So $n_f(x) = n_f(y)$ for all $f \in F$. Therefore every player in S chooses a new facility already chosen by a member of S and obtains a higher utility. Among the utilities assigned to members of S there is a maximum, since S is finite. Any player in S rewarded with this maximum cannot get more in the new configuration. Hence a contradiction arises. Every Nash equilibrium is strong. □

Based on a switching argument the next lemma shows the similarities in utilities for different Nash equilibria.

Lemma 3.13 *Let $G \in \mathcal{C}(F, n)$ be determined by $((w_f(1), \ldots, w_f(n)))_{f \in F}$ and let x and y be Nash equilibria of G. For all $f, g \in F$ such that $n_f(x) < n_f(y)$ and $n_g(y) < n_g(x)$, and for all $l \in \{n_f(x)+1, \ldots, n_f(y)\}$ and $m \in \{n_g(y)+1, \ldots, n_g(x)\}$ it holds that*

$$w_f(l) = w_f(n_f(y)) = w_g(n_g(x)) = w_g(m).$$

Proof. Let $f, g \in F$ and l, m be as described in the lemma. Both x and y are Nash equilibria, so $w_f(n_f(y)) \geqq w_g(n_g(y)+1) \geqq w_g(m) \geqq w_g(n_g(x)) \geqq w_f(n_f(x)+1) \geqq w_f(l) \geqq w_f(n_f(y))$. □

Our next proposition specifies a potential function for a game in $\mathcal{C}(F, n)$.

Proposition 3.14 *Let the game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \mathcal{C}(F, n)$ be determined by $((w_f(1), \ldots, w_f(n)))_{f \in F}$. Define the function $P : X \to \mathbb{R}$ for all $x \in X$ as*

$$P(x) = \sum_{f \in \cup_{i \in N} x_i} \sum_{l=1}^{n_f(x)} w_f(l).$$

Then P is an exact potential of G.

Proof. Let $i \in N, f, g \in X_i \subseteq F, f \neq g, x_{-i} \in X_{-i}$. For notational convenience, write $x = (f, x_{-i})$ and $y = (g, x_{-i})$. Notice that $n_f(x) = n_f(y) + 1$, $n_g(x) = n_g(y) - 1$, and

$n_h(x) = n_h(y)$ for $h \in F \setminus \{f, g\}$. By definition:

$$\begin{aligned} P(x) - P(y) &= \sum_{h \in \cup_{i\in N} x_i, h \notin \{f,g\}} \sum_{l=1}^{n_h(x)} w_h(l) + \sum_{l=1}^{n_f(x)} w_f(l) + \sum_{l=1}^{n_g(x)} w_g(l) \\ &- \sum_{h \in \cup_{i\in N} x_i, h \notin \{f,g\}} \sum_{l=1}^{n_h(y)} w_h(l) - \sum_{l=1}^{n_f(y)} w_f(l) - \sum_{l=1}^{n_g(y)} w_g(l) \\ &= w_f(n_f(x)) - w_g(n_g(y)) \\ &= u_i(x) - u_i(y). \end{aligned}$$

□

Remark 3.15 Let a game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ satisfy (P1') and (P5). Assumption (P5) again implies that for each $f \in F$ there exists a function $w_f : \{1, \ldots, n\} \to \mathbb{R}$ such that for all $x \in X$, if $x_i = f$, then $u_i(x) = w_f(n_f(x))$. An exact potential for G can then be constructed as in Proposition 3.14. ◁

To compute the potential of Proposition 3.14 it is necessary to add the utilities of the used facilities up to the number of users. This means that in each vector w_f all the first $n_f(x)$ numbers are added.

As a consequence it is clear that by n times consecutively choosing the facilities with highest remaining numbers, from left on, in the set of vectors $\{(w_f(1), ..., w_f(n))\}_{f\in F}$ a potential maximizing profile is found. This is illustrated in the following example.

Example 3.16 Let $G \in \mathcal{C}(\{f, g\}, 4)$ such that

$$\begin{aligned} w_f &= (4, 3, 2, 1) \\ w_g &= (5, 2, 1, 0) \end{aligned}$$

In the first step we take the first cell in w_g, in the second step the first cell in w_f, in the third step the second cell of w_f and, finally, in the fourth step either the third cell of w_f or the second cell of w_g. Consequently, the potential maximizing strategy combinations are those $x \in F^N$ with $n_f(x) = 3$, $n_g(x) = 1$ and those with $n_f(x) = 2$, $n_g(x) = 2$. Notice that for these x, $P(x) = 14$ and that all Nash equilibria are potential maximizing. ◁

Proof. [**Proposition 3.12**] Let $G \in \mathcal{C}(F, n)$ be determined by $((w_f(1), \ldots, w_f(n)))_{f\in F}$. It suffices to show that $P(x) = P(y)$ if x is a Nash equilibrium and y a potential maximizing strategy combination. Let $x \in NE(G)$ and $y \in PM(G)$. Facilities $f \in F$ such that $n_f(x) = n_f(y)$ add as much to $P(x)$ as to $P(y)$. Furthermore, by Lemma 3.13, if $n_f(x) < n_f(y)$ and $n_g(y) < n_g(x)$ for certain $f, g \in F$ then $w_f(l) = w_f(n_f(y)) =$

$w_g(n_g(x)) = w_g(m)$ for all $l \in \{n_f(x)+1, \ldots, n_f(y)\}$ and $m \in \{n_g(y)+1, \ldots, n_g(x)\}$. The total contribution of the facilities in the set $\{f \in F \mid n_f(x) \neq n_f(y)\}$ to the potentials $P(x)$ and $P(y)$ is apparently the same. $\square$

In the last part of this section we consider strictly strong Nash equilibria. Recall that given a game $\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, a strategy profile $x \in X$ is a *strictly strong Nash equilibrium* if for all coalitions $S \subseteq N$ and strategy combinations $y_S \in \prod_{i\in S} X_i$, $u_i(y_S, x_{N\setminus S}) = u_i(x)$ for all $i \in S$ or $u_i(y_S, x_{N\setminus S}) < u_i(x)$ for at least one $i \in S$. The following example illustrates that the properties of $\mathcal{C}$ do not guarantee the existence of strictly strong Nash equilibria.

Example 3.17 Consider the game $G \in \mathcal{C}(\{f, g\}, 3)$ with w_f, w_g given by

$$\begin{aligned} w_f &= (4, \boxed{2}, 0), \\ w_g &= (\boxed{3}, 2, 1), \end{aligned}$$

where the squared numbers depict a strong Nash equilibrium payoff. If the two players choosing f agree that one of them switches to g and the other one sticks to f, the utility will still be 2 for the switching one but increases from 2 to 4 for the remaining player. A similar argument holds for the other type of strong Nash equilibria given by

$$\begin{aligned} w_f &= (\boxed{4}, 2, 0) \\ w_g &= (3, \boxed{2}, 1) \end{aligned}$$

Since these are the only two types of strong Nash equilibria, and neither of them is strictly strong, strictly strong Nash equilibria do not exist. $\triangleleft$

3.5 Extensions of the model

The class $\mathcal{C}$ is characterized by properties (P1), (P4), and (P5). It is obvious that relaxation of those properties will have consequences on the result presented in Section 3.4.

First of all, the classes of congestion games of Quint and Shubik (1994), Milchtaich (1996), and Konishi *et al.* (1997) without (P5) not necessarily admit an exact potential.

Secondly, consider the class $\mathcal{CP}$ of strategic games which satisfy the properties (P1) and (P5). Each n person game G in $\mathcal{CP}$ is a potential game and can be represented by a collection of arbitrary vectors $((w_f(1), \ldots, w_f(n)))_{f\in F} \in (\mathbb{R}^n)^F$. It is obvious that not every game $G \in \mathcal{CP}$ has a strong Nash equilibrium. For instance, the Prisoner's Dilemma in Example 2.2 is an element of $\mathcal{CP}$ with $F = \{c, d\}$, $w_c = (4, 1)$ and $w_d = (0, 3)$, but does not have a strong Nash equilibrium. But even the existence of a strong Nash equilibrium for a game $G \in \mathcal{CP}$ does not guarantee that each Nash equilibrium is strong too, nor that a strong equilibrium is a potential maximizer. The next example gives a game $G \in \mathcal{CP}$ such that $\emptyset \neq SNE(G) \subset NE(G)$ and $SNE(G) \cap PM(G) = \emptyset$.

Example 3.18 Let $G \in \mathcal{CP}(\{f,g\},3)$ with

$$\begin{aligned} w_f &= (4, 0, \boxed{5}) \\ w_g &= (4, 2, 0) \end{aligned}$$

The unique strong Nash equilibrium in which all three players chooses facility f is indicated. By Remark 3.15, the potential can be computed as in Proposition 3.14. The maximal potential arises at the non strong equilibria which are given by

$$\begin{aligned} w_f &= (\boxed{4}, 0, 5) \\ w_g &= (4, \boxed{2}, 0) \end{aligned}$$

◁

Finally, consider the class of strategic games $\mathcal{C}'$ satisfying (P1'),(P4), and (P5). Similarly to Proposition 3.11 one can show

Theorem 3.19 *For every game $G \in \mathcal{C}'$, $NE(G) = SNE(G)$.*

In this class of games, however, the set of potential maximizing strategy combinations need not coincide with the set of Nash equilibria, as can be seen in the following example.

Example 3.20 Consider the game $G \in \mathcal{C}'(\{f,g,h\},5)$ in which three players have strategy set $\{f,h\}$ and two $\{g,h\}$. The payoff vectors are

$$\begin{aligned} w_f &= (4, 2, \boxed{1}, -, -) \\ w_g &= (\boxed{3}, 2, -, -, -) \\ w_h &= (\boxed{2}, 1, 1, 0, 0) \end{aligned}$$

where the squared numbers depict a Nash equilibrium payoff. It represents strategy combinations in which the three players with strategy set $\{f,h\}$ all play f. Consider now the equilibrium in which two of those three play f and the other plays h.

$$\begin{aligned} w_f &= (4, \boxed{2}, 1, -, -) \\ w_g &= (3, \boxed{2}, -, -, -) \\ w_h &= (\boxed{2}, 1, 1, 0, 0) \end{aligned}$$

The potential can be computed as in Proposition 3.14 (see Remark 3.15). For the first type of equilibrium in this example, the potential value equals $4 + 2 + 1 + 3 + 2 = 12$, which is less than $4 + 2 + 3 + 2 + 2 = 13$, the potential value associated to the second type of equilibrium. ◁

Chapter 4

Sequential Production Situations and Potentials

4.1 Introduction

In recent years there has been a growing effort in the study of specific, practically relevant classes of noncooperative games possessing pure-strategy Nash equilibria. Several instances of congestion situations with pure Nash equilibria were considered in previous chapters.

The purpose of the present chapter, based on Voorneveld, Tijs, and Mallozzi (1998), is to describe sequential production games, a type of production games that is motivated by production situations in practice. In a sequential production game, raw materials are transformed into a product. The value of the product depends on the activities performed on the raw materials and is divided equally over the production departments. The production consists of several stages. In each stage, production departments observe the production techniques chosen in the earlier stages and simultaneously perform some activities on the intermediate product (or on the raw materials, if we look at the first stage). The fact that within a stage departments simultaneously and independently choose a production technique introduces imperfect information into the game. Since the state of the intermediate product strongly depends on the production techniques or activities conducted during the preceding stages, the production departments incur set-up and production costs depending on the previous stages and — of course — on the production strategies of the departments simultaneously performing their activities.

The model is introduced by means of a practical example, based on the processing of rough diamonds. The use of diamond essentially falls into two categories. First of all, properly processed diamond as loose gemstones or part of jewelry has an ornamental function. Secondly, since diamond is the hardest naturally occurring substance, it has an important industrial application: it forms part of cutting and sawing tools, as well as drilling equipment, for instance in mining industry.

In this simplified example, production takes place in two stages and is conducted by three departments. During the first stage, department 1 decides whether a unit of diamond is used for ornamental or industrial purposes, strategies O and I, respectively.

In the second stage, two departments simultaneously perform an activity. In case the unit of diamond was designated for ornamental use, it has to be faceted (cutting flat facets over the entire surface of the stone, usually in a highly symmetrical pattern) and polished to a mirror-like finish to aid light reflection from the surface of the stone or refraction of light through the stone. This is done by department 2, which can use modern equipment to do this (action M), or do the job mostly by relatively old machinery (action O). During the faceting and polishing, department 3 takes care of cooling and lubricating. Department 3 can decide to use high or low quality products to do this, actions Hi and Lo, respectively.

In case the unit of diamond was designated for industrial use, the second department pulverizes the diamond to produce diamond grit for saw blades. Using the modern action (M) produces grit with a higher mesh (i.e., finer grid, more adequate for precision work) than the old machinery (O). During this process, department 3 takes care of removing debris, again by choosing either high or low quality measures.

The first department operates at negligible costs. In the second stage, departments 2 and 3 incur set-up costs depending on whether processing takes places for ornamental or industrial purposes. These set-up costs are given in Figure 4.1. Given the industrial

Purpose	Set-up dept.2	Set-up dept.3
I	1	1
O	2	3

Figure 4.1: Set-up costs

or ornamental purpose decided on in the first stage and the technique (either Hi or Lo) chosen by the third department, the operating costs of department 2 are given in Figure 4.2, with a similar specification of the production costs of department 3. Finally, Figure

Purpose, tech. of dept. 3	Prod. costs of dept. 2
(I, Hi)	1
(I, Lo)	1
(O, Hi)	2
(O, Lo)	3

Purpose, tech. of dept. 2	Prod. costs of dept. 3
(I, M)	2
(I, O)	1
(O, M)	1
(O, O)	3

Figure 4.2: Production costs

4.3 specifies the value of the end product as a function of the production techniques. Assuming that the value of the end product is divided equally over the three production

Production profile	Value
(I, M, Hi)	27
(I, M, Lo)	15
(I, O, Hi)	27
(I, O, Lo)	12
(O, M, Hi)	21
(O, M, Lo)	18
(O, O, Hi)	30
(O, O, Lo)	18

Figure 4.3: Value of end product

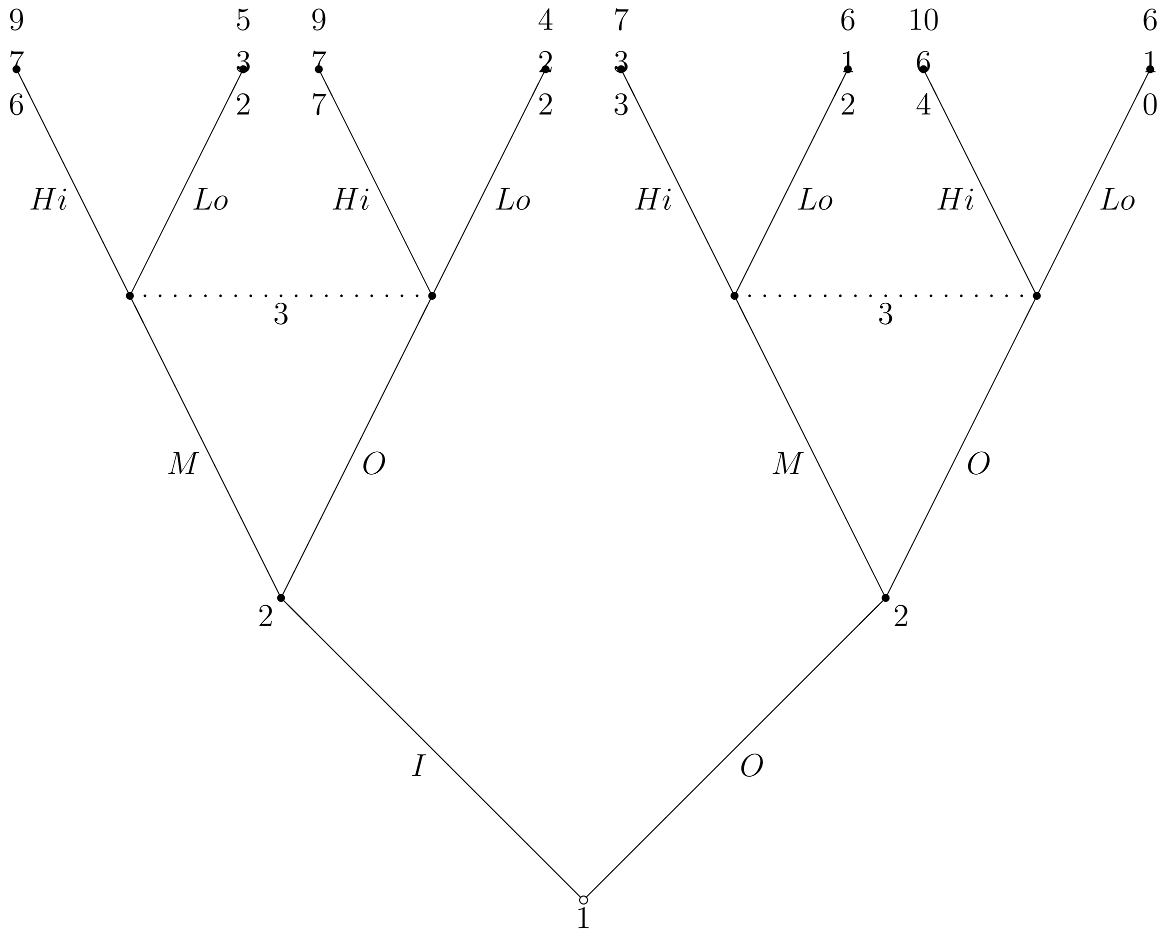


Figure 4.4: The diamond game

departments and subtracting the costs, one obtains the extensive form game in Figure 4.4, where w.l.o.g. department 2 moves before department 3, but department 3 does not observe the choice of department 2, thus capturing the fact that these departments operate in the same stage and hence only observe the action taken in stage 1: histories (I, M) and (I, O) are in one information set, just like histories (O, M) and (O, O).

For instance, if the production profile is (I, M, Hi), each department receives one third of 27. Department 1 incurs no costs, so the payoff to this department equals 9. Department 2 incurs set-up costs 1 and production costs 1, so the payoff to this department equals $9 - 1 - 1 = 7$. Similarly, department 3 has payoff $9 - 1 - 2 = 6$ since its set-up costs are 1 and its processing costs are 2.

Sequential production situations give rise to a special class of extensive form games with imperfect information, since players at a certain stage observe the production techniques chosen in previous stages, but not those of the departments in the same and later stages. Thus, in general, the existence of pure-strategy Nash equilibria is not guaranteed. In the diamond game of Figure 4.4, however, there are several. We show that this is no coincidence and that these games are closely related to exact potential games. A more formal description of the model is provided in Section 4.2. Section 4.3 contains the results. Section 4.4 concludes with remarks concerning extensions of the model.

4.2 Model

This section contains a formal description of the model. The games arising from sequential production situations are hierarchical games.

Definition 4.1 A *hierarchical game* is an extensive form game described by a tuple $H = \langle N = (N_1, \ldots, N_m), (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$. Adopting a slight abuse of notation, the finite player set N is an ordered partition $(N_1, \ldots, N_m)$. The number $m \in \mathbb{N}$ denotes the number of stages of the game. For $k \in \{1, \ldots, m\}$, N_k denotes the set of players operating at stage k. Each player $i \in N$ has a finite set A_i of actions containing at least two elements and a payoff function $u_i : \prod_{j \in N} A_j \to \mathbb{R}$. The game is played in such a way that for each stage $k \in \{1, \ldots, m\}$, the players in N_k observe only the action choices of the players in $N_1 \cup \cdots \cup N_{k-1}$ operating in the previous stages and then simultaneously and independently choose an action. $\triangleleft$

Notice that a hierarchical game is a specific type of extensive form game with imperfect information. The players in N_1, operating in the first stage, make no observations prior to simultaneously and independently choosing their action. The players in N_2, operating in the second stage, observe the actions of the players in N_1 and then simultaneously and independently choose their actions, thus having no information about the action choices of the other players in the same stage and the players in later stages. The same reasoning applies to later stages of the game. Strategic games are a special case, since they can be modelled by a hierarchical game with only one stage: all players simultaneously and independently make a strategy choice.

The players are assumed to be numbered from 1 through $|N|$; players with a low number play in early stages, i.e., if $i \in N_k, j \in N_l$, and $i < j$, then $k \leqq l$.

The following notation is used. The *predecessors* $Pr(i)$ of a player $i \in N$ are those players operating at an earlier stage than i:

$$\forall k \in \{1, \ldots, m\}, \forall i \in N_k : Pr(i) := \cup_{l \in \{1,\ldots,m\}, l<k} N_l.$$

The *colleagues* $C(i)$ of a player $i \in N$ are those players operating at the same stage as player i:

$$\forall k \in \{1, \ldots, m\}, \forall i \in N_k : C(i) := N_k \setminus \{i\}.$$

The *followers* $F(i)$ of a player $i \in N$ are those players operating at a later stage than player i:

$$\forall k \in \{1, \ldots, m\}, \forall i \in N_k : F(i) := \cup_{l \in \{1,\ldots,m\}, l>k} N_l.$$

For instance, in the diamond game of Figure 4.4, the player set $N = \{1, 2, 3\}$ is described by the ordered partition (N_1, N_2) with $N_1 = \{1\}$ and $N_2 = \{2, 3\}$. Department 1 has no predecessors, no colleagues, and followers 2 and 3. Department 2 has predecessor 1, colleague 3, and no followers.

Definition 4.2 A *sequential production situation* is a tuple

$$\langle N = (N_1, \ldots, N_m), (A_i)_{i \in N}, \rho, (c_i)_{i \in N} \rangle,$$

where the set N of production departments or players is described by an ordered partition $(N_1, \ldots, N_m)$. The number $m \in \mathbb{N}$ denotes the number of production stages. Each player $i \in N$ has a finite set A_i of production techniques (containing at least two elements). The function $\rho : \prod_{i \in N} A_i \to \mathbb{R}$ specifies for each production profile $a = (a_i)_{i \in N} \in \prod_{i \in N} A_i$ the value $\rho(a)$ of the end product. Each player $i \in N$ has a cost function c_i denoting the set-up and operating costs of this player. This cost function depends on the predecessors (set-up) and colleagues (operating) (if any), i.e.,

$$\forall i \in N \quad : \quad c_i : \textstyle\prod_{j \in Pr(i) \cup C(i)} A_j \to \mathbb{R}.$$

Production takes place in such a way that for each stage $k \in \{1, \ldots, m\}$, the players in N_k observe only the production techniques of the players in $N_1 \cup \cdots \cup N_{k-1}$ operating in the previous stages and then simultaneously and independently choose a production technique. $\triangleleft$

Remark 4.3 The definition of c_i for players $i \in N_1$ in the first stage deserves special attention. In this case, the set of predecessors $Pr(i)$ of i is empty by definition, so c_i is a function only of i's colleagues. If this set also happens to be empty, i.e., if there is only one department i in the first stage, we allow $c_i \in \mathbb{R}$ to be an arbitrary constant. $\triangleleft$

Remark 4.4 Two main assumptions underlie the production process captured by a sequential production situation. In this remark, some motivation for these assumptions is provided.

- The first assumption is that the departments within a production stage independently and simultaneously choose a production technique. Many modern firms are decentralized: departments act as autonomous units with their own decision power. In such environments this assumption seems reasonable.
- The second assumption is that the production costs of a production department do not depend on its own technique. This is equivalent with stating that a production department has fixed costs given the state of the intermediate product and the production techniques of the colleagues.

$\triangleleft$

Given a sequential production situation and assuming for now that the value of the end product is split equally over the departments or players (for a relaxation of this assumption, see Section 4), one can easily define its associated hierarchical game.

Definition 4.5 Let $\langle N = (N_1, \ldots, N_m), (A_i)_{i \in N}, \rho, (c_i)_{i \in N} \rangle$ be a sequential production situation. The associated *sequential production game* is the hierarchical game $\langle N = (N_1, \ldots, N_m), (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with for all $i \in N$ and all $a \in \prod_{i \in N} A_i$:

$$u_i(a) = \begin{cases} \frac{1}{|N|}\rho(a) - c_i & \text{if } i \in N_1 \text{ and } C(i) = \emptyset \\ \\ \frac{1}{|N|}\rho(a) - c_i((a_j)_{j \in Pr(i) \cup C(i)}) & \text{otherwise} \end{cases}$$

That is, the payoff to a production department is an equal share of the value of the end product minus the costs it incurs. $\triangleleft$

Let $H = \langle N = (N_1, \ldots, N_m), (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a hierarchical game. The *normalization* of H is defined — in the usual way — to be the strategic game $\mathcal{N}(H) = \langle N, (S_i)_{i \in N}, (U_i)_{i \in N} \rangle$, where the strategy space S_i of player $i \in N$ prescribes an action choice in every contingency that a player may be called upon to act and the payoff function associates to each strategy profile the payoff in the outcome of the hierarchical game induced by this strategy. Formally,

$$S_i = \begin{cases} A_i & \text{if } i \in N_1, \\ \\ \{\sigma_i \mid \sigma_i : \prod_{j \in Pr(i)} A_j \to A_i\} & \text{if } i \in N_k, k \geqq 2. \end{cases}$$

Inductively, one can define the realized play of the game by means of a function $r : \prod_{i \in N} S_i \to \prod_{i \in N} A_i$ as follows.

$$r_i(\sigma) = \begin{cases} \sigma_i \in A_i & \text{if } i \in N_1, \\ \\ \sigma_i((r_j(\sigma))_{j \in Pr(i)}) & \text{if } i \in N_k, k \geqq 2. \end{cases}$$

Player i's payoff function U_i assigns to every strategy profile $\sigma = (\sigma_i)_{i \in N} \in \prod_{i \in N} S_i$ the payoff associated with the outcome realized by σ: $U_i(\sigma) = u_i(r(\sigma))$.

For instance, in the strategic game corresponding to the diamond game of Figure 4.4, player 1 has two strategies: $S_1 = A_1 = \{I, O\}$. Player 2 has 4 strategies: $S_2 = \{(M, M), (M, O), (O, O), (O, M)\}$, where the first coordinate specifies the action choice if player 1 chose I and the second coordinate specifies the action choice if player 1 chose O. Similarly, the strategy space of player 3 equals $\{(Hi, Hi), (Hi, Lo), (Lo, Lo), (Lo, Hi)\}$. The strategic game is given in Figure 4.5.

	(Hi, Hi)	(Hi, Lo)	(Lo, Hi)	(Lo, Lo)
(M, M)	9,7,6	9,7,6	5,3,2	5,3,2
(M, O)	9,7,6	9,7,6	5,3,2	5,3,2
(O, M)	9,7,7	9,7,7	4,2,2	4,2,2
(O, O)	9,7,7	9,7,7	4,2,2	4,2,2

Department 1 plays I

	(Hi, Hi)	(Hi, Lo)	(Lo, Hi)	(Lo, Lo)
(M, M)	7,3,3	6,1,2	7,3,3	6,1,2
(M, O)	10,6,4	6,1,0	10,6,4	6,1,0
(O, M)	7,3,3	6,1,2	7,3,3	6,1,2
(O, O)	10,6,4	6,1,0	10,6,4	6,1,0

Department 1 plays O

Figure 4.5: The normalization of the diamond game

Some matters of notation. In the normalization of a hierarchical game, the strategy "always choose a_i" is denoted $\overline{a}_i$. Furthermore, conventional game theoretic notation is used. For instance, $S := \prod_{j \in N} S_j$ denotes the set of strategy profiles for all players in N, $S_{-i} := \prod_{j \in N \setminus \{i\}} S_j$ denotes the set of strategy profiles of i's opponents. Similar notation is adopted for elements of these sets: $\sigma \in S, \sigma_{-i} \in S_{-i}$, and for profiles of actions, rather than strategies.

4.3 Results

In this section the sequential production games are related to exact potential games introduced by Monderer and Shapley (1996). Hierarchical potential games are defined and, analogous to the isomorphism between congestion games à la Rosenthal (1973) and exact potential games (see Theorem 2.17), it is shown that not only every sequential production game is a hierarchical potential game, but conversely, every hierarchical potential game can be seen as a well-chosen sequential production game. This result has an important implication: sequential production games have pure-strategy equilibria. So-called potential-maximizing strategies, introduced in Monderer and Shapley (1996) and studied in more detail by Peleg, Potters, and Tijs (1996), form an interesting equilibrium refinement and are studied for this class of games.

First, recall the definition of exact potential games:

Definition 4.6 A strategic game $G = \langle N, (S_i)_{i \in N}, (U_i)_{i \in N} \rangle$ is an *exact potential game* if there exists a function $P : \prod_{i \in N} S_i \to \mathbb{R}$ such that for each player $i \in N$, each profile $\sigma_{-i} \in \prod_{j \in N \setminus \{i\}} S_j$ of strategies of the opponents, and each pair of strategies $\sigma_i, \tau_i \in S_i$ of player i:

$$U_i(\sigma_i, \sigma_{-i}) - U_i(\tau_i, \sigma_{-i}) = P(\sigma_i, \sigma_{-i}) - P(\tau_i, \sigma_{-i}),$$

i.e., if the change in the payoff to a unilaterally deviating player is equal to the change in the value of the function P. P is called an *(exact) potential* of the game. $\triangleleft$

It is easy to see that the set of Nash equilibria of the game G coincides with the set of Nash equilibria of the game $\langle N, (S_i)_{i \in N}, (P)_{i \in N} \rangle$ with all payoff functions replaced by the potential function P. Finite exact potential games consequently have pure-strategy Nash equilibria: the potential P achieves a maximum over the finite set $\prod_{i \in N} S_i$, which is easily seen to be a pure strategy Nash equilibrium (Proposition 2.4).

Theorem 2.6 showed that a game is an exact potential game if and only if there exists a real-valued function P on the strategy space such that for each player i, the difference between his payoff and the function P does not depend on the strategy choice of player i himself. That is, an exact potential game can be seen as the 'sum' of a coordination game, in which the payoff to all players is given by the function P, and a dummy game, in which the payoff to a player is independent of his own strategy choice. This result is used later, so we summarize it in a lemma.

Lemma 4.7 *A strategic game $G = \langle N, (S_i)_{i \in N}, (U_i)_{i \in N} \rangle$ is an exact potential game if and only if there exists a function $P : \prod_{i \in N} S_i \to \mathbb{R}$ and for each player $i \in N$ a function $D_i : \prod_{j \in N \setminus \{i\}} S_j \to \mathbb{R}$ such that*

$$\forall i \in N, \forall \sigma \in \prod_{j \in N} S_j : U_i(\sigma) - P(\sigma) = D_i(\sigma_{-i}).$$

The function P in Lemma 4.7 is easily seen to be an exact potential of the game.

If the normalization of a hierarchical game is a potential game, then the potential depends on the realized outcome, but not on the strategies leading to this outcome:

Lemma 4.8 *Let H be a hierarchical game. If its normalization $\mathcal{N}(H)$ is an exact potential game with potential function P, and σ, τ are strategy profiles such that $r(\sigma) = r(\tau)$, then $P(\sigma) = P(\tau)$.*

Proof. If $r(\sigma) = r(\tau) = (a_j)_{j \in N}$, then $\sigma_i = \tau_i$ for all $i \in N_1$ and for players $i \in N_k, k \geqq 2$, σ_i and τ_i differ only in their behavior off the play path $(a_j)_{j \in N}$. Thus, the payoff in

$\mathcal{N}(H)$ to deviating players along the path from $\sigma = (\sigma_1, \ldots, \sigma_n)$ to $(\tau_1, \sigma_2, \ldots, \sigma_n)$ to ... to $(\tau_1, \ldots, \tau_{n-1}, \sigma_n)$ to $\tau = (\tau_1, \ldots, \tau_n)$ does not change. Hence $P(\sigma) = P(\tau)$. □

Definition 4.9 A hierarchical game $H = \langle N = (N_1, \ldots, N_m), (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is called a *hierarchical potential game* if there exist functions $p : \prod_{j \in N} A_j \to \mathbb{R}$ and $(d_i)_{i \in N}$ with

$$\forall i \in N : \quad d_i : \prod_{j \in Pr(i) \cup C(i)} A_j \to \mathbb{R},$$

or $d_i \in \mathbb{R}$ if $i \in N_1$ and $C(i) = \emptyset$ (Analogous to Remark 4.3.), such that for each player $i \in N$ and each action profile $a \in \prod_{i \in N} A_i$:

$$u_i(a) = \begin{cases} p(a) + d_i & \text{if } i \in N_1 \text{ and } C(i) = \emptyset, \\ \\ p(a) + d_i((a_j)_{j \in Pr(i) \cup C(i)}) & \text{otherwise.} \end{cases}$$

The function p is called a *potential* for H. ◁

The reason for this definition is the following:

Theorem 4.10 *A hierarchical game $H = \langle N = (N_1, \ldots, N_m), (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a hierarchical potential game if and only if its normalization $\mathcal{N}(H) = \langle N, (S_i)_{i \in N}, (U_i)_{i \in N} \rangle$ is an exact potential game.*

Proof. If H is a hierarchical potential game with $p, (d_i)_{i \in N}$ as in Definition 4.9, then by definition of the normalized game one has that for each $\sigma \in \prod_{i \in N} S_i$:

$$U_i(\sigma) = u_i(r(\sigma)) = p(r(\sigma)) + d_i((r_j(\sigma))_{j \in Pr(i) \cup C(i)}).$$

Lemma 4.7 implies that $\mathcal{N}(H)$ is an exact potential game.

To prove the converse, assume $\mathcal{N}(H)$ is an exact potential game with potential P. We have to show the existence of functions p and $(d_i)_{i \in N}$ as in Definition 4.9. For each $a \in \prod_{i \in N} A_i$, recall that $\overline{a}_i \in S_i$ is the strategy in which player i always chooses a_i. Denote $\overline{a} = (\overline{a}_i)_{i \in N}$. Define $p(a) = P(\overline{a})$. The definition of the functions $(d_i)_{i \in N}$ is split up into two cases.

Case 1: $i \in N_m$. Lemma 4.7 implies the existence of a function $D_i : \prod_{j \in N \setminus \{i\}} S_j \to \mathbb{R}$ such that $U_i(\sigma) = P(\sigma) + D_i(\sigma_{-i})$ for each $\sigma \in \prod_{j \in N} S_j$. Define for each $a_{-i} \in A_{-i}$: $d_i(a_{-i}) = D_i(\overline{a}_{-i})$. Then, for each $a \in \prod_{j \in N} A_j$, $u_i(a) = U_i(\overline{a}) = P(\overline{a}) + D_i(\overline{a}_{-i}) = p(a) + d_i(a_{-i}) = p(a) + d_i((a_j)_{j \in Pr(i) \cup C(i)})$.

Case 2: $i \in N_k, k < m$. To prove the existence of d_i as in Definition 4.9, it suffices to show that $u_i - p$ does not depend on the actions chosen by player i himself and i's followers, since we can then take d_i equal to this difference. Formally, it is shown that for all $a \in A$ and $(b_j)_{j \in F(i) \cup \{i\}} \in \prod_{j \in F(i) \cup \{i\}} A_j$:

$$u_i(a) - p(a) = u_i((a_j)_{j \in Pr(i) \cup C(i)}, (b_j)_{j \in F(i) \cup \{i\}}) - p((a_j)_{j \in Pr(i) \cup C(i)}, (b_j)_{j \in F(i) \cup \{i\}}). \quad (4.1)$$

Let $a \in A$ and $(b_j)_{j \in F(i) \cup \{i\}} \in \prod_{j \in F(i) \cup \{i\}} A_j$.

CASE 2A: Suppose $a_i \neq b_i$. Define

- $\sigma_i = \overline{a}_i$ and $\tau_i = \overline{b}_i$,
- for each player $j \in Pr(i) \cup C(i) : \sigma_j = \overline{a}_j$,
- for $j \in F(i)$, let σ_j be the strategy that always chooses b_j, unless the history is $(a_k)_{k \in Pr(j)}$, in which case j chooses a_j.

Notice that $r(\sigma_i, \sigma_{-i}) = a, r(\tau_i, \sigma_{-i}) = ((a_j)_{j \in Pr(i) \cup C(i)}, (b_j)_{j \in F(i) \cup \{i\}})$. By Lemma 4.7: $U_i(\sigma) - P(\sigma) = U_i(\tau_i, \sigma_{-i}) - P(\tau_i, \sigma_{-i})$. By Lemma 4.8: $P(\eta) = P(\tau)$ if $r(\eta) = r(\tau)$. Hence

$$\begin{aligned}
u_i(a) - p(a) &= u_i(r(\sigma_i, \sigma_{-i})) - p(r(\sigma_i, \sigma_{-i})) \\
&= U_i(\sigma_i, \sigma_{-i}) - P(\overline{a}) \\
&= U_i(\sigma_i, \sigma_{-i}) - P(\sigma_i, \sigma_{-i}) \\
&= U_i(\tau_i, \sigma_{-i}) - P(\tau_i, \sigma_{-i}) \\
&= U_i(\tau_i, \sigma_{-i}) - P((\overline{a}_j)_{j \in Pr(i) \cup C(i)}, (\overline{b}_j)_{j \in F(i) \cup \{i\}}) \\
&= u_i(r(\tau_i, \sigma_{-i})) - p((a_j)_{j \in Pr(i) \cup C(i)}, (b_j)_{j \in F(i) \cup \{i\}}) \\
&= u_i((a_j)_{j \in Pr(i) \cup C(i)}, (b_j)_{j \in F(i) \cup \{i\}}) - p((a_j)_{j \in Pr(i) \cup C(i)}, (b_j)_{j \in F(i) \cup \{i\}})
\end{aligned}$$

which proves that (4.1) holds if $b_i \neq a_i$.

CASE 2B: Suppose $a_i = b_i$. By assumption (cf. Definition 4.1), A_i contains at least two elements. Let $c_i \in A_i$ with $c_i \neq a_i$. Applying the result of case 2A twice yields:

$$\begin{aligned}
u_i(a) - p(a) &= u_i((a_j)_{j \in Pr(i) \cup C(i)}, c_i, (b_j)_{j \in F(i)}) - p((a_j)_{j \in Pr(i) \cup C(i)}, c_i, (b_j)_{j \in F(i)}) \\
&= u_i((a_j)_{j \in Pr(i) \cup C(i)}, b_i, (b_j)_{j \in F(i)}) - p((a_j)_{j \in Pr(i) \cup C(i)}, b_i, (b_j)_{j \in F(i)}) \\
&= u_i((a_j)_{j \in Pr(i) \cup C(i)}, (b_j)_{j \in F(i) \cup \{i\}}) - p((a_j)_{j \in Pr(i) \cup C(i)}, (b_j)_{j \in F(i) \cup \{i\}}),
\end{aligned}$$

which proves that (4.1) holds if $b_i = a_i$. $\square$

The assumption in Definition 4.1 that each player $i \in N$ has an action set A_i containing at least two elements is relatively innocent: players having to make a choice from a singleton set of options are not extremely interesting. Notice, however, that in the proof above we explicitly made use of this assumption. In fact, the following example shows that the 'if'-part of Theorem 4.10 *breaks down* if some of the players have only one action.

Example 4.11 Consider the extensive form game in Figure 4.6 where player 1 has only one action S and player 2 in the next stage chooses either L or R. Payoffs are $u_1(S,L) = 2, u_1(S,R) = 1, u_2(S,L) = u_2(S,R) = 0$. Its normalization is clearly an exact potential game. But $p : \{S\} \times \{L,R\} \to \mathbb{R}, d_1 \in \mathbb{R}$, and $d_2 : \{S\} \to \mathbb{R}$ as in Definition 4.9 would have to satisfy the following inconsistent system of linear equations:

$$\begin{cases} u_1(S,L) &= p(S,L) + d_1 \\ u_1(S,R) &= p(S,R) + d_1 \\ u_2(S,L) &= p(S,L) + d_2(S) \\ u_2(S,R) &= p(S,R) + d_2(S) \end{cases}$$

The last two equations imply that p has to be a constant function. But then $u_1 - p$ depends on the action choice of player 2. In hierarchical potential games, the difference between the payoff function and a potential was assumed to be independent of the action choices of followers. $\triangleleft$

The following theorem relates hierarchical games and sequential production games.

Theorem 4.12 *Every sequential production game is a hierarchical potential game. For every hierarchical potential game there is a sequential production situation that induces this game.*

Proof. To see that a sequential production game as in Definition 4.5 is a hierarchical potential game, take

$$\begin{aligned} p &= \tfrac{1}{|N|}\rho, \\ d_i &= -c_i \quad \text{for each } i \in N. \end{aligned}$$

Conversely, consider a hierarchical potential game $\langle N = (N_1, \ldots, N_m), (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ with p and $(d_i)_{i\in N}$ as in Definition 4.9. Then the sequential production situation $\langle N = (N_1, \ldots, N_m), (A_i)_{i\in N}, \rho, (c_i)_{i\in N}\rangle$ with

$$\begin{aligned} \rho &= |N|p, \\ c_i &= -d_i \quad \text{for each } i \in N \end{aligned}$$

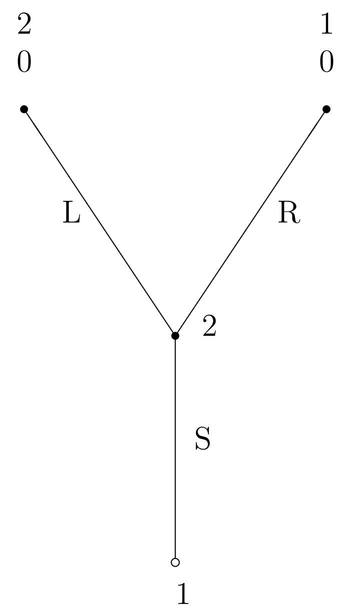


Figure 4.6: Player 1 has only one action

induces exactly the same game. □

Notice that a potential of a sequential production game equals the value function ρ divided by the number of players.

After defining hierarchical games we observed that every finite strategic game can be seen an a hierarchical game with only one stage. The theorem above establishes that every exact potential game is essentially a hierarchical potential game or a sequential production game.

It follows from the remark after the definition of potential games that every hierarchical potential game and thus every sequential production game has a pure-strategy Nash equilibrium. One can even extend this result to subgame-perfect equilibria, as is done below.

Subgames of imperfect information games are defined as usual. In hierarchical games, this implies that the game itself is a subgame, and that for each number k of stages, each profile of actions of the players in the first k stages induces a subgame. Formally,

Definition 4.13 Let $H = \langle N = (N_1, \ldots, N_m), (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a hierarchical game. Then H itself is a subgame and, moreover, for each $k \in \{1, \ldots, m-1\}$ and each profile or history $h = (a_i)_{i \in N_1 \cup \ldots \cup N_k} \in \prod_{i \in N_1 \cup \ldots \cup N_k} A_i$, the subgame $H(h)$ is the hierarchical game $\langle N(h) = (N_{k+1}, \ldots, N_m), (A_i)_{i \in N(h)}, (\tilde{u}_i)_{i \in N(h)} \rangle$ with $\tilde{u}_i(\cdot) = u_i(h, \cdot)$ for each player $i \in N(h)$. ◁

For instance, the subgame $H(I)$ that arises if department 1 chooses action I in the diamond game is given in Figure 4.7. Realize that since the remaining departments both operate at the same stage, the subgame $H(I)$ has no subgame other than itself.

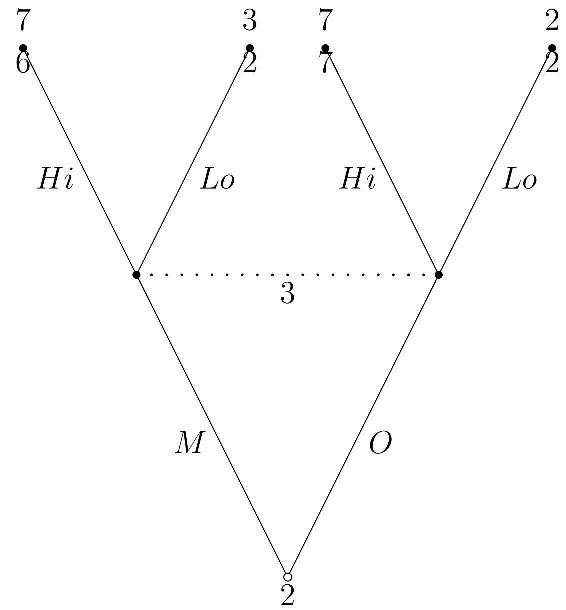


Figure 4.7: The subgame $H(I)$ of the diamond game

Corollary 4.14 *Every subgame of a hierarchical potential game is a hierarchical potential game.*

This corollary can be proven either directly, using Definition 4.9, or indirectly, using Theorem 4.10. The details are left to the reader. Notice that if p is a potential of the hierarchical game H, then for each subgame $H(h)$, the function $\tilde{p}$ with $\tilde{p}(\cdot) = p(h, \cdot)$ is a potential for the subgame $H(h)$.

Recall that a strategy profile σ in the normalized game $\mathcal{N}(H)$ is a *subgame-perfect Nash equilibrium* if it induces a Nash equilibrium in each subgame, i.e., if behavior outside the play path is also credible. For instance, $(O, (O, O), (Hi, Hi))$ is a subgame-perfect Nash equilibrium of the diamond game, but $(O, (M, O), (Lo, Hi))$ is not, since in the subgame $H(I)$ of Figure 4.7 player 3 would rather play Hi than Lo.

Potential-maximizing strategies form a refinement of the Nash equilibrium concept in strategic games with a potential. This refinement was introduced by Monderer and Shapley (1996). It was studied axiomatically in Peleg, Potters, and Tijs (1996) and used in Voorneveld (1997) to derive equilibrium existence results in infinite games. In hierarchical potential games H the notion of potential maximizing strategies can be extended to *subgame potential-maximizing strategies*, being those strategy profiles σ in the normalization $\mathcal{N}(H)$ that select actions maximizing the potential in every subgame. Corollary 4.14 guarantees that subgame potential-maximizers are well-defined.

The next theorem establishes one of the main results of this chapter: hierarchical potential games, and in particular sequential production games, have subgame perfect Nash equilibria in pure strategies, despite the presence of imperfect information.

Theorem 4.15 *Let H be a hierarchical potential game and $\mathcal{N}(H)$ its normalization.*

- *$\mathcal{N}(H)$ has a subgame potential maximizing strategy profile in pure strategies;*
- *each such pure-strategy subgame potential-maximizing profile is a pure-strategy subgame-perfect Nash equilibrium;*
- *not every pure-strategy subgame-perfect Nash equilibrium is a pure-strategy subgame potential maximizer.*

Proof. The proof of the first claim proceeds by induction on the number of stages of the game and closely mimics the existence proof of pure-strategy Nash equilibria in standard perfect information games. It is therefore left to the reader.

Strategies maximizing the potential of a subgame are easily seen to be Nash equilibrium strategies for the subgame by using Definition 4.9: the only difference between u_i and the potential p is a function d_i not depending on the choices of player i. This proves the second claim.

The final claim already follows from the insights in potential games in strategic form. Consider the single stage hierarchical potential game H with player set $N = N_1 = \{1, 2\}$, action sets $A_1 = A_2 = \{\alpha, \beta\}$, potential $p : A_1 \times A_2 \to \mathbb{R}$ with $p(\alpha, \alpha) = 2, p(\alpha, \beta) = p(\beta, \alpha) = 0, p(\beta, \beta) = 1$ and with $d_1 : A_2 \to \mathbb{R}$ and $d_2 : A_1 \to \mathbb{R}$ equal to the zero function. This is just the 2×2 exact potential game in Figure 4.8. Notice that (β, β) is

	α	β
α	2,2	0,0
β	0,0	1,1

Figure 4.8: (β, β) subgame perfect, not potential maximizing.

a pure-strategy subgame-perfect Nash equilibrium (there is only one subgame, namely the game itself), but not potential maximizing. □

In the diamond game of Figure 4.4, the pure strategy subgame potential maximizers are $(O, (M, O), (Hi, Hi))$ and $(O, (O, O), (Hi, Hi))$. The profile $(O, (M, O), (Lo, Hi))$ is potential maximizing, but does not select a potential maximizing outcome in the subgame $H(I)$.

4.4 Conclusions and extensions of the model

Practical situations can sometimes be studied using game theoretic tools. The topic of this chapter has been the study of an important type of production problems in which production takes place in several stages. These problems were modeled as sequential production games, a specific class of extensive form games with imperfect information. These games were related to potential games. In fact, it was shown that the class of sequential production games coincides with the class of hierarchical potential games (cf. Theorem 4.12).

Firms seeking the help from game theorists want clear-cut recommendations. Extensive form games with incomplete information typically do not have pure-strategy equilibria, which makes it hard to provide such easily adoptable recommendations. A significant feature of sequential production games is the existence of pure-strategy subgame-perfect Nash equilibria. Using subgame potential-maximizing profiles, we were able to identify a subset of these equilibria.

In Definition 4.5, payoffs to departments in a sequential production game were determined by giving each department an *equal* share of the value of the end product and then subtracting the costs. A possible extension of the model is to consider unequal division of the value over the departments. Introduce a vector $(w_i)_{i \in N}$ of weights satisfying $w_i \geqq 0$ for each department i and such that $\sum_{i \in N} w_i = 1$. The payoff functions u_i in Definition 4.5 can then be changed to $u_i = w_i \rho - c_i$.

Such unequal splitting of the value of the end product might be reasonable in the following sequential production situation. Students of a graduate school, the 'raw materials', receive an education in three 'production stages': there are preliminary or refresher courses in the first stage, the core courses in the second stage, and specialized courses in the third stage. The value of the 'end product', the PhD student successfully finishing the three stages, is usually considered to be the result of the specialized, advanced courses, to a lesser degree of the core courses, and hardly of the preliminary and refresher courses. In this teaching example, it appears reasonable to measure the contribution to the end product in such a way that a larger weight is assigned to lecturers teaching more advanced material.

Making the necessary modifications, the main results of this chapter still hold for sequential production games with unequal splitting of the value over the production departments. In particular, pure-strategy subgame-perfect Nash equilibria still exist.

The class of games generated in this way is closely related to weighted potential games, a class of ordinal potential games introduced in Monderer and Shapley (1996). Ordinal potential games were characterized in Voorneveld and Norde (1997). For another practical class of ordinal potential games, refer to Voorneveld, Koster, and Reijnierse (1998), Chapter 6 in this thesis, who consider schemes to finance public goods in a voluntary contribution game.

Chapter 5

Ordinal Potential Games

5.1 Introduction

Monderer and Shapley (1996) introduced several classes of potential games. Exact potential games were studied in the previous three chapters. As an example of an exact potential game, consider the two-person game with its exact potential function in Figure 5.1. In Theorem 2.11, exact potential games were characterized by the property that the

	L	R
T	$0, 2$	$-1, 3$
B	$1, 0$	$0, 1$

	L	R
T	0	1
B	1	2

Figure 5.1: An exact potential game

changes in payoff to deviating players along a cycle sum to zero, where a cycle in the strategy space is a closed sequence of strategy combinations in which players unilaterally deviate from one point to the next. Exact potential games are therefore extremely sensitive to small changes in the payoff functions: the slightest perturbation of payoffs can make this cycle property break down. In the next chapters we therefore look at more general classes of potential games in which not the precise change in payoff to a unilaterally deviating player matters, but rather the direction of the change in payoff. This chapter focuses on ordinal potential games. The game in Figure 5.2a is an example of

	L	R
T	$0, 2$	$0, 3$
B	$1, 0$	$0, 1$

a

	L	R
T	0	2
B	1	2

b

Figure 5.2: An ordinal potential game

an ordinal potential game. It is obtained from Figure 5.1 by changing $u_1(T, R)$ from -1

to 0. Consider the function in Figure 5.2b and notice that the sign of the change in the payoff to a unilaterally deviating player exactly matches the sign of the corresponding change in this function. For instance, if the second player deviates from (T, L) to (T, R), his payoff increases, just like the function in Figure 5.2b. Since deviating from (T, R) to (B, R) does not change player 1's payoff, the value of the function remains the same. For this reason, the function in Figure 5.2b is called an ordinal potential of the game.

Monderer and Shapley do not give a characterization of ordinal potential games. The class of finite ordinal potential games was characterized in Voorneveld (1996) through the absence of weak improvement cycles, i.e., cycles along which a unilaterally deviating player never incurs a lower payoff and at least one such player increases his payoff. The necessity of this condition is immediate, since a potential function would never decrease along a weak improvement cycle, but increases at least once. This gives a contradiction, because a cycle ends up where it started. Proving sufficiency is harder. In this chapter, a modified version of Voorneveld and Norde (1997), the general class of ordinal potential games is characterized. It turns out that countable ordinal potential games are still characterized by the absence of weak improvement cycles, but that for uncountable ordinal potential games an additional order condition on the strategy space is required.

The organization of this chapter is as follows: In Section 5.2 ordinal potential games are defined; some of its properties are studied. In Section 5.3 we provide a full characterization of ordinal potential games. In Section 5.4 we indicate that the absence of weak improvement cycles characterizes ordinal potential games with a countable strategy space, but not necessarily ordinal potential games in which the strategy space is uncountable.

5.2 Ordinal potential games

In this section we define ordinal potential games and study some of its properties.

Definition 5.1 A strategic game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is an *ordinal potential game* if there exists a function $P : X \to \mathbb{R}$ such that

$$\forall i \in N, \forall x_{-i} \in X_{-i}, \forall x_i, y_i \in X_i : u_i(x_i, x_{-i}) > u_i(y_i, x_{-i}) \Leftrightarrow P(x_i, x_{-i}) > P(y_i, x_{-i}).$$

The function P is called an *(ordinal) potential* of the game G. $\triangleleft$

In other words, if P is an ordinal potential function for G, the sign of the change in payoff to a unilaterally deviating player matches the sign of the change in the value of P.

Again, it is easy to see that strategy profiles maximizing an ordinal potential function of the game yield Nash equilibria and that — as a consequence — finite ordinal potential games have pure Nash equilibria. As opposed to exact potential games, however, the strategy profiles that maximize an ordinal potential function do depend on the particular potential function that is chosen. Consider for instance the two-player game in Figure 5.3. In this game both payoff functions u_1 and u_2 are ordinal potentials. If u_1 is chosen

	L	R
T	1,2	0,0
B	0,0	3,1

Figure 5.3: Potential maximizers

as a potential, (B, R) is the potential maximizing strategy, if u_2 is chosen, it is (T, L).

Notice that every order-preserving transformation of an ordinal potential function is again an ordinal potential function of the game.

The set of exact potential games, given a fixed set of players and strategy space, was seen to be a vector space. The set of ordinal potential games is not as well-behaved. In fact, it is not even closed under addition. The game in Figure 5.4a is an ordinal

a

	L	R
T	0,0	1,1
B	2,0	0,1

b

	L	R
T	1,2	1,0
B	0,0	0,1

c

	L	R
T	1,2	2,1
B	2,0	0,2

Figure 5.4: Set of ordinal potential games: not closed under addition

potential game with potential $P(T, L) = 0, P(T, R) = 3, P(B, L) = 1, P(B, R) = 2$. The game in Figure 5.4b is an ordinal potential game with potential $Q(T, L) = 3, P(T, R) = 2, P(B, L) = 0, P(B, R) = 1$. The sum of these games is the game in Figure 5.4c, which is not an ordinal potential game. Suppose it had an ordinal potential function U. Then it would have to satisfy $U(T, L) > U(T, R) > U(B, R) > U(B, L) > U(T, L)$, a contradiction.

A subset of the set of ordinal potential games is the set of weighted potential games.

Definition 5.2 A strategic game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a *weighted potential game* if there exists a function $P : X \to \mathbb{R}$ and a vector $(w_i)_{i \in N}$ of positive numbers such that

$$\forall i \in N, \forall x_{-i} \in X_{-i}, \forall x_i, y_i \in X_i : u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = w_i[P(x_i, x_{-i}) - P(y_i, x_{-i})].$$

The function P is called a *(weighted) potential* of the game G. $\triangleleft$

Without going into details, it is not difficult to see that weighted potential games — like exact potential games — can be decomposed into a dummy game and a coordination-type game.

Proposition 5.3 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. G is a weighted potential game if and only if there exist positive numbers $(w_i)_{i \in N}$, functions $(c_i)_{i \in N}$ and $(d_i)_{i \in N}$ such that*

- *$u_i = w_i c_i + d_i$ for all $i \in N$,*
- *$\langle N, (X_i)_{i \in N}, (c_i)_{i \in N} \rangle$ is a coordination game, and*
- *$\langle N, (X_i)_{i \in N}, (d_i)_{i \in N} \rangle$ is a dummy game.*

5.3 Characterization of ordinal potential games

This section contains a characterization of ordinal potential games. Similar to Theorem 2.11, it is shown that a particular requirement on cycles in the strategy space plays a central role.

Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. Recall that a *path* in the strategy space X is a sequence $(x^1, x^2, \ldots)$ of elements $x^k \in X$ such that for all $k = 1, 2, \ldots$ the strategy combinations x^k and x^{k+1} differ in exactly one, say the $i(k)$-th, coordinate. A path is *non-deteriorating* if $u_{i(k)}(x^k) \leqq u_{i(k)}(x^{k+1})$ for all $k = 1, 2, \ldots$. Non-deteriorating paths have restrictions only on consecutive strategy profiles, so by definition the trivial path (x^1) consisting of a single strategy profile $x^1 \in X$ is non-deteriorating. A finite path $(x^1, \ldots, x^m)$ is called a *weak improvement cycle* if it is non-deteriorating, $x^1 = x^m$, and $u_{i(k)}(x^k) < u_{i(k)}(x^{k+1})$ for some $k \in \{1, \ldots, m-1\}$.

Define a binary relation $\lhd$ on the strategy space X as follows: $x \lhd y$ if there exists a non-deteriorating path from x to y. Notice that $x \lhd x$ for each $x \in X$, since (x) is a non-deteriorating path from x to x. The binary relation $\approx$ on X is defined by $x \approx y$ if $x \lhd y$ and $y \lhd x$.

By checking reflexivity, symmetry, and transitivity, one sees that the binary relation $\approx$ is an equivalence relation. Denote the equivalence class of $x \in X$ with respect to $\approx$ by $[x]$, i.e., $[x] = \{y \in X \mid y \approx x\}$, and define a binary relation $\prec$ on the set $X_{\approx}$ of equivalence classes as follows: $[x] \prec [y]$ if $[x] \neq [y]$ and $x \lhd y$. To show that this relation is well-defined, observe that the choice of representatives in the equivalence classes is of no concern:

$$\forall x, \tilde{x}, y, \tilde{y} \in X \text{ with } x \approx \tilde{x} \text{ and } y \approx \tilde{y} : x \lhd y \Leftrightarrow \tilde{x} \lhd \tilde{y}.$$

Notice, moreover, that the relation $\prec$ on $X_{\approx}$ is irreflexive and transitive. The equivalence relation $\approx$ plays an important role in the characterization of ordinal potential games.

A tuple $(A, \prec)$ consisting of a set A and an irreflexive and transitive binary relation $\prec$ is *properly ordered* if there exists a function $F : A \to \mathbb{R}$ that preserves the order $\prec$:

$$\forall x, y \in A : x \prec y \Rightarrow F(x) < F(y).$$

Properly ordered sets are a key topic of study in utility theory. Not every tuple $(A, \prec)$ with $\prec$ irreflexive and transitive is properly ordered. A familiar example is the lexicographic order on $\mathbb{R}^2$. See, e.g., Fishburn (1979) for more details. However, if the set A is countable, i.e. if A is finite or if there exists a bijection between A and $\mathbb{N}$, then $(A, \prec)$ is properly ordered. The proof of this lemma is based on Bridges (1983).

Lemma 5.4 *Let A be a countable set and $\prec$ a binary relation on A that is irreflexive and transitive. Then $(A, \prec)$ is properly ordered.*

Proof. Since A is countable, we can label its elements and write $A = \{x_1, x_2, \ldots\}$. Let $\ll$ denote the transitive closure of $\prec$, i.e., $x \ll y$ iff there exist finitely many (at least two) elements $y_1, \ldots, y_n$ of A such that $y_1 = x, y_n = y$ and $y_1 \prec \ldots \prec y_n$.

For each $x \in A$, let $S(x) = \{n \in \mathbb{N} \mid x_n \ll x\}$ and define $F(x) = \sum_{n \in S(x)} 2^{-n}$. To see that F preserves the order $\prec$, let $x, y \in A, x \prec y$. Then $S(x) \subseteq S(y)$. Moreover, $x \in S(y)$, but $x \notin S(x)$ since $\prec$ is irreflexive and transitive, ruling out the possibility that $x \ll x$. So $S(x) \subset S(y)$, which implies $F(x) < F(y)$. □

Example 5.10 in Section 5.4 provides a game in which $(X_\approx, \prec)$ is not properly ordered. A sufficient condition for an uncountable set $(A, \prec)$ to be properly ordered is the existence of a countable subset B of A such that if $x \prec z, x \notin B, z \notin B$, there exists a $y \in B$ such that $x \prec y, y \prec z$. Such a set B is $\prec$-*order dense* in A.

Lemma 5.5 *Let A be a set and $\prec$ a binary relation on A that is irreflexive and transitive. If there exists a countable subset of A that is $\prec$-order dense in A, then $(A, \prec)$ is properly ordered.*

Proof. This is a corollary of Theorem 3.2 in Fishburn (1979). □

The following theorem characterizes ordinal potential games.

Theorem 5.6 *A strategic game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is an ordinal potential game if and only if the following two conditions are satisfied:*

1. *X contains no weak improvement cycles;*
2. *$(X_\approx, \prec)$ is properly ordered.*

Proof.
($\Rightarrow$): Assume P is an ordinal potential for G. Suppose that $(x^1, \ldots, x^m)$ is a weak improvement cycle. By definition, $u_{i(k)}(x^k) \leqq u_{i(k)}(x^{k+1})$ for all $k \in \{1, \ldots, m-1\}$ with strict inequality for at least one such k. But then $P(x^k) \leqq P(x^{k+1})$ for all and strict inequality for at least one $k \in \{1, \ldots, m-1\}$, implying $P(x^1) < P(x^m) = P(x^1)$, a contradiction. So X contains no weak improvement cycles.

Define $F : X_\approx \to \mathbb{R}$ by taking for all $[x] \in X_\approx : F([x]) = P(x)$. To see that F is well-defined, let $y, z \in [x]$. Since $y \approx z$ there is a non-deteriorating path from y to z and vice versa. But since the game has no weak improvement cycles, all changes in the payoff to the deviating players along these paths must be zero: $P(y) = P(z)$.

Now take $[x], [y] \in X_\approx$ with $[x] \prec [y]$. Since $x \lhd y$, there is a non-deteriorating path from x to y, so $P(x) \leqq P(y)$. Moreover, since x and y are in different equivalence classes, some player must have gained from deviating along this path: $P(x) < P(y)$. Hence $F([x]) < F([y])$.

($\Leftarrow$): Assume that the two conditions hold. Since $(X_{\approx}, \prec)$ is properly ordered, there exists a function $F : X_{\approx} \to \mathbb{R}$ that preserves the order $\prec$. Define $P : X \to \mathbb{R}$ by $P(x) = F([x])$ for all $x \in X$. Let $i \in N$, $x_{-i} \in X_{-i}$, and $x_i, y_i \in X_i$.

- If $u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) > 0$, then $(y_i, x_{-i}) \lhd (x_i, x_{-i})$, and by the absence of weak improvement cycles: not $(x_i, x_{-i}) \lhd (y_i, x_{-i})$. Hence $[(y_i, x_{-i})] \prec [(x_i, x_{-i})]$, which implies $P(x_i, x_{-i}) - P(y_i, x_{-i}) = F([(x_i, x_{-i})]) - F([(y_i, x_{-i})]) > 0$.

- If $P(x_i, x_{-i}) - P(y_i, x_{-i}) > 0$, then $[(x_i, x_{-i})] \neq [(y_i, x_{-i})]$, so $u_i(x_i, x_{-i}) \neq u_i(y_i, x_{-i})$. If $u_i(x_i, x_{-i}) < u_i(y_i, x_{-i})$, then $(x_i, x_{-i}) \lhd (y_i, x_{-i})$, and hence $[(x_i, x_{-i})] \prec [(y_i, x_{-i})]$. But then $P(x_i, x_{-i}) - P(y_i, x_{-i}) = F([(x_i, x_{-i})]) - F([(y_i, x_{-i})]) < 0$, a contradiction. Hence $u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) > 0$.

Conclude that P is an ordinal potential for the game G. □

The first condition in Theorem 5.6 involving cycles closely resembles the characterization of exact potential games in Theorem 2.11: a strategic game is an exact potential game if and only if the payoff changes to deviating players along a cycle sum to zero. In fact, in exact potential games it suffices to look at cycles involving only four deviations. The next example indicates that the absence of weak improvement cycles involving four deviations only is not sufficient to characterize ordinal potential games.

Example 5.7 Suppose P is an ordinal potential of the game below. Then P has to satisfy: $P(T, L) > P(T, R) = P(M, R) = P(M, M) = P(B, M) = P(B, L) = P(T, L)$, which is clearly impossible: this is not an ordinal potential game. Finiteness of the strategy space and Lemma 5.4 imply that the order condition is satisfied. Moreover, there exist no weak improvement cycles involving exactly four deviations. ◁

	L	M	R
T	0,1	1,2	0,0
M	1,1	0,0	0,0
B	0,0	0,0	1,1

5.4 Countable and uncountable games

Lemmas 5.4 and 5.5 give sufficient conditions for $(X_{\approx}, \prec)$ to be properly ordered. A consequence of Lemma 5.4 is that a game G with a countable strategy space X is an ordinal potential game if and only if it contains no weak improvement cycles. The strategy space X is countable if the set N of players is finite and every player $i \in N$ has a countable set X_i of strategies.

Theorem 5.8 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. If X is countable, then G is an ordinal potential game if and only if X contains no weak improvement cycles.*

Proof. If X is countable, $X_{\approx}$ is countable. According to Lemma 5.4, $(X_{\approx}, \prec)$ is properly ordered, so the result now follows from Theorem 5.6. $\square$

Theorem 5.8 generalizes the analogous result from Voorneveld (1996) for finite games.

A consequence of Lemma 5.5 is that if $(X_{\approx}, \prec)$ contains a countable $\prec$-order dense subset, then the absence of weak improvement cycles is once again enough to characterize ordinal potential games.

Theorem 5.9 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. If $(X_{\approx}, \prec)$ contains a countable $\prec$-order dense subset, then G is an ordinal potential game if and only if X contains no weak improvement cycles.*

Proof. By Lemma 5.5, $(X_{\approx}, \prec)$ is properly ordered. The result follows from Theorem 5.6. $\square$

This section is concluded with two examples of games with uncountable strategy spaces. The first is an example of a game in which no weak improvement cycles exist, but which is not an ordinal potential game since $(X_{\approx}, \prec)$ is not properly ordered. The second example is the only example in this thesis of a game with an infinite number of players; it shows that a Prisoner's Dilemma game with countably many players is an ordinal potential game.

Example 5.10 Consider the two-player game G with $X_1 = \{0, 1\}, X_2 = \mathbb{R}$, and payoff functions defined by

$$u_1(x, y) = \begin{cases} x & \text{if } y \in \mathbb{Q} \\ \\ -x & \text{if } y \notin \mathbb{Q} \end{cases}$$

and $u_2(x, y) = y$ for all $(x, y) \in \{0, 1\} \times \mathbb{R}$.

This game has no weak improvement cycles, since every weak improvement cycle trivially has to include deviations by at least two players. But if the second player deviates once and improves his payoff, he has to return to his initial strategy eventually, thereby reducing his payoff.

This game nevertheless is not an ordinal potential game. Suppose, to the contrary, that P is an ordinal potential for G. We show that this implies the existence of an injective function f from the uncountable set $\mathbb{R} \setminus \mathbb{Q}$ to the countable set $\mathbb{Q}$, a contradiction.

For each $y \in \mathbb{R} \setminus \mathbb{Q}$, $u_1(0, y) = 0 > -1 = u_1(1, y)$, so $P(0, y) > P(1, y)$. Fix $f(y) \in [P(1, y), P(0, y)] \cap \mathbb{Q}$. In order to show that $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{Q}$ is injective, let $x, z \in \mathbb{R} \setminus \mathbb{Q}, x < z$. Then there exists a number $y \in (x, z) \cap \mathbb{Q}$. However:

$$\begin{cases} u_2(0, x) & < & u_2(0, y) \\ u_1(0, y) & < & u_1(1, y) \\ u_2(1, y) & < & u_2(1, z) \end{cases} \Rightarrow \begin{cases} P(0, x) & < & P(0, y) \\ & < & P(1, y) \\ & < & P(1, z) \end{cases}$$

Since $f(x) \in [P(1, x), P(0, x)]$ and $f(z) \in [P(1, z), P(0, z)]$, it follows that $f(x) < f(z)$. So f is injective, a contradiction. $\triangleleft$

Finally, as an example of an ordinal potential game with an uncountable strategy space, let us extend the Prisoner's Dilemma of Example 2.2 to a possibly infinite but countable number of players. Consider a game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where

- $N \subseteq \mathbb{N}$;
- $\forall i \in N : X_i = \{0, 1\}$;
- 1 is a dominant strategy for every player:

$$\forall i \in N, \forall x_{-i} \in X_{-i} : u_i(1, x_{-i}) > u_i(0, x_{-i});$$

- Every player is better off in $(0, \ldots, 0)$ where all players choose the 0 strategy than in the Nash equilibrium $(1, \ldots, 1)$ where all players choose the 1 strategy:

$$\forall i \in N : u_i(0, \ldots, 0) > u_i(1, \ldots, 1).$$

If $N \subseteq \mathbb{N}$ is infinite, $\{0, 1\}^N$ is not countable. Yet, this game is an ordinal potential game — a result that is implicit in Basu (1994).

Proposition 5.11 *The game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ as described above is an ordinal potential game.*

Proof. Take for all $x \in X = \{0, 1\}^N$:

$$P(x) = \sum_{\{i \in N | x_i = 1\}} 2^{-i}.$$

Let $i \in N, x_{-i} \in X_{-i}$. Then $u_i(1, x_{-i}) > u_i(0, x_{-i})$ by definition. Also $P(1, x_{-i}) - P(0, x_{-i}) = 2^{-i} > 0$. Hence P is an ordinal potential for G. $\square$

Chapter 6

Voluntary Contribution to Multiple Facilities; A Class of Ordinal Potential Games

6.1 Introduction

The object of this chapter, which is based on Voorneveld, Koster, and Reijnierse (1998), is to study games arising from a class of problems in which players make private contributions for the eventual funding of a collection of facilities, or — as we call them — machines. The machines are considered public goods: once a machine has been built, all players can use it. Specifically, in the contribution problem there are finitely many players. Each of these players is interested in a subset of the finite set of machines. Realization of these machines is necessary for him to derive a benefit: only if a superset of them is realized, he receives a reward. Associated with each machine are its costs. We focus on two decision making processes, differing in the possibilities for cooperation.

In the cooperative situation — in presence of the possibility to enforce general agreement — we focus on the naturally related cooperative TU-game, the *realization game*. The game is determined by associating to each coalition of players the aggregate profits that it is capable of generating itself independent from the others, just by making an optimal choice between the feasible combinations of machines.

In the noncooperative mode, i.e., in absence of the possibility to make binding agreements, an additional component, the *realization scheme*, determines the strategic *contribution game*. The players are assumed to submit a contribution independently of the other players, and given the profile of contributions the realization scheme determines which machines are realized, and consequently also the individual payoffs. The strategy space of each player is his set of possible contributions. This set is taken to be the interval from zero (inclusive) to a player's reward (exclusive), meaning that each player contributes a nonnegative amount, but strictly less than his reward. The payoff function of a player is a player's reward if all machines he is interested in are realized, minus his contribution.

Up to now, not much has been said about the realization function: after all contributions have been made, what machines will be built? In fact, many possible realization functions come to mind. But considering that the players behave noncooperatively to subsidize public goods, it is of obvious significance to investigate whether a realization scheme can be defined that induces the contributors to play the contribution game — perhaps without them being aware of it — in the interest of the collective player set. In this chapter, a simple measure of collective welfare is used: the sum of the individual player's payoff functions, often referred to as the classical utilitarian collective welfare function (cf. Moulin, 1988).

It is indeed possible to define a realization scheme in such a way that the contribution game is best-response equivalent with a coordination game in which each contributor's payoff is the utilitarian welfare function. In terms of Monderer and Shapley (1996), this realization scheme makes the contribution game an ordinal potential game, where one of the ordinal potential functions is the utilitarian welfare function. The realization scheme takes into account that each contributor is willing to pay only for machines he is interested in and that the money allocated to a machine is never more than its costs. Remaining contributions in excess of the costs of the realized machines go to waste. Under these restrictions, there may still be several ways to allocate as much of the contributions as possible to the machines. Our realization scheme builds only those machines that are completely financed by *each* such maximal allocation. The realization scheme uses maximal flows and minimum cuts in certain flow networks.

The existence of Nash equilibria of the contribution game is established and several of its properties are studied. In a Nash equilibrium, a player makes a positive contribution only if all machines he is interested in are realized. Moreover, the contributions in a Nash equilibrium exactly suffice to pay for the machines of the players making a positive contribution, so no money goes to waste.

Now that it has been established that the players at least implicitly act in the interest of utilitarian welfare and that the game has a nonempty collection of Nash equilibria, one can derive that there is a Nash equilibrium maximizing utilitarian welfare. Hence, single players have no incentive to deviate since the profile is a Nash equilibrium, and the entire player set has no incentive to deviate since the profile maximizes utilitarian welfare. But one can show more. Such strategy profiles are in fact strong Nash equilibria of the contribution game: there is no coalition of players with an incentive to deviate from a strategy profile maximizing utilitarian welfare.

In particular this means that each strong Nash equilibrium defines a pre-imputation of the cooperative realization game, and — as will be shown — it determines a core element. There exists a strong relation between the concept of the core and the concept of strong Nash equilibrium: there is a 1-1 correspondence between the set of strong Nash equilibria of the contribution game and the payoffs in the core except those that give zero payoff to non-null players.

Summarizing, by choosing a particular realization scheme, one can guarantee that the players of a noncooperative contribution game act in common interest, in the sense that maximizing a player's payoff function given the strategy profile of his opponents is equivalent with maximizing utilitarian welfare given the strategy profile of his opponents. Not only do the players act in common interest, but there exist profiles maximizing utilitarian welfare, which turn out to be strong Nash equilibria of the contribution game and core elements of the realization game.

6.2 Model

In this section the model is specified and some preliminary results are provided. A *realization problem* is represented by a tuple

$$\mathcal{G} = \langle N, M, m \in (2^M)^N, \omega \in \mathbb{R}^N_{++}, c \in \mathbb{R}^M_{++} \rangle,$$

where

- N is the nonempty, finite set of players;
- M is the nonempty, finite set of public goods or machines;
- $m = (m_i)_{i \in N} \in (2^M)^N$ specifies the set of machines required by each player: player $i \in N$ needs the machines in $m_i \subseteq M, m_i \neq \emptyset$;
- $\omega = (\omega_i)_{i \in N} \in \mathbb{R}^N_{++}$ specifies the reward to each player $i \in N$ if (a superset of) all machines in m_i are realized;
- $c = (c_j)_{j \in M} \in \mathbb{R}^M_{++}$ specifies for each machine $j \in M$ the costs c_j to provide this machine.

The machines are considered to be public goods: once a machine has been built, all players can make use of it. Each realization problem corresponds to a TU-game in a natural way. The value of a coalition of players $S \subseteq N$ is the total of net benefits that it is able to collect by building the right combination of machines. That is, the *cooperative realization game* associated with a realization problem $\mathcal{G} = \langle N, M, m, \omega, c \rangle$ is the TU-game $(N, v_{\mathcal{G}})$ defined through

$$v_{\mathcal{G}}(S) = \max_{L \subseteq M} \left\{ \sum_{i \in S:\, m_i \subseteq L} \omega_i - \sum_{j \in L} c_j \right\} \text{ for all } S \subseteq N.$$

Example 6.1 Let $\mathcal{G} = \langle N, M, m, \omega, c \rangle$ be the realization problem with $N = \{1, 2, 3\}$, $M = \{p, q, r\}$, $m_1 = \{p\}, m_2 = \{p, q\}, m_3 = \{q, r\}$, $\omega = (10, 10, 20)$, and $c = (9, 5, 10)$. The values of the different coalitions of the corresponding 3-player cooperative realization game $(N, v_{\mathcal{G}})$ are listed in Figure 6.1 $\triangleleft$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v_{\mathcal{G}}(S)$	0	1	0	5	6	6	6	16

Figure 6.1: The values for $v_{\mathcal{G}}$.

Some additional notation: for $S \subseteq N$ write $\omega(S) = \sum_{i \in S} \omega_i$ and $m_S = \cup_{i \in S} m_i$.

Theorem 6.2 *The cooperative realization game* $(N, v_{\mathcal{G}})$ *is convex, i.e. for every* $i \in N, S \subset T \subseteq N \backslash \{i\}$*:*

$$v_{\mathcal{G}}(T \cup \{i\}) - v_{\mathcal{G}}(T) \geqq v_{\mathcal{G}}(S \cup \{i\}) - v_{\mathcal{G}}(S).$$

Proof. Let $S_i \subseteq S \cup \{i\}$ be such that $v_{\mathcal{G}}(S \cup \{i\}) = \omega(S_i) - c(m_{S_i})$ and let $T_0 \subseteq T$ be such that $v_{\mathcal{G}}(T) = \omega(T_0) - c(m_{T_0})$. Then:

$$\begin{aligned}
v_{\mathcal{G}}(T \cup \{i\}) - v_{\mathcal{G}}(T) &\geqq \{\omega(T_0 \cup S_i) - c(m_{T_0 \cup S_i})\} - \{\omega(T_0) - c(m_{T_0})\} \\
&= \omega(S_i) - \omega(S_i \cap T_0) - c(m_{S_i}) + c(m_{S_i} \cap m_{T_0}) \\
&= v_{\mathcal{G}}(S \cup \{i\}) - \{\omega(S_i \cap T_0) - c(m_{S_i \cap T_0})\} \\
&\geqq v_{\mathcal{G}}(S \cup \{i\}) - v_{\mathcal{G}}(S).
\end{aligned}$$

□

The convexity of cooperative realization games expresses that there is an incentive for the players to cooperate. Given the cooperation of the grand coalition the problem of allocating $v_{\mathcal{G}}(N)$ over the individual players remains. A preferable allocation is stable in the sense that no coalition of players has an incentive to split off. The corresponding solution concept for TU-games incorporating this collective rationality principle is the *core*.

Definition 6.3 The *core* $C(N, v)$ of a TU-game (N, v) consists of all vectors $x \in \mathbb{R}^N$ satisfying the following conditions:

$$(i) \quad \sum_{i \in S} x_i \geqq v(S) \text{ for all } S \subseteq N$$

$$(ii) \quad \sum_{i \in N} x_i = v(N).$$

◁

Condition (ii) combines the feasibility requirement $\sum_{i\in N} x_i \leqq v(N)$ with the rationality constraint for the grand coalition, $\sum_{i\in N} x_i \geqq v(N)$. It is common to refer to (ii) as *efficiency*; it assures that all profits of cooperation are allocated. A well-known relation between the convexity of games and the existence of core elements is the following:

Theorem 6.4 (Shapley, 1971) *If (N, v) is convex, then $C(N, v) \neq \emptyset$.*

This means that $C(N, v_{\mathcal{G}}) \neq \emptyset$ for each realization problem $\mathcal{G}$.

The values $v_{\mathcal{G}}(S)$ can be calculated in polynomial time by determining minimal cuts of certain flow networks that are defined subsequently. For $S \subseteq N$ construct a flow network Γ_S as follows. Γ_S has a node set V consisting of a source, a sink, S, and $m_S = \cup_{i\in S} m_i$. The nodes are called So, Si, $node(i)$ $(i \in S)$, and $node(j)$ $(j \in m_S)$. Γ_S has arc set A consisting of directed arcs. For each player $i \in S$ there is an arc $arc(i)$ from the source So to player i's node $node(i)$ with capacity $cap(i) = \omega_i$. When machine $j \in m_S$ is an element of m_i, there is an arc $arc(ij)$ from $node(i)$ to $node(j)$ with a capacity strictly larger than the individual benefits ω_i, say for instance $cap(ij) = \omega_i + 1$. For each machine $j \in m_S$ there is an arc $arc(j)$ from $node(j)$ to the sink Si with capacity $cap(j) = c_j$.

Example 6.5 The flow network Γ_N corresponding to the realization problem in Example 6.1 has the form of Figure 6.2. ◁

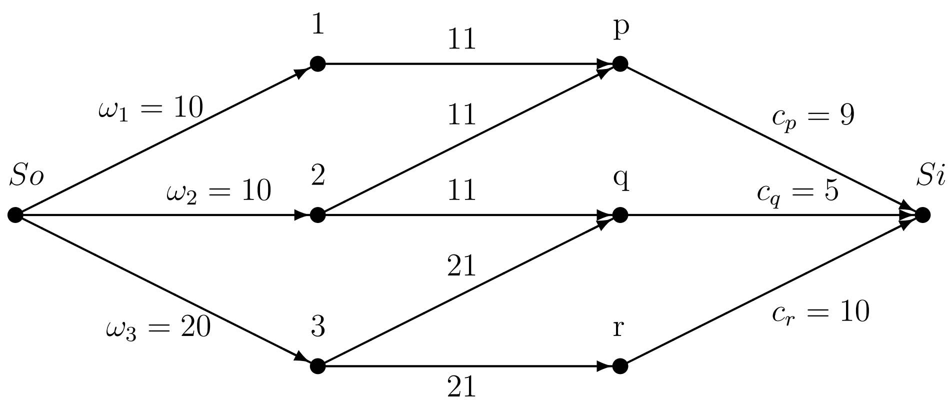


Figure 6.2: A flow network

Theorem 6.9 shows that the construction of precisely those machines that appear in some minimum cut of Γ_S maximizes the aggregate payoffs for coalition S. Definitions concerning flows and cuts in a flow network (V, A) with a source and a sink are briefly reviewed. For a more detailed study, see for instance Rockafellar (1984). A flow is a function $f : A \to \mathbb{R}$ such that for each directed arc (i,j) from node i to node j, $f(i,j) \in [0, cap(i,j)]$, and flow is conserved at every node, except possibly at the source and the sink. One can understand a flow as an amount of water transported from the source, through the network, to the sink, without flooding the arcs. A cut is a set of arcs such that each positive flow from source to sink uses at least one of these arcs. Intuitively, it is called a cut because removal of the arcs in a cut would disconnect all possible channels for a positive flow. The maximal amount of flow in a network $\Gamma = (V, A)$ is denoted max flow(Γ). The capacity of a cut is the sum of the capacities of the arcs in this cut. A cut is minimal if there is no cut in the network with a smaller capacity. The capacity of a minimum cut of Γ is denoted min cut(Γ). The following results are often used.

Lemma 6.6 *In a network* $\Gamma = (V, A)$*,*

1. max flow(Γ) = min cut(Γ).
2. *an arc is used to full capacity in each maximal flow if and only if it is contained in some minimum cut.*

The first part of the lemma is the famous max flow-min cut theorem of Ford and Fulkerson (1956). The proof of the second part is straightforward: an arc is used to full capacity in each maximal flow if and only if reducing its capacity reduces the value of the flow, if and only if the arc is in some minimum cut. Consider a flow network Γ_S arising from a realization problem $\mathcal{G}$. Notice that the capacity of an arc $arc(ij)$ with $i \in S, j \in m_S$ is chosen so large, that arcs of this type are never in a minimum cut of Γ_S. Thus, for every minimum cut C in a flow network Γ_S there exist $S' \subseteq S$ and $T \subseteq m_S$ such that $C = (\bigcup_{i \in S'} arc(i)) \cup \left(\bigcup_{j \in T} arc(j)\right)$. With a slight abuse of notation, this cut C is denoted (S', T) with $S' \subseteq S, T \subseteq m_S$. The set of minimum cuts of a flow network Γ_S is denoted $MC(\Gamma_S)$.

The following example illustrates these definitions.

Example 6.7 Consider a flow network similar to that in the previous example. Let $\omega_1 = 10, \omega_2 = 6, \omega_3 = 8$, and take $c_p = 9, c_q = 5, c_r = 10$. This gives the flow network in Figure 6.3. The capacities of intermediary arcs are considered to be high and are omitted for notational convenience.

There are infinitely many maximal flows by taking convex combinations of the maximal

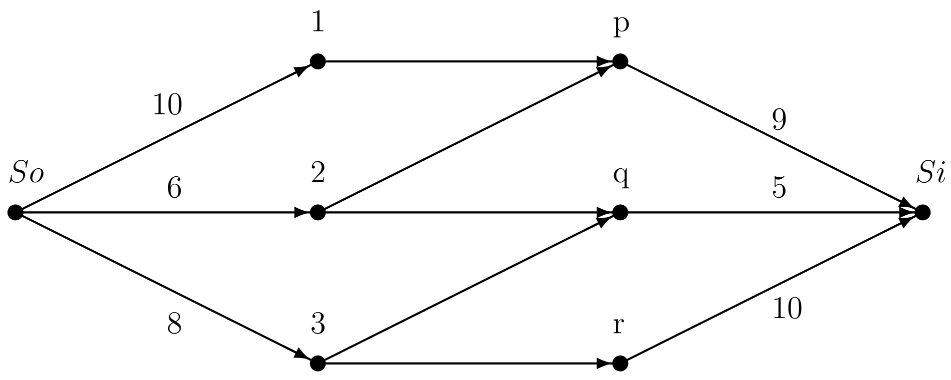


Figure 6.3: Illustrating flows and cuts

flows f and g defined as follows:

arc	f	g
$arc(1)$	8	9
$arc(2)$	6	5
$arc(3)$	8	8
$arc(1p)$	8	9
$arc(2p)$	1	0
$arc(2q)$	5	5
$arc(3q)$	0	0
$arc(3r)$	8	8
$arc(p)$	9	9
$arc(q)$	5	5
$arc(r)$	8	8

There is one minimum cut, namely $C = (S, T)$ with $S = \{3\}$ and $T = \{p, q\}$. Notice that the maximal amount of flow from source to sink equals 22, which is exactly the capacity of the cut (S, T). $\triangleleft$

Lemma 6.8 *Let $\mathcal{G}$ be a realization problem and Γ_S the corresponding flow network for a coalition of players S. If $C_1 = (S_1, T_1)$ and $C_2 = (S_2, T_2)$ are minimum cuts of Γ_S, then so are $C_3 = (S_1 \cap S_2, T_1 \cup T_2)$ and $C_4 = (S_1 \cup S_2, T_1 \cap T_2)$.*

Proof. Each directed path from source to sink is uniquely described by a pair (i, j) with $i \in S$ and $j \in m_i$. By definition of a cut, for each such path (i, j) either $i \in S_k$ or $j \in T_k$ $(k = 1, 2)$. It follows easily that $i \in S_1 \cap S_2$ or $j \in T_1 \cup T_2$ and that $i \in S_1 \cup S_2$

or $j \in T_1 \cap T_2$. As a consequence, C_3 and C_4 are cuts. Moreover,

$$\begin{aligned}\sum_{\ell \in C_1} cap(\ell) + \sum_{\ell \in C_2} cap(\ell) &= \sum_{i \in S_1} cap(i) + \sum_{j \in T_1} cap(j) + \sum_{i \in S_2} cap(i) + \sum_{j \in T_2} cap(j) \\ &= \sum_{i \in S_1 \cap S_2} cap(i) + \sum_{j \in T_1 \cup T_2} cap(j) + \sum_{i \in S_1 \cup S_2} cap(i) + \sum_{j \in T_1 \cap T_2} cap(j) \\ &= \sum_{\ell \in C_3} cap(\ell) + \sum_{\ell \in C_4} cap(\ell).\end{aligned}$$

Since both C_1 and C_2 are minimum cuts, C_3 and C_4 are minimum cuts. □

Theorem 6.9 *Let $\mathcal{G} = \langle N, M, m, \omega, c\rangle$ be a realization problem and $(N, v_{\mathcal{G}})$ the corresponding cooperative realization game. Let $S \subseteq N$ and let $(S_1, Q) \in \mathrm{MC}(\Gamma_S)$. Then*

$$v_{\mathcal{G}}(S) = \sum_{i \in S:\, m_i \subseteq Q} \omega_i - \sum_{j \in Q} c_j.$$

Proof. Every cut of Γ_S that is minimal w.r.t. set inclusion is of the form $(\{i \in S \mid m_i \not\subseteq P\}, P)$ for some $P \subseteq M$. Hence $S_1 = \{i \in S \mid m_i \not\subseteq Q\}$ and

$$\sum_{i \in S:\, m_i \not\subseteq Q} cap(i) + \sum_{j \in Q} cap(j) = \min_{P \subseteq M} \left\{ \sum_{i \in S:\, m_i \not\subseteq P} cap(i) + \sum_{j \in P} cap(j) \right\}.$$

This gives

$$\begin{aligned}\sum_{i \in S:\, m_i \subseteq Q} \omega_i - \sum_{j \in Q} c_j &= \sum_{i \in S} cap(i) - \left\{ \sum_{i \in S:\, m_i \not\subseteq Q} cap(i) + \sum_{j \in Q} cap(j) \right\} \\ &= \sum_{i \in S} \omega_i - \min_{P \subseteq M} \left\{ \sum_{i \in S:\, m_i \not\subseteq P} cap(i) + \sum_{j \in P} cap(j) \right\} \\ &= \sum_{i \in S} \omega_i + \max_{P \subseteq M} \left\{ - \sum_{i \in S:\, m_i \not\subseteq P} \omega_i - \sum_{j \in P} c_j \right\} \\ &= \max_{P \subseteq M} \left\{ \sum_{i \in S:\, m_i \subseteq P} \omega_i - \sum_{j \in P} c_j \right\} \\ &= v_{\mathcal{G}}(S).\end{aligned}$$

□

6.3 Contribution games

In the cooperative model for realizing a set of machines, as discussed in the previous section, a collective of players is able to decide upon the optimal set of machines to be constructed. This section formulates the realization problem as a *contribution problem* where no binding agreements can be made and the different players have to decide individually how much they want to spend on having their machines realized. After the individual contributions have been made, an independent arbitrator is supposed to decide upon the machinery to buy. This task involves a lot of ambiguity, since in general a profile of contributions can be associated with more than one feasible set of machines. Therefore the arbitrator makes use of a decision rule, a so-called *realization scheme.* A realization scheme maps each profile of contributions to an affordable combination of machines. A realization problem $\mathcal{G} = \langle N, M, m \in (2^M)^N, \omega \in \mathbb{R}^N_{++}, c \in \mathbb{R}^M_{++}\rangle$ together with a realization scheme R is called a *contribution problem* and is denoted

$$\mathcal{C} = \langle N, M, m \in (2^M)^N, \omega \in \mathbb{R}^N_{++}, c \in \mathbb{R}^M_{++}, R\rangle.$$

The arbitration procedure is not a black box: before the players make their bids known to the arbitrator the realization scheme is publicly announced.

It makes sense to require from the arbitrator that he puts forward a “reasonable” realization scheme. For instance, it may be perceived as “unfair” if the arbitrator decides to use the contribution of a player to buy other machines than he is interested in, especially if these are machines for zero contributors. Also the realization scheme should give the players the right incentives. Those players who profit a lot by having the desired set of machines should be pushed to contribute. The realization scheme defined in this section combines a number of desirable features in this respect. The formal definition requires some additional work. First we formally define the strategic game that corresponds to the above noncooperative setting.

The contribution problem $\mathcal{C} = \langle N, M, m \in (2^M)^N, \omega \in \mathbb{R}^N_{++}, c \in \mathbb{R}^M_{++}, R\rangle$ induces a *contribution game* $G(\mathcal{C}) = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where the strategy space of player $i \in N$, the set of possible contributions, is $X_i = [0, \omega_i)$. The realization scheme $R : \prod_{i\in N} X_i \to 2^M$ specifies for each profile of contributions of the players which machines are built. Player i's payoff function $u_i : X \to \mathbb{R}$ is defined, for each profile $x = (x_i)_{i\in N} \in X$ as

$$u_i(x) = \begin{cases} -x_i & \text{if} \quad m_i \not\subseteq R(x) \\ \\ \omega_i - x_i & \text{if} \quad m_i \subseteq R(x). \end{cases}$$

That is: he gets his reward ω_i only if all of his machines are realized and his contribution x_i causes disutility.

By taking $X_i = [0, \omega_i)$, it is assumed that each player $i \in N$ contributes a nonnegative amount, but strictly less than his reward ω_i. This is not a very restrictive assumption: it makes no sense to contribute more than the benefit you can derive from the realization of your machines and contributing ω_i is weakly dominated by contributing 0. A different approach — not influencing the results in the present chapter — would be to endow each player $i \in N$ with an initial amount $e_i \in \mathbb{R}_+$ of money such that $e_i < \omega_i$ and to take $X_i = [0, e_i]$. This approach is not taken in this chapter.

The promised realization scheme R is inspired by the techniques that were used to find the values of the characteristic function of the realization problem. We define in a similar way a flow network $\Gamma(x)$ for each profile $x \in X$ of contributions. $\Gamma(x)$ has a node set V consisting of a source, a sink, N, and M. The nodes are called So, Si, $node(i)$ $(i \in N)$, and $node(j)$ $(j \in M)$. $\Gamma(x)$ has arc set A consisting of directed arcs. For each player $i \in N$ there is an arc $arc(i)$ from the source So to player i's node $node(i)$ with capacity $cap(i) = x_i$. When machine $j \in M$ is an element of m_i, there is an arc $arc(ij)$ from $node(i)$ to $node(j)$ with a capacity strictly larger than any possible contribution by player i, for instance $cap(ij) = \omega_i + 1$. For each machine $j \in M$ there is an arc $arc(j)$ from $node(j)$ to the sink Si with capacity $cap(j) = c_j$. Notice that the underlying network (V, A) is the same for each $\Gamma(x)$; only the capacities of the player arcs are different.

Example 6.10 In a contribution problem with player set $N = \{1, 2, 3\}$, machine set $M = \{p, q, r\}$, and $m_1 = \{p\}, m_2 = \{p, q\}, m_3 = \{q, r\}$, the flow network $\Gamma(x)$ given contributions $x = (x_1, x_2, x_3)$ has the form of Figure 6.4. $\triangleleft$

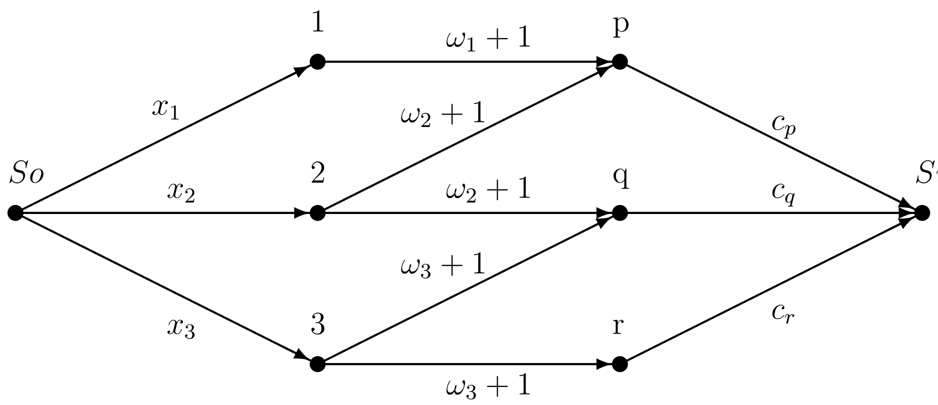


Figure 6.4: The flow network $\Gamma(x)$

Recall the definitions concerning flows and cuts in a flow network (V, A) with a source and a sink. Take a flow network $\Gamma(x)$ arising from some contribution problem $\mathcal{C}$. The set of minimum cuts of $\Gamma(x)$ will be denoted by $MC(x)$. Notice that the capacity of an arc $arc(ij)$ with $i \in N, j \in M$ is chosen so large, that arcs of this type are never in a minimum cut of $\Gamma(x)$. Thus, for every minimum cut $C \in MC(x)$ there exist $S \subseteq N$ and $T \subseteq M$ such that $C = (S, T)$.

What insight does the flow network $\Gamma(x)$ defined above give us in the problem under consideration? Given the constraints that each player $i \in N$ is willing to contribute only to the cost of machines in his desired set m_i and the money allocated to a machine does not exceed its costs, a maximal flow f describes exactly

- how much of the total contribution $\sum_{i \in N} x_i$ can be used for the provision of the machines, namely max flow$(\Gamma(x))$,
- which machines can be financed using this particular maximal flow, namely those with arcs used to maximum capacity, and
- who contributes how much to the costs of these machines in the maximal flow f: player i contributes to machine j the amount of flow through $arc(ij)$, $f(arc(ij))$.

Since selecting a particular maximal flow would strongly favor some of the players, the realization scheme R is defined as follows: in a contribution problem $\mathcal{C}$, for each profile $x \in X$ of contributions the set $R(x)$ of realized machines equals the set of machines used to maximal capacity by *each* maximal flow in $\Gamma(x)$. By Lemma 6.6.2, this is equivalent with stating that a machine is realized if and only if it is contained in some minimum cut of $\Gamma(x)$. Formally,

$$\forall x \in \prod_{i \in N} X_i : R(x) = \bigcup_{(S,T) \in MC(x)} T.$$

Many of the proofs use the fact that for each $x \in X$ there exists a minimum cut (S, T) in $\Gamma(x)$ such that $R(x) = T$. This result follows immediately from the next lemma.

Lemma 6.11 *Let $\mathcal{C}$ be a contribution problem, $x \in X$ a profile of contributions, and $\Gamma(x)$ the corresponding flow network. If $C_1 = (S_1, T_1)$ and $C_2 = (S_2, T_2)$ are minimum cuts, then $C_3 = (S_1 \cap S_2, T_1 \cup T_2)$ and $C_4 = (S_1 \cup S_2, T_1 \cap T_2)$ are also minimum cuts.*

Proof. See the proof of Lemma 6.8. □

The realization scheme R uses personalized contributions, i.e., each individual contribution x_i is used for machines in m_i. No player is subsidizing others at the cost of the realization of his own plan. The next sections will also show that in equilibrium the players together act on behalf of the desires of the society of players by maximizing utilitarian welfare.

6.4 Contribution games are ordinal potential games

Applications of potential games to economic situations were mentioned in Chapters 2, 3, and 4. In this section, contribution games are shown to be ordinal potential games. Since every order-preserving transformation of an ordinal potential function is again an ordinal potential function, it follows that an ordinal potential game has infinitely many different potential functions (whereas for exact potential games the potential function was determined up to an additive constant; see Proposition 2.8). It will be shown that one of the potential functions of a contribution game is the classical utilitarian collective welfare function, defined as the sum of the individual players' utility functions. See Moulin (1988) for a survey of this and other welfare functions.

Theorem 6.12 *Let $G(\mathcal{C}) = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a contribution game. The utilitarian welfare function $U : X \to \mathbb{R}$ defined by $U = \sum_{i \in N} u_i$ is an ordinal potential of $G(\mathcal{C})$.*

Proof. Let $i \in N$, $x_{-i} \in X_{-i}$, and $x_i, y_i \in X_i$. Assume without loss of generality that $x_i < y_i$. For notational convenience, write $x = (x_i, x_{-i})$ and $y = (y_i, x_{-i})$. Discern three cases:

Case 1: $m_i \not\subseteq R(y)$.
Since some arcs that correspond with machines in m_i are not a member of any minimum cut of the flow network $\Gamma(y)$, it must be that $arc(i) \in C$ for every $C \in MC(y)$. By decreasing $cap(i)$ from y_i to x_i the collection of minimum cuts does not change. So $R(y) = R(x)$. This implies $u_i(y) - u_i(x) = U(y) - U(x) = x_i - y_i$.

Case 2: $m_i \subseteq R(x)$.
By Lemma 6.11, there exists a minimum cut (S, T) in the flow network $\Gamma(x)$ such that $T = R(x)$. Since $m_i \subseteq T$, $arc(i) \notin S$. By increasing $cap(i)$ from x_i to y_i, (S, T) remains a minimum cut; no new minimum cuts appear, although some may disappear. So $R(y) \subseteq R(x)$. Because $T \subseteq R(y)$ it follows that $R(y) = R(x)$ and that $u_i(y) - u_i(x) = U(y) - U(x) = x_i - y_i$.

Case 3: $m_i \not\subseteq R(x)$ and $m_i \subseteq R(y)$.
In this case, $u_i(y) - u_i(x) = \omega_i - y_i + x_i > 0$. When player i spends the amount x_i, $arc(i) \in C$ for every $C \in MC(x)$. Let $z_i \in (x_i, y_i]$ be the smallest contribution of player i for which $arc(i)$ is no longer in every minimum cut of the flow network $\Gamma(z_i, x_{-i})$. Case 1 shows that $R(x) = R(t, x_{-i})$ for every $t \in (x_i, z_i)$. Case 2 shows that $R(y) = R(t, x_{-i})$ for every $t \in [z_i, y_i]$. By increasing $cap(i)$ from x_i to z_i, no minimum cut disappears, whereas some minimum cuts will appear, at least one of them not containing $arc(i)$. Therefore $R(x)$ is a proper subset of $R(z_i, x_{-i}) = R(y)$ and as a consequence

$$U(y) - U(x) = \sum_{l:\, m_l \subseteq R(y)} \omega_l - y_i - \sum_{l:\, m_l \subseteq R(x)} \omega_l + x_i \geqq \omega_i - y_i + x_i > 0.$$

This concludes our proof. □

Consequently, a contribution game is best-response equivalent with a coordination game where the payoff functions of the players are replaced by the utilitarian welfare function U. This is a significant insight: even though the players play a noncooperative game, utilitarian social welfare enters their game in the sense that — given the strategy profile of the opponents — a player maximizes his payoff if and only if he maximizes utilitarian welfare. Therefore, it is of interest to investigate the relation between equilibria of the contribution game and strategies that maximize social welfare.

6.5 Equilibria of contribution games

The existence of Nash equilibria of contribution games is established in the first theorem of this section. This result is not straightforward, taking into account the fact that payoff functions u_i are discontinuous and the strategy set of player i equals $[0, \omega_i)$, which is not closed. Two properties of Nash equilibria are derived: no money is wasted in an equilibrium and if a player is not satisfied since not all of his machines are realized, then he contributes nothing. These two properties are used to establish the existence of strategy profiles that maximize utilitarian welfare in a contribution game. Utilitarian welfare maximizing strategy profiles are proven to be strong Nash equilibria: no coalition of players has an incentive to deviate from such a profile.

Theorem 6.13 *Each contribution game $G(\mathcal{C})$ has a Nash equilibrium.*

Proof. The proof is based on an algorithm which is shown to terminate in finitely many steps with a Nash equilibrium of the game. Initially, set $k = 0$ and $x^0 = 0$: each player contributes zero. The general step of the algorithm is as follows. After k iterations, we are given a strategy profile x^k such that

$$\sum_{i \in N} x_i^k = \sum_{j \in R(x^k)} c_j = \max \text{flow}(\Gamma(x^k)) = \min \text{cut}(\Gamma(x^k)), \tag{6.1}$$

$$\{i \in N \mid x_i^k > 0, m_i \not\subseteq R(x^k)\} = \emptyset, \tag{6.2}$$

$$\{i \in N \mid m_i \subseteq R(x^{k-1})\} \subset \{i \in N \mid m_i \subseteq R(x^k)\} \text{ if } k \geqq 1. \tag{6.3}$$

The profile $x^0 = 0$ trivially satisfies these conditions. Define

$$\begin{aligned} C^k &= \{i \in N \mid x_i^k > 0, m_i \subseteq R(x^k)\}, \\ F^k &= \{i \in N \mid x_i^k = 0, m_i \subseteq R(x^k)\}, \\ N^k &= \{i \in N \mid x_i^k = 0, m_i \not\subseteq R(x^k)\}. \end{aligned}$$

The algorithm stops after k iterations if $N^k = \emptyset$ or if $N^k \neq \emptyset$ and x^k is a Nash equilibrium of $G(\mathcal{C})$. If the algorithm does not stop after k iterations, some player $i(k+1) \in N$ can improve by unilaterally changing his contribution. We claim that $i(k+1) \in N^k$. To prove this claim, notice that by (6.2) N is the union of the pairwise disjoint sets C^k, F^k, and N^k.

Clearly, $i(k+1) \notin F^k$, since players $i \in F^k$ achieve their payoff maximum ω_i by contributing nothing and therefore cannot possibly increase their payoff.

To show that $i(k+1) \notin C^k$, consider $h \in C^k$. By definition $x_h^k > 0$ and $m_h \subseteq R(x^k)$. Player h cannot benefit from increasing his contribution: for $y_h \in X_h$ with $y_h > x_h^k$ we have $u_h(y_h, x_{-h}^k) \leqq \omega_h - y_h < \omega_h - x_h^k = u_h(x^k)$. Player h also cannot benefit from decreasing his contribution: Property (6.1) implies that each maximal flow f in $\Gamma(x^k)$ uses every arc $arc(i)$ with $i \in N$ to full capacity x_i^k and each arc $arc(j)$ with $j \in R(x^k)$ to full capacity c_j. If player h decreases his contribution, say to λx_h^k with $\lambda \in [0,1)$, a maximal flow f' in the new flow network can be constructed as follows:

$$\text{For } i \in N: \qquad f'(arc(i)) = \begin{cases} \lambda f(arc(i)) & \text{if } i = h \\ f(arc(i)) & \text{otherwise.} \end{cases}$$

$$\text{For } i \in N, j \in m_i: \quad f'(arc(ij)) = \begin{cases} \lambda f(arc(ij)) & \text{if } i = h, j \in m_h \\ f(arc(ij)) & \text{otherwise.} \end{cases}$$

$$\text{For } j \in M: \qquad f'(arc(j)) = \textstyle\sum_{i \in N:\, j \in m_i} f'(arc(ij)).$$

If $j \in m_h$ is such that $f(arc(hj)) > 0$, then $f'(arc(j)) < f(arc(j)) = c_j$, so j is not used to full capacity by the maximal flow f' in the new flow network: not all machines in m_h are used to full capacity by every maximal flow, so player h will lose his reward ω_h if he decreases his contribution, thus decreasing his payoff from $\omega_h - x_h^k > 0$ to something nonpositive, namely $-\lambda x_h^k$.

Consequently, $i(k+1) \in N^k$, which implies $x_{i(k+1)}^k = 0$. The fact that he can improve, means that $\sum_{j \in m_{i(k+1)} \setminus R(x^k)} c_j < \omega_{i(k+1)}$: he can pay the costs necessary to finance that part of his machines that is not realized in $\Gamma(x^k)$. Set

$$x_i^{k+1} = \begin{cases} x_i^k & \text{if } \quad i \neq i(k+1) \\ \\ \sum_{j \in m_i \setminus R(x^k)} c_j & \text{if } \quad i = i(k+1). \end{cases}$$

Notice that a maximal flow f in $\Gamma(x^k)$ can easily be extended to a maximal flow in $\Gamma(x^{k+1})$ by adding a flow via player $i(k+1)$ that pays exactly for his machines in $m_{i(k+1)} \setminus R(x^k)$. Since such an extended maximal flow exactly finances the machines in $R(x^k) \cup m_{i(k+1)}$ and no others, it follows that $R(x^{k+1}) = R(x^k) \cup m_{i(k+1)}$. Increasing k by one, this also means that the input again satisfies (6.1) – (6.3), so that the general step can be repeated.

Two things remain to be shown: that the algorithm ends and that — if it ends after k iterations — x^k is indeed a Nash equilibrium of the game.

By construction, the algorithm ends after k iterations if $N^k \neq \emptyset$ and x^k is a Nash equilibrium, or if $N^k = \emptyset$. If $N^k = \emptyset$, x^k must be a Nash equilibrium, since it was shown above that any player that could benefit from unilateral deviation had to be a member of N^k. By (6.3), $|N^k| < |N^{k-1}|$ if $k \geqq 1$, so the algorithm terminates. □

Now that the existence of Nash equilibria in contribution games has been established, it becomes interesting to study their properties. The next proposition makes clear that players whose machine sets are not completely realized do not contribute anything in an equilibrium. Moreover, no money is wasted: in an equilibrium, the contributions of the players exactly suffice to pay for the realized machines.

Proposition 6.14 *Let $G(\mathcal{C})$ be a contribution game and $x \in NE(G(\mathcal{C}))$.*

1. *Let $i \in N$. If $m_i \not\subseteq R(x)$, then $x_i = 0$.*
2. $\sum_{j \in R(x)} c_j = \min \text{cut}(\Gamma(x)) = \max \text{flow}(\Gamma(x)) = \sum_{i \in N} x_i$.

Proof.

1. Assume $m_i \not\subseteq R(x)$ and suppose that $x_i > 0$. By definition of R, $m_i \not\subseteq R(x)$ implies that there is no minimum cut (S, T) in $\Gamma(x)$ such that $m_i \subseteq T$. Hence, $arc(i)$ is in each minimum cut of $\Gamma(x)$. Reducing i's contribution slightly does not change the set of minimum cuts and thus increases i's payoff, contradicting $x \in NE(G(\mathcal{C}))$. Hence $m_i \not\subseteq R(x)$ implies $x_i = 0$.

2. Obviously
$$\sum_{j \in R(x)} c_j \leqq \min \text{cut}(\Gamma(x)) = \max \text{flow}(\Gamma(x)) \leqq \sum_{i \in N} x_i.$$
By Lemma 6.11, $\Gamma(x)$ has a minimum cut (S, T) such that $R(x) = T$. If $m_i \subseteq T = R(x)$ and $x_i > 0$, then $arc(i) \notin S$. If $m_i \not\subseteq T = R(x)$, then $x_i = 0$ by Proposition 6.14.1. Hence S contains no arcs $arc(i)$ with $cap(i) = x_i > 0$. Thus
$$\sum_{j \in R(x)} c_j = \min \text{cut}(\Gamma(x)) = \max \text{flow}(\Gamma(x)).$$
Suppose
$$\sum_{j \in R(x)} c_j = \min \text{cut}(\Gamma(x)) = \max \text{flow}(\Gamma(x)) < \sum_{i \in N} x_i.$$
Then there exists an $i \in N$ with $x_i > 0$ such that $arc(i)$ is not used to full capacity in some maximal flow in $\Gamma(x)$. According to Lemma 6.6.2, $arc(i)$ is in no minimum cut. Then i can reduce his contribution slightly without affecting the set of minimum cuts and thus increase his payoff, contradicting $x \in NE(G(\mathcal{C}))$. □

The next proposition shows that a strategy profile maximizing utilitarian welfare U exists in each contribution game $G(\mathcal{C})$. Notice that the collection $\arg\max_{x \in X} U(x)$ is a subset of $NE(G(\mathcal{C}))$; otherwise, some player could increase his payoff by deviating, but then the ordinal potential U would increase as well.

Proposition 6.15 *Let $G(\mathcal{C})$ be a contribution game and $U = \sum_{i \in N} u_i$. Then the utilitarian welfare function achieves its maximum:* $\arg\max_{x \in X} U(x) \neq \emptyset$.

Proof. Observe that $\arg\max_{x \in X} U(x) = \arg\max\{U(x) \mid x \in NE(G(\mathcal{C}))\}$. Let $x \in NE(G(\mathcal{C}))$. By Proposition 6.14.2:

$$\begin{aligned} U(x) &:= \sum_{i \in N} u_i(x) \\ &= \sum_{i \in N:\, m_i \subseteq R(x)} \omega_i - \sum_{i \in N} x_i \\ &= \sum_{i \in N:\, m_i \subseteq R(x)} \omega_i - \sum_{j \in R(x)} c_j. \end{aligned}$$

There are finitely many machines, so the collection $\{R(x) \mid x \in NE(G(\mathcal{C}))\} \subseteq 2^M$ has finitely many elements. This implies that $\{U(x) \mid x \in NE(G(\mathcal{C}))\}$ also has finitely many elements. Consequently, this set has a maximum: $\arg\max_{x \in X} U(x) \neq \emptyset$. □

In a potential game, the collection of strategy profiles at which there is a potential achieving its maximum is called the potential maximizer. The potential maximizer is suggested as an equilibrium refinement tool by Monderer and Shapley (1996) and Peleg, Potters, and Tijs (1996). In ordinal potential games, different potentials give rise to different maximizers (as opposed, for instance, to exact potential games). Hence the collection of strategies maximizing utilitarian welfare in a contribution game may be a proper subset of the potential maximizer of the game.

Strong Nash equilibria were defined in Aumann (1959). In a game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a strategy combination $x \in X$ is a *strong Nash equilibrium* if for every $\emptyset \neq S \subseteq N$ and every $y_S \in X_S$ there exists an $i \in S$ such that $u_i(x) \geqq u_i(x_{N \setminus S}, y_S)$. In other words, $x \in X$ is a strong Nash equilibrium if there is no coalition $\emptyset \neq S \subseteq N$ of players and no alternative strategy $y_i \in X_i \setminus \{x_i\}$ for the members $i \in S$ such that $u_i(x_{N \setminus S}, y_S) > u_i(x)$ for each player $i \in S$. A slightly weaker definition would be to require that there is no coalition of players that can deviate and make each of its members not worse off and at least one of its members better off. In contribution games, however, the two definitions are equivalent, since each payoff function u_i satisfies

$$\forall x, y \in X : x_i \neq y_i \Rightarrow u_i(x) \neq u_i(y).$$

The set of strong Nash equilibria of a game G is denoted $SNE(G)$.

Although the set of Nash equilibria is nonempty in a wide class of noncooperative games, existence of strong Nash equilibria is much rarer. Existence of strong Nash equilibria in contribution games is established in the next theorem by showing that a strategy profile maximizing utilitarian welfare is a strong Nash equilibrium.

Theorem 6.16 *Let $G(\mathcal{C}) = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a contribution game, $U = \sum_{i \in N} u_i$. Then* $\arg\max_{x \in X} U(x) \subseteq SNE(G(\mathcal{C}))$. *Hence $SNE(G(\mathcal{C})) \neq \emptyset$.*

Proof. As soon as the inclusion is established, existence of strong Nash equilibria follows from Proposition 6.15. Let $x \in \arg\max_{x \in X} U(x)$. Individual players cannot profitably deviate from x, since $x \in NE(G(\mathcal{C}))$. The entire player set N cannot profitably deviate from x, since x maximizes $\sum_{i \in N} u_i$. Suppose that $x \notin SNE(G(\mathcal{C}))$. Then there exists a coalition $S \subset N$ with $1 < |S| < |N|$ and strategies $y_i \in X_i \setminus \{x_i\}$ for each $i \in S$ such that $u_i(x_{N \setminus S}, y_S) > u_i(x)$ for each $i \in S$. For notational convenience, define $y = (x_{N \setminus S}, y_S)$. Below it is shown that there is a strategy profile $z \in X$ such that $U(z) > U(x)$, contradicting the assumption that x maximizes U.

A player $i \in N$, in general, belongs to one of four types:

$$\text{(type 1)} \quad x_i > 0 \quad , \quad m_i \subseteq R(x);$$

$$\text{(type 2)} \quad x_i = 0 \quad , \quad m_i \not\subseteq R(x);$$

$$\text{(type 3)} \quad x_i > 0 \quad , \quad m_i \not\subseteq R(x);$$

$$\text{(type 4)} \quad x_i = 0 \quad , \quad m_i \subseteq R(x).$$

Since x is a Nash equilibrium, Proposition 6.14.1 implies that there are no players of the third type in $\Gamma(x)$. If a player is of the fourth type, he achieves his payoff maximum ω_i without contributing: such players cannot belong to S. Hence, members of S are either of type 1 or of type 2.

Write $S = S_1 \cup S_2$ with $S_k = \{i \in S \mid i \text{ is of type } k\}, k = 1, 2$. The fact that the members of S deviate from x and improve their payoff implies

$$y_i < x_i \text{ and } m_i \subseteq R(y) \quad \text{if} \quad i \in S_1, \tag{6.4}$$

$$y_i > 0 \text{ and } m_i \subseteq R(y) \quad \text{if} \quad i \in S_2. \tag{6.5}$$

It is impossible that $S_2 = \emptyset$. Suppose, to the contrary, that all members of S are of type 1: $S = S_1$. By (6.4), there exists for each $i \in S = S_1$ a $\lambda_i \in [0, 1)$ such that $y_i = \lambda_i x_i$. Take any maximal flow f in $\Gamma(x)$. Then a maximal flow f' in $\Gamma(y)$ is obtained as follows:

$$\text{For } i \in N: \qquad f'(arc(i)) = \begin{cases} \lambda_i f(arc(i)) & \text{if } i \in S \\ f(arc(i)) & \text{otherwise.} \end{cases}$$

$$\text{For } i \in N, j \in m_i: \quad f'(arc(ij)) = \begin{cases} \lambda_i f(arc(ij)) & \text{if } i \in S, j \in m_i \\ f(arc(ij)) & \text{otherwise.} \end{cases}$$

$$\text{For } j \in M: \qquad f'(arc(j)) = \textstyle\sum_{i \in N:\, j \in m_i} f'(arc(ij)).$$

According to Proposition 6.14.2, the contributions in $\Gamma(x)$ are exactly sufficient to pay for the machines in $R(x)$. By definition of R: $f(arc(j)) = c_j$ for all $j \in R(x)$. If $i \in S$

pays part of the costs of $j \in m_i$ according to f, i.e., $f(arc(ij)) > 0$, then this flow decreases by a factor λ_i in $\Gamma(y)$, so that $f'(arc(j)) < f(arc(j)) = c_j$. Hence $arc(j)$ is not used to full capacity by the maximal flow f' in $\Gamma(y)$, implying that $j \notin R(y)$. Then $m_i \not\subseteq R(y)$, contradicting (6.4). This completes the proof that $S_2 \neq \emptyset$.

Define $V = \{i \in N \mid x_i > 0\}$ and the nonempty machine set $M' = \bigcup_{i \in S_2} m_i \setminus R(x) = \bigcup_{i \in S_2} m_i \setminus \bigcup_{i \in V} m_i$.

Let f be a maximal flow in $\Gamma(x)$. By definition of R, every arc $arc(j)$ with $j \in R(x)$ is used to full capacity c_j by f. By Proposition 6.14.2:

$$\sum_{i \in N} f(arc(i)) = \sum_{i \in N} x_i = \sum_{j \in R(x)} c_j = \sum_{j \in R(x)} f(arc(j)).$$

Let g be a maximal flow in $\Gamma(y)$. By (6.5), $M' \subseteq R(y)$. By definition of R, every arc $arc(j)$ with $j \in M'$ is used to full capacity by g:

$$\text{for } j \in M' : c_j = g(arc(j)).$$

Since $M' = \bigcup_{i \in S_2} m_i \setminus \bigcup_{i \in V} m_i$, the flow in arcs $arc(j)$ with $j \in M'$ is generated entirely by members of S_2:

$$\text{for } j \in M' : c_j = g(arc(j)) = \sum_{i \in S_2 \colon j \in m_i} g(arc(ij)).$$

The total flow through the arcs $arc(j)$ with $j \in M'$ then equals

$$\sum_{j \in M'} c_j = \sum_{j \in M'} \sum_{i \in S_2 \colon j \in m_i} g(arc(ij))$$

and is generated entirely by the members of S_2. Given flow g, an arbitrary player $i \in S_2$ pays $\sum_{j \in M' \cap m_i} g(arc(ij))$ for the machines in M' . Summing over the players in S_2 yields

$$\sum_{j \in M'} c_j = \sum_{i \in S_2} \sum_{j \in M' \cap m_i} g(arc(ij)) = \sum_{j \in M'} \sum_{i \in S_2 \colon j \in m_i} g(arc(ij)).$$

Define a strategy profile $z \in X$ as follows:

$$z_i = \begin{cases} \sum_{j \in M' \cap m_i} g(arc(ij)) & \text{if } i \in S_2 \\ \\ x_i & \text{otherwise.} \end{cases}$$

Combine flows f and g to a feasible flow h in $\Gamma(z)$ as follows:

$$\text{For } i \in N : \quad h(arc(i)) = \begin{cases} z_i & \text{if } i \in S_2 \\ f(arc(i)) = x_i = z_i & \text{otherwise.} \end{cases}$$

$$\text{For } i \in N, j \in m_i : \quad h(arc(ij)) = \begin{cases} g(arc(ij)) & \text{if } i \in S_2, j \in M' \\ f(arc(ij)) & \text{otherwise.} \end{cases}$$

$$\text{For } j \in M : \quad h(arc(j)) = \begin{cases} g(arc(j)) = c_j & \text{if } j \in M' \\ f(arc(j)) = c_j & \text{if } j \in R(x) \\ f(arc(j)) = 0 & \text{otherwise.} \end{cases}$$

Notice that

$$\begin{aligned}
\sum_{i\in N} z_i &= \sum_{i\in S_2} z_i + \sum_{i\in N\setminus S_2} z_i \\
&= \sum_{i\in S_2}\sum_{j\in M'\cap m_i} g(arc(ij)) + \sum_{i\in N\setminus S_2} x_i \\
&= \sum_{j\in M'} c_j + \sum_{j\in R(x)} c_j \\
&= \sum_{j\in M} h(arc(j)).
\end{aligned}$$

Thus, h is a maximal flow in $\Gamma(z)$ and $\{arc(j) \mid j \in R(x) \cup M'\}$ is a minimum cut of $\Gamma(z)$. Hence $R(z) = R(x) \cup M'$. But then $u_i(z) \geqq u_i(x)$ for each $i \in N \setminus S_2$ and $u_i(z) = \omega_i - z_i > 0 = u_i(x)$ for each $i \in S_2$, implying $U(z) > U(x)$. This contradicts $x \in \arg\max_{x\in X} U(x)$. Conclude that x is indeed a strong Nash equilibrium. □

The converse inclusion of Theorem 6.16 holds as well. The set of realized machines is the same in each strong Nash equilibrium and — as a consequence — every strong Nash equilibrium maximizes utilitarian welfare.

Theorem 6.17 *Let $G(\mathcal{C})$ be a contribution game and $U = \sum_{i\in N} u_i$. If $x, y \in SNE(G(\mathcal{C}))$, then $R(x) = R(y)$. Hence $SNE(G(\mathcal{C})) \subseteq \arg\max_{z\in X} U(z)$.*

Proof. As soon as the implication is established, the inclusion of the set of strong Nash equilibria in the set of maximizers of utilitarian welfare can be shown as follows: let $x \in SNE(G(\mathcal{C}))$ and $y \in \arg\max_{z\in X} U(z)$. Then $y \in SNE(G(\mathcal{C}))$ by Theorem 6.16 and $R(x) = R(y)$ by the implication. Then Proposition 6.14.2 implies

$$\begin{aligned}
U(x) &= \sum_{i\in N} u_i(x) = \sum_{i\in N:\, m_i\subseteq R(x)} \omega_i - \sum_{i\in N} x_i \\
&= \sum_{i\in N:\, m_i\subseteq R(x)} \omega_i - \sum_{j\in R(x)} c_j = \sum_{i\in N:\, m_i\subseteq R(y)} \omega_i - \sum_{j\in R(y)} c_j \\
&= \sum_{i\in N:\, m_i\subseteq R(y)} \omega_i - \sum_{i\in N} y_i = \sum_{i\in N} u_i(y) \\
&= U(y) = \max_{z\in X} U(z).
\end{aligned}$$

So $x \in \arg\max_{z\in X} U(z)$, as was to be shown.

To show the implication, let $x, y \in SNE(G(\mathcal{C}))$ and suppose $R(x) \neq R(y)$. Without loss of generality, $R(y) \setminus R(x) \neq \emptyset$. Below it is shown that the coalition

$$D = \{i \in N \mid y_i > 0, m_i \cap [R(y) \setminus R(x)] \neq \emptyset\}$$

can profitably deviate from x, contradicting $x \in SNE(G(\mathcal{C}))$.

If $i \in D$, then $m_i \not\subseteq R(x)$, so Proposition 6.14.1 implies that $x_i = 0$ and $u_i(x) = 0$.

Let f be a maximal flow in $\Gamma(y)$. By definition of R, every arc $arc(j)$ with $j \in R(y) \setminus R(x)$ is used to full capacity c_j by f. Since this flow is generated entirely by the members of D, one finds

$$\forall j \in R(y) \setminus R(x) \ : \ c_j = \sum_{i \in D\colon j \in m_i} f(arc(ij)).$$

Player $i \in D$ contributes $\sum_{j \in m_i \cap [R(y) \setminus R(x)]} f(arc(ij))$ to the machines in $R(y) \setminus R(x)$ in the maximal flow f. Define $z \in X$ by

$$z_i = \begin{cases} x_i & \text{if } i \notin D \\ \\ \sum_{j \in m_i \cap [R(y) \setminus R(x)]} f(arc(ij)) & \text{if } i \in D. \end{cases}$$

It is shown that this deviation from x by the members of D will guarantee the realization of $R(x) \cup R(y)$, which is an improvement for the members of D. Let g be a maximal flow in $\Gamma(x)$. A flow h in $\Gamma(z)$ that extends the flow g in such a way that the machines in $R(y) \setminus R(x)$ can be financed by the members of D is defined as follows.

$$\begin{array}{llcl} \text{For } i \in N: & h(arc(i)) & = & z_i \\ \\ \text{For } i \in N, j \in m_i: & h(arc(ij)) & = & \begin{cases} f(arc(ij)) & \text{if } i \in D, j \in R(y) \setminus R(x) \\ g(arc(ij)) = 0 & \text{if } i \in D, j \notin R(y) \setminus R(x) \\ g(arc(ij)) & \text{otherwise.} \end{cases} \\ \\ \text{For } j \in M: & h(arc(j)) & = & \begin{cases} g(arc(j)) = 0 & \text{if } j \in M \text{ and} \\ & j \notin R(x) \cup R(y) \\ \sum_{i \in D\colon j \in m_i} f(arc(ij)) = c_j & \text{if } j \in R(y) \setminus R(x) \\ g(arc(j)) = c_j & \text{if } j \in R(x). \end{cases} \end{array}$$

Notice that h is a maximal flow in $\Gamma(z)$ and max flow$(\Gamma(z)) = \sum_{i \in N} z_i = \sum_{j \in R(x) \cup R(y)} c_j =$ min cut$(\Gamma(z))$. Hence $R(z) = R(x) \cup R(y)$. Then for each $i \in D$: $u_i(z) = \omega_i - z_i > 0 = u_i(x)$, contradicting $x \in SNE(G(\mathcal{C}))$. □

6.6 Strong Nash equilibria and the core

Since by Theorem 6.17 each strong Nash equilibrium of the noncooperative contribution game induces maximal utilitarian welfare, the corresponding profile of individual net-payoffs defines a pre-imputation of the cooperative realization game. These pre-imputations are in fact core allocations of the realization game. To be more precise, there is a one-to-one correspondence between the set of strong Nash equilibria of the contribution game and the subset of the core of the realization game where players with zero payoffs must be null players. In other words, the set of strong Nash equilibria naturally corresponds to the largest subset of the core that maximizes the number of players with positive rewards. Recall that $i \in N$ is a *null player* for a TU-game (N, v) if for all $S \subseteq N \setminus \{i\}$ it holds $v(S \cup \{i\}) = v(S)$. So a null player i is a dummy player with $v(\{i\}) = 0$ (see e.g. Shapley, 1953).

Theorem 6.18 *Consider a realization problem $\mathcal{G} = \langle N, M, m, \omega, c\rangle$, its associated contribution problem $\mathcal{C} = \langle N, M, m, \omega, c, R\rangle$, and the corresponding cooperative realization game $(N, v_{\mathcal{G}})$ and contribution game $G(\mathcal{C}) = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$. There exists a one-to-one correspondence from the set of strong Nash equilibria of $G(\mathcal{C})$ to the following subset K of $C(N, v_{\mathcal{G}})$:*

$$K := \{u \in C(N, v_{\mathcal{G}}) \mid \textit{ for all } i \in N, u_i = 0 \textit{ implies that } i \textit{ is a null player of } v_{\mathcal{G}}\}.$$

Proof. Theorem 6.17 states that in each strong Nash equilibrium of $G(\mathcal{C})$ the same subset of M is realized. Call this subset R. Define $N_+ = \{i \in N \mid m_i \subseteq R\}$. For every $u \in K$, $N_+ = \{i \in N \mid u_i > 0\}$. To see this, let $u \in K$.

- If $i \in N_+$, then m_i is realized in every strong Nash equilibrium y, so the reward ω_i of player i contributes to $U(y)$. Hence, $v_{\mathcal{G}}(N) > v_{\mathcal{G}}(N \setminus \{i\})$. So i is not a null player of $(N, v_{\mathcal{G}})$, which implies $u_i > 0$ by definition of K.
- If $i \notin N_+$, then ω_i does not contribute to $U(y)$ for any strong Nash equilibrium y and therefore $v_{\mathcal{G}}(N) = v_{\mathcal{G}}(N \setminus \{i\})$. Then $z_i = 0$ for every core element z; in particular $u_i = 0$.

Let $u \in K$. Define $x(u) \in \mathbb{R}^N$ as follows:

$$x(u)_i = \begin{cases} \omega_i - u_i & \text{if } i \in N_+ \text{ (so } u_i > 0) \\ \\ 0 & \text{if } i \notin N_+ \text{ (so } u_i = 0). \end{cases}$$

We prove that $x := x(u)$ is a strong Nash equilibrium. First, we show that x_i is a strategy of player i, i.e., that $x_i \in [0, \omega_i)$. This is the case since for all $i \in N_+$, $u_i > 0$ and $u_i \leqq v_{\mathcal{G}}(N) - v_{\mathcal{G}}(N \setminus \{i\}) \leqq \omega_i$. Consequently,

$$\sum_{i\in N} x_i = \sum_{i\in N_+} \omega_i - \sum_{i\in N} u_i = \sum_{i\in N_+} \omega_i - v_{\mathcal{G}}(N) = \sum_{i\in N_+} \omega_i - (\sum_{i\in N_+} \omega_i - \sum_{j\in R} c_j) = \sum_{j\in R} c_j.$$

Hence, in order to prove that $U(x)$ is maximal (and thus, by Theorem 6.16, x is a strong Nash equilibrium), it remains to show that $R(x) = R$. That is, for every $j \in R$, there must exist a minimum cut in $\Gamma(x)$ containing $arc(j)$. There exists a cut with capacity $\sum_{i\in N} x_i$ of which $arc(j)$ is a member: take all arcs of the players $i \in N$ with $x_i = 0$ and all arcs of elements in R. Hence, it suffices to show that a minimum cut in $\Gamma(x)$

has (at least) capacity $\sum_{i \in N} x_i$. Let (S, Q) be a minimum cut (with $S \subseteq N$, $Q \subseteq M$). W.l.o.g. $i \in S$ if $x_i = 0$. Then $m_i \subseteq Q$ for all $i \in N \setminus S$. The capacity of (S, Q) equals $\sum_{i \in S} x_i + \sum_{j \in Q} c_j$. We have:

$$\begin{aligned}
\sum_{i \in S} x_i + \sum_{j \in Q} c_j &\geqq \sum_{i \in S} x_i + \sum_{j \in \bigcup_{i \in N \setminus S} m_i} c_j \\
&\geqq \sum_{i \in S} x_i + \sum_{i \in N \setminus S} \omega_i - v_{\mathcal{G}}(N \setminus S) \\
&\geqq \sum_{i \in S} x_i + \sum_{i \in N \setminus S} (\omega_i - u_i) \\
&\geqq \sum_{i \in S} x_i + \sum_{i \in N \setminus S} x_i \\
&= \sum_{i \in N} x_i.
\end{aligned}$$

The first inequality holds because (S, Q) is a cut, the second follows from the definition of $v_{\mathcal{G}}$, the third follows from the assumption that u is a core element, and the fourth follows from the definition of x.

Now let $x \in SNE(G(\mathcal{C}))$. Then $u = (u_i(x))_{i \in N}$ can be considered as an allocation of $(N, v_{\mathcal{G}})$. By Theorems 6.16 and 6.17, U achieves its maximum at x, so the allocation u is efficient: $\sum_{i \in N} u_i = v_{\mathcal{G}}(N)$ $(= \max_{y \in X} U(y))$.

Let $S \subseteq N$. To show that $u \in C(N, v_{\mathcal{G}})$, we must prove that $\sum_{i \in S} u_i \geqq v_{\mathcal{G}}(S)$. Since $u \geqq 0$, assume that $v_{\mathcal{G}}(S) > 0$.

Let $S_+ \subseteq S$ be a *smallest* subcoalition of S such that $v_{\mathcal{G}}(S) = v_{\mathcal{G}}(S_+)$. Then $v_{\mathcal{G}}(S) = \sum_{i \in S_+} \omega_i - \sum_{j \in Q} c_j$, where $Q = \bigcup_{i \in S_+} m_i$. We prove that $Q \subseteq R$. Since $v_{\mathcal{G}}(S_+) > 0$, $S_+ \neq \emptyset$. Let $i \in S_+$. Then $v_{\mathcal{G}}(S_+) - v_{\mathcal{G}}(S_+ \setminus \{i\}) > 0$ by the minimality assumption on S_+. By convexity of $(N, v_{\mathcal{G}})$, we get $v_{\mathcal{G}}(N) - v_{\mathcal{G}}(N \setminus \{i\}) > 0$. Hence, the grand coalition strictly benefits from the fact that i is one of its members, so ω_i contributes to the value of N. Therefore, $m_i \subseteq R$.

In equilibrium no money is wasted and a coalition pays only for machines it needs (see Proposition 6.14), so it follows that $\sum_{i \in S_+} x_i \leqq \sum_{j \in Q} c_j$. Hence:

$$\sum_{i \in S} u_i \geqq \sum_{i \in S_+} u_i = \sum_{i \in S_+} (\omega_i - x_i) \geqq \sum_{i \in S_+} \omega_i - \sum_{j \in Q} c_j = v_{\mathcal{G}}(S_+) = v_{\mathcal{G}}(S).$$

Conclude that $u \in C(N, v_{\mathcal{G}})$. To show that $u \in K$, consider a player $i \in N$ with $u_i = 0$. Then $m_i \not\subseteq R$. Hence $v_{\mathcal{G}}(N) = v_{\mathcal{G}}(N \setminus \{i\})$. This gives that player i is a null player, by convexity of $(N, v_{\mathcal{G}})$.

To prove the one-to-one correspondence, one has to prove that

1. for each $y \in K$: $u(x(y)) = y$ and

2. for each $y \in SNE(G(\mathcal{C}))$: $x(u(y)) = y$.

The proof of these claims is straightforward, since for each $u \in K$ and $y \in SNE(G(\mathcal{C}))$: $\{i \in N \mid u_i > 0\} = \{i \in N \mid m_i \subseteq R(y)\}$. $\square$

Chapter 7

Best-Response Potential Games

7.1 Introduction

In potential games, introduced by Monderer and Shapley (1996), information concerning Nash equilibria can be incorporated into a single real-valued function on the strategy space. All classes of potential games that Monderer and Shapley defined share the finite improvement property: start with an arbitrary strategy profile. Each time, let a player that can improve deviate to a better strategy. Under the finite improvement property, this process eventually ends, obviously in a Nash equilibrium.

The purpose of this chapter, which is based on Voorneveld (1998), is to introduce and study best-response potential games, a new class of potential games. The main distinctive feature is that it allows infinite improvement paths, by imposing restrictions only on paths in which players that can improve actually deviate to a best response. The definition of best-response potential games is given in Section 7.2. A characterization of these games is provided in Section 7.3. Relations with the potential games of Monderer and Shapley (1996) are indicated in Section 7.4. Section 7.5 contains a discussion and motivation for the concept of best-response potential games.

7.2 Best-response potential games

This section contains the definition of best-response potential games and some preliminary results.

Definition 7.1 A strategic game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a *best-response potential game* if there exists a function $P : X \to \mathbb{R}$ such that

$$\forall i \in N, \forall x_{-i} \in X_{-i} \quad : \quad \arg\max_{x_i \in X_i} u_i(x_i, x_{-i}) = \arg\max_{x_i \in X_i} P(x_i, x_{-i}).$$

The function P is called a *(best-response) potential* of the game G. $\triangleleft$

In other words, a game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a best-response potential game if there exists a coordination game $\langle N, (X_i)_{i\in N}, (P)_{i\in N}\rangle$ where the payoff to each player is given by function P such that the best-response correspondence of each player $i \in N$ in G coincides with his best-response correspondence in the coordination game.

Recall that mixed extensions are not considered in Part I of this thesis and that 'Nash equilibrium' should be read as '*pure-strategy* Nash equilibrium'.

Proposition 7.2 *Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a best-response potential game with best-response potential P.*

1. *The Nash equilibria of $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ and $G = \langle N, (X_i)_{i\in N}, (P)_{i\in N}\rangle$, the coordination game with all payoff functions replaced by the potential P, coincide.*
2. *If P has a maximum over X (e.g. if X is finite), G has a Nash equilibrium.*

7.3 Characterization

This section contains a characterization of best-response potential games, similar to Theorem 5.6, the main result of Voorneveld and Norde (1997).

Let $\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a strategic game. A *path* in the strategy space X is a sequence $(x^1, x^2, \ldots)$ of elements $x^k \in X$ such that for all $k = 1, 2, \ldots$ the strategy combinations x^k and x^{k+1} differ in exactly one, say the $i(k)$-th, coordinate. A path is *best-response compatible* if the deviating player moves to a best response:

$$\forall k = 1, 2, \ldots \quad : \quad u_{i(k)}(x^{k+1}) = \max_{y_i \in X_i} u_{i(k)}(y_i, x^k_{-i(k)}).$$

Best-response compatible paths have restrictions only on consecutive strategy profiles, so by definition the trivial path (x^1) consisting of a single strategy profile $x^1 \in X$ is best-response compatible. A finite path $(x^1, \ldots, x^m)$ is called a *best-response cycle* if it is best-response compatible, $x_1 = x_m$, and $u_{i(k)}(x^k) < u_{i(k)}(x^{k+1})$ for some $k \in \{1, \ldots, m-1\}$.

Define a binary relation $\sqsubset$ on the strategy space X as follows: $x \sqsubset y$ if there exists a best-response compatible path from x to y, i.e., there is a best-response compatible path $(x^1, \ldots, x^m)$ with $x^1 = x, x^m = y$. Notice that $x \sqsubset x$ for each $x \in X$, since (x) is a best-response compatible path from x to x. The binary relation $\sim$ on X is defined by $x \sim y$ if $x \sqsubset y$ and $y \sqsubset x$.

By checking reflexivity, symmetry, and transitivity, one sees that the binary relation $\sim$ is an equivalence relation. Denote the equivalence class of $x \in X$ with respect to $\sim$ by $[x]$, i.e., $[x] = \{y \in X \mid y \sim x\}$, and define a binary relation $\prec$ on the set $X_\sim$ of equivalence classes as follows: $[x] \prec [y]$ if $[x] \neq [y]$ and $x \sqsubset y$. To show that this relation is well-defined, observe that the choice of representatives in the equivalence classes is of no concern:

$$\forall x, \tilde{x}, y, \tilde{y} \in X \text{ with } x \sim \tilde{x} \text{ and } y \sim \tilde{y} : x \sqsubset y \Leftrightarrow \tilde{x} \sqsubset \tilde{y}.$$

Notice, moreover, that the relation $\prec$ on $X_\sim$ is irreflexive and transitive.

A tuple $(A, \prec)$ consisting of a set A and an irreflexive and transitive binary relation $\prec$ on A is *properly ordered* if there exists a function $F : A \to \mathbb{R}$ that preserves the order $\prec$:

$$\forall x, y \in A \quad : \quad x \prec y \Rightarrow F(x) < F(y).$$

Theorem 7.3 *A strategic game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a best-response potential game if and only if the following two conditions are satisfied:*

1. *X contains no best-response cycles;*
2. *$(X_\sim, \prec)$ is properly ordered.*

Proof.
($\Rightarrow$): Assume P is a best-response potential for G. Suppose that $(x^1, \ldots, x^m)$ is a best-response cycle. By best-response compatibility, $P(x^k) \leqq P(x^{k+1})$ for each $k = 1, \ldots, m-1$. Since $u_{i(k)}(x^k) < u_{i(k)}(x^{k+1})$ for some $k \in \{1, \ldots, m-1\}$, it follows that for such k: $P(x^k) < P(x^{k+1})$. Conclude that $P(x^1) < P(x^m) = P(x^1)$, a contradiction. This shows that X contains no best-response cycles.

To prove that $(X_\sim, \prec)$ is properly ordered, define $F : X_\sim \to \mathbb{R}$ by taking for all $[x] \in X_\sim : F([x]) = P(x)$. To see that F is well-defined, let $y, z \in [x]$. Since $y \sim z$ there is a best-response compatible path from y to z and vice versa. But since the game has no best-response cycles, all changes in the payoff to the deviating players along these paths must be zero: $P(y) = P(z)$.

Now take $[x], [y] \in X_\sim$ with $[x] \prec [y]$. Since $x \sqsubset y$, there is a best-response compatible path from x to y, so $P(x) \leqq P(y)$. Moreover, since x and y are in different equivalence classes, some player must have gained from deviating along this path: $P(x) < P(y)$. Hence $F([x]) < F([y])$.

($\Leftarrow$): Assume that the two conditions hold. Since $(X_\sim, \prec)$ is properly ordered, there exists a function $F : X_\sim \to \mathbb{R}$ that preserves the order $\prec$. Define $P : X \to \mathbb{R}$ by $P(x) = F([x])$ for all $x \in X$. Let $i \in N, x_{-i} \in X_{-i}$.

- Let $y_i \in \arg\max_{x_i \in X_i} \; u_i(x_i, x_{-i})$ and $z_i \in X_i \setminus \{y_i\}$.
  - If $u_i(y_i, x_{-i}) = u_i(z_i, x_{-i})$, then $(y_i, x_{-i}) \sim (z_i, x_{-i})$, so $P(y_i, x_{-i}) = F([(y_i, x_{-i})])$ $= F([(z_i, x_{-i})]) = P(z_i, x_{-i})$.
  - If $u_i(y_i, x_{-i}) > u_i(z_i, x_{-i})$, then $(z_i, x_{-i}) \sqsubset (y_i, x_{-i})$. By the absence of best-response cycles, not $(y_i, x_{-i}) \sqsubset (z_i, x_{-i})$. Hence $[(z_i, x_{-i})] \prec [(y_i, x_{-i})]$, which implies $P(z_i, x_{-i}) = F([(z_i, x_{-i})]) < F([(y_i, x_{-i})]) = P(y_i, x_{-i})$.

  The above observations imply that $y_i \in \arg\max_{x_i \in X_i} \; P(x_i, x_{-i})$. This concludes the proof that

$$\forall i \in N, \forall x_{-i} \in X_{-i} \quad : \quad \arg\max_{x_i \in X_i} \; u_i(x_i, x_{-i}) \subseteq \arg\max_{x_i \in X_i} \; P(x_i, x_{-i}). \tag{7.1}$$

- Let $y_i \in \arg\max_{x_i \in X_i} P(x_i, x_{-i})$ and $z_i \in X_i \setminus \{y_i\}$. Suppose $u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})$. Then $(y_i, x_{-i}) \sqsubset (z_i, x_{-i})$. By the absence of best-response cycles, not $(z_i, x_{-i}) \sqsubset (y_i, x_{-i})$. Hence $[(y_i, x_{-i})] \prec [(z_i, x_{-i})]$, which implies $P(y_i, x_{-i}) = F([(y_i, x_{-i})]) < F([(z_i, x_{-i})]) = P(z_i, x_{-i})$, contradicting $y_i \in \arg\max_{x_i \in X_i} P(x_i, x_{-i})$. This finishes the proof that

$$\forall i \in N, \forall x_{-i} \in X_{-i} \quad : \quad \arg\max_{x_i \in X_i} u_i(x_i, x_{-i}) \supseteq \arg\max_{x_i \in X_i} P(x_i, x_{-i}). \tag{7.2}$$

Conclude from (7.1) and (7.2) that P is a best-response potential for the game G. □

If the strategy space X is countable, i.e., X is finite or there exists a bijection between $\mathbb{N}$ and X, the proper order condition is redundant.

Theorem 7.4 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. If X is countable, then G is a best-response potential game if and only if X contains no best-response cycles.*

Proof. If X is countable, $X_\sim$ is countable. It follows from Lemma 5.4 that $(X_\sim, \prec)$ is properly ordered. The result now follows from Theorem 7.3. □

This theorem, together with Proposition 7.2 generalizes Theorem 4.2 in Jurg *et al.* (1993).

7.4 Relations with other potential games

Monderer and Shapley (1996) introduce exact, weighted, ordinal, and generalized ordinal potential games. The relations between these classes of games (indicated by E, W, O, and G, respectively) and best-response potential games (indicated by BR) are indicated in Figure 7.1. For easy reference, their definitions are as follows. A strategic game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is

- an *exact potential game* if there exists a function $P : X \to \mathbb{R}$ such that for all $i \in N$, for all $x_{-i} \in X_{-i}$, and all $y_i, z_i \in X_i$:

$$u_i(y_i, x_{-i}) - u_i(z_i, x_{-i}) = P(y_i, x_{-i}) - P(z_i, x_{-i}).$$

- a *weighted potential game* if there exists a function $P : X \to \mathbb{R}$ and a vector $(w_i)_{i \in N}$ of positive numbers such that for all $i \in N$, for all $x_{-i} \in X_{-i}$, and all $y_i, z_i \in X_i$:

$$u_i(y_i, x_{-i}) - u_i(z_i, x_{-i}) = w_i(P(y_i, x_{-i}) - P(z_i, x_{-i})).$$

- an *ordinal potential game* if there exists a function $P : X \to \mathbb{R}$ such that for all $i \in N$, for all $x_{-i} \in X_{-i}$, and all $y_i, z_i \in X_i$:

$$u_i(y_i, x_{-i}) - u_i(z_i, x_{-i}) > 0 \Leftrightarrow P(y_i, x_{-i}) - P(z_i, x_{-i}) > 0.$$

- a *generalized ordinal potential game* if there exists a function $P : X \to \mathbb{R}$ such that for all $i \in N$, for all $x_{-i} \in X_{-i}$, and all $y_i, z_i \in X_i$:

$$u_i(y_i, x_{-i}) - u_i(z_i, x_{-i}) > 0 \Rightarrow P(y_i, x_{-i}) - P(z_i, x_{-i}) > 0.$$

Since Monderer and Shapley already indicated the relations between their classes of games, we stress the relation with best-response potential games.

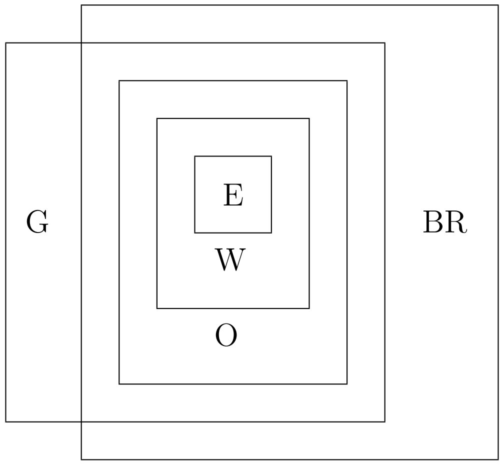


Figure 7.1: Relations between classes of potential games

That an ordinal potential game is a best-response potential game follows immediately from their definitions. Example 7.5 indicates that a generalized ordinal potential game is not necessarily a best-response potential game. Example 7.6 indicates that a best-response potential game is not necessarily a generalized ordinal potential game. Example 7.7 indicates that the intersection of the set of best-response potential games and generalized ordinal potential games properly includes the set of ordinal potential games, i.e., there are games which are both a best-response and a generalized ordinal potential game, but not an ordinal potential game.

Example 7.5 The game in Figure 7.2a has a generalized ordinal potential as given in Figure 7.2b. However, a best-response potential (and ordinal potential) would have to satisfy $P(T, L) = P(B, L) > P(B, R) > P(T, R) > P(T, L)$, which is a contradiction. $\triangleleft$

Example 7.6 The game in Figure 7.3a has a best-response potential as given in Figure 7.3b. However, a generalized ordinal (or ordinal) potential would have to satisfy $P(T, M) > P(B, M) > P(B, R) > P(T, R) > P(T, M)$, a contradiction. $\triangleleft$

	L	R
T	0,0	0,1
B	0,1	1,0

a

	L	R
T	0	1
B	3	2

b

Figure 7.2: Not a best-response potential game

	L	M	R
T	2,2	1,0	0,1
B	0,0	0,1	1,0

a

	L	M	R
T	4	3	0
B	0	2	1

b

Figure 7.3: Not a generalized ordinal potential game

Example 7.7 The game in Figure 7.4a has a best-response and generalized ordinal potential as given in Figure 7.4b. However, an ordinal potential would have to satisfy $P(T,M) > P(B,M) > P(B,R) > P(T,R) = P(T,M)$, a contradiction. $\triangleleft$

	L	M	R
T	0,2	1,0	0,0
B	0,2	0,1	1,0

a

	L	M	R
T	4	3	0
B	4	2	1

b

Figure 7.4: Not an ordinal potential game

7.5 Discussion

There are several reasons for introducing best-response potential games. In the first place, they are based on a simple insight: to determine Nash equilibria, what matters are best-responses. It is quite natural, in trying to find out whether a finite game has a Nash equilibrium, to look at the best situation a player can achieve by changing his strategy choice. This idea is at the root of fictitious play (Brown, 1951). Moreover, this is exactly what Milchtaich (1996) does to prove the existence of an equilibrium in his congestion games.

Best-response potential games differ from the potential games of Monderer and Shapley in an important aspect: they allow the presence of infinite improvement paths even in finite games. The games of Monderer and Shapley have equilibria because one could look at an improvement path and notice that it stopped somewhere. Best-response potential games give sufficient conditions for the existence of equilibria even if infinite improvement paths exist, as is the case in Example 7.6.

The obvious next step would be to consider games in which the Nash equilibrium set corresponds with the Nash equilibrium set of a suitably chosen coordination game. Formally,

Definition 7.8 A strategic game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a *Nash potential game* if there exists a function $P : X \to \mathbb{R}$ such that for all $x \in X$:

$$x \text{ is a Nash equilibrium of } G \Leftrightarrow x \text{ is a Nash equilibrium of } \langle N, (X_i)_{i \in N}, (P)_{i \in N} \rangle.$$

The function P is called a *(Nash) potential* of the game G. ◁

It turns out that in finite games, the set of Nash potential games is exactly the set of games with pure Nash equilibria. But in the infinite case, this concept makes no distinction whatsoever: every infinite game is a Nash potential game.

Theorem 7.9 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game.*

- *If G is finite, G is a Nash potential game if and only if it has a pure Nash equilibrium;*
- *If G is infinite, G is a Nash potential game.*

Proof. Clearly, a finite Nash potential game has a pure Nash equilibrium.

Now assume that G has a pure Nash equilibrium. It is shown that G is a Nash potential game, irrespective of the cardinality of the strategy space X.

Define the function $p : X \times NE(G) \to \{0, 1, \ldots, |N|\}$ for each strategy profile $x \in X$ and each Nash equilibrium y of G as

$$p(x, y) = \mid \{i \in N : x_i \neq y_i\} \mid,$$

i.e., $p(x, y)$ is the number of players that need to switch strategies to turn x into the Nash equilibrium y. Define the function $P : X \to \mathbb{R}$ for each $x \in X$ as

$$P(x) = - \min_{y \in NE(G)} p(x, y),$$

i.e., $P(x)$ equals minus the minimal number of strategy changes that is required to go from x to a Nash equilibrium.

To see that P is a Nash potential for the game G, define $H = \langle N, (X_i)_{i \in N}, (P)_{i \in N} \rangle$ and notice that

$$x \in NE(H) \Leftrightarrow P(x) = 0 \Leftrightarrow x \in NE(G).$$

All implications are trivial, except the fact that $P(x) = 0$ if $x \in NE(H)$. To see this, let $x \in X$ be such that $P(x) < 0$. Take $y \in NE(G)$ such that $P(x) = -p(x, y)$ and select $i \in N$ such that $x_i \neq y_i$. Then $P(x) < P(y_i, x_{-i})$, so $x \notin NE(H)$.

This settles the proof that every game with a Nash equilibrium is a Nash potential game. Remains to show that an infinite game without Nash equilibria is also a Nash potential game. In that case, at least one player $i \in N$ has an infinite strategy set X_i. This set has a countable subset with elements indexed $x_{i,s}$, $s \in \mathbb{N}$. Define the function $P : X \to \mathbb{N}_0$ as

$$P(x) = \begin{cases} s & \text{if } x_i = x_{i,s} \\ \\ 0 & \text{otherwise.} \end{cases}$$

To see that the coordination game with payoff functions P has no Nash equilibria, let $x \in X$. Now either $x_i = x_{i,s}$ for some $s \in \mathbb{N}$ in which case $P(x) = s$, or $P(x) = 0$. In both cases, player $i \in N$ would do better by deviating to $x_{i,s+1}$. $\square$

Chapter 8

Equilibria and Approximate Equilibria in Infinite Potential Games

8.1 Introduction

In strategic games where each player has only finitely many pure strategies, the existence of Nash equilibria is not guaranteed, unless mixed strategies are allowed (Nash, 1950a, 1951). In games where two or more players have infinitely many pure strategies, this result breaks down: not even mixed strategies yield equilibrium existence. A famous example is the $\infty \times \infty$ zero-sum game of Wald (1945) where both players choose a natural number and the player choosing the smallest number pays one dollar to the other player. Norde and Potters (1997) prove that approximate equilibria exist in bimatrix games where one player has a finite number of pure strategies and the other player infinitely (but countably) many pure strategies.

Since maxima of potential functions coincide with Nash equilibria of the corresponding game and a potential function achieves its maximum over a finite set of strategy profiles, it follows that finite potential games have Nash equilibria in pure strategies. This need no longer be the case if infinite games are considered.

If a Nash equilibrium does not exist, there may be strategy profiles in which players either receive a large payoff that satisfies them or cannot gain too much from deviating. Such an instance is an approximate equilibrium. Approximate equilibria are defined in Section 8.2.

The main purpose of this chapter is to provide some results on the existence of Nash equilibria or approximate equilibria in infinite potential games. Norde and Tijs (1998) provided results for exact potential games. These results are summarized in Section 8.3. Voorneveld (1997) looks at more general classes of potential games. In Section 8.4 we look at approximate equilibria for such general classes of potential games. We show that generalized ordinal potential games in which at most one player has an infinite set of strategies always have approximate equilibria. This generalizes a theorem from Norde and Tijs (1998) on exact potential games to ordinal and generalized ordinal potential games.

An open problem from Peleg, Potters, and Tijs (1996) is solved in Section 8.5 by showing that an ordinal potential game where all players have compact strategy sets and continuous payoff functions may not have a continuous ordinal potential function.

8.2 Definitions and preliminary results

First, recall the definitions of the several classes of potential games as summarized in Chapter 7, in particular Section 7.4. If $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a potential game, i.e. admits any type of potential, the *potential maximizer* is the set of strategy combinations $x \in X$ for which some potential P achieves a maximum. The following proposition summarizes the existence result for pure Nash equilibria in finite potential games.

Proposition 8.1 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a potential game and P a potential for G. If $x \in X$ is a Nash equilibrium of $\langle N, (X_i)_{i \in N}, (P)_{i \in N} \rangle$, i.e., of the coordination game with all payoff functions replaced by P, then x is a Nash equilibrium of G. In particular, every finite potential game has at least one pure Nash equilibrium, since the potential maximizer is nonempty.*

If G is an exact or ordinal potential game and x is a Nash equilibrium of G, then x is also a Nash equilibrium of $\langle N, (X_i)_{i \in N}, (P)_{i \in N} \rangle$. This is not necessarily true for generalized ordinal potential games.

Example 8.2 Consider a one-player game with strategy space X_1 and $u_1(x) = 0$ for all $x \in X_1$. Then any function $P : X_1 \to \mathbb{R}$ is a generalized ordinal potential function, since in generalized ordinal potential games there are no requirements on the potential function if the deviating player's payoff does not change. So the maxima of P w.r.t. unilateral deviations not necessarily pick out all pure Nash equilibria of the game. $\triangleleft$

Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game. Recall that a path in the strategy space is a sequence of strategy profiles generated by unilateral deviations and that a cycle is a nontrivial path that ends where it started. Call a cycle $(x^1, \ldots, x^m)$ in the strategy space X an *improvement cycle* if at each step $k \in \{1, \ldots, m-1\}$ the unique deviating player $i(k) \in N$ increases his payoff: $u_{i(k)}(x^k) < u_{i(k)}(x^{k+1})$. The proof of the following lemma is straightforward and therefore omitted.

Lemma 8.3 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a generalized ordinal potential game. Then G contains no improvement cycles.*

Let $\varepsilon > 0, k \in \mathbb{R}$. A strategy $x_i \in X_i$ of player $i \in N$ is called an *ε-best response* to $x_{-i} \in X_{-i}$ if

$$u_i(x_i, x_{-i}) \geqq \sup_{y_i \in X_i} u_i(y_i, x_{-i}) - \varepsilon$$

and a *k-guaranteeing response* to $x_{-i} \in X_{-i}$ if

$$u_i(x_i, x_{-i}) \geqq k.$$

By playing an ε-best response, a player makes sure that he cannot gain more than ε by deviating. Playing a k-guaranteeing response gives him a payoff of at least k. If x_i is either an ε-best or k-guaranteeing response (or both) to x_{-i}, it is called an *(ε, k)-best response* to x_{-i}. Notice that an (ε, k)-best response to x_{-i} always exists. A strategy combination $x \in X$ is called an *ε-equilibrium* of the game G if for each $i \in N$, x_i is an ε-best response to x_{-i}. It is called an *(ε, k)-equilibrium* if x_i is an (ε, k)-best response to x_{-i} for all $i \in N$. In such an equilibrium, each player can gain at most ε from deviating or receives at least a utility of k.

A game is called *weakly determined* if it has an (ε, k)-equilibrium for every $\varepsilon > 0$ and every $k \in \mathbb{R}$.

This section is concluded with some examples to illustrate these definitions. Notice that a one-person game is trivially a potential game.

Example 8.4 Consider a one-person game with the player having strategy space $\mathbb{Z}$ and $u(x) = x$ for all $x \in \mathbb{Z}$. This game has no Nash equilibria, but is weakly determined, since for every $k \in \mathbb{R}$, $x = \lceil k \rceil$ is a k-guaranteeing response, where for $r \in \mathbb{R}$, $\lceil r \rceil$ is the smallest integer greater than or equal to r. ◁

Example 8.5 Consider a one-person game with the player having strategy space $(0, \infty)$ and $u(x) = -\frac{1}{x}$ for all $x \in (0, \infty)$. This game has no Nash equilibria, but for every $\varepsilon > 0$, $x > \frac{1}{\varepsilon}$ is an ε-equilibrium. ◁

The following example from Norde and Tijs (1998) shows that infinite potential games may not be weakly determined.

Example 8.6 Consider the $\infty \times \infty$-bimatrix game with payoff functions $u_1(i, j) = i - j$ and $u_2(i, j) = j - i$, where $i, j \in \mathbb{N}$. This is an exact potential game, with a potential $P(i, j) = i + j$ for all $i, j \in \mathbb{N}$. Clearly, this game does not have (ε, k)-equilibria whenever $k > 0$. ◁

8.3 Infinite exact potential games

The results concerning weak determinateness of exact potential games that were obtained by Norde and Tijs (1998) rely heavily on the fact that differences in the value of the potential coincide with the difference in utility to deviating players or on the decomposition of exact potential games into coordination games and dummy games. These results are summarized in this section. In the next section, more general classes of potential games are considered.

Call a game *continuous* if the strategy spaces are topological spaces and all payoff functions are continuous with respect to the product topology. Continuous exact potential games have continuous exact potential functions and continuous functions on a compact set achieve a maximum. Hence (cf. Monderer and Shapley, 1996, Lemma 4.3):

Proposition 8.7 *Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a continuous exact potential game with compact strategy sets. Then G has a pure Nash equilibrium.*

Moreover,

Proposition 8.8 *Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be an exact potential game with upper bounded potential P. Then G is weakly determined.*

Proof. Let $\varepsilon > 0$. Choose $x \in X$ such that $P(x) > \sup_{y\in X} P(y) - \varepsilon$. Then x is an ε-equilibrium of G. □

Exact potential games where at most one player has a non-compact set of pure strategies are — under some continuity assumptions — weakly determined. Recall that a real-valued function f on a topological space T is lower semi-continuous if for each $c \in \mathbb{R}$ the set $\{x \in T \mid f(x) \leqq c\}$ is closed.

Theorem 8.9 *Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be an exact potential game. If*

- *$X_1, \ldots, X_{n-1}$ are compact topological spaces,*
- *$x_{-n} \mapsto u_i(x_n, x_{-n})$ is continuous for all $i \in N \setminus \{n\}$ and $x_n \in X_n$, and*
- *$x_{-n} \mapsto u_n(x_n, x_{-n})$ is lower semi-continuous for all $x_n \in X_n$,*

then G is weakly determined.

Proof. According to Proposition 8.8 it suffices to look at exact potentials P which are not upper bounded. Let $x_n \in X_n$ and $(y_1, \ldots, y_{n-1}) \in X_{-n}$. By definition of an exact potential function it follows that for every $(x_1, \ldots, x_{n-1}) \in X_{-n}$:

$$\begin{aligned} P(x_1, \ldots, x_n) &= u_1(x_1, x_2, \ldots, x_n) - u_1(y_1, x_2, \ldots, x_n) \\ &+ u_2(y_1, x_2, \ldots, x_n) - u_2(y_1, y_2, \ldots, x_n) \\ &\vdots \end{aligned}$$

$$+ \quad u_{n-1}(y_1, y_2, \ldots, x_{n-1}, x_n) - u_{n-1}(y_1, y_2, \ldots, y_{n-1}, x_n)$$

$$+ \quad P(y_1, \ldots, y_{n-1}, x_n),$$

which shows that $x_{-n} \mapsto P(x_n, x_{-n})$ is continuous. Let $k \in \mathbb{R}$ and define $d_n = u_n - P$. Then $x_{-n} \mapsto d_n(x_n, x_{-n})$ is lower semi-continuous for every $x_n \in X_n$. Moreover, d_n does not depend on x_n, so we may define $l = \min_{x \in X} d_n(x)$. Choose $y = (y_1, \ldots, y_n) \in X$ such that $P(y) \geqq k - l$, which is possible since P is not upper bounded. Since $x_{-n} \mapsto P(y_n, x_{-n})$ is continuous and X_{-n} is compact, we may choose $z_{-n} \in X_{-n}$ such that $P(y_n, z_{-n}) = \max_{x_{-n} \in X_{-n}} P(x_n, z_{-n})$. Then players $i \in N \setminus \{n\}$ cannot at all improve upon (y_n, z_{-n}) and $u_n(y_n, z_{-n}) = P(y_n, z_{-n}) + d_n(y_n, z_{-n}) \geqq P(y_n, y_{-n}) + l \geqq k$, so (y_n, z_{-n}) is an (ε, k)-equilibrium for every $\varepsilon > 0$. □

Consider an exact potential game G in which all but one player have a finite set of pure strategies. Endow these finite sets with the discrete topology. An immediate corollary of Theorem 8.9 is that G is weakly determined. This result is generalized in Theorem 8.11. But what happens if two players have infinite sets of pure strategies? Then a remarkable phenomenon occurs: there may be games with the same exact potential function, of which one game is weakly determined and the other not.

Example 8.10 Consider $\infty \times \infty$-bimatrix game where $X_1 = X_2 = \mathbb{N}$ and $u_1(i,j) = u_2(i,j) = i + j$ for all $i, j \in \mathbb{N}$. This is an exact potential game with potential $P(i,j) = i + j$ for all $i, j \in \mathbb{N}$. Let $k \in \mathbb{R}, \varepsilon > 0$. Let $r = \lceil k \rceil \in \mathbb{N}$ be the smallest integer greater than or equal to k. Then (r, r) is an (ε, k)-equilibrium, so this game is weakly determined.

Now change the payoff functions to those in Example 8.6. Again $P : (i,j) \mapsto i + j$ is an exact potential of this game, but the game is not weakly determined. ◁

8.4 Infinite potential games

The results in the previous section concerned exact potential games. In this section we look at other classes of potential games. If at most one player in a generalized ordinal potential game has an infinite set of strategies, the game has (ε, k)-equilibria for all $\varepsilon > 0, k \in \mathbb{R}$.

Theorem 8.11 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a generalized ordinal potential game. If $X_1, \ldots, X_{n-1}$ are finite sets, then G is weakly determined.*

Proof. Let P be a potential for G. Fix $\varphi(x_n) \in \arg\max_{x_{-n} \in X_{-n}} P(x_n, x_{-n})$ for each $x_n \in X_n$. Let $\varepsilon > 0, k \in \mathbb{R}$. Construct a sequence $\gamma = (x^1, x^2, \ldots)$ in X as follows: Take $x_n \in X_n$, define $x^1 = (x_n, \varphi(x_n))$. Let $m \in \mathbb{N}$. Suppose x^m is defined. If m is odd, and

- x_n^m is not an (ε, k)-best response to x_{-n}^m, take $x^{m+1} = (x_n, x_{-n}^m)$ with x_n an (ε, k)-best response to x_{-n}^m;
- otherwise, stop.

If m is even, and

- $x_{-n}^m \notin \arg\max_{x_{-n} \in X_{-n}} P(x_n^m, x_{-n})$, take $x^{m+1} = (x_n^m, \varphi(x_n^m))$;
- otherwise, stop.

If the sequence γ is finite, the terminal point is clearly an (ε, k)-equilibrium. So now assume this sequence is infinite.

Since the sets $X_1, \ldots, X_{n-1}$ are finite, there exist $l, m \in \mathbb{N}$ such that l is even, m is odd, $l < m$, and $x_{-n}^l = x_{-n}^m$. By construction, $P(x^l) < P(x^m)$, which implies $u_n(x^l) \leqq u_n(x^m)$. But x_n^l is an (ε, k)-best response to $x_{-n}^l = x_{-n}^m$, so x_n^m is an (ε, k)-best response to x_{-n}^m. Since $x_{-n}^m = \varphi(x_n^m)$, the other players cannot improve at all. Hence x^m is an (ε, k)-equilibrium. □

Example 8.6 indicates that this result cannot be extended to include two or more players with an infinite strategy set.

Under different assumptions one can also establish existence of Nash equilibria, like in the following theorem. Recall that a real-valued function f on a topological space T is called *upper semi-continuous (u.s.c.)* if for each $c \in \mathbb{R}$ the set $\{x \in T \mid f(x) \geqq c\}$ is closed.

Theorem 8.12 *Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a generalized ordinal potential game or a best-response potential game. If $X_1, \ldots, X_{n-1}$ are finite, X_n is a compact topological space and u_n is u.s.c. in the n-th coordinate, then G has a Nash equilibrium.*

Proof. Fix for each $x_{-n} \in X_{-n}$ an element $\varphi(x_{-n}) \in \Phi(x_{-n}) = \arg\max_{z \in X_n} u_n(z, x_{-n})$, which is possible by the upper semi-continuity and compactness conditions.

Suppose that G does not have a Nash equilibrium. Let $x_{-n} \in X_{-n}$. Take $x^1 = (\varphi(x_{-n}), x_{-n})$. Then there exists an infinite path $(x^1, x^2, \ldots)$ such that for each $k \in \mathbb{N}$, if $x_n^k \notin \Phi(x_{-n}^k)$, then $x^{k+1} = (\varphi(x_{-n}^k), x_{-n}^k)$, and otherwise $x^{k+1} = (y_i, x_{-i}^k)$ for some player $i \in N \setminus \{n\}$ not playing a best response against x_{-i}^k and $y_i \in \arg\max_{x_i \in X_i} u_i(x_i, x_{-i}^k)$ a best response to x_{-i}^k.

Since X_{-n} is finite and player n uses only strategies from $\{\varphi(x_{-n}) \mid x_{-n} \in X_{-n}\}$, there exist $k, l \in \mathbb{N}, k < l$, such that $x^k = x^l$. Hence $(x^k, x^{k+1}, \ldots, x^l)$ is a best-response cycle and in particular an improvement cycle. However, Theorem 7.3 and Lemma 8.3 show that the absence of such cycles is necessary for the existence of a best-response or generalized ordinal potential function, which yields the desired contradiction. □

8.5 Continuity of potential functions

Peleg, Potters, and Tijs (1996) study properties of the potential maximizer. It was left as an open problem in their paper whether ordinal potential games on a compact strategy space with payoff functions u_i which are continuous in the i-th coordinate have a non-empty potential maximizer or, even stronger, whether all such ordinal potential games possess a continuous potential. The result from this section indicates that this is not the case, even if payoff functions are continuous in each coordinate.

Theorem 8.13 *There exists an ordinal potential game with compact strategy spaces and continuous payoff functions for which no potential achieves a maximum and which consequently has no continuous ordinal potential function.*

Proof. Consider the game with $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, and payoff functions defined as

$$u_1(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy^6}{(x^2+y^2)^3} & \text{otherwise} \end{cases}$$

and

$$u_2(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{x^6 y}{(x^2+y^2)^3} & \text{otherwise.} \end{cases}$$

Clearly, these payoff functions are continuous. Moreover,

$$P(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy}{(x^2+y^2)^3} & \text{otherwise} \end{cases}$$

is a non-continuous (consider the image of the sequence $\{(\frac{1}{n}, \frac{1}{n})\}_{n=1}^{\infty}$) ordinal potential for the game. This follows easily from $u_1(x, y) = y^5 P(x, y)$ and $u_2(x, y) = x^5 P(x, y)$.

Now consider any ordinal potential Q for this game and the path C in the strategy space from $(1, 1)$ to $(\frac{1}{2}, 1)$ to $(\frac{1}{2}, \frac{1}{2}) \ldots (\frac{1}{2^n}, \frac{1}{2^n})$ to $(\frac{1}{2^{n+1}}, \frac{1}{2^n}) \ldots$ This path is depicted in Figure 8.1.

For $n \in \mathbb{N}_0$ and $y = \frac{1}{2^n}$ the functions $u_1(\cdot, y)$ and (hence) $Q(\cdot, y)$ are strictly decreasing on $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$. We will work out this case and leave other similar cases to the reader. The partial derivative of u_1 with respect to x equals

$$\frac{\partial u_1(x, y)}{\partial x} = y^5 \frac{\partial P(x, y)}{\partial x} = \frac{y^6(y^2 - 5x^2)}{(x^2 + y^2)^4}.$$

Since $\frac{1}{2^{n+1}} \leqq x \leqq \frac{1}{2^n}$, we have that $\frac{1}{2^{2n}} - \frac{5}{2^{2n}} \leqq y^2 - 5x^2 \leqq \frac{1}{2^{2n}} - \frac{5}{2^{2n+2}}$, which is equivalent to $\frac{-4}{2^{2n}} \leqq y^2 - 5x^2 \leqq \frac{2^2}{2^{2n+2}} - \frac{5}{2^{2n+2}} = \frac{-1}{2^{2n+2}} < 0$.

Similarly, for $n \in \mathbb{N}$ and $x = \frac{1}{2^n}$ the functions $u_2(x, \cdot)$ and (hence) $Q(x, \cdot)$ are strictly decreasing on $[\frac{1}{2^n}, \frac{1}{2^{n-1}}]$. This implies that Q must strictly increase along the path C from $(1, 1)$ to $(0, 0)$.

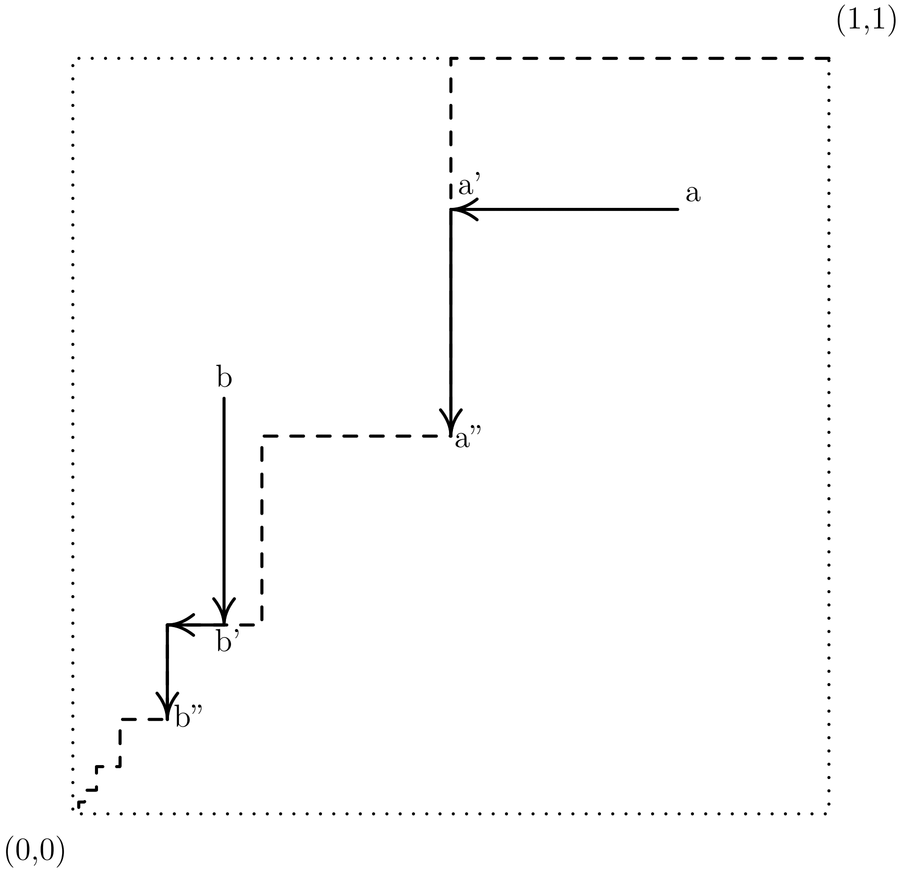


Figure 8.1: The strategy space and path C from Theorem 8.13.

Also $Q(x, 0) = Q(1, 0) < Q(1, 1)$ and $Q(0, y) = Q(0, 1) < Q(1, 1)$. Once again using the above, if (x, y) lies to the right of C, like the point a in Figure 8.1, and (x', y) is on C, like the point a', then $Q(x, y) < Q(x', y)$, since given $y \in (0, 1)$, there exists a $n \in \mathbb{N}$ such that $\frac{1}{2^n} \leqq y < \frac{1}{2^{n-1}}$. Then by definition $(\frac{1}{2^n}, y)$ is on C and $u_1(\cdot, y)$ is strictly decreasing on $[\frac{1}{2^n}, 1]$.

Also, if (x, y) lies to the left of C, like the point b, and (x, y') is on C, like the point b', then $Q(x, y) < Q(x, y')$, since, given $x \in (0, 1)$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{2^{n+1}} \leqq x < \frac{1}{2^n}$. Then by definition $(x, \frac{1}{2^n})$ is on C and $u_2(x, \cdot)$ is strictly decreasing on $[\frac{1}{2^n}, 1]$.

Therefore, for any $(x, y) \in [0, 1]^2$, we have $Q(x, y) < Q(\frac{1}{2^n}, \frac{1}{2^n})$ for some $n \in \mathbb{N}_0$. For the points a and b in Figure 8.1, such points are denoted by a'' and b'', respectively. Since the sequence $\{Q(\frac{1}{2^n}, \frac{1}{2^n})\}_{n=0}^{\infty}$ is strictly increasing, Q has no maximum, which is what we had to prove.

The continuity of a potential function for this game together with the compactness of the strategy space in the product topology would imply the existence of a maximum, contradicting our proof. Hence this game has no continuous potential. $\square$

Notice that continuity, however, is too strong a requirement. Reasonable conditions may exist under which a potential turns out to be upper semi-continuous, which given the compactness of the strategy space would still result in a maximum.

Chapter 9

Ordinal Games and Potentials

9.1 Potential functions and utility functions

In the previous chapters we studied several classes of potential games and gave applications to economic problems. The present chapter concludes our discussion of potential games.

We saw that potential functions are a handy tool for establishing results concerning Nash equilibria; a central result was the existence of pure Nash equilibria in finite games. But is a potential *just* a handy tool? Is there no exact meaning we can attach to a potential function? Several authors (Slade, 1994, Monderer and Shapley, 1996) have asked themselves this question, but did not come up with an answer. This can be due to the fact that there are many different types of potential functions, which tends to blur the overall picture. Norde and Patrone (1999) are motivated by Voorneveld and Norde (1997) to extend the notion of potentials to ordinal games. Although seemingly ignoring the question of attaching a meaning to the potential function, they nevertheless implicitly provide an answer.

Consider a game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$. If the aim is to check whether or not a given strategy profile $x \in X$ is a Nash equilibrium and a player $i \in N$ unilaterally deviates, then the *only* factor of interest is how this affects player i's payoff; the effect of i's deviation on another player $j \neq i$ is of no concern whatsoever. Abstracting from such irrelevant information, one can say that the preferences of the unilaterally deviating players define an *overall preference relation*, a binary relation $\prec$ on the strategy space X such that for each pair $x, y \in X$ of strategy profiles, $x \prec y$ if and only if x and y differ in exactly one — say the i-th — coordinate and $u_i(x) < u_i(y)$. All types of potential functions P introduced by Monderer and Shapley (1996) have in common that

$$\forall x, y \in X : \quad x \prec y \Rightarrow P(x) < P(y).$$

So P is essentially a utility function representing the relation $\prec$; summarizing:

> A potential function as defined by Monderer and Shapley is an *overall utility function* in the sense that it represents the preferences of unilaterally deviating players.

Notice that the binary relation $\prec$ is neither complete nor transitive; no comparisons are made between strategy profiles that differ in more than one coordinate. The reason for this is illustrated in the following example.

Consider the game in Figure 9.1, a simple example of a potential game. The overall

	L	R
T	1,1	0,0
B	0,0	2,2

Figure 9.1: A potential game

preference relation $\prec$ is given by

$$(T, R) \prec (T, L), (B, L) \prec (T, L), (T, R) \prec (B, R), (B, L) \prec (B, R).$$

It is clear that the following two functions P and Q are (ordinal) potentials of this game:

$$P(T, L) = 1, \quad P(T, R) = P(B, L) = 0, \quad P(B, R) = 1,$$

$$Q(T, L) = 2, \quad Q(T, R) = Q(B, L) = 0, \quad Q(B, R) = 1.$$

P ranks the two pure Nash equilibria of the game equally; but no meaning should be attached to the difference between the value of the potential at (T, L) and (B, R): even though both players prefer the equilibrium (B, R) to the equilibrium (T, L), the potential function Q attaches a lower value to (B, R). This justifies looking at the incomplete and nontransitive order $\prec$.

Having established that a potential function is a utility function representing preferences of unilaterally deviating players, some intuition arises for the specific requirements that make a game a potential game. In utility theory, it is common that the absence of certain cycles in a binary relation is necessary and sufficient for the existence of a utility function representing this relation (Bridges, 1983).

In the remainder of this chapter, we consider ordinal games: games in which a player is characterized by a general type of preferences on the strategy space. The main question will be whether we can still find something like a potential function and what type of cycles must be excluded. We start by taking a step back. Instead of looking at games with multiple players, the case of a single decision maker is treated first. The decision maker is endowed with a preference structure specifying his strict preference and indifference relation over a countable set of outcomes.

To handle nontransitive indifference relations, Luce (1956) introduced his well-known threshold model in which a decision maker prefers one outcome over another if and only if the increase in utility exceeds a certain nonnegative threshold. Formally, denoting the set of outcomes by X, strict preference by $\succ$ and indifference by $\sim$, each $x \in X$ is assigned a utility $u(x)$ and a threshold $t(x) \geqq 0$ such that for all $x, y \in X$:

$$x \succ y \quad \Leftrightarrow \quad u(x) > u(y) + t(y)$$

$$x \sim y \quad \Leftrightarrow \quad \begin{cases} u(x) \leqq u(y) + t(y), \\ u(y) \leqq u(x) + t(x). \end{cases}$$

In this chapter, most of which is based on Voorneveld (1999b), also incomparability between outcomes and nontransitivity of strict preferences is allowed. Incomparabilities arise if the decision maker is not capable to compare outcomes, finds it unethical to do so, or thinks that outcomes are comparable, but lacks the information to do so. Fishburn (1991) motivates nontransitive preferences. In this case the double implications above are replaced by single implications, so that we want for all $x, y \in X$:

$$x \succ y \quad \Rightarrow \quad u(x) > u(y) + t(y)$$

$$x \sim y \quad \Rightarrow \quad \begin{cases} u(x) \leqq u(y) + t(y), \\ u(y) \leqq u(x) + t(x) \end{cases}$$

Our main theorem gives necessary and sufficient conditions for the existence of functions u and t as above on a broad class of preference structures over a countable set of alternatives. As a corollary, a representation theorem of interval orders (See Bridges, 1983, and Fishburn, 1970) is obtained.

Section 9.2 provides definitions of preference structures, Section 9.3 formulates the main representation theorem. The difficulty of extending the theorem to uncountable sets is illustrated in Section 9.4. In Section 9.5 the representation theorem from Section 9.3 and Lemma 5.4 are used to characterize two types of potential functions for ordinal games through the absence of certain cycles in the strategy space.

9.2 Preference structures

A *preference structure* on a set X is a pair $(\succ, \sim)$ of binary relations on X such that

- For each $x, y \in X$, at most one of the following is true: $x \succ y, y \succ x, x \sim y$;
- The relation $\sim$ is reflexive and symmetric.

The first condition implies that $\succ$ is anti-symmetric (if $x \succ y$, then not $y \succ x$). With $\succ$ interpreted as strict preference and $\sim$ as indifference, this leads to a very general type of preferences in which neither strict preference, nor indifference is assumed to be transitive and in which a decision maker may have pairs $x, y \in X$ which he cannot compare; this imposes much less rationality restrictions on the decision maker than usual. Let us give an example of such a preference structure.

Example 9.1 An agent intends to invest in one of four sports teams: $X = \{a, b, c, d\}$. He has to base his decision on a limited amount of information: the number of scored points of each team in the matches played in the last three weeks; so each of the four teams can be represented by a vector in $\mathbb{R}^3$. For instance, $b = (1, 3, 0) \in \mathbb{R}^3$ indicates that team b scored one point one week ago, three points two weeks ago, and no points three weeks ago. His preferences are based on coordinate-wise comparisons. Being convinced that a one-point difference can be based on pure luck rather than quality, he finds a difference between two scores noticeable if it exceeds one. He bases his judgment between to teams x and y in X on the most recent pair of consecutive matches $i, i+1$ in which the teams scored a noticeably different number of points.

- if no such pair exists, he is indifferent between x and y;
- if such a pair does exist, then

$$x \succ y \Leftrightarrow \begin{cases} x_i & > \quad y_i + 1 \\ \\ x_{i+1} & > \quad y_{i+1} + 1 \end{cases}$$

- otherwise he cannot compare.

To illustrate this, assume

$$a = (a_1, a_2, a_3) = (4, 4, 0), b = (1, 3, 0), c = (2, 2, 2), d = (3, 0, 0).$$

By definition, his equivalence relation is reflexive: $\forall x \in X : x \sim x$. Since $a_2 = 4$ and $b_2 = 3$ are not noticeably different, there is no pair of consecutive matches in which teams a and b scored a noticeably different number of points. Therefore $a \sim b$ and $b \sim a$. Similarly $b \sim c, c \sim b, a \sim d$, and $d \sim a$. But team a scored a noticeably higher number of points in the most recent two matches than team c: $a_1 > c_1 + 1$ and $a_2 > c_2 + 1$. Hence $a \succ c$. Similarly $c \succ d$: the teams did not score a noticeably different number of points in the most recent match, but c performed noticeably better than d two and three weeks ago: $c_2 > d_2 + 1$ and $c_3 > d_3 + 1$. Finally, notice that b and d are incomparable: the number of points of both teams in the most recent two matches is noticeably different, but one week ago d was noticeably better than b ($d_1 > b_1 + 1$), whereas two weeks ago b was noticeably better than d ($b_2 > d_2 + 1$).

This relation satisfies the conditions on a preference structure. However, neither $\sim$ nor $\succ$ is transitive, since

$$a \sim b, b \sim c, a \succ c,$$

and

$$a \succ c, c \succ d, a \sim d.$$

$\triangleleft$

Consider a set X with preference structure $(\succ, \sim)$. A *path* in X is a finite sequence $(x_1, \ldots, x_m)$ of elements of X such that for each $k = 1, \ldots, m-1$, either $x_k \succ x_{k+1}$ or $x_k \sim x_{k+1}$. In the first case, we speak of a $\succ$-*connection* between x_k and x_{k+1}, in the second case of a $\sim$-*connection* between x_k and x_{k+1}. A *cycle* in X is a path $(x_1, \ldots, x_m)$ in X with at least two different elements of X and $x_1 = x_m$.

A path $(x_1, \ldots, x_m)$ in X has two consecutive $\sim$-connections if for some $k = 1, \ldots, m-2$: $x_k \sim x_{k+1}$ and $x_{k+1} \sim x_{k+2}$ or — in case the path is a cycle — if $x_1 \sim x_2$ and $x_{m-1} \sim x_m = x_1$.

Denote by $\rhd$ the composition of $\succ$ and $\sim$, i.e., for each $x, y \in X$:

$$x \rhd y \quad \Leftrightarrow \quad (\exists z \in X \;:\; x \succ z, \text{ and } z \sim y).$$

Since $\sim$ is reflexive, $x \succ y$ implies $x \rhd y$. The relation $\rhd$ is *acyclic* if its transitive closure is irreflexive, i.e., if there is no finite sequence $(x_1, \ldots, x_m)$ of elements of X such that $x_1 = x_m$ and for each $k = 1, \ldots, m-1$: $x_k \rhd x_{k+1}$.

A special case of a preference structure is an interval order (Fishburn, 1970). The preference structure $(\succ, \sim)$ is an *interval order* if for each $x, y \in X$

$$x \sim y \quad \Leftrightarrow \quad (\text{ not } x \succ y \text{ and not } y \succ x), \tag{9.1}$$

and for each $x, x', y, y' \in X$

$$(x \succ y \text{ and } x' \succ y') \quad \Rightarrow \quad (x \succ y' \text{ or } x' \succ y).$$

In interval orders, exactly one of the claims $x \succ y, y \succ x, x \sim y$ is true. Define the binary relation $\succeq$ on X by taking for each $x, y \in X$:

$$x \succeq y \quad \Leftrightarrow \quad \text{not } y \succ x.$$

Then it is easily seen that a preference structure satisfying (9.1) is an interval order if and only if for each $x, x', y, y' \in X$:

$$x \succ x' \succeq y' \succ y \quad \Rightarrow \quad x \succ y. \tag{9.2}$$

Hence, interval orders have transitive strict preference $\succ$. The preference structure of an interval order can be identified with the relation $\succ$, since the relations $\sim$ and $\succeq$ follow from $\succ$.

Lemma 9.2 *Let $\succ$ be an interval order on a set X. Then the relation $\rhd$ is acyclic.*

Proof. Suppose, to the contrary, that X contains a cycle $(x_1, y_1, x_2, y_2, \ldots, x_{m-1}, y_{m-1}, x_m)$ such that for each $k = 1, \ldots, m-1$: $x_k \succ y_k$ and $y_k \sim x_{k+1}$. Then $x_1 \succ y_1$ by definition. Moreover, $x_1 \succ y_1 \sim x_2 \succ y_2$, so (9.2) implies $x_1 \succ y_2$. Similarly, one shows that $x_1 \succ y_k$ for each $k = 1, \ldots, m-1$. In particular, $x_1 \succ y_{m-1}$. However, by definition of the cycle, $y_{m-1} \sim x_m = x_1$, so $x_1 \sim y_{m-1}$ by symmetry of $\sim$. But at most one of the two possibilities $x_1 \succ y_{m-1}$ and $x_1 \sim y_{m-1}$ is true, a contradiction. $\square$

9.3 The representation theorem

This section contains the main theorem and an application of this theorem to obtain a well-known characterization of interval orders.

Theorem 9.3 *Let X be a countable set and $(\succ, \sim)$ a preference structure on X. The following claims are equivalent.*

(a) *There exist functions $u : X \to \mathbb{R}$ and $t : X \to \mathbb{R}_+$ such that for all $x, y \in X$:*

$$x \succ y \quad \Rightarrow \quad u(x) > u(y) + t(y)$$

$$x \sim y \quad \Rightarrow \quad \begin{cases} u(x) \leqq u(y) + t(y), \\ \\ u(y) \leqq u(x) + t(x) \end{cases}$$

(b) *The relation $\rhd$ is acyclic;*

(c) *Every cycle in X contains at least two consecutive $\sim$-connections.*

Proof.
(a) $\Rightarrow$ (b): Assume (a) holds and suppose that $\rhd$ is cyclic. Take a sequence $(x_1, \ldots, x_m)$ of points in X such that $x_1 = x_m$ and for each $k = 1, \ldots, m-1 : x_k \rhd x_{k+1}$. Then for each such k there exists a $y_k \in X$ such that $x_k \succ y_k$ and $y_k \sim x_{k+1}$, which implies $u(x_k) > u(y_k) + t(y_k) \geqq u(x_{k+1})$. Hence $u(x_1) > u(x_2) > \ldots > u(x_m) = u(x_1)$, a contradiction.

(b) $\Rightarrow$ (c): Suppose $(x_1, \ldots, x_m)$ is a cycle in X without two consecutive $\sim$-connections. W.l.o.g. $x_1 \succ x_2$. Let $(y_1, \ldots, y_n)$ with $n \leqq m$ be the sequence of points in X obtained by removing from $(x_1, \ldots, x_m)$ all those points x_k $(k = 1, \ldots, m-1)$ satisfying $x_k \sim x_{k+1}$, i.e., all those points that are indifferent to the next point in the cycle. Notice that by construction $y_1 = x_1$, $y_n = x_m = x_1$, and for each $k = 1, \ldots, n-1$ there exists an $l \in \{1, \ldots, m-1\}$ such that

- either $y_k = x_l$ and $y_{k+1} = x_{l+1}$, in which case $y_k \succ y_{k+1}$, which implies $y_k \rhd y_{k+1}$,

- or $y_k = x_l$ and $y_{k+1} = x_{l+2}$, in which case $y_k \succ x_{l+1}$ and $x_{l+1} \sim y_{k+1}$, which also implies $y_k \rhd y_{k+1}$.

But then the sequence $(y_1, \ldots, y_n)$ indicates that $\rhd$ is cyclic.

(c) $\Rightarrow$ (a): Assume (c) holds. Since X is countable, write $X = \{x_k \mid k \in \mathbb{N}\}$. Call a path from x to y a good path if it does not contain two consecutive $\sim$-connections. Define for each $x \equiv x_k \in X$:

$$\begin{aligned}
S(x) &:= \{n \in \mathbb{N} \mid \text{there is a good path from } x \text{ to } x_n \text{ starting with a } \succ\text{-connection}\},\\
T(x) &:= \{n \in \mathbb{N} \mid \text{there is a good path from } x \text{ to } x_n\},\\
u(x) &:= \textstyle\sum_{n \in S(x)} 2^{-n},\\
v(x) &:= \textstyle\sum_{n \in T(x)} 2^{-n},\\
t(x) &:= 2^{-k-1} + v(x) - u(x).
\end{aligned}$$

We proceed to prove that u and t defined above give the desired representation.

- Clearly $S(x) \subseteq T(x)$, so $v \geqq u$ and $t > 0$.
- Let $x, x_k \in X, x \succ x_k$. Then $T(x_k) \subseteq S(x)$. Moreover, $k \in S(x)$, but $k \notin T(x_k)$, since by assumption every cycle in X has two consecutive $\sim$-connections. Hence $T(x_k) \subset S(x)$ and $k \in S(x) \setminus T(x_k)$. So $u(x) = v(x_k) + \sum_{n \in S(x) \setminus T(x_k)} 2^{-n} \geqq v(x_k) + 2^{-k} > v(x_k) + 2^{-k-1} = u(x_k) + t(x_k)$.
- Let $x, y \in X, x \sim y$. Then $S(y) \subseteq T(x)$. Hence $u(x) + t(x) > v(x) \geqq u(y)$ and similarly $u(y) + t(y) \geqq u(x)$.

This completes the proof. □

Remark 9.4 Luce (1956) considers nonnegative threshold functions, Fishburn (1970) and Bridges (1983) consider positive threshold functions. Our statement of (c) involves nonnegative threshold functions $t : X \to \mathbb{R}_+$. However, in the proof that (c) implies (a) we actually construct a positive function. Clearly, the proof that (a) implies (b) — and hence the theorem — also holds if t were required to be positive rather than nonnegative. The theorem was formulated with nonnegative threshold functions for intuitive reasons: there seems to be no reason to require that sufficiently perceptive decision makers need to have a positive threshold above which they can perceive changes in utility. ◁

An immediate corollary of this theorem is a well-known representation theorem of interval orders. See Fishburn (1970, Theorem 4) and Bridges (1983, Theorem 2).

Theorem 9.5 *Let X be a countable set and $\succ$ a binary relation on X. The following claims are equivalent.*

(a) *The relation $\succ$ is an interval order;*

(b) *There exist functions $u, v : X \to \mathbb{R}, v \geqq u$, such that for each $x, y \in X$, $x \succ y$ if and only if $u(x) > v(y)$;*

(c) *There exist functions $u, t : X \to \mathbb{R}, t > 0$, such that for each $x, y \in X$, $x \succ y$ if and only if $u(x) > u(y) + t(y)$.*

Proof. Obviously (c) $\Rightarrow$ (b) $\Rightarrow$ (a). That (a) $\Rightarrow$ (c) follows from Lemma 9.2, Remark 9.4, and Theorem 9.3. That $u(x) > u(y) + t(y)$ implies $x \succ y$ is clear: $y \succ x$ implies $u(y)+t(y) > u(y) > u(x)+t(x) > u(x)$ and $x \sim y$ implies $u(y)+t(y) \geqq u(x)$. In interval orders exactly one of the claims $x \succ y, y \succ x$, or $x \sim y$ holds, so one must have that $x \succ y$. □

9.4 Uncountable sets

In Theorem 9.3, the proof that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) holds for arbitrary, not necessarily countable, sets X. Moreover, it is easy to see that also (c) implies (b) for arbitrary sets. However, acyclicity of $\rhd$ does not imply the existence of the desired functions u, t if the set X is uncountable. This is not surprising: it is usually necessary to require additional assumptions to guarantee the existence of preference representing functions on uncountable sets. The purpose of this section is to indicate that such assumptions are not straightforward. Fishburn (1973) discusses representations of interval orders on uncountable sets.

The existence of functions u, t as in part (a) of Theorem 9.3 implies that

$$\forall x, y \in X : \quad x \rhd y \quad \Rightarrow \quad u(x) > u(y). \tag{9.3}$$

Hence, the existence of a function $u : X \to \mathbb{R}$ satisfying (9.3) is a necessary condition for the existence of functions u, t satisfying the conditions in Theorem 9.3a. However, it is not sufficient. Suppose such a function u exists. Without loss of generality, u is bounded (take $x \mapsto \arctan(u(x))$ if necessary). The function $t : X \to \mathbb{R}_+$ has to satisfy for each $x, y \in X$, if $y \succ x$, then $u(y) - u(x) > t(x)$ and if $y \sim x$, then $u(y) - u(x) \leqq t(x)$.

Define $\mathcal{S}(x) := \sup\{u(y) - u(x) \mid y \sim x\}$ and $\mathcal{I}(x) := \inf\{u(y) - u(x) \mid y \succ x\}$. Let $y \succ x, z \sim x$. Then $u(y) > u(z)$, so $\mathcal{S}(x) \leqq \mathcal{I}(x)$. Notice also that $\mathcal{S}(x) \geqq u(x) - u(x) = 0$. So if $\mathcal{S}(x) < \mathcal{I}(x)$, one can take $t(x) \in [\mathcal{S}(x), \mathcal{I}(x))$. However, if $\mathcal{S}(x) = \mathcal{I}(x)$, then the only candidate for $t(x)$ equals $\mathcal{S}(x)$. But to make sure that $u(y) - u(x) > t(x)$ for all y with $y \succ x$, we need the additional property that the infimum $\mathcal{I}(x)$ is not achieved.

The next example shows that in some cases there exists a function $u : X \to \mathbb{R}$ satisfying (9.3), but in which the last property is not satisfied.

Example 9.6 Take $X = \mathbb{R}$ and define for each $x, y \in \mathbb{R}$:

$$x \succ y \quad \Leftrightarrow \quad x \geqq y + 1,$$

$$x \sim y \quad \Leftrightarrow \quad |x - y| < 1.$$

Then

$$x \rhd y \Leftrightarrow \exists z \in \mathbb{R} : (x \geqq z + 1, |z - y| < 1) \Leftrightarrow \exists z \in \mathbb{R} : x \geqq z + 1 > y > z - 1 \Leftrightarrow x > y.$$

So $\rhd$ is acyclic and the set of functions preserving the order $\rhd$ is the set of strictly increasing functions $u : \mathbb{R} \to \mathbb{R}$. For every strictly increasing function u and every $x \in X$ we have that $\mathcal{I}(x) = \inf_{y \geqq x+1} u(y) - u(x) = u(x+1) - u(x)$. Hence the infimum is achieved. This means that a function t as in Theorem 9.3 exists if and only if there is an increasing function u such that

$$\forall x \in \mathbb{R} : \quad \mathcal{S}(x) < u(x + 1) - u(x),$$

i.e., an increasing function $u : \mathbb{R} \to \mathbb{R}$ such that for each $x \in \mathbb{R}$: $\sup_{y<x} u(y) < u(x)$. Suppose such a function u exists. We derive a contradiction by constructing an injective function f from the uncountable set $\mathbb{R} \setminus \mathbb{Q}$ to the countable set $\mathbb{Q}$. For each $x \in \mathbb{R} \setminus \mathbb{Q}$, take $f(x) \in \mathbb{Q}$ such that $\sup_{y<x} u(y) < f(x) < u(x)$. To show that f is injective, let $x, y \in \mathbb{R} \setminus \mathbb{Q}, x < y$. Then $f(x) < u(x) < \sup_{z<y} u(z) < f(y)$. $\lhd$

9.5 Ordinal games and potentials

To conclude this chapter, we return to the game theoretic set-up. Consider an *ordinal game* $G = \langle N, (X_i)_{i \in N}, (\succ_i, \sim_i)_{i \in N} \rangle$, where

- N is finite;
- for each player $i \in N$: X_i is countable, and
- for each player $i \in N$: $(\succ_i, \sim_i)$ is a preference structure over X.

Examples of games in which players may be easier characterized by means of preference structures instead of single real-valued payoff functions include multicriteria games, the topic of the second part of this thesis.

Using the representation theorem 9.3, it follows that in certain ordinal games the information concerning Nash equilibria can be summarized in a utility/potential function and a threshold function.

Theorem 9.7 *Let $G = \langle N, (X_i)_{i \in N}, (\succ_i, \sim_i)_{i \in N} \rangle$ be an ordinal game. The following claims are equivalent:*

(a) *There exist functions* $P : X \to \mathbb{R}$ *and* $T : X \to \mathbb{R}_+$ *such that for each* $i \in N$*, each* $x_{-i} \in X_{-i}$ *and each* $x_i, y_i \in X_i$*:*

$$(x_i, x_{-i}) \succ_i (y_i, x_{-i}) \Rightarrow P(x_i, x_{-i}) > P(y_i, x_{-i}) + T(y_i, x_{-i}).$$

(b) *Every cycle of unilateral deviations contains at least two consecutive deviations to strategies which the deviating players find equivalent.*

One of the main motivations for this chapter was to study preference structures that could be represented by means of a utility function and a threshold function. Without invoking threshold functions, one can sharpen the above theorem. The proof is completely analogous to that of Lemma 5.4.

Theorem 9.8 *Let* $G = \langle N, (X_i)_{i \in N}, (\succ_i, \sim_i)_{i \in N} \rangle$ *be an ordinal game. The following claims are equivalent:*

(a) *There exist functions* $P : X \to \mathbb{R}$ *such that for each* $i \in N$*, each* $x_{-i} \in X_{-i}$ *and each* $x_i, y_i \in X_i$*:*

$$(x_i, x_{-i}) \succ_i (y_i, x_{-i}) \Rightarrow P(x_i, x_{-i}) > P(y_i, x_{-i}).$$

(b) *There are no cycles of unilateral deviations in which each deviating player changes to an outcome he strictly prefers.*

Part II

Multicriteria Games

Chapter 10

Introduction to Part II

10.1 Multicriteria optimization

Multicriteria optimization extends optimization theory by permitting several — possibly conflicting — objective functions, which are to be 'optimized' simultaneously. By now an important branch of Operations Research (see Steuer *et al.*, 1996), it ranges from highly verbal approaches like Larichev and Moshkovich (1997) to highly mathematical approaches like Sawaragi *et al.* (1985), and is known by various other names, including: Pareto optimization, vector optimization, efficient optimization, and multiobjective optimization.

Formally, a multicriteria optimization problem can be formulated as

$$\begin{aligned} &\text{Optimize} && f_1(x), \ldots, f_r(x) \\ &\text{subject to} && x \in F, \end{aligned} \tag{10.1}$$

where F denotes the feasible set of alternatives and $r \in \mathbb{N}$ the number of separate criterion functions $f_k : F \to \mathbb{R}$ $(k = 1, \ldots, r)$.

The simultaneous optimization of multiple objective functions suggests the question: what does it mean to optimize, i.e., *what is a good outcome*? Different answers to this question lead to different ways of solving multicriteria optimization problems. The exact distinction between the methods is not always clear. For a detailed description and good introductions to the area, see White (1982), Yu (1985), and Zeleny (1982). Figure 10.1 lists several approaches. Below, their main ideas are briefly discussed.

Suppose a feasible set of outcomes is evaluated on the basis of two criterion functions, f_1 and f_2, each of which is desired to be as large as possible. Let the feasible set S in the objective space be the polytope in Figure 10.2. That is, for every point $s \in S$ there exists a feasible alternative x such that $(f_1(x), f_2(x)) = s$.

In finding *Pareto-optimal points*, there is a common distinction between *strongly* and *weakly* Pareto-optimal points. A feasible point in $\mathbb{R}^n$ is strongly Pareto-optimal if there

- Find the Pareto-optimal outcomes;
- Hierarchical optimization method;
- Trade-off method;
- Scalarization method, including
  - Weighted objectives method;
  - Distance function method;
  - Minmax optimization method;
- Goal programming method;

Figure 10.1: Methods of multicriteria optimization

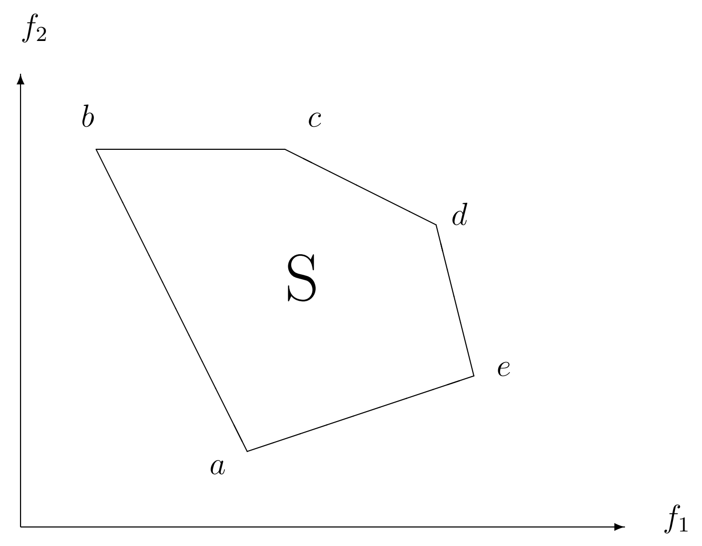


Figure 10.2: A 2-criterion problem

is no other feasible point which is larger in at least one coordinate and not smaller in all other coordinates. A feasible point in $\mathbb{R}^n$ is weakly Pareto-optimal if there is no other feasible point which is larger in each coordinate. In Figure 10.2, for instance, a is neither weakly nor strongly Pareto-optimal, b is weakly Pareto-optimal, but not strongly, since $b_1 < c_1$ and $b_2 = c_2$, and c is strongly Pareto-optimal. The set of weakly Pareto-optimal points consists of the line-segments (b, c), (c, d), and (d, e), whereas the set of strongly

Pareto-optimal points consists of the line-segments (c, d) and (d, e).

The *hierarchical optimization method* allows the decision maker to rank his criteria in order of importance. Starting with the most important criterion function, each function is then optimized individually, subject to possible additional constraints that restrict the feasible domain to points giving rise to values in the previously optimized functions that are not too far away from their optimal level. For instance, if in Figure 10.2 the first criterion is most important, the decision maker would start with maximizing f_1; the maximum of f_1 is e_1. In the next step, he would maximize f_2 subject to the feasibility constraints and the additional constraint that f_1 cannot be more than say 5 percent below e_1. If no such slack is allowed, i.e., if the optima of the k-th ranking objective function has to be determined subject to the constraint that the $k-1$ previous objective functions remain at their optimal level, one speaks about *lexicographic optimization.*

The *trade-off method*, also known as the constraint method, essentially chooses one of the objective functions as the function to optimize and imposes additional constraints on the remaining objective functions, restricting them to lie in a desirable range, for instance:

$$\begin{aligned} &\text{Optimize} \quad f_k(x) \\ &\text{subject to} \quad x \in F \\ &\qquad\qquad f_m(x) \in D_m \quad \forall m \in \{1, \ldots, k-1, k+1, \ldots, r\} \end{aligned}$$

There are several *scalarization methods*, where the multicriteria problem to be solved is reduced to a standard optimization problem with a single objective function by, first, defining a global criterion function $g : \mathbb{R}^r \to \mathbb{R}$ reflecting a suitable aggregate of the separate objective functions $f_1, \ldots, f_r$ in the multicriteria problem (10.1) and, second, solving

$$\begin{aligned} &\text{Optimize} \quad g(f_1(x), \ldots, f_r(x)) \\ &\text{subject to} \quad x \in F \end{aligned}$$

One scalarization method is the *weighted objectives method*, which assigns nonnegative weights $w_k \geqq 0$ $(k = 1, \ldots, r)$ to each of the objective functions f_k in the multicriteria optimization problem (10.1), reflecting the relative importance of the criteria and solves the scalarized problem

$$\begin{aligned} &\text{Optimize} \quad \textstyle\sum_{k=1}^{r} w_k f_k(x) \\ &\text{subject to} \quad x \in F, \end{aligned}$$

thereby reducing the problem to a standard optimization problem with a single objective function. If the feasible set in the objective space satisfies certain convexity conditions, it can be shown that all Pareto-optimal points can be found by suitable weightings of the criteria functions. See Theorem 10.1 in Section 10.3.

Another scalarization method is the *distance function method*. Here, the distance between the feasible set and an *ideal solution* is minimized. Consider Figure 10.2. In this problem, the optimal level of the first criterion is e_1 and the optimal level of the second criterion is $b_2 = c_2$. Hence a good candidate for the ideal solution would be the point (e_1, c_2). So let $I = (I_1, \dots, I_r)$ denote the ideal solution in (10.1), where for $k \in \{1, \dots, r\}$:

$$I_k := \max_{x \in F} \ f_k(x).$$

The distance function method then solves

$$\begin{array}{ll} \text{Minimize} & \left[\sum_{k=1}^{r} |f_k(x) - I_k|^p\right]^{1/p} \\ \\ \text{subject to} & x \in F, \end{array}$$

where $p \in [1, \infty]$ is a chosen parameter reflecting the actual norm that is optimized. Variants include:

1. Different types of distance functions;
2. Considering relative distances, i.e., minimizing

$$\left[\sum_{k=1}^{r} \left|\frac{f_k(x) - I_k}{I_k}\right|^p\right]^{1/p}. \tag{10.2}$$

Although usually treated separately in the literature, the *minmax optimization method* is essentially a distance function method which minimizes the maximal relative deviation of the individual objective functions from the ideal solution. Formally, it coincides with choosing the Tchebyshev norm ($p = \infty$) in (10.2):

$$\begin{array}{ll} \text{Minimize} & \max_{k \in \{1,\dots,r\}} \left|\frac{f_k(x) - I_k}{I_k}\right| \\ \\ \text{subject to} & x \in F. \end{array}$$

The *goal programming method*, introduced by Charnes and Cooper (1961), requires the decision maker to set goals for each of the objective functions. Let f_k^0 denote the goal for the k-th objective function in (10.1). Next, weighting factors $w_k \geqq 0$ $(k = 1, \dots, r)$ are assigned to rank the goals in order of importance. Finally, a single objective function is constructed by minimizing the deviations from the stated goals:

$$\begin{array}{lll} \text{Minimize} & \sum_{k=1}^{r} w_k(d_k^+ + d_k^-) & \\ \\ \text{subject to} & x \in F & \\ \\ & f_k(x) + d_k^- - d_k^+ = f_k^0 & \forall k \in \{1, \dots, r\} \\ \\ & d_k^- \geqq 0, d_k^+ \geqq 0 & \forall k \in \{1, \dots, r\} \end{array}$$

10.2 Multicriteria games

In matters of conflict, players frequently evaluate situations on the basis of several criteria. Selten (1994, p.42), for instance, regards any decision procedure as "guided by multiple goals, which are not easily comparable". Moreover, "[s]uch procedures seek to avoid tradeoffs among different goal dimensions", i.e., should not be modelled on the basis of a single goal function, but should explicitly take into account the multiple criteria that are of relevance to the decision.

The second part of this thesis studies multicriteria games, also appearing in the literature as 'games with vector payoffs' or 'multiobjective games'. The main focus is on noncooperative games. Multicriteria games were first studied by Blackwell (1956), who provides an analog of the minimax theorem for repeated zero-sum games with vector payoffs in terms of approachable and excludable sets. These notions reflect the extent to which a player can control the trajectory of the average payoffs. A subset of the payoff space is approachable if a player through repeated play of a zero-sum game can force the average payoff to approach this set and excludable if the average payoff can be kept away from this set. Blackwell's theorem is one of the central results in the theory of repeated games with incomplete information. See Aumann and Maschler (1995).

Formally, a *(noncooperative) multicriteria game* is a tuple $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where

- $N \subset \mathbb{N}$ is a nonempty, finite set of players;
- each player $i \in N$ has a nonempty set X_i of pure strategies, and
- for each player $i \in N$ the function $u_i : \prod_{j \in N} X_j \to \mathbb{R}^{r(i)}$ maps each strategy combination to a point in $r(i)$-dimensional Euclidean space.

The interpretation of the function u_i is that player $i \in N$ considers not just one, but $r(i) \in \mathbb{N}$ different criteria.

A player $i \in N$ can for instance be seen as a set of individuals, as an organization with $r(i)$ members, each having his own utility function. Under this interpretation of a multicriteria game we have an interesting aggregate of conflicts: the *organizations* $i \in N$ are engaged in a noncooperative game in which the *members* $k = 1, \dots, r(i)$ of an organization i jointly have to agree on a strategy choice.

In fact, this is a feature that all games in the second part of this thesis have in common: they are based on an aggregate of conflicts, namely conflicts between the players, but also between the criteria a specific player takes into account, i.e., the relevant characteristics by which a decision maker evaluates his strategic possibilities.

Notice that strategic games can be seen as multicriteria games in which each of the players has exactly one criterion. The Nash equilibrium concept for strategic games requires that each player plays a best response against the strategy combination of his opponents. If a player has a real-valued utility function, best responses are unambiguously defined as those strategies to which there is no alternative strategy yielding a higher utility. As observed in the previous section, the selection of good outcomes in the presence of multiple goal/utility functions is less clear. In fact, different solution concepts can be defined by giving different answers to the question 'What is a best response?' This topic is taken up in Chapters 11 and 13.

The Pareto equilibrium concept of Chapter 11 defines best responses as those strategies yielding a Pareto-optimal outcome. Pareto equilibria were first introduced by Shapley (1959) in the context of two-player zero-sum games with vector payoffs. The definition easily extends to more general classes of multicriteria games. See Borm *et al.* (1989), Kruś and Bronisz (1994), Wang (1991, 1993), Zeleny (1975). Chapter 11 studies the properties of the Pareto equilibrium concept and provides several axiomatic characterizations. Chapter 12 considers the structure of the set of Pareto equilibria in two-person multicriteria games, a topic that is, among other things, of computational interest.

In Chapter 13 three other solution concepts for noncooperative multicriteria games are defined:

- *compromise equilibria*, where players choose those strategies as best responses that are closest to the ideal outcome. This concept is closely related to the distance function method described in the previous section.

- *Nash bargaining equilibria*, where players choose those strategies as best responses that yield a bargaining solution far away from a disagreement point. This concept is closely related to the game theoretic literature on bargaining.

- *perfect equilibria*, a refinement of Pareto equilibria that is motivated by the refinement literature in noncooperative game theory.

Chapter 14 considers Pareto-optimal security strategies in two-person zero-sum games with multiple criteria. Pareto-optimal security strategies were introduced by Ghose and Prasad (1989) as 'cautious' strategies, in the sense that a player checks, for each of his strategy choices, what is the worst that can happen to him in each criterion separately. In this way, a player assigns to each strategy a 'security vector' that specifies the worst-case scenario if this strategy is chosen. A Pareto-optimal security strategy is a strategy that gives rise to the most agreeable worst-case scenario. Several characterizations of Pareto-optimal security strategies are provided. In particular, Pareto-optimal security strategies will be seen to coincide with minimax strategies of a standard matrix game, a two-person zero-sum game with only one criterion in which each player chooses a mixed strategy.

Cooperative multicriteria games are studied in Chapter 15. A distinction is made between indivisible, public criteria that take the same value for all members of a coalition of players, and divisible, private criteria, that can be freely divided over the coalition members. The chapter mainly focuses on a core concept for cooperative multicriteria games. This core concept is axiomatized and additional motivation for the core concept is provided by showing that core elements naturally arise as strong equilibria of associated noncooperative claim games, in which players independently state coalitions they want to form and the payoff they want to receive.

Chapter 16 proposes and analyzes a model for boundedly rational behavior of players in interactive situations that can be modelled as an ordinal game. The model focuses on best replies, the set of actions a player cannot improve upon given the action profile of his opponents. If a player ends up playing an action that is not a best reply against the actions taken by his opponents, he may feel regret for not having done the right thing. The anticipation of regret may influence the decision making and determine the behavior of players. Chapter 16 suggests matching behavior as an explanation of how this influence may work. Matching is observed in numerous experimental situations of decision making under uncertainty and essentially means that an alternative is chosen with a probability proportional to its value. A common explanation of the matching phenomenon involves agents who do not believe that the mechanism causing the uncertainty is entirely random and try to 'outguess' the mechanism by trying to decipher the pattern. This explanation is particularly appealing in interactive situations where players are confronted with other players, rather than with nature.

Chapter 17 studies aggregate conflicts that arise through the uncertainty of players about the exact game that is played. The random games that are introduced incorporate uncertainty about all characteristics of the game: its player set, the action sets, as well as the preferences of the involved players. Having to decide upon a course of action in such an environment allows unforeseen contingencies to frustrate the implementation of action choices. Maximum likelihood equilibria are introduced, a solution concept that selects those actions that are most likely to end up in a good outcome of the random game.

The above description of the chapters in the second part of this thesis was purposely kept short. For more detailed introductions refer to the first sections of the respective chapters.

10.3 Preliminaries

This section contains definitions and matters of notation, additional to those provided in Section 1.4.

Three classes of noncooperative games are considered in the following chapters. We write

- Γ_{finite} for the set of finite multicriteria games, i.e., multicriteria games with finitely many players, each player having finitely many pure strategies, and in which mixed strategies are not allowed;

- Γ for the set of mixed extensions of finite multicriteria games;
- $\Gamma_{strategic}$ for the set of mixed extensions of finite games in strategic form, where $\Gamma_{strategic} \subset \Gamma$, since strategic games are multicriteria games in which each player has only one criterion.

In all three cases, one needs to specify the set N of players, sets $(X_i)_{i \in N}$ of pure strategies, and payoff functions $(u_i)_{i \in N}$, so we adopt the generic notation $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ for games in all three classes and indicate where necessary with notation like $G \in \Gamma_{finite}$, $G \in \Gamma$, and/or $G \in \Gamma_{strategic}$ whether or not mixed strategies and multidimensional payoffs are allowed. The number of criteria of player $i \in N$ is denoted by $r(i) \in \mathbb{N}$. The mixed strategy set of player $i \in N$ is denoted $\Delta(X_i)$ or Δ_i if the set of pure strategies is understood. Mixed extensions are defined as in Section 1.4. Conventional game theoretic notation is used:

$$\Delta_{-i} = \prod_{j \in N \setminus \{i\}} \Delta_j \qquad \Delta = \prod_{i \in N} \Delta_i$$

Write $X_i = \{1, \ldots, m(i)\}$, where $m(i) = |X_i|$ is the number of pure strategies. The mixed strategy that assigns probability one to pure strategy $k \in X_i$ is denoted by $e_k \in \Delta_i$. Mixed strategies of player i are sometimes denoted by σ_i and sometimes by $x_i \in \Delta(X_i)$. In the second case, x_{ik} denotes the probability assigned to the k-th strategy in X_i. The *carrier* of $x_i \in \Delta(X_i)$ is the set $\{k \in X_i \mid x_{ik} > 0\}$ of pure strategies that are played with positive probability in x_i.

For a real number $x \in \mathbb{R}$,

$$|x| = \begin{cases} x & \text{if} \quad x \geqq 0, \\ -x & \text{if} \quad x < 0, \end{cases}$$

denotes the absolute value of x. For two vectors $x, y \in \mathbb{R}^n$, $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$ denotes the inner product of x and y. Let $m, n \in \mathbb{N}$. The set of $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m \times n}$.

The unit simplex in $\mathbb{R}^n$ is denoted by Δ_n, its relative interior by Δ_n^0:

$$\Delta_n := \{x \in \mathbb{R}^n_+ \mid \sum_{k=1}^n x_i = 1\} \qquad \Delta_n^0 := \{x \in \mathbb{R}^n_{++} \mid \sum_{k=1}^n x_i = 1\}$$

For two subsets A and B of a vector space V we define $A + B = \{a + b \mid a \in A, b \in B\}$.

Let $A \subseteq \mathbb{R}^n$ be a finite set. Its convex hull, consisting of all convex combinations of elements in A, is denoted conv (A) and is called a polytope. The comprehensive hull of a set $A \subseteq \mathbb{R}^n$ is denoted compr (A):

$$\text{compr } (A) := \{b \in \mathbb{R}^n \mid b \leqq a \text{ for some } a \in A\}.$$

Semi-algebraic sets in $\mathbb{R}^n$ are solution sets to systems of polynomial inequalities: a set $A \subseteq \mathbb{R}^n$ is *semi-algebraic* if it is the finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_1(x) \leqq 0, \ldots, f_m(x) \leqq 0\},$$

where $f_k : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function for each $k = 1, \ldots, m$.

For vectors $a, b \in \mathbb{R}^n$, we write

$$\begin{aligned} a = b &\Leftrightarrow \forall k \in \{1, \ldots, n\} : a_k = b_k \\ a \geqq b &\Leftrightarrow \forall k \in \{1, \ldots, n\} : a_k \geqq b_k \\ a \geq b &\Leftrightarrow a \geqq b, \text{ and } a \neq b \\ a > b &\Leftrightarrow \forall k \in \{1, \ldots, n\} : a_k > b_k \end{aligned}$$

Relations $\leqq, \leq, <$ are defined analogously.

Let $A \subseteq \mathbb{R}^n$ and $a, b \in A$. Then a *weakly (Pareto) dominates* b if $a \geq b$ and a *strongly (Pareto) dominates* b if $a > b$. The *weak Pareto edge* of A is the set

$$\{a \in A \mid \nexists b \in A : b > a\}.$$

The *strong Pareto edge* of A is the set

$$\{a \in A \mid \nexists b \in A : b \geq a\}.$$

Be cautious: elements of the *weak* Pareto edge are those points that are not *strongly* dominated; elements of the *strong* Pareto edge are those points that are not *weakly* dominated. Elements of the weak (strong) Pareto edge of A are called weakly (strongly) Pareto optimal.

Under convexity conditions, Pareto-optimal points can be found by assigning weights to the separate criteria and maximizing the weighted sum of the coordinates.

Theorem 10.1 *Let $C \subseteq \mathbb{R}^n$ and $c \in C$.*

(i) *If C is convex, then c is weakly Pareto-optimal if and only if there exists a $\lambda \in \Delta_n$ such that for all $d \in C$: $\langle c, \lambda \rangle \geqq \langle d, \lambda \rangle$;*

(ii) *If C is a polytope, then c is strongly Pareto-optimal if and only if there exists a $\lambda \in \Delta_n^0$ such that for all $d \in C$: $\langle c, \lambda \rangle \geqq \langle d, \lambda \rangle$.*

Proof. The 'if' parts of the theorem are straightforward. We start by proving the 'only if' part of (i).

Take $B = \{x \in \mathbb{R}^n \mid \exists y \in C : y > x\}$ and $A = B \cup C$. Then A is convex, c is weakly Pareto-optimal in A and c lies on the boundary of A. Hence there exists a hyperplane with normal $\lambda \in \mathbb{R}^n \setminus \{0\}$ supporting A at c:

$$\forall d \in A : \quad \langle c, \lambda \rangle \geqq \langle d, \lambda \rangle.$$

Without loss of generality $\sum_{k=1}^n \lambda_k = 1$. To see that $\lambda \geq 0$, let $d \in \mathbb{R}^n, d < c$, (so $d \in B$), and $k \in \{1, \ldots, n\}$. Define for all $m \in \mathbb{N} : d_m = d - me_k$. Then $c > d \geq d_m$ implies that $d_m \in B \subseteq A$, so

$$\langle c, \lambda \rangle \geqq \langle d_m, \lambda \rangle = \langle d, \lambda \rangle - m\lambda_k$$

for all $m \in \mathbb{N}$. So $\lambda_k \geqq 0$. Since this holds for all $k \in \{1, \ldots, n\}$ and $\lambda \in \mathbb{R}^n \setminus \{0\}$, it follows that $\lambda \in \Delta_n$.

Next, the proof of the 'only if' part of (ii). Since C is a polytope, there exist finitely many vectors $v^1, \ldots, v^s \in \mathbb{R}^n$ and real numbers $\alpha_1, \ldots, \alpha_s$ such that $C = \cap_{k=1}^s \{x \in \mathbb{R}^n \mid \langle v^k, x\rangle \leqq \alpha_k\}$. Let $I = \{k \in \{1, \ldots, s\} \mid \langle v^k, c\rangle = \alpha_k\}$. This set is nonempty, otherwise there would be a sufficiently small $\varepsilon > 0$ such that $(c_1 + \varepsilon, \ldots, c_n + \varepsilon) \in C$, which would strongly Pareto-dominate c.

Let M be the $n \times |I|$-matrix with columns $\{v^k \mid k \in I\}$. Then there is no $x \in \mathbb{R}^n$ solving $xM \leqq 0, x \geq 0$. Otherwise, this would imply that $c + \varepsilon x \in C$ for sufficiently small $\varepsilon > 0$. But then $c + \varepsilon x \geq c$, contradicting strong Pareto-optimality of c.

By the duality theorem (cf. Gale, 1960, p.49, Theorem 2.10), there is a vector $y \in \mathbb{R}^{|I|}$ such that $My > 0$ and $y \geqq 0$. Take $\lambda = My \in \mathbb{R}^n_{++}$ and let $d \in C$. Since $\langle v^k, d\rangle \leqq \alpha^k = \langle v^k, c\rangle$ for all $k \in I$, it follows that $dM \leqq cM$. Consequently, $\langle c, \lambda\rangle = cMy \geqq dMy = \langle d, \lambda\rangle$. It is clear that λ can be normalized to add up to one, finishing the proof. □

For two nonempty sets $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^\ell$, a correspondence $F : X \twoheadrightarrow Y$ is a function that assigns to each element $x \in X$ an element of $2^Y \setminus \{\emptyset\}$, i.e., a nonempty subset $F(x)$ of Y. F is called *upper semicontinuous (u.s.c.)* in $x \in X$ if for every open neighborhood V of $F(x)$ there exists an open neighborhood U of x with $F(x') \subseteq V$ for every $x' \in U$. F is *upper semicontinuous (u.s.c.)* on X if F is u.s.c. in each $x \in X$.

Kakutani's fixed point theorem is a often used to prove the existence of equilibria.

Theorem 10.2 [Kakutani's fixed point theorem] *Let C be a nonempty, compact, convex subset of $\mathbb{R}^p$ and let $F : C \twoheadrightarrow C$ be an u.s.c. correspondence such that $F(x)$ is nonempty, compact, convex for each $x \in C$. Then there exists an element $c \in C$ such that $c \in F(c)$.*

The maximum theorem shows that the maxima of parametric optimization problems are well-behaved.

Theorem 10.3 [Maximum theorem] *Let X and Y be metric spaces, Y compact, and $f : X \times Y \to \mathbb{R}$ continuous. Then $m : x \mapsto \max_{y \in Y} f(x, y)$ is continuous and $M : x \mapsto \{y \in Y \mid f(x, y) = m(x)\}$ is u.s.c.*

Chapter 11

Pareto Equilibria in Noncooperative Multicriteria Games

11.1 Introduction

The Nash equilibrium concept relies on the stability property that single players — given the strategy profile of their opponents — cannot deviate to a better outcome. In single-criterion games, this is a clear statement: each player's incentives are unambiguously described by his real-valued utility function. An incentive to deviate just means having an alternative that yields a higher utility. In multicriteria games, the question 'What is a *good* outcome?' does not have such a clear answer. In fact, several extensions of the Nash equilibrium concept to multicriteria games can be introduced, depending on the answer to this question.

This chapter considers two extensions of the Nash equilibrium concept to noncooperative multicriteria games. They are based on weak and strong Pareto dominance. Shapley (1959) was the first to introduce such equilibrium points in two-person zero-sum games with multiple criteria. Zeleny (1975) addresses the same issue. Borm *et al.* (1989) extend the analysis of Shapley to general two-person multicriteria games.

Definitions of weak and strong Pareto equilibria are given in Section 11.2. Pareto equilibria are characterized as Nash equilibria in suitably weighted single-criterion games, thus providing a simple existence proof. Other properties are mentioned in the same section.

In a recent manifesto Bouyssou *et al.* (1993) observe that within multicriteria decision making '[a] systematic axiomatic analysis of decision procedures and algorithms is yet to be carried out'. In the second part of this chapter, based on Voorneveld, Vermeulen, and Borm (1999), several axiomatizations of the Pareto equilibrium concept for multicriteria games are provided.

Axiomatic properties of the Nash equilibrium concept based on the notion of consistency have been studied in several articles, including Peleg and Tijs (1996), Peleg, Potters, and Tijs (1996), and Norde *et al.* (1996). Informally, consistency requires that if a strategy combination x is a solution of a game with player set N, and players outside a coalition S of players commit to playing according to $x_{N \setminus S}$, i.e. the strategy combination restricted to the players in $N \setminus S$, then x_S is a solution of the reduced game. Several of these axiomatizations carry over to multicriteria games. The strong result of Norde *et al.* (1996), characterizing the Nash equilibrium concept on the set of mixed extensions of finite strategic form games by nonemptiness, the selection of utility maximizing strategies in one-person games, and consistency, does not have such an analogon in multicriteria games: we show that nonemptiness, consistency and an immediate extension of utility maximization are not sufficient to axiomatize the Pareto equilibrium concept. An additional property is provided to establish an axiomatization.

11.2 Pareto equilibria

Weak and strong Pareto equilibria are relatively straightforward extensions of the Nash equilibrium concept that rule out unilateral deviations to strategies that are better in the sense of the orders $>$ and $\geq$ on a finite dimensional Euclidean space. This section provides definitions of Pareto equilibria, points out some properties that are different from the Nash equilibrium concept, and characterizes Pareto equilibria as Nash equilibria of weighted games.

Definition 11.1 Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ be a multicriteria game. A strategy profile $x \in \prod_{i \in N} \Delta(X_i)$ is a

- *weak Pareto equilibrium* if for each player $i \in N$ there does not exist a $y_i \in \Delta(X_i)$ such that $u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$;
- *strong Pareto equilibrium* if for each player $i \in N$ there does not exist a $y_i \in \Delta(X_i)$ such that $u_i(y_i, x_{-i}) \geq u_i(x_i, x_{-i})$.

The set of weak and strong Pareto equilibria of G are denoted by $WPE(G)$ and $SPE(G)$, respectively. ◁

It is clear that every strong Pareto equilibrium is a weak Pareto equilibrium, but not the other way around. For a concrete example, refer to Figure 11.2 in Section 11.5. Weak and strong Pareto equilibria of multicriteria games in Γ_{finite}, in which mixed strategies are not allowed, are — of course — defined in a similar way by restricting attention to pure strategies. A multicriteria game in which each of the players has only one criterion is simply a strategic game. In the case of strategic games, the sets of weak and strong Pareto equilibria coincide with the set of Nash equilibria.

Alternatively, Pareto equilibria can be characterized as fixed points of certain best-response correspondences. Formally, let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a multicriteria game, $x \in \prod_{i \in N} \Delta(X_i)$ a strategy profile and $i \in N$ a player. Define

$$\begin{aligned} WB_i(x_{-i}) &= \{x_i \in \Delta(X_i) \mid \nexists y_i \in \Delta(X_i) : u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})\}, \\ WB : x &\mapsto \prod_{i \in N} WB_i(x_{-i}), \\ SB_i(x_{-i}) &= \{x_i \in \Delta(X_i) \mid \nexists y_i \in \Delta(X_i) : u_i(y_i, x_{-i}) \geq u_i(x_i, x_{-i})\}, \\ SB : x &\mapsto \prod_{i \in N} SB_i(x_{-i}), \end{aligned}$$

the natural counterparts of the best-response correspondence for weak and strong Pareto equilibria. The fixed points of WB and SB are exactly the weak and strong Pareto equilibria. In some cases, when the game G needs to be stressed to avoid confusion, we write $WB_i(G, \cdot)$, etc.

The following example is taken from Van Megen *et al.* (1999).

Example 11.2 Consider a game G with an inspector (player 1) who has to decide whether or not to inspect a factory (player 2) to check if its production is hygienical. The inspector has two objectives: to minimize inspection costs and to guarantee an acceptable level of hygiene in production. The factory also has two objectives: to minimize production costs and to achieve some level of hygienical production. The strategies the inspector can take are Inspection (I) and No Inspection (NI); the factory chooses between Hygienical (H) or Non-Hygienical (NH) production. Payoffs are as given below. Here $c > 1$ denotes the penalty that is imposed if the inspected production fails to

	H	NH
I	$(-1, 1)$	$(c-1, \frac{1}{2})$
NI	$(0, 1)$	$(0, 0)$

Payoffs to inspector

	H	NH
I	$(-1, 1)$	$(-c-1, 1)$
NI	$(-1, 1)$	$(0, 0)$

Payoffs to factory

be hygienical. The first coordinate of the payoff to player 1 denotes the negative costs of inspection, the second coordinate specifies satisfaction with the hygienical situation. The first coordinate for the factory depicts extra negative production costs, the second represents the hygiene satisfaction level.

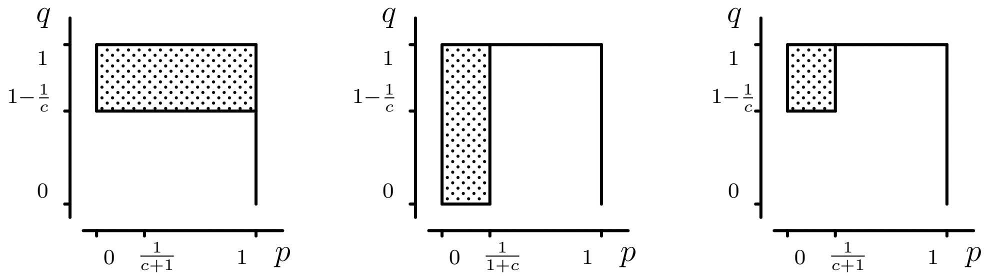
Let $p \in [0, 1]$ denote the probability of player 1 playing I and let $q \in [0, 1]$ denote the probability of player 2 playing H. Then $u_1(1, q) = (-qc + c - 1, \frac{1}{2} + \frac{1}{2}q)$ and $u_1(0, q) = (0, q)$. Hence

$$WB_1(q) = \begin{cases} \{1\} & \text{if} \quad 0 \leqq q < 1 - \frac{1}{c} \\ \\ [0, 1] & \text{if} \quad 1 - \frac{1}{c} \leqq q \leqq 1 \end{cases}$$

Further, $u_2(p,1) = (-1,1)$ and $u_2(p,0) = (p(-c-1), p)$. Hence

$$WB_2(p) = \begin{cases} [0,1] & \text{if} \quad 0 \leqq p \leqq \frac{1}{c+1} \text{ or } p = 1 \\ \{1\} & \text{if} \quad \frac{1}{c+1} < p < 1 \end{cases}$$

This implies that $WPE(G) = ([0, \frac{1}{c+1}] \times [1 - \frac{1}{c}]) \cup ((\frac{1}{c+1}, 1) \times \{1\}) \cup (\{1\} \times [0,1])$. The



$WB_1(q)$ $WB_2(p)$ $WPE(G)$

analysis above shows that weak Pareto equilibria in this model are those in which there is full inspection, those in which the factory produces in a hygienical way with probability 1, and those in which the chance upon inspection is small, but the production is nevertheless hygienical with a relatively high probability. Moreover, it is seen that a higher penalty c shrinks the equilibrium set to equilibria that favor more hygienical production.

Similar computations show

$$SB_1(q) = \begin{cases} \{1\} & \text{if} \quad 0 \leqq q \leqq 1 - \frac{1}{c} \\ [0,1] & \text{if} \quad 1 - \frac{1}{c} < q < 1 \\ \{0\} & \text{if} \quad q = 1 \end{cases}$$

and

$$SB_2(p) = \begin{cases} [0,1] & \text{if} \quad 0 \leqq p < \frac{1}{c+1} \\ \{1\} & \text{if} \quad \frac{1}{c+1} \leqq p \leqq 1 \end{cases}$$

This implies that $SPE(G) = (\{0\} \times (1 - \frac{1}{c}, 1]) \cup ((0, \frac{1}{c+1}) \times (1 - \frac{1}{c}, 1))$. ◁

A first peculiar feature of Pareto equilibria that is worth noting, is the following. In single-criterion games, every pure Nash equilibrium is also a mixed Nash equilibrium. This is no longer the case when multiple criteria are taken into account.

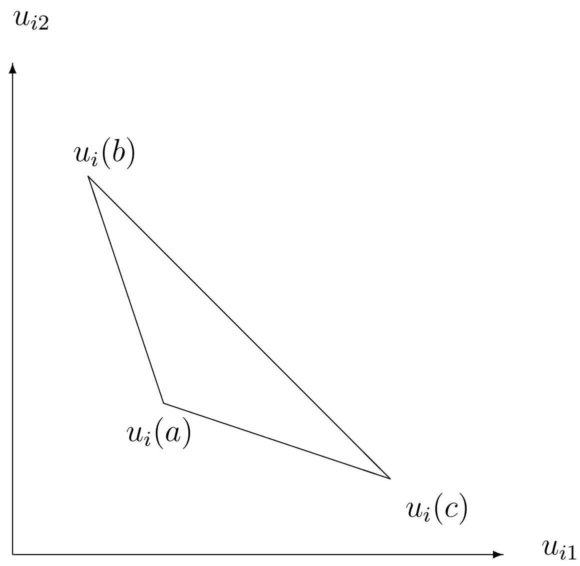


Figure 11.1: Pure, not mixed equilibria

Example 11.3 Consider a one-player game in which the unique player i has two criteria, three pure strategies a, b, and c and payoffs $u_i = (u_{i1}, u_{i2})$ as in Figure 11.1. All three pure strategies are both weak and strong Pareto equilibria when only pure strategies are allowed. But when mixtures are taken into account, strategy a is dominated. $\triangleleft$

A second point of interest is that a mixture of two points that are not Pareto dominated may be Pareto dominated. Refer for instance to Figure 10.2 and take a convex combination of points c and e. This implies that the well-known characterization of Nash equilibria, according to which a strategy profile is a Nash equilibrium if and only if each pure strategy that is played with positive probability is a pure best reply against the strategy profile of the opponents, does not hold for Pareto equilibria. An analogous characterization exists, however, when carriers are restricted to the faces of the payoff polytope that are contained in the set of Pareto optimal points.

Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a multicriteria game, $x \in \prod_{i\in N} \Delta(X_i)$ a strategy profile and $i \in N$ a player. Take $C(G, x_i) = \{k \in X_i \mid x_{ik} > 0\}$, the *carrier* of x_i, as the set of pure strategies k in X_i that are played with positive probability. A set $I \subseteq X_i$ of pure strategies is called *weakly efficient* for player i against x_{-i} if for all $x_i \in \Delta(X_i)$ with $C(G, x_i) \subseteq I$ it holds that $x_i \in WB_i(x_{-i})$. A set $I \subseteq X_i$ is a *weakly efficient pure best reply set* for player i against x_{-i} if it is weakly efficient and there is no $K \subseteq X_i$ with $I \subset K$ such that K is weakly efficient. Let $\mathcal{E}_i(G, x_{-i})$ be the set of weakly efficient pure best reply sets for player i against x_{-i}. The following result in terms of weak Pareto equilibria is stated without proof. The analogon for strong Pareto equilibria is straightforward.

Proposition 11.4 *Let* $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ *be a multicriteria game and* $x \in \prod_{i \in N} \Delta(X_i)$. *Then for all* $i \in N$:

$$x_i \in WB_i(x_{-i}) \quad \Leftrightarrow \quad C(G, x_i) \subseteq I \text{ for some } I \in \mathcal{E}_i(G, x_{-i}).$$

The weighted objectives method for solving multicriteria problems involves assigning weights to each of the criteria, reflecting their relative importance. Consider a multicriteria game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ in which player i has $r(i) \in \mathbb{N}$ criteria. For each $i \in N$, let $\lambda_i \in \Delta_{r(i)}$ be a vector of weights for the criteria, $\lambda := (\lambda_i)_{i \in N}$. The λ-*weighted game* G_λ is the strategic form game with player set N, mixed strategy spaces $(\Delta(X_i))_{i \in N}$, and payoff functions $(v_i)_{i \in N}$ defined for all $i \in N$ and $x \in \prod_{i \in N} \Delta(X_i)$ by $v_i(x) = \langle \lambda_i, u_i(x) \rangle = \sum_{k=1}^{r(i)} \lambda_{ik} u_{ik}(x)$. If each player assigns equal weight to all his criteria, i.e., $\lambda_i = \frac{1}{r(i)}(1, \ldots, 1) \in \mathbb{R}^{r(i)}$ for all $i \in N$, the weighted game is denoted by G_e. The following theorem, stating that Pareto equilibria are exactly the Nash equilibria of weighted games for suitable weight vectors, is due to Shapley (1959). Shapley stated the theorem for two-person zero-sum multicriteria games, but the result extends immediately to more general games.

Theorem 11.5 *Let* $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ *be a multicriteria game and* $x \in \prod_{i \in N} \Delta(X_i)$. *Then*

- $x \in WPE(G)$ *if and only if there exists for each* $i \in N$ *a vector of weights* $\lambda_i \in \Delta_{r(i)}$ *such that* $x \in NE(G_\lambda)$;
- $x \in SPE(G)$ *if and only if there exists for each* $i \in N$ *a vector of weights* $\lambda_i \in \Delta^0_{r(i)}$ *such that* $x \in NE(G_\lambda)$.

The proof of this theorem follows easily from Theorem 10.1. See also Zeleny (1975), Borm *et al.* (1989), and Kruś and Bronisz (1994). As a corollary, Pareto equilibria always exist in mixed extensions of finite multicriteria games, since for any vector of weights the game G_λ has Nash equilibria in mixed strategies (Nash, 1950a, 1951). Wang (1991, 1993) provides existence results for Pareto equilibria in a larger class of games, mainly based on the Kakutani fixed point theorem.

11.3 The consistency axiom

The next two sections are devoted to axiomatizing the weak Pareto equilibrium concept. The main axiom is consistency, which requires that if a strategy combination x is a solution of a game with player set N and each player i that is not a member of a coalition $S \subseteq N$ commits to playing his strategy x_i, then x_S, the strategy profile restricted to the remaining players, is a solution of the reduced game.

To avoid confusion about the players involved in a game, multicriteria games are sometimes denoted by $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle$, where N^G is the player set of the game G. Another matter of crucial importance in the remainder of this chapter is whether or not mixed strategies and multiple criteria are allowed. Therefore, special care is taken to specify whether a game is contained in Γ, Γ_{finite}, or $\Gamma_{strategic}$.

Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma$ be a multicriteria game, let $x \in \prod_{i \in N^G} \Delta(X_i)$ be a strategy profile, and let $S \in 2^{N^G} \setminus \{\emptyset, N^G\}$ be a proper subcoalition of the player set N^G. The *reduced game* $G^{S,x}$ of G with respect to S and x is the multicriteria game in Γ in which

- the player set is S;
- each player $i \in S$ has the same set X_i of pure strategies as in G;
- the payoff functions $(u'_i)_{i \in S}$ are defined by $u'_i(y_S) := u_i(y_S, x_{N^G \setminus S})$ for all $y_S \in \prod_{i \in S} \Delta(X_i)$.

Notice that this is the game that arises if the players in $N^G \setminus S$ commit to playing according to $x_{N^G \setminus S}$, the strategy combination restricted to the players in $N^G \setminus S$. Definitions for reduced games on Γ_{finite} and $\Gamma_{strategic}$ are completely analogous.

A *solution concept* on Γ is a function ρ which assigns to each element $G \in \Gamma$ a subset $\rho(G) \subseteq \prod_{i \in N^G} \Delta(X_i)$ of strategy combinations. Analogously one defines a solution concept on $\Gamma_{strategic}$ or Γ_{finite}. Clearly, WPE and SPE, the functions that assign to a multicriteria game its set of weak and strong Pareto equilibria, respectively, are solution concepts.

For strategic form games, we recall the following axioms. A solution concept ρ on $\Gamma_{strategic}$ satisfies:

- **Nonemptiness (NEM)**, if $\rho(G) \neq \emptyset$ for all $G \in \Gamma_{strategic}$;
- **Utility Maximization (UM)**, if for each one-player game $G = \langle \{i\}, X_i, u_i \rangle \in \Gamma_{strategic}$ we have that $\rho(G) \subseteq \{x \in \Delta(X_i) \mid u_i(x) \geq u_i(y) \ \forall y \in \Delta(X_i)\}$;
- **Consistency (CONS)**, if for each game $G \in \Gamma_{strategic}$, each proper subcoalition $S \in 2^{N^G} \setminus \{\emptyset, N^G\}$, and each element $x \in \rho(G)$, we have that $x_S \in \rho(G^{S,x})$.

Norde *et al.* (1996) prove:

Proposition 11.6 *A solution concept ρ on $\Gamma_{strategic}$ satisfies NEM, UM, and CONS if and only if $\rho = NE$, the Nash equilibrium concept.*

This yields the conclusion that there is no proper refinement of the Nash equilibrium concept that satisfies NEM, UM, and CONS.

11.4 Finite multicriteria games

Peleg and Tijs (1996) and Peleg, Potters, and Tijs (1996) provide several axiomatizations of the Nash equilibrium concept for finite strategic form games. In this section two of these axiomatizations are extended to weak Pareto equilibria of finite multicriteria games. Remark 11.13 at the end of this chapter points out how axiomatizations for weak Pareto equilibria can be adapted to axiomatizations for strong Pareto equilibria.

We use the following axioms. A solution concept ρ on Γ_{finite} satisfies:

- **Restricted Nonemptiness (r-NEM)**, if for every $G \in \Gamma_{finite}$ with $WPE(G) \neq \emptyset$ we have $\rho(G) \neq \emptyset$;
- **One-Person Efficiency (OPE)**, if for each one-player game $G = \langle \{i\}, X_i, u_i \rangle \in \Gamma_{finite}$ we have that $\rho(G) = \{x \in X_i \mid \nexists y \in X_i : u_i(y) > u_i(x)\}$;
- **Consistency (CONS)**, if for each $G \in \Gamma_{finite}$, each proper subcoalition $S \in 2^{N^G} \setminus \{\emptyset, N^G\}$, and each element $x \in \rho(G)$, we have that $x_S \in \rho(G^{S,x})$;
- **Converse Consistency (COCONS)**, if for each $G \in \Gamma_{finite}$ with at least two players, we have that $\widetilde{\rho}(G) \subseteq \rho(G)$, where
$$\widetilde{\rho}(G) = \{x \in \prod_{i \in N^G} X_i \mid \forall S \in 2^{N^G} \setminus \{\emptyset, N^G\} : x_S \in \rho(G^{S,x})\}.$$

According to restricted nonemptiness, the solution concept provides a nonempty set of strategies whenever weak Pareto equilibria exist. One-person efficiency claims that in games with only one player, the solution concept picks out all strategies which yield a maximal payoff with respect to the > - order. Consistency means that a solution x of a game is also a solution of each reduced game in which the players that leave the game play according to the strategies in x. Converse consistency prescribes that a strategy combination which gives rise to a solution in every reduced game is also a solution of the original game.

Our first result indicates that the axiomatization of the Nash equilibrium concept on finite strategic games of Peleg, Potters, and Tijs (1996, Thm. 3) in terms of restricted nonemptiness, one-person efficiency, and consistency can be generalized to multicriteria games.

Theorem 11.7 *A solution concept ρ on Γ_{finite} satisfies r-NEM, OPE, and CONS if and only if $\rho = WPE$.*

Proof. It is clear that WPE satisfies the axioms. Let ρ be a solution concept on Γ_{finite} satisfying r-NEM, OPE, and CONS. Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma_{finite}$. We first show that $\rho(G) \subseteq WPE(G)$. Let $x \in \rho(G)$. If $|N^G| = 1$, then $x \in WPE(G)$ by OPE. If $|N^G| > 1$, then CONS implies that for each $i \in N^G$: $x_i \in \rho(G^{\{i\},x})$, so

$x_i \in \{y_i \in X_i \mid \nexists\, z_i \in X_i : u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})\}$ by OPE. Hence x is a weak Pareto equilibrium: $x \in WPE(G)$. Since $G \in \Gamma_{finite}$ was chosen arbitrarily, conclude that $\rho \subseteq WPE$.

To prove that $WPE \subseteq \rho$, again let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma_{finite}$ and let $\hat{x} \in WPE(G)$. Construct a multicriteria game $H \in \Gamma_{finite}$ as follows:

- let $m \in \mathbb{N} \setminus N^G$; the player set is $N^G \cup \{m\}$;
- players $i \in N^G$ have the same strategy set X_i as in G;
- player m has strategy set $\{0, 1\}$;
- payoff functions v_i to players $i \in N^G$ are defined, for all $(x_m, x) \in \{0,1\} \times \prod_{i \in N^G} X_i$, by:
$$v_i(x_m, x) = \begin{cases} u_i(x) & \text{if} \quad x_m = 1 \\ -e^{r(i)} & \text{if} \quad x_m = 0, x_i \neq \hat{x}_i \\ e^{r(i)} & \text{if} \quad x_m = 0, x_i = \hat{x}_i \end{cases}$$
where $e^{r(i)} \in \mathbb{R}^{r(i)}$ is the vector with each component equal to one.
- the payoff function v_m to player m is defined, for all $(x_m, x) \in \{0,1\} \times \prod_{i \in N^G} X_i$, by:
$$v_m(x_m, x) = \begin{cases} 0 & \text{if} \quad x_m = 0 \\ -1 & \text{if} \quad x_m = 1, x \neq \hat{x} \\ 1 & \text{if} \quad x_m = 1, x = \hat{x} \end{cases}$$

Simple verification indicates that $(1, \hat{x})$ is the unique weak Pareto equilibrium of H. Since $\rho(H) \subseteq WPE(H)$ by the previous part of the proof, we conclude by r-NEM that $(1, \hat{x}) \in \rho(H)$. Then by CONS, $\hat{x} \in \rho(H^{N^G,(1,\hat{x})}) = \rho(G)$, since by definition $H^{N^G,(1,\hat{x})} = G$. Hence $\hat{x} \in \rho(G)$, finishing our proof. □

Our second result shows that the axiomatization of the Nash equilibrium concept on finite strategic games of Peleg and Tijs (1996, Thm. 2.12) in terms of one-person efficiency, consistency, and converse consistency can also be generalized to multicriteria games.

Theorem 11.8 *A solution concept ρ on Γ_{finite} satisfies OPE, CONS, and COCONS if and only if $\rho = WPE$.*

Proof. WPE satisfies the axioms. Let ρ be a solution concept on Γ_{finite} that also satisfies them. As in the proof of Theorem 11.7, we have that $\rho(G) \subseteq WPE(G)$ for each $G \in \Gamma_{finite}$ by OPE and CONS. To prove that $WPE(G) \subseteq \rho(G)$ for each $G \in \Gamma_{finite}$, we use induction on the number of players. In one-player games, the claim follows from OPE. Now assume the claim holds for all finite multicriteria games with at most n players and let $G \in \Gamma_{finite}$ be an $(n+1)$-player game. By CONS of WPE: $WPE(G) \subseteq \widetilde{WPE}(G)$.

By induction: $\widetilde{WPE}(G) \subseteq \widetilde{\rho}(G)$. By COCONS of ρ: $\widetilde{\rho}(G) \subseteq \rho(G)$. Combining these three inclusions: $WPE(G) \subseteq \rho(G)$. □

These results seem to illustrate that the axiomatizations that exist in the literature for the Nash equilibrium concept generalize to the Pareto equilibrium concept for multicriteria games. This analogy, however, breaks down when we consider mixed extensions of finite multicriteria games, as is done in the next section.

11.5 Mixed extensions of finite multicriteria games

Norde *et al.* (1996) characterize the Nash equilibrium concept on mixed extensions of finite strategic form games by nonemptiness, utility maximization, and consistency (cf. Proposition 11.6). In this section it is shown that analogons of these properties are not sufficient to characterize the weak Pareto equilibrium concept in mixed extensions of finite multicriteria games.

First, we list some of the axioms used in this section. A solution concept ρ on Γ satisfies:

- **Nonemptiness (NEM)**, if $\rho(G) \neq \emptyset$ for each $G \in \Gamma$;
- **Weak One-Person Efficiency (WOPE)**, if for each game $G = \langle \{i\}, X_i, u_i \rangle \in \Gamma$ with one player we have that $\rho(G) \subseteq \{x \in \Delta(X_i) \mid \nexists y \in \Delta(X_i) : u_i(y) > u_i(x)\}$;
- **Consistency (CONS)**, if for each $G \in \Gamma$, each proper subcoalition $S \in 2^{N^G} \setminus \{\emptyset, N^G\}$, and each element $x \in \rho(G)$, we have that $x_S \in \rho(G^{S,x})$;
- **Converse Consistency (COCONS)**, if for each $G \in \Gamma$ with at least two players, we have that $\widetilde{\rho}(G) \subseteq \rho(G)$, where
$$\widetilde{\rho}(G) = \{x \in \prod_{i \in N^G} \Delta(X_i) \mid \forall S \in 2^{N^G} \setminus \{\emptyset, N^G\} : x_S \in \rho(G^{S,x})\}.$$

It is easy to see that WPE on Γ satisfies NEM (See Theorem 11.5), WOPE, and CONS. Moreover,

Lemma 11.9 *If a solution concept ρ on Γ satisfies WOPE and CONS, then $\rho \subseteq WPE$.*

Proof. Let ρ be a solution concept on Γ, satisfying WOPE and CONS. Let $G \in \Gamma$ and $x \in \rho(G)$. If $|N^G| = 1$, then $x \in WPE(G)$ by WOPE. If $|N^G| > 1$, then for each player $i \in N^G$: $x_i \in \rho(G^{\{i\},x})$ by CONS, so $x_i \in \{y_i \in \Delta(X_i) \mid \nexists z_i \in \Delta(X_i) : u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})\}$ by WOPE. Hence x is a weak Pareto equilibrium: $x \in WPE(G)$. □

Obviously, WPE is the largest solution concept on Γ satisfying NEM, WOPE, and CONS, but not the only one, as our next result shows.

Theorem 11.10 *There exists a solution concept ρ on Γ which satisfies NEM, WOPE, and CONS, such that $\rho \neq WPE$.*

Proof. Define ρ as follows. Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma$. Then

$$\rho(G) = \begin{cases} \{x \in \Delta(X_i) \mid \nexists y \in \Delta(X_i) : u_i(y) > u_i(x)\} = WPE(G) & \text{if} \quad |N^G| = 1 \\ SPE(G) & \text{if} \quad |N^G| > 1 \end{cases}$$

The definition of ρ for one-player games guarantees that ρ satisfies WOPE. Theorem 11.5 establishes that ρ satisfies NEM. It is easy to see that ρ is also consistent. To show that $\rho \neq WPE$, consider the game G in Figure 11.2, where both players have two pure strategies and two criteria.

	L	R
T	(1,1),(1,0)	(0,0),(0,2)
B	(1,0),(0,0)	(0,0),(0,0)

Figure 11.2: Consistent refinement of WPE

Obviously $(B, L) \in WPE(G)$, but $(B, L) \notin \rho(G)$, since $u_1(T, L) \geq u_1(B, L)$. $\square$

A more interesting class of refinements of the Pareto equilibrium concept on Γ that satisfy NEM, WOPE, and CONS are the compromise equilibria introduced in Chapter 13. In order to arrive at an axiomatization of WPE, we require an additional axiom. A solution concept ρ on Γ satisfies:

- **WEIGHT** if for every game $G \in \Gamma$ and each vector $\lambda = (\lambda_i)_{i \in N^G} \in \prod_{i \in N^G} \Delta_{r(i)}$ of weights: $\rho(G_\lambda) \subseteq \rho(G)$.

The solution concept ρ satisfies WEIGHT if for every weight vector, the solutions of the associated weighted strategic form game are solutions of the underlying multicriteria game.

Our main result, using the strong theorems of Norde *et al.* (1996) and Shapley (1959), shows that the weak Pareto equilibrium concept is the unique solution concept on Γ satisfying NEM, WOPE, CONS, and WEIGHT.

Theorem 11.11 *A solution concept ρ on Γ satisfies NEM, WOPE, CONS, and WEIGHT if and only if $\rho = WPE$.*

Proof. Straightforward verification and application of Theorem 11.5 indicates that WPE indeed satisfies the four axioms. Now let ρ be a solution concept on Γ satisfying NEM, WOPE, CONS, and WEIGHT. By Lemma 11.9, $\rho \subseteq WPE$. Now let $G \in \Gamma$, and $x \in WPE(G)$. Remains to show that $x \in \rho(G)$. By Theorem 11.5, there exists a vector $\lambda = (\lambda_i)_{i \in N^G} \in \prod_{i \in N^G} \Delta_{r(i)}$ of weights such that $x \in NE(G_\lambda)$. Notice that ρ restricted to $\Gamma_{strategic}$, the set of mixed extensions of strategic form games, satisfies NEM, UM, and CONS, and hence by Proposition 11.6, $\rho(H) = NE(H)$ for all $H \in \Gamma_{strategic}$. Consequently, $\rho(G_\lambda) = NE(G_\lambda) \ni x$. So by WEIGHT: $x \in \rho(G)$. □

Finally, without proof, we mention that the analogon of Theorem 11.8 also holds when we consider mixed extensions:

Theorem 11.12 *A solution concept ρ on Γ satisfies OPE, CONS, and COCONS if and only if $\rho = WPE$.*

It is an easy exercise to show that the axioms used in our theorems are logically independent.

Remark 11.13 In the proof of Theorem 11.10 we mentioned the strong Pareto equilibrium concept. By slight modifications in the axioms (in particular, to (weak) one-person strong efficiency and a weight axiom concerning strictly positive, rather than nonnegative, weights), all axiomatizations in Sections 11.4 and 11.5 have analogons for the strong Pareto equilibrium concept. Also, a result analogous to Theorem 11.10 holds. To see this, define a solution concept ψ on Γ as follows. Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma$.

- If $|N^G| = 1$, take $\psi(G) = \{x \in \Delta(X_i) \mid \nexists y \in \Delta(X_i) : u_i(y) \geq u_i(x)\}$. This guarantees that ψ satisfies (weak) one-person strong efficiency.
- If $|N^G| > 1$, take $\psi(G) = NE(G_e)$, the set of Nash equilibria of the scalarized game in which the players assign equal weight to their criteria. By the existence of Nash equilibria in mixed extensions, ψ satisfies NEM.

It is easy to see that ψ is also consistent. To show that ψ is not equal to the strong Pareto equilibrium concept, refer again to the game G in Figure 11.2. (T, L) is a strong Pareto equilibrium of G, but the weighted payoff to player 2 increases from $\frac{1+0}{2}$ to $\frac{0+2}{2}$ if he deviates to R, indicating that $(T, L) \notin \psi(G) = NE(G_e)$. ◁

Chapter 12

The Structure of the Set of Equilibria for Two-Person Multicriteria Games

12.1 Introduction

Nash introduced the notion of an equilibrium for noncooperative games in strategic form in his papers in 1950a and 1951. Since then the Nash equilibrium concept and its refinements have been and still are studied extensively. One of the topics in this investigation concerns the structure of the set of equilibria of bimatrix games, noncooperative two-player games in strategic form. Over the last decades a fair number of papers has been published on this topic. It turned out that the set of equilibria of a bimatrix game is a finite union of polytopes. Proofs of this fact can for example be found in Winkels (1979), Jansen (1981) and Jansen and Jurg (1990).

These results are of considerable importance from a computational point of view. One reason for this is that the original proofs by Nash of the existence of equilibria is not constructive. The 1950 paper uses the Kakutani fixed point theorem, whereas the 1951 paper applies the Brouwer fixed point theorem to establish existence. These proofs, therefore, do not tell you how to find an equilibrium for a given game. Also the basic inequalities in the definition of the equilibrium concept are not of much help. In general (without further assumptions on the structure of the game) these inequalities are polynomial and it is not clear how one can actually calculate one single solution given these inequalities, let alone how to find a parametric representation of the complete set of equilibria.

In the case of bimatrix games life is much simpler. For such a game it is possible to show that the set of equilibria is a finite union of polytopes and it is moreover possible to derive a polyhedral description of each of these polytopes. Hence, by using some theory of linear inequalities, it is possible to compute all extremal points of such a polytope and in this way find a parametric description of the set of equilibria. There are also a number of exact algorithms for the computation of one specific equilibrium, such as the algorithm of Lemke and Howson (1964), that are based on the special structure of the set of equilibria for bimatrix games.

This chapter, based on Borm, Vermeulen, and Voorneveld (1998), investigates to what extent the results on the structure of the set of equilibria of a bimatrix game carry over to the Pareto equilibrium concept introduced by Shapley (1959) for two-person multicriteria games. This concept was discussed in more depth in Chapter 11. Unfortunately, most results are on the negative side of the spectrum. The specific results are specified below.

- Section 12.4 provides an example to show that the set of weak Pareto equilibria may have a quadratic component whenever both players have three or more pure strategies and one of the players has more than one criterion.
- In Section 12.5 we show that the set of equilibria is indeed a finite union of polytopes if one of the players has two pure strategies.

In order to make the chapter closer to the existing literature on the structure of equilibrium sets in bimatrix games, notation is used that differs slightly from that in the previous chapters. Most of this notation is settled in the next section. Section 12.3 contains general results on the structure of the set of weak Pareto equilibria.

12.2 Preliminaries

In a (two-person multicriteria) game the first player has a finite set M of pure strategies and player two has a finite set N of pure strategies. The players are supposed to choose their strategies simultaneously. Given their choices $m \in M$ and $n \in N$, player one has a finite set S of criteria to evaluate the pure strategy pair (m, n). For each criterion $s \in S$ the evaluation is a real number $(A_s)_{mn} \in \mathbb{R}$. Of course we also have an evaluation $(B_t)_{mn} \in \mathbb{R}$ for each criterion $t \in T$ of player two. Thus the game is specified by the two sequences

$$A := (A_s)_{s \in S} \qquad \text{and} \qquad B := (B_t)_{t \in T}$$

of matrices

$$A_s := [(A_s)_{mn}]_{(m,n) \in M \times N} \qquad \text{and} \qquad B_t := [(B_t)_{mn}]_{(m,n) \in M \times N}.$$

Despite the fact that the players may have more than one criterion, we will refer to A and B as payoff matrices. The game is denoted by (A, B). The players of the game are also allowed to use mixed strategies. Given such mixed strategies $p \in \Delta(M)$ and $q \in \Delta(N)$ for players one and two respectively, the vectors

$$pAq := (pA_sq)_{s \in S} \qquad \text{and} \qquad pBq := (pB_tq)_{t \in T}$$

are called *payoff vectors* (for player one and two, resp.).

A very convenient way to define equilibria, certainly when one wants to analyze their structure, is by means of best replies. In the notation for bimatrix games, the best-response correspondence that gives rise to weak Pareto equilibria is defined as follows:

Definition 12.1 Let (A, B) be a game and let $q \in \Delta(N)$ be a strategy of player two. A strategy $p \in \Delta(M)$ of player one is a *best reply* of player one against q if there is no other strategy $p' \in \Delta(M)$ such that the payoff vector $p'Aq$ strongly dominates the payoff vector pAq. The set of best replies of player one against q is denoted by $WB_1(q)$. ◁

The best-response correspondence WB_2 of player 2 against strategies $p \in \Delta(M)$ is defined analogously. In equilibrium, both players play a best response.

Definition 12.2 A strategy pair (p, q) is an *equilibrium* of (A, B) if $p \in WB_1(q)$ and $q \in WB_2(p)$. ◁

Notice that we restrict attention to weak Pareto equilibria in this chapter. Since the more restrictive notion of strong Pareto equilibria does not necessarily yield a closed set of equilibria (see Example 11.2), we decided to use the weaker version.

12.3 Stability regions and structure

In case of bimatrix games, the proof that the set of Nash equilibria is a finite union of polytopes is based on the fact that this set of equilibria can be chopped up into a finite number of sets. Then each of these sets can easily be shown to be a polytope. It turns out to be worthwhile to execute this procedure for multicriteria games as well.

First of all, recall that according to Theorem 11.5 weak Pareto equilibria coincide with Nash equilibria of weighted games where nonnegative weight is assigned to each of the criteria. Recall that for each criterion $t \in T$ the real number $e_i B_t e_j$ is the payoff of player two according to his criterion t and B_t is the matrix whose entry on place i, j is this number $e_i B_t e_j$. Now suppose that player two decides to assign a weight $\mu_t \geqq 0$ to each criterion $t \in T$ available to him (we assume that $\sum_{t \in T} \mu_t$ equals one). The vector $\mu = (\mu_t)_{t \in T}$ is called a *weight vector*. According to the criterion associated with this weight vector the evaluation of the outcome (e_i, e_j) is the real number

$$\sum_{t \in T} \mu_t e_i B_t e_j = e_i \left(\sum_{t \in T} \mu_t B_t \right) e_j.$$

So, given the weight vector $(\mu_t)_{t \in T}$, player two in effect uses the matrix

$$B(\mu) := \sum_{t \in T} \mu_t B_t$$

to calculate his payoff. With this terminology, the result of Shapley (1959) can be rephrased as follows.

Lemma 12.3 *Let (A, B) be a two-person multicriteria game. Let p be a strategy of player one and let q be a strategy of player two. Then the following two statements are equivalent.*

- *q is a best reply of player two against p*
- *There exists a weight vector $\mu := (\mu_t)_{t \in T}$ such that q is a best reply of player two against p in the single-criterion game $B(\mu)$.*

In words, the lemma states that q is a best reply of player two against p if and only if player two can assign to each criterion $t \in T$ a nonnegative weight μ_t such that the resulting weighted payoff is maximal in q, given that player one plays p. For a proof, refer to Theorem 10.1.

We decompose the set of equilibria of the game (A, B) into a finite number of sets that are easier to handle. This decomposition is in fact the multicriteria equivalent of the technique that is used to prove that the set of equilibria of a bimatrix game is a finite union of polytopes. In order to give the reader some background concerning the line of reasoning employed here, we will first give an informal discussion of this technique.

Suppose that we have a bimatrix game and a subset I of the set of pure strategies of player one. Then we can associate two areas with this set, one in the set of mixed strategies of player one and one in the set of mixed strategies of player two. For player one, this is the set $\Delta(I)$ of mixed strategies that put all weight exclusively on the pure strategies in I, and for player two this is the set $U(I)$ of mixed strategies of player two against which (at least) all strategies in $\Delta(I)$ are best replies. Such a set $U(I)$ is called a *stability region*. Obviously we can do the same for a subset J of the set of pure strategies of player two.

Now the crucial point is that for a bimatrix game all these sets $\Delta(I)$, $\Delta(J)$, $U(I)$, and $U(J)$ are polytopes (and for each of these polytopes it is even possible to find a describing system of linear inequalities). So, also the set

$$(\Delta(I) \cap U(J)) \times (\Delta(J) \cap U(I))$$

is a polytope. Moreover there is only a finite number of such sets *and* it can be shown that their union equals the set of Nash equilibria of the given bimatrix game.

Although the sets $U(I)$ and $U(J)$ not necessarily need to be polytopes in the multicriteria case, we can still carry out this procedure for two-person multicriteria games. To this end, let v be an element of $\mathbb{R}^n$ and let P be a polytope in $\mathbb{R}^n$. The vector v is said to *attain its maximum over* P in the point $x \in P$ if

$$\langle v, x \rangle \geqq \langle v, y \rangle \quad \text{for all } y \in P.$$

Then we have the following well-known lemma.

Lemma 12.4 *Let v be a vector in $\mathbb{R}^n$. Further, let P be a polytope in $\mathbb{R}^n$ and let F be a face of P. If v attains its maximum over P in some relative interior point x of F, then it also attains its maximum over P in any other point of F.*

Now let I be a subset of M. Slightly abusing notation we write $\Delta(I)$ for the set of strategies $p \in \Delta(M)$ whose carrier is a subset of I. Further, the *stability region* $U(I)$ (of player two) is defined as

$$U(I) := \{q \in \Delta(N) \mid \Delta(I) \subseteq WB_1(q)\}.$$

Similarly we can define sets $\Delta(J)$ and $U(J)$ for a subset J of N.

Theorem 12.5 *Let (A, B) be a two-person multicriteria game. The set of equilibria of the game (A, B) equals the union over all $I \subseteq M$ and $J \subseteq N$ of the sets*

$$(\Delta(I) \cap U(J)) \times (\Delta(J) \cap U(I)).$$

Proof.

(a) Assume that a strategy pair (p^*, q^*) is an element of a set $(\Delta(I) \cap U(J)) \times (\Delta(J) \cap U(I))$ for some subset I of M and subset J of N. We will only show that p^* is a best reply against q^*.

Since q^* is an element of $U(I)$, we know that any strategy in $\Delta(I)$ is a best reply against q^*. Now p^* is an element of $\Delta(I)$ by assumption. Hence, p^* is a best reply against q^*.

(b) Conversely, let (p^*, q^*) be an equilibrium. Take $I = C(p^*)$ and $J = C(q^*)$. We will show that p^* is an element of $\Delta(I) \cap U(J)$.

Obviously p^* is an element of $\Delta(I)$. So it remains to show that p^* is also an element of $U(J)$. In other words, we need to show that each strategy $q \in \Delta(J)$ is a best reply against p^*. To this end, take a $q \in \Delta(J)$. Since q^* is a best reply against p^* we know by Lemma 12.3 that there exists a weight vector $\mu = (\mu_t)_{t \in T}$ such that q^* is a best reply against p^* in the single-criterion game $B(\mu)$. In other words, the vector $p^*B(\mu)$ attains its maximum over $\Delta(N)$ in q^*. However, since q^* is an element of the relative interior of $\Delta(J)$, $p^*B(\mu)$ must also attain its maximum in q by Lemma 12.4. Hence, q is a best reply against p^* according to $B(\mu)$, and, again by Lemma 12.3, q is a best reply against p^*. □

Clearly the sets $\Delta(I)$ and $\Delta(J)$ are polytopes for all subsets I of M and J of N. So, from the previous theorem it follows that the set of equilibria of the game (A, B) is a finite union of polytopes as soon as the sets $U(I)$ and $U(J)$ are polytopes. Unfortunately this need not be the case. In the next section we provide a counterexample.

12.4 An example

We will give a fairly elaborate analysis of the counterexample. This is done because the calculations involved in the determination of best replies and stability regions for this game are exemplary for such calculations in general.

There are two players in the game. Player one is the row player and player two is the column player. Both players have three pure strategies. The pure strategies of player one are called T, M, and B, the pure strategies of player two are called L, C, and R. Further, player one has two criteria and player two has only one criterion. The payoff for player two according to his criterion is always zero. The payoff matrix A of player one is given in Figure 12.1.

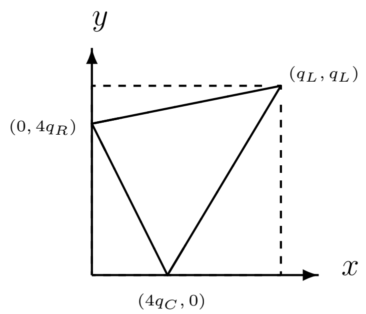
	L	C	R
T	$(1,1)$	$(0,0)$	$(0,0)$
M	$(0,0)$	$(4,0)$	$(0,0)$
B	$(0,0)$	$(0,0)$	$(0,4)$

Figure 12.1: The payoff matrix A of player one

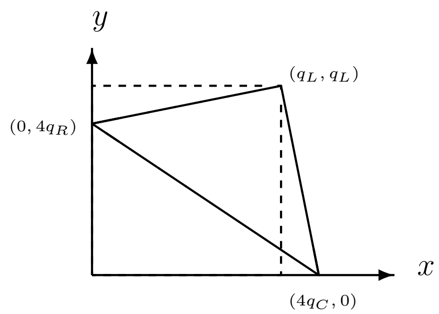
Since player two is completely indifferent between his strategies, it is immediately clear that a strategy pair $(p^*, q^*) \in \Delta(M) \times \Delta(N)$ is an equilibrium of the game if and only if p^* is an element of $WB_1(q^*)$. In other words, the set of equilibria equals the graph of the best-reply correspondence WB_1. In order to calculate this graph we will first compute the areas in the strategy space of the second player where the best reply correspondence WB_1 is constant. In other words, we need to compute the stability regions of player two.

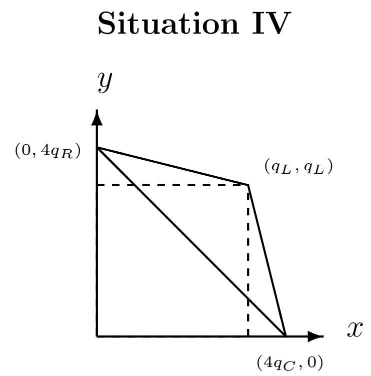
First of all note that if player two plays strategy $q = (q_L, q_C, q_R)$ and player one plays his pure strategy e_T, the payoff for player one is $e_T A q = (q_L, q_L)$. This is a point on the line $x = y$ when plotted in the xy-plane. Similarly, $e_M A q = (4q_C, 0)$ is a point on the line $y = 0$ and $e_B A q = (0, 4q_R)$ is a point on the line $x = 0$. Now there are five possible situations as is depicted below.

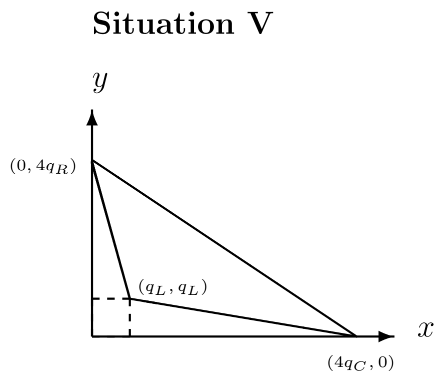
Situation I



Situations II and III







In situation **I** both $e_M Aq$ and $e_B Aq$ are dominated by $e_T Aq$. In situation **II** $e_T Aq$ dominates $e_B Aq$, but does not dominate $e_M Aq$. (Situation **III** is the symmetric situation with the roles of the second and third pure strategy of player one interchanged.) In situation **IV** $e_T Aq$ is itself undominated and dominates neither $e_M Aq$ nor $e_B Aq$, and **V** depicts the situation in which $e_T Aq$ is dominated by some convex combination of $e_M Aq$ and $e_B Aq$.

Now if we calculate exactly where in the strategy space of player two these five situations occur we get Figure 12.2 below. The boldface roman numbers in the various areas in this picture correspond to the roman numbers assigned to the situations depicted above. Notice that an area in the strategy space of player two corresponding to one of the five situations above is necessarily of full dimension by the graphics above. Further, one cannot jump from situation **V** to situations **I**, **II** or **III** without crossing the area where situation **IV** occurs (except on the boundary of the strategy space).

The boundary line between areas **I** and **II** and areas **III**, **IV** and **V** is given by the equality $q_L = 4q_R$. Similarly, $q_L = 4q_C$ is the boundary between areas **I** and **III** and areas **II**, **IV** and **V**.

Finally, it can be seen in the graphics above that the boundary between area **V** and the others is exactly the set of strategies where $e_T Aq$ is an element of the line segment between $e_M Aq$ and $e_B Aq$. This means that it is the set of strategies for which (q_L, q_L) satisfies the linear equation $q_R x + q_C y = 4q_C q_R$. Hence it must be the set of strategies that satisfy the quadratic equation

$$q_L q_R + q_L q_C = 4q_C q_R$$

(except the solution $(q_L, q_C, q_R) = (1, 0, 0)$ of this equation). This gives us enough information to write down the stability regions of player two.

$$\begin{aligned} U(\{T\}) &= \mathbf{I} \cup \mathbf{II} \cup \mathbf{III} \cup \mathbf{IV} \\ U(\{M\}) &= \mathbf{II} \cup \mathbf{IV} \cup \mathbf{V} \\ U(\{B\}) &= \mathbf{III} \cup \mathbf{IV} \cup \mathbf{V} \end{aligned}$$

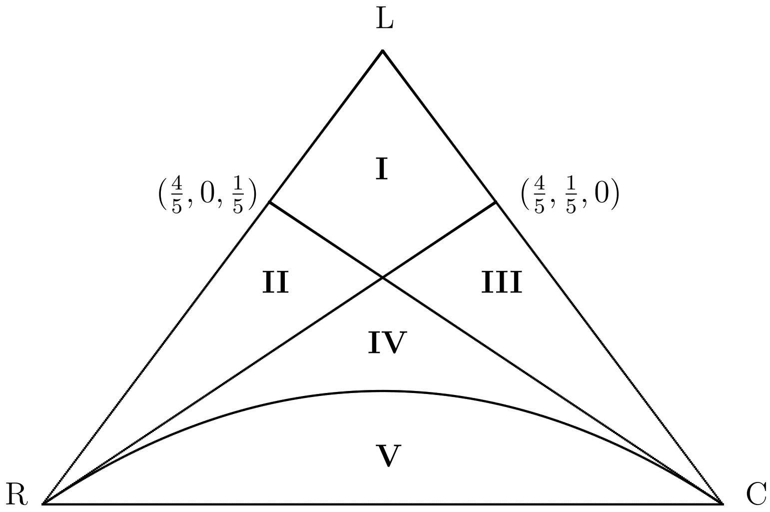


Figure 12.2: Stability regions of player two

$$\begin{aligned} U(\{T,M\}) &= \quad \mathbf{II} \quad \cup \mathbf{IV} \\ U(\{T,B\}) &= \quad \mathbf{III} \cup \mathbf{IV} \\ U(\{M,B\}) &= \quad \mathbf{V} \\ U(\{T,M,B\}) &= \quad \mathbf{IV} \cap \mathbf{V} \end{aligned}$$

Note the essential differences with the structure of stability regions for bimatrix games. For a bimatrix game we would for example have the equality

$$U(\{M,B\}) = U(\{M\}) \cap U(\{B\}).$$

The example shows that this is no longer true for multicriteria games. In this case the set

$$U(\{M\}) \cap U(\{B\}) = \mathbf{IV} \cup \mathbf{V}$$

subdivides into the areas **IV**, on whose relative interior

$$\Delta(\{T,M\}) \cup \Delta(\{T,B\})$$

is the set of best replies, and **V**, on whose relative interior the set of best replies is indeed $\Delta(\{T,M,B\})$. An area like **IV** simply cannot occur for bimatrix games.

The second essential difference, and the main one in this section, is the fact that $U(\{T,M,B\})$ is a quadratic curve. This means that the subset

$$\Delta(\{T,M,B\}) \times U(\{T,M,B\})$$

of the set of equilibria cannot be written as a finite union of polytopes. This concludes the example.

Remark 12.6 The observation that the set of equilibria of a multicriteria game in general cannot be written as a finite union of polytopes implies that in Theorem 11.5 it will usually be necessary to invoke infinitely many vectors of weights to compute the Pareto equilibria as Nash equilibria of weighted games. ◁

12.5 Multicriteria games of size $\mathbf{2} \times \mathbf{n}$

The previous example shows that, in case at least one of the players has more than one criterion, the set of equilibria may have a quadratic component as soon as both players have at least three pure strategies. The degenerate case in which one of the players has only one pure strategy immediately yields that the set of equilibria is a finite union of polytopes. This case is not considered in the remainder of this chapter. So, in the multicriteria case it is necessary to have (at least) one player who has exactly two pure strategies to guarantee that the set of equilibria is indeed a finite union of polytopes. In this section we will show that this assumption is also sufficient.

Without loss of generality, we assume that every two-person multicriteria game (A, B) considered in this section is a $2 \times n$-game in which player one's set of pure strategies M equals $\{T, B\}$.

This section is ordered as follows: first we establish that the stability regions of player two are finite unions of polytopes. Next, the same result is proven for the stability regions of the first player. The computational aspects are considered in the final part of this section.

First, the stability regions of player two. In this special case the analysis of the dominance relation on the possible payoff vectors for player one for a fixed strategy q of player two is quite straightforward. Since player one has only two pure strategies e_T and e_B, the set of possible payoff vectors is a line segment (or a singleton in case $e_T Aq = e_B Aq$) in $\mathbb{R}^S$. Given this observation it is easy to check

Lemma 12.7 *The following two statements are equivalent.*

- $e_T Aq$ *is dominated by* pAq *for some* $p \in \Delta(M)$.
- $e_T Aq$ *is dominated by* $e_B Aq$.

Given this lemma we can show that each stability region $U(I)$ of player two is a finite union of polytopes. Two cases are considered.

Case 1. For $|I| = 1$. Assume w.l.o.g. that $I = \{T\}$. Then

$$U(I) \quad = \quad \{q \in \Delta(N) \mid \Delta(I) \subseteq WB_1(q)\}$$

$$
\begin{aligned}
&= \{q \in \Delta(N) \mid e_T \in WB_1(q)\} \\
&= \{q \in \Delta(N) \mid e_T Aq \text{ is not dominated by } pAq \text{ for any } p \in \Delta(M)\} \\
&= \{q \in \Delta(N) \mid e_T Aq \text{ is not dominated by } e_B Aq\} \\
&= \bigcup_{s \in S} \{q \in \Delta(N) \mid e_T A_s q \geqq e_B A_s q\}
\end{aligned}
$$

where the fourth equality follows from the previous lemma. Clearly this last expression is a finite union of polytopes.

Case 2. For $I = \{T, B\}$. Using the previous lemma it is easy to check that $U(I)$ is the set of strategies q for which $e_T Aq$ does not dominate $e_B Aq$ and $e_B Aq$ does not dominate $e_T Aq$. So, $U(I) = U(\{T\}) \cap U(\{B\})$. Thus, since both $U(\{T\})$ and $U(\{B\})$ are finite unions of polytopes as we saw in Case 1, $U(I)$ is also a finite union of polytopes.

This finishes the proof of

Theorem 12.8 *Let (A, B) be a $2 \times n$ two-person multicriteria game. Then for each $I \subseteq M = \{T, B\}$, the stability region $U(I)$ of player two is a finite union of polytopes.*

Now that we have come this far, the only thing left to prove is that the stability region

$$U(J) = \{p \in \Delta(M) \mid \Delta(J) \subseteq WB_2(p)\}$$

is a finite union of polytopes for each set $J \subseteq N$ of pure strategies of player two. In order to do this we need to do some preliminary work.

Let the subset $V(J)$ of $\Delta(M) \times \mathbb{R}^T$ be defined by

$$
\begin{aligned}
V(J) := \{(p, \mu) \mid\ & \Delta(J) \text{ is included in the set of best replies against } p \\
& \text{according to the criterion } B(\mu)\} \\
= \{(p, \mu) \mid\ & \Delta(J) \text{ is included in the set of strategies where} \\
& \text{the vector } pB(\mu) \text{ attains its maximum over } \Delta(N)\}.
\end{aligned}
$$

Note that we allow $pB(\mu)$ to attain its maximum in points outside $\Delta(J)$ as well. We only require that $\Delta(J)$ is indeed a subset of the set of points where $pB(\mu)$ attains its maximum over $\Delta(N)$.

Further, let the projection $\pi : \mathbb{R}^2 \times \mathbb{R}^T \to \mathbb{R}^2$ be defined by

$$\pi(p, v) := p \quad \text{for all } (p, v) \in \mathbb{R}^2 \times \mathbb{R}^T.$$

Now we can prove

Lemma 12.9 *The stability region $U(J)$ equals the projection $\pi(V(J))$ of the set $V(J)$.*

Proof.
(a) Let p be an element of $U(J)$. We will show that p is also an element of $\pi(V(J))$. Let q^* be an element of the relative interior of $\Delta(J)$. Since p is an element of $U(J)$ we know that q^* is a best reply to p. Then we know, by Lemma 12.3, that there is a weight vector $\mu = (\mu_t)_{t \in T}$ such that the vector $pB(\mu)$ attains its maximum over $\Delta(N)$ in q^*. So, since q^* is a relative interior point of $\Delta(J)$, $pB(\mu)$ also attains its maximum over $\Delta(N)$ in any other point of $\Delta(J)$ by Lemma 12.4. Therefore (p, μ) is an element of $V(J)$ and $p = \pi(p, \mu)$ is an element of $\pi(V(J))$.
(b) Conversely, let $p = \pi(p, \mu)$ be an element of $\pi(V(J))$ and let q be an element of $\Delta(J)$. Then we know that the vector $pB(\mu)$ attains its maximum over $\Delta(N)$ in q. Again by Lemma 12.3, this means that q is a best reply against p. Hence, since q was chosen arbitrarily in $\Delta(J)$, p is an element of $U(J)$. □

This enables us to show

Theorem 12.10 *Let (A, B) be a $2 \times n$ two-person multicriteria game. Then for each $J \subseteq N$, the stability region $U(J)$ of player one is a finite union of polytopes.*

Proof. Observe that the set $V(J)$ is the collection of points $(p, \mu) \in \mathbb{R}^2 \times \mathbb{R}^T$ that satisfy the system of polynomial (in)equalities

$$\begin{aligned}
p_i &\geqq 0 \quad \text{for } i = 1, 2 \\
p_1 + p_2 &= 1 \\
\mu_t &\geqq 0 \quad \text{for all } t \in T \\
\sum_{t \in T} \mu_t &= 1 \\
\sum_{t \in T} \mu_t p B_t e_j &\geqq \sum_{t \in T} \mu_t p B_t e_k \quad \text{for all } j \in J \text{ and } k \in N.
\end{aligned}$$

Therefore, $V(J)$ is a semi-algebraic set. Furthermore, by Lemma 12.9, $U(J)$ equals the set of vectors $p \in \mathbb{R}^2$ such that there exists a $\mu \in \mathbb{R}^T$ for which

$$(p, \mu) \in V(J).$$

Hence, by the Theorem of Tarski (1951) and Seidenberg (1954) (see Blume and Zame, 1994, for a clear discussion of this theorem) $U(J)$ is also a semi-algebraic set. Further, $U(J)$ is compact, since $V(J)$ is compact and π is continuous. So, $U(J)$ is the union of a finite collection $\{S_\alpha\}_{\alpha \in A}$ of sets S_α in $\Delta(M)$ and each S_α is described by a finite number of polynomial inequalities

$$p_{\alpha,k}(x) \geqq 0 \quad (k = 1, \ldots, m(\alpha)).$$

However, $\Delta(M)$ is a line segment in $\mathbb{R}^2$. So the set of points in $\Delta(M)$ that satisfies one particular inequality is the finite union of (closed) line segments (singletons also count as line segments). So, since each S_α is the intersection of such finite unions, S_α is itself the finite union of closed line segments. Therefore, since $U(J)$ is the finite union over all sets S_α, it is the finite union of closed line segments. Hence, $U(J)$ is a finite union of polytopes. □

Combination of the previous two theorems yields the result we set out to prove:

Theorem 12.11 *In two-person multicriteria games of size* $2 \times n$*, the set of equilibria is a finite union of polytopes.*

Finally, we consider the case where the second player has only one criterion: $|T| = 1$. In this case we have a complete polyhedral description of the polytopes involved in the union. Notice that we already know that the sets $\Delta(I)$ and $\Delta(J)$ are polytopes, and the sets $U(I)$ and $U(J)$ are finite unions of polytopes. We will now show that a polyhedral description of all these polytopes can be found.

For the polytopes $\Delta(I)$, $\Delta(J)$ this polyhedral description is trivial. For $U(I)$ we saw in Case 1 below Lemma 12.7 that it is the finite union of polytopes of the form

$$\{q \in \Delta(N) \mid e_T A_s q \geqq e_B A_s q\}.$$

So, in Case 1 the polytopes involved in the union are already given by linear inequalities. This implies that also in Case 2 we can find the linear inequalities that describe the polytopes involved. Finally, for $J \subseteq N$, we get

$$\begin{aligned} U(J) &= \{p \in \Delta(M) \mid \Delta(J) \subseteq WB_2(p)\} \\ &= \{p \in \Delta(M) \mid pBe_j \geqq pBe_k \text{ for all } j \in J \text{ and } k \in N\}. \end{aligned}$$

The assumption that $|T| = 1$ is used in the second equality. The last expression in the display now shows that $U(J)$ is itself a polytope that can be written as the solution set of a finite number of linear inequalities. This concludes the argumentation.

Chapter 13

Compromise, Nash Bargaining, and Perfect Equilibria

13.1 Introduction

In Chapter 10 we suggested the following interpretation of a noncooperative multicriteria game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$: each player $i \in N$ represents an organization or group of players and the function u_i into $\mathbb{R}^{r(i)}$ represents the $r(i) \in \mathbb{N}$ separate utility functions of its members. This interpretation induces an aggregate conflict: there is a noncooperative game being played *between* the organizations and a cooperative game *within* each organization, where its members jointly have to decide on a strategy choice that is 'optimal' given their utility functions.

Different solution concepts can be defined, depending on the answer to the following central question:

$$\text{What is a 'best response'?} \tag{Q}$$

The Pareto equilibrium concept was studied in the previous two chapters. In this chapter, three other solution concepts are proposed. The first two concepts provide different answers to question (Q). Compromise equilibria, introduced in Section 13.2, answer question (Q) by requiring to be as close as possible to an ideal solution. Nash bargaining equilibria, introduced in Section 13.3, answer question (Q) by suggesting a bargaining solution far away from an undesirable solution. A more standard approach to equilibrium refinements is taken in Section 13.4, where the analogon of perfect equilibria à la Selten (1975) is defined for multicriteria games. Section 13.5 contains some concluding remarks.

13.2 Compromise equilibria

The distance function method described in Chapter 10 as a solution method for multicriteria problems was based on the idea of finding the feasible point(s) that are closest to an ideal outcome. Zeleny (1976) even states this — rather informally — as an axiom of choice:

> 'Alternatives that are closer to the ideal are preferred to those that are farther away. To be as close as possible to the perceived ideal is the rationale of human choice.'

The distance function method as a way to find compromises for group conflicts was popularized by Yu (1973; see also Freimer and Yu, 1976). It suggests two questions:

- What is an ideal point, and
- What is the meaning of 'close to'?

Suppose that there are two individuals, called $i1$ and $i2$, and that their feasible set of utilities is the polytope U in Figure 13.1. Individual $i1$ can at most hope for x, whereas individual $i2$ can at most hope for y. The *ideal point*, in which they will both receive their maximal utility, is therefore the point $I = (x, y)$. Unfortunately, this point is infeasible. To find a compromise, it is desirable to find a point close to I that *is* feasible. Yu (1973) proposes to measure distances using l_p-norms. Let $p \in [1, \infty]$. The l_p-norm on $\mathbb{R}^n$ assigns to each $x \in \mathbb{R}^n$ the real number

$$\|x\|_p := \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p},$$

where the l_∞-norm, also called the *Tchebyshev norm*, is defined by

$$\|x\|_\infty := \max_{i=1,\ldots,n} |x_i|.$$

These norms induce distance functions on $\mathbb{R}^n$ that map $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ to $\|x - y\|_p$. Using for instance the standard Euclidean distance l_2, the compromise solution would be the feasible outcome yielding the point c in Figure 13.1.
View each player $i \in N$ of a noncooperative multicriteria game $\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ as an organization consisting of $r(i) \in \mathbb{N}$ members, each having his own utility function. Confronted with a strategy profile $x_{-i} \in \prod_{j\in N\setminus\{i\}} \Delta(X_j)$, the members of organization i can answer the question 'what is a best response against x_{-i}?' or, equivalently 'what strategy yields a good outcome in the payoff polytope $\{u_i(x_i, x_{-i}) \mid x_i \in \Delta(X_i)\} = \text{conv}\,\{u_i(x_i, x_{-i}) \mid x_i \in X_i\}$?' by proposing a compromise solution. Defining a best response in this way, and imposing the stability condition that each player should play a best response given the strategy profile of his opponents, one obtains compromise equilibria.

Formally, let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a multicriteria game, $i \in N$, and $x_{-i} \in \prod_{j\in N\setminus\{i\}} \Delta(X_j)$. Player i's *ideal point*, given x_{-i}, is the point $I_i(x_{-i}) \in \mathbb{R}^{r(i)}$ where for

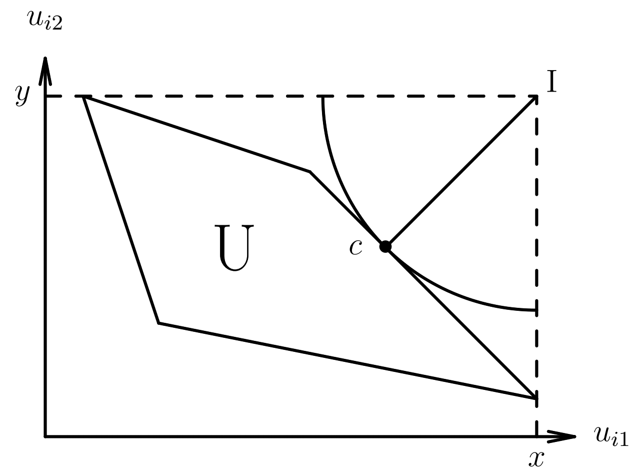


Figure 13.1: A compromise solution

each $k \in \{1, \ldots, r(i)\}$ the k-th coordinate is the highest feasible utility according to criterion u_{ik}:

$$I_{ik}(x_{-i}) = \max_{x_i \in \Delta(X_i)} u_{ik}(x_i, x_{-i}) = \max_{x_i \in X_i} u_{ik}(x_i, x_{-i}).$$

Next, let each player $i \in N$ select an l_p-norm, i.e, let $p = (p(i))_{i \in N} \in [1, \infty]^N$. A strategy profile $x \in \prod_{i \in N} \Delta(X_i)$ is a *compromise equilibrium* of the multicriteria game G, given p, notation $x \in CE_p(G)$, if each player $i \in N$ chooses a strategy profile that yields a utility vector closest to the ideal point according to the $l_{p(i)}$-norm:

$$\forall i \in N: \quad x_i \in \arg \min_{y_i \in \Delta(X_i)} \|u_i(y_i, x_{-i}) - I_i(x_{-i})\|_{p(i)}.$$

Theorem 13.1 *Let* $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ *and* $p = (p(i))_{i \in N} \in [1, \infty]^N$. *Then* $CE_p(G) \neq \emptyset$.

Proof. The proof is based on Kakutani's fixed point theorem. Let $i \in N$. $\Delta_{-i} = \prod_{j \in N \setminus \{i\}} \Delta(X_j)$ is a metric space. $\Delta_i = \Delta(X_i)$ is a compact metric space. The function $f_i : \Delta_{-i} \times \Delta_i \to \mathbb{R}$ with $f_i(x_{-i}, x_i) = -\|u_i(x_i, x_{-i}) - I_i(x_{-i})\|_{p(i)}$ is continuous. By the Maximum theorem

$$m_i : x_{-i} \mapsto \max_{x_i \in \Delta_i} f_i(x_{-i}, x_i)$$

is continuous and

$$M_i : x_{-i} \mapsto \arg \max_{x_i \in \Delta_i} f_i(x_{-i}, x_i) = \arg \min_{x_i \in \Delta(X_i)} \|u_i(x_i, x_{-i}) - I_i(x_{-i})\|_{p(i)}$$

is u.s.c. Nonemptiness of $M_i(x_{-i})$ is immediate from the fact that every continuous function on a compact set achieves its maximum.

Moreover, for each $x_{-i} \in \Delta_{-i}$ the set $M_i(x_{-i}) \subseteq \Delta_i$ is bounded and equals the inverse image of $\{m_i(x_{-i})\}$ under the continuous function $f_i(x_{-i}, \cdot)$, which implies that $M_i(x_{-i})$ is closed. Hence $M_i(x_{-i})$ is compact for each $x_{-i} \in \Delta_{-i}$.

Also, for each $x_{-i} \in \Delta_{-i}$ the set $M_i(x_{-i}) = \arg\min_{x_i \in \Delta(X_i)} \|u_i(x_i, x_{-i}) - I_i(x_{-i})\|_{p(i)}$ is convex. To see this, let $x_{-i} \in \Delta_{-i}$, y_i and z_i in $M_i(x_{-i})$, and $\lambda \in (0,1)$. Then

$$
\begin{aligned}
&\min_{x_i \in \Delta_i} \|u_i(x_i, x_{-i}) - I_i(x_{-i})\|_{p(i)} \\
\leqq \quad & \|u_i(\lambda y_i + (1-\lambda) z_i, x_{-i}) - I_i(x_{-i})\|_{p(i)} \\
= \quad & \|\lambda \left[u_i(y_i, x_{-i}) - I_i(x_{-i})\right] + (1-\lambda)\left[u_i(z_i, x_{-i}) - I_i(x_{-i})\right]\|_{p(i)} && (13.1) \\
\leqq \quad & \lambda \|u_i(y_i, x_{-i}) - I_i(x_{-i})\|_{p(i)} + (1-\lambda)\|u_i(z_i, x_{-i}) - I_i(x_{-i})\|_{p(i)} && (13.2) \\
= \quad & \lambda \min_{x_i \in \Delta_i} \|u_i(x_i, x_{-i}) - I_i(x_{-i})\|_{p(i)} && (13.3) \\
+ \quad & (1-\lambda) \min_{x_i \in \Delta_i} \|u_i(x_i, x_{-i}) - I_i(x_{-i})\|_{p(i)} \\
= \quad & \min_{x_i \in \Delta_i} \|u_i(x_i, x_{-i}) - I_i(x_{-i})\|_{p(i)},
\end{aligned}
$$

where equality (13.1) follows from the multilinearity of u_i, inequality (13.2) follows from the triangle inequality, and equality (13.3) follows from the fact that $y_i, z_i \in M_i(x_{-i})$. But then all weak inequalities above are in fact equalities, proving that $\lambda y_i + (1-\lambda) z_i \in M_i(x_{-i}) = \arg\min_{x_i \in \Delta_i} \|u_i(x_i, x_{-i}) - I_i(x_{-i})\|_{p(i)}$.

This completes the preliminary work. Notice that Δ is nonempty, compact, and convex, that $M : \Delta \twoheadrightarrow \Delta$ with $M(x) = \prod_{i \in N} M_i(x_{-i})$ is u.s.c., and that $M(x)$ is nonempty, compact, and convex for each $x \in \Delta$. Kakutani's fixed point theorem implies the existence of a point $x \in \Delta$ satisfying $x \in M(x)$, which is a compromise equilibrium. □

Formally, CE_p is not a solution concept, since the vector $p = (p(i))_{i \in N}$ depends on the player set of the game being played. This can be remedied in a trivial way: recall that the set of potential players is $\mathbb{N}$ and that each multicriteria game $G \in \Gamma$ has a finite player set $N^G \subset \mathbb{N}$. Fix a function $p : \mathbb{N} \to [1, \infty]$ that specifies for each potential player $i \in \mathbb{N}$ a norm $l_{p(i)}$. Define a solution concept $\mathcal{CE}_p$ on Γ as follows:

$$\forall G \in \Gamma : \quad \mathcal{CE}_p(G) = CE_{(p(i))_{i \in N^G}}(G),$$

i.e., $\mathcal{CE}_p$ assigns to each game $G \in \Gamma$ its set of compromise equilibria given that each player $i \in N^G$ uses the $l_{p(i)}$-norm. It is a trivial exercise to show that the l_p-norms satisfy the following monotonicity condition:

$$
\begin{aligned}
&\forall x, y \in \mathbb{R}^n_+, \forall p \in [1, \infty) &: \quad x \geq y \;\Rightarrow\; \|x\|_p &> \|y\|_p \\
&\forall x, y \in \mathbb{R}^n_+ &: \quad x \geq y \;\Rightarrow\; \|x\|_\infty &\geqq \|y\|_\infty
\end{aligned}
$$

Therefore, if $x \in CE_p(G)$ is a compromise equilibrium and $i \in N$ has $p(i) \in [1, \infty)$, strategy x_i yields a payoff $u_i(x_i, x_{-i})$ on the strong Pareto edge of $\{u_i(y_i, x_{-i}) \mid y_i \in \Delta_i\}$, whereas a player $i \in N$ with $p(i) = \infty$ selects a point on the weak Pareto edge. It is not difficult to check that $\mathcal{CE}_p$ is a consistent solution concept. As a consequence, we found a nontrivial, nonempty, consistent refinement of weak and strong Pareto equilibria.

Theorem 13.2 *Let $p : \mathbb{N} \to [1, \infty]$. If $p(i) \neq \infty$ for each player $i \in \mathbb{N}$, then $\mathcal{CE}_p$ is a nonempty, consistent refinement of SPE. Otherwise, $\mathcal{CE}_p$ is a nonempty, consistent refinement of WPE.*

13.3 Nash bargaining equilibria

There is an interesting duality between the multicriteria literature that suggests a compromise approach to finding a desirable alternative from a feasible set and the game theoretic approach on bargaining. The compromise approach entails formulating a desirable, ideal solution solution and then 'working your way down' to a feasible solution as close as possible to the ideal. The bargaining approach entails formulating a typically undesirable disagreement point and then 'working your way up' to a feasible point dominating this disagreement outcome. Mixtures of the two approaches, like the Kalai-Smorodinsky (1975) solution, exist as well.

In this section, Nash meets Nash. Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$. Confronted with a strategy profile $x_{-i} \in \prod_{j \in N \setminus \{i\}} \Delta(X_j)$, the members of organization i can answer the question 'what strategy yields a good outcome in the payoff polytope $\{u_i(x_i, x_{-i}) \mid x_i \in \Delta(X_i)\} = \text{conv}\,\{u_i(x_i, x_{-i}) \mid x_i \in X_i\}$?' by finding an appropriate disagreement point in $\mathbb{R}^{r(i)}$ and proposing the bargaining solution proposed by Nash (1950b). Defining a best response in this way, and imposing the stability condition of the Nash equilibrium concept (Nash, 1950a) that each player should play a best response given the strategy profile of his opponents, one obtains Nash bargaining equilibria.

Hence, in Nash bargaining equilibria we have

- for the noncooperative conflict *between* players/organizations the Nash condition that each player plays a best response against the strategy profile of the opponents, and
- the Nash bargaining solution to settle the cooperative conflict *within* an organization.

Formally, let $i \in N$, and $x_{-i} \in \prod_{j \in N \setminus \{i\}} \Delta(X_j)$. Player i's *disagreement point*, given x_{-i}, is the point $d_i(x_{-i}) \in \mathbb{R}^{r(i)}$ where for each $k \in \{1, \ldots, r(i)\}$ the k-th coordinate is the lowest possible utility according to criterion u_{ik}:

$$d_{ik}(x_{-i}) = \min_{x_i \in \Delta(X_i)} u_{ik}(x_i, x_{-i}) = \min_{x_i \in X_i} u_{ik}(x_i, x_{-i}).$$

A strategy profile $x \in \prod_{i \in N} \Delta(X_i)$ is a *Nash bargaining equilibrium* of the multicriteria game G, notation $x \in NBE(G)$, if each player $i \in N$ chooses a strategy profile that yields a utility vector coinciding with the Nash bargaining solution given feasible set $\{u_i(y_i, x_{-i}) \mid y_i \in \Delta(X_i)\}$ and disagreement point $d_i(x_{-i})$:

$$\forall i \in N : \quad x_i \in \arg \max_{y_i \in \Delta(X_i)} \prod_{k=1}^{r(i)} \left(u_{ik}(y_i, x_{-i}) - d_{ik}(x_{-i})\right).$$

Theorem 13.3 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$. Then $NBE(G) \neq \emptyset$.*

Proof. Define for each $i \in N$ the function $f_i : \Delta_{-i} \times \Delta_i \to \mathbb{R}$ as follows:

$$\forall (x_{-i}, x_i) \in \Delta_{-i} \times \Delta_i : f_i(x_{-i}, x_i) = \prod_{k=1}^{r(i)} \left(u_{ik}(x_i, x_{-i}) - d_{ik}(x_{-i})\right).$$

Then f_i is obviously continuous. By the Maximum theorem we know that

$$m_i : x_{-i} \mapsto \max_{x_i \in \Delta_i} f_i(x_{-i}, x_i)$$

is continuous and that

$$M_i : x_{-i} \mapsto \arg \max_{x_i \in \Delta_i} f_i(x_{-i}, x_i)$$

is u.s.c. Moreover, by continuity of f_i, the set $M_i(x_{-i})$ is nonempty and compact for each $x_{-i} \in \Delta_{-i}$.

To prove that $M_i(x_{-i})$ is convex, discern two cases. Either $m_i(x_{-i}) = 0$, in which case $M_i(x_{-i}) = \Delta_i$ is convex, or $m_i(x_{-i}) > 0$, in which case convexity of $M_i(x_{-i})$ follows from the fact that maximizing f_i is then equivalent with maximizing $\log f_i$, which is easily seen to be a strictly concave function of x_i.

Since Δ is nonempty, convex, compact, the function $M : \Delta \twoheadrightarrow \Delta$ with

$$M : x \mapsto M_1(x_{-1}) \times \cdots \times M_n(x_{-n})$$

is u.s.c., and $M(x)$ is nonempty, convex, and compact for each $x \in \Delta$, the Kakutani fixed point theorem implies the existence of a strategy profile $x \in \Delta$ satisfying $x \in M(x)$. Such a profile x is a Nash bargaining equilibrium of the game G. □

Some caution should be applied here. In a Nash bargaining equilibrium $x \in NBE(G)$ it holds for each player $i \in N$ that

$$\begin{array}{ll}
\prod_{k=1}^{r(i)} \left(u_{ik}(x_i, x_{-i}) - d_{ik}(x_{-i})\right) > 0 & \Leftrightarrow \\
\exists y_i \in \Delta(X_i) : \quad \prod_{k=1}^{r(i)} \left(u_{ik}(y_i, x_{-i}) - d_{ik}(x_{-i})\right) > 0 & \Leftrightarrow \\
\forall k \in \{1, \ldots, r(i)\}\ \exists y_i \in \Delta(X_i) : \quad u_{ik}(y_i, x_{-i}) - d_{ik}(x_{-i}) > 0 & \Leftrightarrow \\
\forall k \in \{1, \ldots, r(i)\} : \quad u_{ik}(\cdot, x_{-i}) \text{ is not a constant function.} &
\end{array}$$

The second equivalence follows from multilinearity of u_i. The nonnegative function $\prod_{k=1}^{r(i)} (u_{ik}(\cdot, x_{-i}) - d_{ik}(x_{-i}))$ therefore equals the zero function if and only if for some criterion $k \in \{1, \ldots, r(i)\}$ the function $u_{ik}(\cdot, x_{-i})$ is a constant function. In this case, the strategy x_i of player i in the Nash bargaining equilibrium x may yield a point on the weak Pareto edge of the payoff polytope $\{u_i(y_i, x_{-i}) \mid y_i \in \Delta_i\}$, rather than on the strong Pareto edge. In the literature on bargaining situations this problem is usually avoided by making nonlevelness assumptions. Consequently:

Theorem 13.4 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$. Then $NBE(G) \subseteq WPE(G)$. Moreover, if $x \in NBE(G)$ and for each player $i \in N$ and each criterion $k \in \{1, \ldots, r(i)\}$ the function $u_{ik}(\cdot, x_{-i})$ is not constant, then $x \in SPE(G)$.*

13.4 Perfect equilibria

This section, based on Van Megen *et al.* (1999), takes a more conventional game theoretic approach to equilibrium refinements by defining the analogon of perfect equilibria (Selten, 1975) for multicriteria games. A perfect equilibrium point is defined as a limit point of a sequence of weak Pareto equilibria of perturbed multicriteria games. Perturbed games are derived from the original game by demanding that every pure strategy is played with positive probability.

Formally, let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ be a multicriteria game. Denote the finite set X_i of pure strategies of player $i \in N$ by $X_i = \{x_{i1}, \ldots, x_{im(i)}\}$, where $m(i) = |X_i|$. A vector $\varepsilon = (\varepsilon^i)_{i \in N} \in \prod_{i \in N} \mathbb{R}^{m(i)}$ is a *mistake vector* if $\sum_{k=1}^{m(i)} \varepsilon^i_k < 1$ and $\varepsilon > 0$. The ε-perturbed game associated with G is the game $G(\varepsilon) = \langle N, (X_i(\varepsilon))_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$, where the pure strategy set of player $i \in N$ is $X_i(\varepsilon) = \{x_{i1}(\varepsilon), \ldots, x_{im(i)}(\varepsilon)\}$, where $x_{ik}(\varepsilon)$ denotes the mixed strategy in $\Delta(X_i)$ which gives probability ε^i_t to x_{it} if $t \neq k$ and probability $1 - \sum_{t \neq k} \varepsilon^i_t$ to x_{ik}. With a slight abuse of notation, the payoff functions in the game $G(\varepsilon)$ are just the functions u_i restricted to the new domain.

Since $G(\varepsilon)$ is itself an element of Γ, carriers, payoff polytopes, and weakly efficient pure best reply sets (see Section 11.2), are well-defined. Carriers and weakly efficient sets are defined in terms of strategy indices. For instance, the set of pure strategies of player $i \in N$ is indexed with labels $1, \ldots, m(i)$ in both G and $G(\varepsilon)$. In G, strategy $k \in \{1, \ldots, m(i)\}$ of player i refers to $x_{ik} \in X_i$, whereas in $G(\varepsilon)$ it refers to $x_{ik}(\varepsilon) \in X_i(\varepsilon)$. Each mixed strategy in the perturbed game can be identified with a mixed strategy in the original game, so that — with a minor abuse of notation — one obtains $\Delta(X_i(\varepsilon)) \subset \Delta(X_i)$.

In Proposition 13.5 it is shown that the weakly efficient pure best reply sets of a player i w.r.t. a mixed strategy $\sigma_{-i} \in \prod_{j \in N \setminus \{i\}} \Delta_j(X_j(\varepsilon))$ in G and $G(\varepsilon)$ coincide. In the proof we use (for each $i \in N$) the function $f_i : \Delta(X_i) \to \Delta(X_i(\varepsilon))$ defined for each $\sigma_i \in \Delta(X_i)$ and each pure strategy $k \in \{1, \ldots, m(i)\}$ as

$$f_i(\sigma_i)(x_{ik}(\varepsilon)) := \sigma_i(x_{ik}). \tag{13.4}$$

Alternatively, $f_i(\sigma_i)$ can be expressed for each $\sigma_i \in \Delta(X_i)$ and each pure strategy $k \in \{1, \ldots, m(i)\}$ as

$$f_i(\sigma_i)(x_{ik}) = \varepsilon_k^i + \left(1 - \sum_{t=1}^{m(i)} \varepsilon_t^i\right) \sigma_i(x_{ik}).$$

Clearly, f_i is *continuous, dominance preserving* and *bijective* where $f_i^{-1} : \Delta(X_i(\varepsilon)) \to \Delta(X_i)$ is given by

$$f_i^{-1}(\tilde{\sigma}_i)(x_{ik}) = \frac{\tilde{\sigma}_i(x_{ik}) - \varepsilon_k^i}{(1 - \sum_{t=1}^{m(i)} \varepsilon_t^i)} \text{ for all } \tilde{\sigma}_i \in \Delta(X_i(\varepsilon)) \text{ and } k \in \{1, \ldots, m(i)\}.$$

Furthermore, (13.4) immediately implies that σ_i assigns a positive probability to pure strategy x_{ik} in G if and only if $f_i(\sigma_i)$ assigns positive probability to pure strategy $x_{ik}(\varepsilon)$ in $G(\varepsilon)$: $C(G, \sigma_i) = C(G(\varepsilon), f_i(\sigma_i))$ for all $\sigma_i \in \Delta(X_i)$.

Proposition 13.5 *Let $G \in \Gamma$, ε a mistake vector in $\prod_{i \in N} \mathbb{R}^{m(i)}$, and $\sigma \in \prod_{i \in N} \Delta(X_i(\varepsilon))$. Then $\mathcal{E}_i(G, \sigma_{-i}) = \mathcal{E}_i(G(\varepsilon), \sigma_{-i})$ for all $i \in N$.*

Proof. Let $i \in N$. It suffices to show that any weakly efficient set I w.r.t. σ_{-i} in G is also weakly efficient w.r.t. σ_{-i} in $G(\varepsilon)$ and conversely. Take $I \in \mathcal{E}_i(G, \sigma_{-i})$ (refer to Section 11.2 for the definition of $\mathcal{E}_i(G, \sigma_{-i})$) and suppose that I is not weakly efficient w.r.t. σ_{-i} in $G(\varepsilon)$. Hence, there exists a $\tilde{\sigma}_i \in \Delta(X_i(\varepsilon))$ with $C(G(\varepsilon), \tilde{\sigma}_i) \subseteq I$ which is dominated by another strategy, i.e., a strategy $\hat{\sigma}_i \in \Delta(X_i(\varepsilon))$ such that $u_i(\hat{\sigma}_i, \sigma_{-i}) - u_i(\tilde{\sigma}_i, \sigma_{-i}) > 0$. Consequently also

$$u_i(f_i^{-1}(\hat{\sigma}_i), \sigma_{-i}) - u_i(f_i^{-1}(\tilde{\sigma}_i), \sigma_{-i}) = \frac{1}{1 - \sum_{t=1}^{m(i)} \varepsilon_t^i} (u_i(\hat{\sigma}_i, \sigma_{-i}) - u_i(\tilde{\sigma}_i, \sigma_{-i})) > 0.$$

This contradicts the fact that I is weakly efficient w.r.t. σ_{-i} in G since $C(G, f_i^{-1}(\tilde{\sigma}_i)) = C(G(\varepsilon), \tilde{\sigma}_i) \subseteq I$. The proof of the converse is similar and therefore omitted. □

Definition 13.6 Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$. A strategy profile $\sigma = (\sigma_i)_{i \in N} \in \prod_{i \in N} \Delta(X_i)$ is a *perfect equilibrium* of G if there exists a sequence $(\varepsilon(k))_{k=1}^{\infty}$ of mistake vectors converging to 0 and a sequence $(\sigma(k))_{k=1}^{\infty}$ of strategy profiles such that $\sigma(k) \in WPE(G(\varepsilon(k)))$ for each $k \in \mathbb{N}$ and $\lim_{k \to \infty} \sigma(k) = \sigma$. The set of perfect equilibria of G is denoted by $PERF(G)$. ◁

The following observations can be made.

Theorem 13.7 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$. Then*

(1) if $r(i) = 1$ for all $i \in N$, then perfect equilibrium points correspond to perfect Nash equilibria;

(2) $PERF(G) \neq \emptyset$;

(3) $PERF(G) \subseteq WPE(G)$.

Proof. Claim (1) is obvious and claim (2) follows easily from the compactness of the strategy space $\prod_{i\in N}\Delta(X_i)$. To prove claim (3), let $\sigma \in PERF(G)$, $(\varepsilon(k))_{k=1}^{\infty}$ a sequence of mistake vectors and $(\sigma(k))_{k=1}^{\infty}$ a sequence of strategy profiles such that $\varepsilon(k) \to 0$, $\sigma(k) \in WPE(G(\varepsilon(k)))$ for all $k \in \mathbb{N}$, and $\sigma(k) \to \sigma$. By Proposition 11.4 it suffices to show that for every $i \in N$ it holds that $C(G,\sigma_i) \subseteq I$ for some $I \in \mathcal{E}_i(G,\sigma_{-i})$.

Let $i \in N$. For every $t \in C(G,\sigma_i)$ and sufficiently large $k \in \mathbb{N}$ it holds that $\sigma_i(x_{it}) > \varepsilon_t^i(k)$ and hence for sufficiently large k, $\sigma(k)_i(x_{it}) > \varepsilon_t^i(k)$. This implies $C(G,\sigma_i) \subseteq C(G(\varepsilon(k)),\sigma(k)_i)$ for large k. Since $\sigma(k) \in WPE(G(\varepsilon(k)))$, Proposition 11.4 implies the existence of $I(k) \in \mathcal{E}_i(G(\varepsilon(k)),\sigma(k)_{-i})$ such that $C(G(\varepsilon(k)),\sigma(k)_i) \subseteq I(k)$. By Proposition 13.5 it holds that $\mathcal{E}_i(G(\varepsilon(k)),\sigma(k)_{-i}) = \mathcal{E}_i(G,\sigma(k)_{-i})$. Therefore $C(G,\sigma_i) \subseteq C(G(\varepsilon(k)),\sigma(k)_i) \subseteq I(k)$ for large k and for some $I(k) \in \mathcal{E}_i(G,\sigma(k)_{-i})$.

Draw a subsequence $(\sigma(\ell))_{\ell=1}^{\infty}$ of $(\sigma(k))_{k=1}^{\infty}$ such that $I(\ell) = J$ for all ℓ. Since $\lim_{\ell\to\infty}\sigma(\ell)_{-i} = \sigma_{-i}$ and J is weakly efficient for all $\sigma(\ell)_{-i}$ in G, J is weakly efficient w.r.t. σ_{-i} in G. So we can find a set $I \in \mathcal{E}_i(G,\sigma_{-i})$ with $J \subseteq I$. Conclude that there is an $I \in \mathcal{E}_i(G,\sigma_{-i})$ such that $C(G,\sigma_i) \subseteq J \subseteq I$. □

It is not difficult to show that the set $PERF(G)$ of perfect equilibria of G is closed in $\prod_{i\in N}\Delta(X_i)$.

Example 13.8 In the inspection game of Example 11.2 we found $\{1\}\times[0,1] \subseteq WPE(G)$. But for $q \in [0,1)$ the strategy combination $(1,q)$ is not perfect. Any probability distribution $\tilde{p}$ in $\Delta(X_1(\varepsilon))$ close to $p = 1$ has the property that $WB_2(G(\varepsilon),\tilde{p}) = \{1\}$. This implies that for any sequence of mistake vectors $((\varepsilon(k)))_{k=1}^{\infty}$ and any sequence $((p^k,q^k))_{k=1}^{\infty}$ such that $(p^k,q^k) \in WPE(G(\varepsilon(k))), \lim_{k\to\infty}\varepsilon(k) = 0$, it holds that $q^k \to 1$. Notice that all other equilibrium points are perfect. ◁

Perfect equilibria can be characterized in several ways. First, ε-perfectness for completely mixed strategy combinations is defined.

Definition 13.9 Let $G = \langle N,(X_i)_{i\in N},(u_i)_{i\in N}\rangle \in \Gamma$ and $\varepsilon \in \mathbb{R}_{++}$. A strategy profile $\sigma \in \prod_{i\in N}\Delta(X_i)$ with $C(G,\sigma_i) = \{1,\ldots,m(i)\}$ for all $i \in N$ is called *ε-perfect* if for each $i \in N$ there exists an $I_i \in \mathcal{E}_i(G,\sigma_{-i})$ such that $\sigma_i(x_{it}) \leqq \varepsilon$ for all $t \notin I_i$. ◁

An ε-perfect strategy of player i is a completely mixed strategy such that all pure strategies that are not in a certain weakly efficient pure best reply set are played with probability at most ε.

Theorem 13.10 *Let $G = \langle N,(X_i)_{i\in N},(u_i)_{i\in N}\rangle \in \Gamma$ and $\hat{\sigma} \in \prod_{i\in N}\Delta(X_i)$. The following three claims are equivalent.*

(1) $\hat{\sigma} \in PERF(G)$;

(2) There is a sequence $(\varepsilon(k))_{k=1}^{\infty}$ of positive real numbers converging to 0 and a sequence of completely mixed strategy combinations $(\sigma(k))_{k=1}^{\infty}$ in $\prod_{i \in N} \Delta(X_i)$ converging to $\hat{\sigma}$ such that $\sigma(k)$ is $\varepsilon(k)$-perfect for all $k \in \mathbb{N}$.

(3) There is a sequence $(\sigma(k))_{k=1}^{\infty}$ of completely mixed strategies such that for all $k \in \mathbb{N}$ and all $i \in N$: $\hat{\sigma}_i \in WB_i(\sigma(k)_{-i})$ and $\lim_{k \to \infty} \sigma(k) = \hat{\sigma}$.

Proof. We show that (1) implies (2), (2) implies (3), and (3) implies (1).

(1) $\Rightarrow$ (2): Assume $\hat{\sigma} \in PERF(G)$. Take a sequence $(\delta(k))_{k=1}^{\infty}$ of mistake vectors converging to 0 and a sequence $(\sigma(k))_{k=1}^{\infty}$ of weak Pareto equilibria in the perturbed games $G(\delta(k))$ with $\lim_{k \to \infty} \sigma(k) = \hat{\sigma}$. Take $\varepsilon(k) = \max\{(\delta(k))_t^i | i \in N, t \in \{1, \ldots, m(i)\}\}$ for each $k \in \mathbb{N}$. Then $\lim_{k \to \infty} \varepsilon(k) = 0$ and $\sigma(k)$ is a $\varepsilon(k)$-perfect for each k.

(2) $\Rightarrow$ (3): Suppose (2) holds. Take a sequence $(\varepsilon(k))_{k=1}^{\infty}$ of positive real numbers converging to 0 and a sequence $(\sigma(k))_{k=1}^{\infty}$ of $\varepsilon(k)$-perfect strategy profiles tending to $\hat{\sigma}$. Let $i \in N$. For each $k \in \mathbb{N}$, there is an $I(k) \in \mathcal{E}_i(G, \sigma(k)_{-i})$ with $\sigma(k)_i(x_{it}) \leqq \varepsilon(k)$ for all $t \notin I(k)$. If $\ell \in C(G, \hat{\sigma}_i)$, there exists a sufficiently large $N_\ell \in \mathbb{N}$ such that $\hat{\sigma}_i(x_{i\ell}) > \varepsilon(k)$ and $\sigma(k)_i(x_{i\ell}) > \varepsilon(k)$ for all $k \geqq N_\ell$.

Take $M = \max\{N_\ell | \ell \in C(G, \hat{\sigma}_i)\}$. For all $k \geqq M$ and all $\ell \in C(G, \hat{\sigma}_i)$ we have $\hat{\sigma}_i(x_{i\ell}) > \varepsilon(k), \sigma(k)_i(x_{i\ell}) > \varepsilon(k)$ and so $C(G, \hat{\sigma}_i) \subseteq I(k)$. Using Proposition 11.4, it follows that $\hat{\sigma}_i \in WB_i(G, \sigma(k)_{-i})$.

(3) $\Rightarrow$ (1): Let $(\sigma(k))_{k=1}^{\infty}$ be a sequence of completely mixed strategies converging to $\hat{\sigma}$ such that $\hat{\sigma}_i \in WB_i(G, \sigma(k)_{-i})$ for all $k \in \mathbb{N}$ and all $i \in N$. Define for all $k \in \mathbb{N}$, all $i \in N$, and all $t \in \{1, \ldots, m(i)\}$:

$$(\varepsilon(k))_t^i = \begin{cases} \frac{1}{k} & \text{if} \quad t \in C(G, \hat{\sigma}_i) \\ \sigma(k)_i(s_{it}) & \text{if} \quad t \notin C(G, \hat{\sigma}_i) \end{cases}$$

Clearly, $\lim_{k \to \infty} (\varepsilon(k))_t^i = 0$ and $\varepsilon(k) = (\varepsilon(k)^i)_{i \in N} \in \prod_{i \in N} \mathbb{R}^{m(i)}$ is a mistake vector if k is large enough. It suffices to show that $\sigma(k) \in WPE(G(\varepsilon(k)))$ for large k.

Let $i \in N$. For large k, $\sigma(k)_i(x_{it}) > \frac{1}{k} = (\varepsilon(k))_t^i$ if $t \in C(G, \hat{\sigma}_i)$ and $\sigma(k)_i(x_{it}) = (\varepsilon(k))_t^i$ if $t \notin C(G, \hat{\sigma}_i)$. This implies that $C(G(\varepsilon(k)), \sigma(k)_i) = C(G, \hat{\sigma}_i)$ for large k. Since $\hat{\sigma}_i \in WB_i(G, \sigma(k)_{-i})$ we can find $I(k) \in \mathcal{E}_i(G, \sigma(k)_{-i})$ with $C(G, \hat{\sigma}_i) \subseteq I(k)$. Consequently, $C(G(\varepsilon(k)), \sigma(k)_i) \subseteq I(k)$ for large k. This implies that $\sigma(k) \in WPE(G(\varepsilon(k)))$ for large k. □

Weak Pareto equilibria of finite multicriteria games coincided with Nash equilibria of weighted games, where nonnegative weight is assigned to each of the criteria; see Theorem 11.5. For perfect equilibria this equivalence does not hold: a perfect equilibrium of a multicriteria game need not be a perfect Nash equilibrium of a weighted game.

Example 13.11 Strategy profile (p^*, q^*) with $p^* = \frac{1}{1+c}, q^* = 1$ is a perfect equilibrium of the inspection game in Example 11.2. Let $\lambda = (\lambda_1, \lambda_2)$ be a vector of weights. If

$(p^*, q^*) \in NE(G_\lambda)$, then $\lambda_1 = e_2 = (0, 1)$ in order to make the first player indifferent between his pure strategies. Hence, the payoff matrix to player 1 in G_λ is

	H	NH
I	1	1/2
NI	1	0

Since I weakly dominates NI in this payoff matrix, the completely mixed strategy p^* cannot yield a perfect Nash equilibrium of G_λ. $\triangleleft$

The reverse statement *does* hold: every perfect Nash equilibrium of a weighted game is a perfect equilibrium of the corresponding multicriteria game. Denote the set of perfect Nash equilibria of a strategic game G by $PN(G)$.

Proposition 13.12 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ and $\lambda = (\lambda_i)_{i \in N} \in \prod_{i \in N} \Delta_{r(i)}$. Then $PN(G_\lambda) \subseteq PERF(G)$.*

Proof. Let $\sigma \in PN(G_\lambda)$. Take a sequence of mistake vectors $(\varepsilon(k))_{k=1}^\infty$ converging to 0 and a sequence of completely mixed strategy combinations $(\sigma(k))_{k=1}^\infty$ such that $\lim_{k \to \infty} \sigma(k) = \sigma$ and $\sigma(k) \in NE(G_\lambda(\varepsilon(k)))$. Then $\sigma(k) \in WPE(G_\lambda(\varepsilon(k)))$ by Theorem 11.5 and hence $\sigma \in PERF(G)$. $\square$

Observe that perfect equilibria were based on weak Pareto equilibria, rather than strong Pareto equilibria. This was done for a technical reason: the set of strong Pareto equilibria need not be closed. In Example 11.2, for instance, the strategy profile (p, q) with $p = \frac{1}{c+1}$ and $q = 1$ is the limit of totally mixed strong Pareto equilibria, but is not a strong Pareto equilibrium itself.

13.5 Concluding remarks

In this chapter, three refinements of the Pareto equilibrium concept were presented. The roots of these concepts were fundamentally different:

- *Compromise equilibria* were based on the compromise solutions of Yu (1973) that enjoy great popularity in the literature on multicriteria optimization;
- *Nash bargaining equilibria* were based on the bargaining solution of Nash (1950b), part of the literature on cooperative game theory;
- *Perfect equilibria* were based on the perfect equilibria of Selten (1975), part of the literature on refinements of the Nash equilibrium concept for noncooperative games.

Norde *et al.* (1996) showed that there is no proper refinement of the Nash equilibrium concept for strategic games that yields utility maximizing strategies in one-person games, is consistent, and yields a nonempty set of outcomes. Theorem 11.10 in Chapter 11 indicates that the Pareto equilibrium concept does not suffer from this drawback. In fact, compromise equilibria give rise to such nontrivial refinements. Also Nash bargaining equilibria are consistent refinements of the weak Pareto equilibrium concept.

These two concepts, compromise equilibria and Nash bargaining equilibria, differ from the standard game theoretic approach to equilibrium refinements, which usually requires robustness against certain perturbations or trembles in the structure of the game. It is the multicriteria character of the games under consideration that yield new opportunities for refinements, simply by realizing that there is not only a conflict between players/organizations, but also *within* an organization to decide what exactly constitutes a 'best response' against a strategy profile of the other players.

It would be interesting to approach equilibrium refinements in multicriteria games from an axiomatic point of view. To axiomatize compromise equilibria, for instance, one should first try to obtain an axiomatization for this concept in the case of a single organization, rather than the case where several organizations interact. Combining this with axioms like consistency and converse consistency would then quickly yield an axiomatization.

Similarly, to axiomatize Nash bargaining equilibria, one would typically require a combination of axioms that characterize the Nash bargaining solution in standard bargaining problems (Nash, 1950b) and combine this with axioms used in noncooperative game theory. The bargaining literature usually imposes the following nondegeneracy condition: if $S \subset \mathbb{R}^n$ is the feasible set of alternatives and $d \in \mathbb{R}^n$ the disagreement point, then there is a feasible alternative $s \in S, s > d$, that is better for all concerned individuals; cf. Nash (1950b), Roth (1979, 1985), Peters (1992). In the present context, the feasible set that the members of an organization bargain over changes as a function of the strategy profile of the opponents. There will typically be strategy profiles for which this gives rise to a feasible set not satisfying the nondegeneracy condition, thereby placing the problem outside the range of those covered in most of the existing literature on bargaining.

A next step in refining equilibria for multicriteria games that is more in the spirit of the traditional refinement literature is to define the notion of proper equilibria as in Myerson (1978). Intuitively, as in the perfect equilibrium concept, proper equilibria still admit the possibility of making mistakes, but costly mistakes have lower probability. In multicriteria games 'costly mistakes' could be defined by explicitly using the possibility to define levels of best reply sets. For $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \in \Gamma$ and $\sigma \in \prod_{i \in N} \Delta(X_i)$ the first level of best replies of player $i \in N$ w.r.t. σ_{-i} in G is the set of all pure strategies contained in the efficient pure best reply sets w.r.t. σ_{-i}. The second level is constructed by considering the best replies if pure strategies in the first level are not taken into account, and so on. Formally:

$$\begin{cases} M^1(i) & := & \{1, \ldots, m(i)\} \\ \mathcal{E}_i^1(G, \sigma_{-i}) & := & \mathcal{E}_i(G, \sigma_{-i}) \end{cases}$$

and for every $k \in \mathbb{N}, k > 1$: $M^k(i) := \{t \in M^{k-1}(i) | t \notin I \text{ for all } I \in \mathcal{E}_i^{k-1}(G, \sigma_{-i})\}$. A set $I \subseteq M^k(i)$ is *k-th level weakly efficient* if for all strategies $\sigma_i \in \Delta(X_i)$ with $C(G, \sigma_i) \subseteq I$ it holds that $u_i(\sigma)$ is not dominated by any $u_i(\hat{\sigma}_i, \sigma_{-i})$ with $C(G, \hat{\sigma}_i) \subseteq M^k(i)$. $I \subseteq M^k(i)$ is an *k-th level weakly efficient pure best reply set* if I is k-th level weakly efficient and there is no k-th level efficient set $K \subseteq M^k(i)$ with $I \subseteq K$ and $I \neq K$. $\mathcal{E}_i^k(G, \sigma_{-i})$ is the set of k-th level efficient pure best reply sets for player i w.r.t. σ_{-i} in Γ.

In every perturbed game, strategies included in lower levels should be played with probabilities of lower order than those in higher levels. A proper equilibrium would be defined as a limit of such equilibria of perturbed games getting ever closer to the original game. The main problem with this approach is that the the continuity properties of these level sets are not as well-behaved as one would hope, making an existence proof a difficult matter.

Chapter 14

Pareto-Optimal Security Strategies

14.1 Introduction

Multicriteria matrix games are generalizations of the standard matrix games introduced and solved by von Neumann (1928), in the sense that in a multicriteria matrix game each of the two players has a vector-valued payoff function. By a standard matrix game we mean a two-person zero-sum game with only one criterion in which each player chooses a mixed strategy, being a probability distribution over a finite set of pure strategies.

Ghose and Prasad (1989) introduce Pareto-optimal security strategies in multicriteria matrix games. The interpretation behind this concept is that a player, given his strategy choice, considers the worst payoff he may incur in each criterion separately. A Pareto-optimal security strategy is then a strategy for which there is no alternative that yields a weakly more agreeable worst-case scenario.

Ghose (1991) characterized the first player's Pareto-optimal security strategies as minimax strategies in a weighted zero-sum game with only one criterion. His proof is complex, but was simplified by Fernandez and Puerto (1996), who also provide several other characterizations.

The weighted zero-sum game with one criterion introduced by Ghose (1991) is not a standard matrix game: the second player does not choose a probability distribution over a finite set of pure strategies, but selects a tuple of mixed strategies, one strategy for each of the separate criteria.

The purpose of this chapter, based on Voorneveld (1999a), is to make the final additional step to standard matrix games, reducing the problem of finding a Pareto-optimal security strategy to finding a minimax strategy in a matrix game, a problem that lies at the foundation of game theory.

14.2 Definitions and preliminary results

Consider a payoff matrix A with m rows and n columns. Each entry A_{ij} of A is a k-dimensional vector of real numbers. Equivalently, such a payoff matrix is described by a k-tuple $A = (A(1), \ldots, A(k))$ of $m \times n$ matrices with entries in $\mathbb{R}$. In a multicriteria matrix game based on a payoff matrix $A = (A(1), \ldots, A(k))$, the first player chooses rows and the second player chooses columns. The pure strategies or rows for the first player are denoted by $S_1 = \{1, \ldots, m\}$, the pure strategies or columns for the second player are denoted by $S_2 = \{1, \ldots, n\}$. Consequently, the mixed strategies of player 1 and 2 are $\Delta(S_1)$ and $\Delta(S_2)$, respectively. The payoff from player 1 to player 2, if player 1 chooses $x \in \Delta(S_1)$ and his opponent chooses $y \in \Delta(S_2)$, is

$$xAy = (xA(1)y, \ldots, xA(k)y).$$

The first player tries to minimize this vector, the second player to maximize it.

Remark 14.1 It is common in game theory when studying zero-sum games, to assume that payoffs are from the column player to the row player and that the row player *maximizes* his payoff, whereas the column player tries to *minimize* the payoff to the row player. In this chapter we take the opposite view: the matrix A specifies the payoffs to the column player rather than the row player. This is done to make the chapter in line with the existing literature on Pareto-optimal security strategies, where this assumption is made throughout. $\triangleleft$

In the special case that $k = 1$, we have a matrix game, the topic of von Neumann's paper (1928) and one of the starting points of game theory.

Given a strategy $x \in \Delta(S_1)$, the *security level* $v(x)$ of player 1 is given by

$$v(x) = (\max_{y \in \Delta(S_2)} xA(1)y, \ldots, \max_{y \in \Delta(S_2)} xA(k)y).$$

That is, given a strategy $x \in \Delta(S_1)$, player 1 considers the worst payoff he may incur in each criterion separately. Ghose and Prasad (1989) define Pareto-optimal security strategies as follows:

Definition 14.2 A strategy $x^* \in \Delta(S_1)$ is a *Pareto-optimal security strategy (POSS)* for player 1 in the multicriteria matrix game $A = (A(1), \ldots, A(k))$ if there is no $x \in \Delta(S_1)$ such that $v(x^*) \geq v(x)$. $\triangleleft$

Consider a multicriteria matrix game $B = (B(1), \ldots, B(k))$ with k criteria. This induces a *serial (zero-sum) game* $S(B)$ with two players, where player 1 chooses a mixed strategy $x \in \Delta(S_1)$ and player 2 chooses a vector $y = (y_1, \ldots, y_k) \in \prod_{l=1}^{k} \Delta(S_2)$ of mixed strategies, one strategy for each criterion. The payoff to player 1 if he chooses x and his opponent chooses $y = (y_1, \ldots, y_k)$ equals $\sum_{l=1}^{k} xB(l)y_l$, which player 1 tries to minimize and his opponent tries to maximize. Borm *et al.* (1996) refer to serial games as amalgamations of games.

The main result of Ghose (1991, Theorem 3.3) is:

Proposition 14.3 *A strategy $x^* \in \Delta(S_1)$ is a POSS for player 1 in the multicriteria matrix game $A = (A(1), \ldots, A(k))$ if and only if there exists a vector $\alpha \in \Delta_k^0$ such that*

$$\max_{(y_1,\ldots,y_k)\in\prod_{l=1}^k \Delta(S_2)} \sum_{l=1}^{k} x^*\alpha_l A(l) y_l = \min_{x\in\Delta(S_1)} \max_{(y_1,\ldots,y_k)\in\prod_{l=1}^k \Delta(S_2)} \sum_{l=1}^{k} x\alpha_l A(l) y_l,$$

i.e., x^* is a minimax strategy in the serial game $S(\alpha_1 A(1), \ldots, \alpha_k A(k))$.

Ghose's proof takes about eight pages. Following Fernandez and Puerto (1996), who use methods from multicriteria linear programming, a much clearer and shorter proof can be given. See Section 14.3.

Observe that a serial game has only one criterion, but is not a standard matrix game, since the second player does not choose a probability distribution over his pure strategies, but rather a k-tuple of mixed strategies.

We define a function $p : \prod_{l=1}^k \Delta(S_2) \to \Delta(\prod_{l=1}^k S_2)$ from the k-fold Cartesian product of probability distributions on player 2's set of pure strategies S_2 to the set of probability distributions on the k-fold Cartesian product of his pure strategies S_2 as follows:

$$\forall\ (y_1, \ldots, y_k) \in \textstyle\prod_{l=1}^k \Delta(S_2) \quad : \quad p(y_1, \ldots, y_k) \quad = \quad p \in \Delta(\prod_{l=1}^k S_2),$$

where

$$\forall\ c = (c(1), \ldots, c(k)) \in \textstyle\prod_{l=1}^k S_2 \quad : \quad p_c \quad = \quad \prod_{l=1}^k y_{l,c(l)},$$

where $y_{l,c(l)}$ is the probability that mixed strategy y_l assigns to the pure strategy $c(l) \in S_2$. Notice that this function is one-to-one and assigns to each k-tuple of probability distributions over S_2 the probability distribution it induces on the Cartesian product of pure strategies. Using p, we can consider $\prod_{l=1}^k \Delta(S_2)$ as a subset of $\Delta(\prod_{l=1}^k S_2)$. It is clear that this last set includes more probability distributions than those induced by p.

Example 14.4 Take $k = 2$, $S_2 = \{1, 2\}$, $y_1 = (1/4, 3/4)$, and $y_2 = (1/3, 2/3)$. Then $p(y_1, y_2)$ assigns probability 1/12 to (1,1), 2/12 to (1,2), 3/12 to (2,1), and 6/12 to (2,2). The element $q \in \Delta(S_2 \times S_2)$ which assigns probability 1/3 to (1,1), (1,2), and (2,1) and probability 0 to (2,2) is not the image $p(y)$ of any $y = (y_1, y_2) \in \Delta(S_2) \times \Delta(S_2)$. $\triangleleft$

Consider a multicriteria matrix game $B = (B(1), \ldots, B(k))$ where each matrix $B(l)$ has m rows and n columns. This induces a matrix game $M(B)$, where $M(B)$ is a matrix with m rows, labeled $i = 1, \ldots, m$, and n^k columns, labeled $c = (c(1), \ldots, c(k))$ with $c(l) \in \{1, \ldots, n\}$ for each $l = 1, \ldots, k$. The entry in row i and column $c = (c(1), \ldots, c(k))$ of $M(B)$ equals $\sum_{l=1}^k B(l)_{i,c(l)}$. Notice that the order of the columns is not important. Clearly, the set of mixed strategies of player 1 in $M(B)$ is $\Delta(S_1)$ and the set of mixed strategies of player 2 in $M(B)$ is $\Delta(\prod_{l=1}^k S_2)$. If the first player, the minimizer, plays strategy $x \in \Delta(S_1)$ and the second player, the maximizer, plays strategy $q \in \Delta(\prod_{l=1}^k S_2)$, the payoff to the first player equals, with a minor abuse of notation assuming a given order of the columns, $xM(B)q$.

Example 14.5 Consider a two-criterion matrix game $B = (B(1), B(2))$ in which both players have two pure strategies. Let the matrices $B(1)$ and $B(2)$ be as in Figure 14.1. The associated matrix game is given in Figure 14.2.

$$B(1) = \begin{array}{c|c|c|} & 1 & 2 \\ \hline 1 & 0 & 2/3 \\ \hline 2 & 1/3 & 0 \\ \hline \end{array} \qquad B(2) = \begin{array}{c|c|c|} & 1 & 2 \\ \hline 1 & 0 & -2/3 \\ \hline 2 & -4/3 & 0 \\ \hline \end{array}$$

Figure 14.1: A two-criterion matrix game

$$M(B) = \begin{array}{c|c|c|c|c|} & (1,1) & (1,2) & (2,1) & (2,2) \\ \hline 1 & 0 & -2/3 & 2/3 & 0 \\ \hline 2 & -1 & 1/3 & -4/3 & 0 \\ \hline \end{array}$$

Figure 14.2: The associated matrix game

If the first player plays each row with equal probability, i.e. $x = (1/2, 1/2)$ and the second player chooses column (1,1) with probability 1/12, column (1,2) with probability 2/12, column (2,1) with probability 3/12, and column (2,2) with probability 6/12, i.e., $q = (1/12, 2/12, 3/12, 6/12)$, the payoff to player 1 equals $xM(B)q = -11/72$. ◁

Proposition 14.6 *Given a multicriteria matrix game $B = (B(1), \ldots, B(k))$ and a strategy combination $(x, y_1, \ldots, y_k) \in \Delta(S_1) \times \prod_{l=1}^k \Delta(S_2)$, the following two claims are equivalent:*

(i) $(x, y_1, \ldots, y_k) \in \Delta(S_1) \times \prod_{l=1}^k \Delta(S_2)$ *is a Nash equilibrium in the serial game $S(B)$, i.e.,*

$$\forall\, \overline{x} \in \Delta(S_1) \quad : \quad \textstyle\sum_{l=1}^k xB(l)y_l \leqq \sum_{l=1}^k \overline{x}B(l)y_l,$$

$$\forall\, (\overline{y}_1, \ldots, \overline{y}_k) \in \textstyle\prod_{l=1}^k \Delta(S_2) \quad : \quad \sum_{l=1}^k xB(l)y_l \geqq \sum_{l=1}^k xB(l)\overline{y}_l.$$

(ii) $(x, p(y_1, \ldots, y_k)) \in \Delta(S_1) \times \Delta(\prod_{l=1}^k S_2)$ *is a Nash equilibrium in the matrix game $M(B)$, i.e.,*

$$\forall\, \overline{x} \in \Delta(S_1) \quad : \quad xM(B)p(y_1, \ldots, y_k) \leqq \overline{x}M(B)p(y_1, \ldots, y_k),$$

$$\forall\, q \in \Delta(\textstyle\prod_{l=1}^k S_2) \quad : \quad xM(B)p(y_1, \ldots, y_k) \geqq xM(B)q.$$

Proof. See Borm *et al.* (1996), Proposition 1. □

14.3 Characterizations of POSS

This section provides several characterizations of Pareto-optimal security strategies. The main result is Theorem 14.9, in which by combining the results from the previous section, we obtain a characterization of Pareto-optimal security strategies in terms of minimax strategies of suitably weighted matrix games.

Let us start with some remarks on multicriteria linear programming. A general multicriteria linear programming problem is formulated as follows:

$$\begin{aligned} &\text{Minimize} \quad Cx \\ &\text{subject to} \quad Ax \leqq b \\ &\qquad\qquad\qquad x \geqq 0, \end{aligned} \tag{14.1}$$

where $C \in \mathbb{R}^{p\times q}, A \in \mathbb{R}^{r\times q}, b \in \mathbb{R}^r, x \in \mathbb{R}^q$. A feasible solution x^* of (14.1) is an *efficient solution* if there is no feasible x such that $Cx \leq Cx^*$.

Proposition 14.7 *A feasible point x^* is an efficient solution to (14.1) if and only if there exists a vector $\alpha \in \Delta_p^0$ of weights such that x^* solves the following linear programming problem:*

$$\begin{aligned} &\textit{Minimize} \quad \alpha Cx = \langle \alpha, Cx \rangle \\ &\textit{subject to} \quad Ax \leqq b \\ &\qquad\qquad\qquad x \geqq 0 \end{aligned} \tag{14.2}$$

The proof of this proposition is analogous to the proof of Theorem 10.1.

Theorem 14.8 *Consider a multicriteria matrix game $A = (A(1), \ldots, A(k))$. Let $x^* \in \Delta(S_1), v^* = v(x^*)$. Then x^* is a POSS for player 1 if and only if (v^*, x^*) is an efficient solution to the following multicriteria linear programming problem:*

$$\begin{aligned} &\textit{Minimize} \quad v_1, \ldots, v_k \\ &\textit{subject to} \quad xA(l) \leqq (v_l, \ldots, v_l) \quad l = 1, \ldots, k \\ &\qquad\qquad\qquad x \geqq 0, \textstyle\sum_{i=1}^m x_i = 1, v \in \mathbb{R}^k. \end{aligned} \tag{14.3}$$

Proof. Strategy x^* is POSS if and only if (v^*, x^*) is an efficient solution of

$$\begin{aligned} &\text{Minimize} \quad v_1, \ldots, v_k \\ &\text{subject to} \quad v_l = \max\nolimits_{y\in\Delta(S_2)}\ xA(l)y \quad l = 1, \ldots, k \\ &\qquad\qquad\qquad x \geqq 0, \textstyle\sum_{i=1}^m x_i = 1, v \in \mathbb{R}^k. \end{aligned} \tag{14.4}$$

It is easy to see that:

- if (v, x) is feasible in (14.4), then (v, x) is feasible in (14.3).
- if (v, x) is an efficient solution to (14.3), then for each criterion $l \in \{1, \ldots, k\}$ there is a pure strategy $s \in S_2$ such that $xA(l)e_s = v_l$ (otherwise v_l could be decreased), so $v = v(x)$. But then (v, x) is feasible in (14.4).

Hence (v^*, x^*) is efficient in (14.3) if and only if it is efficient in (14.4). $\square$

The following claims are equivalent:

(a) A strategy $x^* \in \Delta(S_1)$ is a POSS for player 1 in the multicriteria matrix game $A = (A(1), \ldots, A(k))$.

(b) $(v(x^*), x^*)$ solves (14.4).

(c) $(v(x^*), x^*)$ solves

$$\begin{array}{ll} \text{Minimize} & \sum_{l=1}^{k} \alpha_l v_l \\ \text{subject to} & v_l = \max_{y \in \Delta(S_2)} \; xA(l)y \quad l = 1, \ldots, k \\ & x \geqq 0, \sum_{i=1}^{m} x_i = 1, v \in \mathbb{R}^k \end{array} \tag{14.5}$$

for some $\alpha \in \Delta_k^0$.

(d) x^* solves

$$\min_{x \in \Delta(S_1)} \sum_{l=1}^{k} \alpha_l \max_{y_l \in \Delta(S_2)} \; xA(l)y_l$$

for some $\alpha \in \Delta_k^0$.

(e) x^* solves

$$\min_{x \in \Delta(S_1)} \; \max_{(y_1, \ldots, y_k) \in \prod_{l=1}^{k} \Delta(S_2)} \; \sum_{l=1}^{k} x\alpha_l A(l) y_l$$

for some $\alpha \in \Delta_k^0$.

Here (a) $\Leftrightarrow$ (b) follows from Definition 14.2, (b) $\Leftrightarrow$ (c) follows from Proposition 14.7, and (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) is a matter of rewriting. This proves Proposition 14.3.

The next theorem is the main result of this chapter. It characterizes Pareto-optimal security strategies as minimax strategies of a standard matrix game.

Theorem 14.9 *A strategy $x^* \in \Delta(S_1)$ is a POSS for player 1 in the multicriteria matrix game $A = (A(1), \ldots, A(k))$ if and only if there exists a vector $\alpha \in \Delta_k^0$ such that x^* is a minimax strategy in the matrix game $M(\alpha_1 A(1), \ldots, \alpha_k A(k))$.*

Proof. By Proposition 14.3, a strategy $x^* \in \Delta(S_1)$ is a POSS for player 1 in the multicriteria matrix game $A = (A(1), \ldots, A(k))$ if and only if there exists a vector $\alpha \in \Delta_k^0$ such that $x^* \in \Delta(S_1)$ is a minimax strategy in the serial game $S(\alpha_1 A(1), \ldots, \alpha_k A(k))$.

By the compactness of the strategy spaces $\Delta(S_1)$ and $\prod_{l=1}^k \Delta(S_2)$ and the structure of the payoff function $(x, y_1, \ldots, y_k) \mapsto \sum_{l=1}^k x\alpha_l A(l) y_l$, the serial game $S(\alpha_1 A(1), \ldots, \alpha_k A(k))$ has a value

$$\begin{aligned} v &= \min_{x \in \Delta(S_1)} \max_{(y_1,\ldots,y_k) \in \prod_{l=1}^k \Delta(S_2)} \sum_{l=1}^k x\alpha_l A(l) y_l \\ &= \max_{(y_1,\ldots,y_k) \in \prod_{l=1}^k \Delta(S_2)} \min_{x \in \Delta(S_1)} \sum_{l=1}^k x\alpha_l A(l) y_l \end{aligned}$$

and the sets of minimax strategies of player 1 and maximin strategies of player 2 are nonempty (cf. Blackwell and Girshick, 1954, Chapter 2).

Let $(y_1, \ldots, y_k) \in \prod_{l=1}^k \Delta(S_2)$ be a maximin strategy of player 2 in the serial game. Then $x^* \in \Delta(S_1)$ is a minimax strategy if and only if $(x^*, y_1, \ldots, y_k)$ is a Nash equilibrium of $S(\alpha_1 A(1), \ldots, \alpha_k A(k))$. Equivalently, by Proposition 14.6: $(x^*, p(y_1, \ldots, y_k))$ is a Nash equilibrium of the matrix game $M(\alpha_1 A(1), \ldots, \alpha_k A(k))$. Since this is a matrix game, this is equivalent to stating that $p(y_1, \ldots, y_k)$ is a maximin strategy of player 2 and $x^* \in \Delta(S_1)$ is a minimax strategy of player 1 in $M(\alpha_1 A(1), \ldots, \alpha_k A(k))$. □

As a corollary, we obtain the Pareto-optimal security strategies as solutions to a parametric linear programming problem.

Corollary 14.10 *A strategy $x^* \in \Delta(S_1)$ is a POSS for player 1 in the multicriteria matrix game $A = (A(1), \ldots, A(k))$ if and only if there exists a vector $\alpha \in \Delta_k^0$ such that (v^*, x^*) solves the linear program LP(α) given below:*

$$\begin{aligned} &\textit{Minimize} && v \\ &\textit{subject to} && xM(\alpha_1 A(1), \ldots, \alpha_k A(k)) \leqq (v, \ldots, v) \\ &&& x \in \Delta(S_1) \\ &&& v \in \mathbb{R}, \end{aligned}$$

where

$$\begin{aligned} v^* &= \min\nolimits_{x \in \Delta(S_1)} \max\nolimits_{q \in \Delta(\prod_{l=1}^k S_2)} xM(\alpha_1 A(1), \ldots, \alpha_k A(k))q \\ &= \max\nolimits_{q \in \Delta(\prod_{l=1}^k S_2)} \min\nolimits_{x \in \Delta(S_1)} xM(\alpha_1 A(1), \ldots, \alpha_k A(k))q \end{aligned}$$

is the (minimax or maximin) value of $M(\alpha_1 A(1), \ldots, \alpha_k A(k))$.

Proof. For multicriteria matrix games with only one criterion, the notion of POSS and minimax are equivalent for player 1. Hence, the result follows from Theorems 14.8 and 14.9. □

Moreover, Ghose (1991, p. 476) observes that in his type of games only finitely many scalarizations suffice to find all Pareto-optimal security strategies. According to the proof of Theorem 14.9, this result carries over to our type of scalarized games, since we use exactly the same vectors of weights.

Example 14.11 Consider the multicriteria matrix game with

$$A(1) = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{|c|c|} \hline 0 & 2 \\ \hline 1 & 0 \\ \hline \end{array} \end{array} \qquad A(2) = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{|c|c|} \hline 0 & -1 \\ \hline -2 & 0 \\ \hline \end{array} \end{array}$$

In Ghose and Prasad (1989), the set of POSS for player 1 was computed to be the set $\{(p, 1-p) \mid p \in [1/3, 2/3]\}$. Consider the vector $\alpha = (1/3, 2/3)$ of weights. Then $(1/3)A(1)$ equals the matrix $B(1)$ and $(2/3)A(2)$ equals the matrix $B(2)$ in Example 14.5. Hence $M((1/3)A(1), (2/3)A(2))$ is the matrix $M(B)$. The minimax strategies of player 1 in $M(B)$ are the strategies $\{(p, 1-p) \mid p \in [1/3, 2/3]\}$, which is the set of Pareto-optimal security strategies of player 1: in this example a single vector of weights suffices. ◁

14.4 Conclusions

Previous papers have introduced the notion of Pareto-optimal security strategies in multicriteria matrix games and obtained a characterization in terms of minimax strategies of weighted games with a single criterion, in which one of the players chooses a mixed strategy for each of the criteria separately. The purpose of the current chapter has been to characterize Pareto-optimal security strategies as minimax strategies of standard matrix games, one of the cornerstones of game theory, where each player is allowed to choose only one mixed strategy.

The aim was not to facilitate computation. In fact, the definition of the matrix game of Theorem 14.9 points out that the strategy space of the second player grows exponentially with the data input: if this player has n pure strategies in the multicriteria game and there are k criteria, he has n^k pure strategies in the weighted matrix game of Theorem 14.9.

Chapter 15

Cooperative Multicriteria Games with Public and Private Criteria

15.1 Introduction

In matters of conflict, players frequently evaluate situations on the basis of several criteria. Still, games with multiple criteria and in particular *cooperative* games with multiple criteria have received relatively little attention in game theoretic literature. Some exceptions are Bergstresser and Yu (1977), Zhao (1991), and Lind (1996).

In the current chapter, based on Voorneveld and van den Nouweland (1998a, 1998b, 1999), we introduce a new class of cooperative multicriteria games. Two fundamentally different types of criteria are considered: *private criteria* and *public criteria*. *Private* or divisible criteria share the characteristics of the criterion one usually works with when studying games with transferable utility, the characteristics of money: the amount obtained can be divided over coalition members so that one member consumes a different quantity than another member, and that which is consumed by one member cannot be consumed by another. In economic terms, these criteria are rival and excludable. *Public* or indivisible criteria have the same value for all members of a coalition; they are non-rival and non-excludable. Examples of such criteria are global warming, investment in medical research, or, on a different scale, the national rate of unemployment and its effect on the economy, political stability, and the safety in your country.

The introduction of public criteria is new to cooperative game theory, presumably because it is assumed that some central authority takes a (socially optimal) decision on such criteria. However, the value of a public criterion is often influenced by decisions made on private criteria by individual agents (think of pollution levels, for example). Hence, it seems that decisions on private and public criteria should not be treated separately. An integrated view on private and public criteria might expose the trade-offs faced by individuals not only between criteria in the same category, but also between criteria in different categories.

The 'value' of a coalition is usually interpreted as that which its members can guarantee themselves by joining forces. If multiple criteria are involved, then improvement in one criterion (number of fish caught) may well have detrimental effects on other criteria (environmental issues like biodiversity). So, the relative importance of different criteria plays a significant role. But the relative importance of two criteria may differ with their values. For example, rich countries attach relatively more importance to controlling pollution levels than to increasing production since production levels and pollution levels are already high. For developing countries with low production levels, however, increasing production is more important than controlling pollution levels. We believe that 'collapsing' the different criteria to one number by means of a utility function ignores some of the most interesting issues associated with multicriteria decision situations. By leaving the different criteria in their own right, one can investigate what kind of trade-offs players face between the criteria. Moreover, such an approach respects the incommensurability of some attributes: in many cases agents may be incapable of or morally opposed against aggregating the value of money and the value of — for instance — a human life to a common scale. In cooperative multicriteria games we therefore consider it natural to assign a set of vector values to each coalition, i.e., we consider characteristic correspondences instead of single valued characteristic functions and an obtainable 'value' is a vector that specifies the value of all the criteria for a particular alternative that is feasible to a coalition.

Cooperative multicriteria games with public and private criteria as defined and studied in the current chapter generalize the games used in Bergstresser and Yu (1977) and Lind (1996). These authors do not discriminate between several types of criteria; they only use what we call private criteria. Moreover, the characteristic functions in their games are single-valued instead of set-valued.

After defining multicriteria cooperative games with public and private criteria, the obvious next step is the search for reasonable solutions to such games. This chapter concentrates on core concepts, which rule out those outcomes which are in a sense unstable because subcoalitions of agents are able to reach agreements that are better for all their members. Taking into account the features of the model, the distinction between private and public criteria and the introduction of set-valued characteristic functions, we define two concepts: the dominance outcome core and the core.

The current chapter differs fundamentally from other papers that study the core concept of cooperative games with externalities such as Shapley and Shubik (1969), Starrett (1973), and Chander and Tulkens (1997). These papers all start with a game in strategic form or an economy and then discuss how to appropriately define a corresponding cooperative game. Because of the externalities, the behavior of the players or agents in complementary coalitions has to be taken into account when deciding on the value of a coalition of players. Different assumptions about the behavior of the other players lead to different formulations of an associated cooperative game and, correspondingly, to different core concepts, such as the α-core and the β-core. The quest for the "right" core concept is the main issue in these papers. In the current chapter, we abstract from this issue and start with a cooperative game. Our contribution is that we provide a framework to explicitly deal with multiple criteria in cooperative games. We do not use a utility function to reduce a decision with many dimensions to a one-dimensional decision but expose the trade-offs between different dimensions faced by the players.

Well-known axiomatizations of core concepts for single-criterion cooperative games (see Peleg, 1985, 1986, 1987) use a consistency or reduced game property. The consistency principle for cooperative games — which is very similar to the consistency principle for noncooperative games discussed in Chapter 11 — essentially means that if the grand coalition of players reaches an agreement, then no subcoalition of players has an incentive to renegotiate within the coalition after giving the players outside the coalition their part of the solution, because the proposed agreement is also a part of the solution of the reduced game played within the subcoalition.

The current chapter investigates consistency properties of the proposed core for cooperative multicriteria games. We provide three axiomatic characterizations of the core that are based on the notion of consistency. One of these characterizations uses converse consistency, a property that postulates that a proposed agreement must be in the solution of a game if for every subcoalition it holds that the restriction of this agreement to the subcoalition is in the solution of the reduced game. A second axiomatization of the core uses a converse consistency requirement that restricts attention to subcoalitions of two players. The two axiomatizations of the core of cooperative multicriteria games that use converse consistency properties are similar to the axiomatizations of core concepts for cooperative games with or without transferable utility by Peleg (1985, 1986, 1987).

The third axiomatization of the core provided in this chapter differs significantly from the previous two. It uses a new definition of reduced games, one that stresses the fact that there are players outside each subcoalition that cannot be ignored altogether by requiring players in a subcoalition to cooperate with at least one outside player. Consistency with respect to this new definition of reduced games is used to give an axiomatic characterization of the core for multicriteria games with an enlightenment property (see Section 15.5) instead of converse consistency. It is shown by means of a counterexample that this characterization does not hold if the old definition of reduced games is used.

The set-up of the chapter is as follows. Cooperative multicriteria games with public and private criteria are defined in Section 15.2, along with the core and the dominance outcome core. In Section 15.3 we prove that the dominance outcome core always contains the core and that both concepts coincide for games satisfying some additional assumptions. In the next two sections, Sections 15.4 and 15.5, we provide several axiomatizations of the core based on the notion of consistency. In Section 15.4 converse consistency is used to characterize the core and in Section 15.5 we give the new definition of reduced games that was mentioned before and use this to characterize the core without requiring converse consistency. The new definition of reduced games is applied to standard transferable utility games in Section 15.6, which is based on Voorneveld and van den Nouweland (1998b).

The final section of this chapter, Section 15.7, based on Voorneveld and van den Nouweland (1999), provides additional motivation for the core concept. This is done by showing that core elements naturally arise as strong equilibria of associated noncooperative claim games in which players independently state coalitions they want to form and a payoff they want to receive. Related work can be found in von Neumann and Morgenstern (1947) who introduce claim games for TU-games, in which players only claim coalitions and the value of a fitting coalition is split equally over its members. Borm and Tijs (1992) introduce claim games for NTU-games.

15.2 Definitions

For a set $A \subseteq \mathbb{R}^m$, we define its Pareto edge by $\mathrm{Par}(A) := \{x \in A \mid \text{there is no } y \in A \text{ with } y > x\}$. Recall that for an arbitrary set A we denote by $\mathbb{R}^A$ the vector space of all real-valued functions on A.

Let U be an infinite set of players. A *cooperative multicriteria game with public and private criteria*, or a *game* for ease of notation, is described by

- A finite set D of divisible or private criteria;
- A finite set P of indivisible or public criteria;
- A finite, nonempty set $N \subset U$ of players;
- A correspondence $v : 2^N \setminus \{\emptyset\} \twoheadrightarrow \mathbb{R}^{D \cup P}$;

such that $D \cap P = \emptyset$, $D \cup P \neq \emptyset$ and $v(S) \neq \emptyset$ for each coalition $S \in 2^N \setminus \{\emptyset\}$. The sets D and P that define a certain game will not be mentioned explicitly and a game is simply denoted (N, v). For one-person coalitions we write $v(i)$ instead of $v(\{i\})$. Let Υ denote the set of games as defined above.

Example 15.1 Two neighboring countries, A and B, negotiate to reduce CO_2 levels in the air. The marginal costs of reducing CO_2 levels increase as abatements increase: there are relatively cheap methods that can be used to reduce CO_2 levels at first, but to effect higher reductions, more expensive methods have to be employed as well. Suppose country A on its own can abate in a low-cost way by spending 100 to reduce the level of CO_2 in the air by 1, and it can abate more, a reduction of 3, at a cost of 600. Country B on its own can reduce the CO_2 level by 2 at a cost of 150 and by 7 at a cost of 900. If the countries cooperate, they can realize all the above mentioned possibilities but also profit from each other's expertise and abate relatively cheaper. They can reduce CO_2 levels by 3 at a cost of 200 and by 10 at a cost of 1200.

The cooperative multicriteria game describing this situation has one private criterion, minus the cost of the abatements, and one public criterion, the decrease in the CO_2 level in the air. The player set is $N = \{A, B\}$, the characteristic function is given by

$$\begin{aligned} v(A) &= \{(0,0), (-100,1), (-600,3)\}, \\ v(B) &= \{(0,0), (-150,2), (-900,7)\}, \\ v(\{A,B\}) &= v(A) \cup v(B) \cup \{(-200,3), (-1200,10)\}. \end{aligned}$$

$\triangleleft$

Several subsets of Υ correspond to well-known classes of games.

Example 15.2 A game (N, v) with $P = \emptyset$, $|D| = 1$, and $|v(S)| = 1$ for each coalition $S \in 2^N \setminus \{\emptyset\}$ is essentially a TU-game. $\triangleleft$

Example 15.3 A game (N, v) with $P = \emptyset$ and $v(S)$ a compact and comprehensive (in the sense that $b \in v(S)$ and $0 \leqq a \leqq b$ implies $a \in v(S)$) subset of $\mathbb{R}^D_+$ is a multi-commodity game as studied by van den Nouweland *et al.* (1989). $\triangleleft$

The characteristic function of NTU-games is also set-valued and vector-valued, but describes for a coalition the payoff for each separate member, so that the value of a coalition S is a subset of $\mathbb{R}^S$. This differs from our cooperative multicriteria games, where the correspondence v maps the coalitions to a fixed vector space $\mathbb{R}^{D \cup P}$.

Cooperative multicriteria games with public and private criteria generalize the cooperative multicriteria games used by Bergstresser and Yu (1977) and Lind (1996) in the sense that these authors do not use set-valued characteristic functions and do not discriminate between different types of criteria.

In what follows, we need a definition of an allocation. In a game $(N, v) \in \Upsilon$, an allocation takes an element of the set of values attainable by the grand coalition N and divides it among the players in accordance with the characteristics of the criteria: when restricted to divisible criteria everything is divided, whereas for indivisible criteria every player gets the same fixed amount. Before formally defining allocations, some more notation is needed.

Consider a game $(N, v) \in \Upsilon$ and a vector $x = (x^i)_{i \in N}$ with $x^i \in \mathbb{R}^{D \cup P}$ for each $i \in N$. Let $S \in 2^N \setminus \{\emptyset\}$. Then x_S denotes the vector $(x^i)_{i \in S}$, i.e., x restricted to the components of the members of coalition S and $x(S)$ denotes the sum of the elements $(x^i)_{i \in S}$, $x(S) := \sum_{i \in S} x^i$. For a vector (or function) $y \in \mathbb{R}^{D \cup P}$ the restriction of y to P is denoted $y_{|P}$ and the restriction of y to D is denoted $y_{|D}$.

Definition 15.4 Given a game (N, v) an *allocation* is a vector $x = (x^i)_{i \in N}$ with $x^i \in \mathbb{R}^{D \cup P}$ for each player $i \in N$ that satisfies the requirement that there exists a $y \in v(N)$ for which

$$\begin{aligned} \textstyle\sum_{i \in N} x^i_{|D} &= y_{|D} \quad \text{and} \\ x^i_{|P} &= y_{|P} \quad \text{for each } i \in N. \end{aligned}$$

The set of allocations of (N, v) is denoted $A(N, v)$. ◁

A coalition can *improve upon* an allocation if there is an outcome it can guarantee itself which — when distributed over its members in a feasible way — is at least as good for each member and better in some criterion for at least one coalition member. Formally, a coalition $S \subseteq N$ can *improve upon* an allocation x if there exists a vector $y \in v(S)$ such that

$$\begin{aligned} \textstyle\sum_{i \in S} x^i_{|D} &\leqq y_{|D} \quad \text{and} \\ x^i_{|P} &\leqq y_{|P} \quad \text{for each } i \in S, \end{aligned}$$

where at least one of the inequalities is strict ($\leq$). Such a vector y is said to *dominate* x *via* S. An allocation in a game (N, v) is *individually rational* if one-player coalitions, i.e. individual players, cannot improve upon it and it is an *imputation* if neither N nor individual players can improve upon it. The set of individually rational allocations and the set of imputations of a game (N, v) are denoted by $IR(N, v)$ and $I(N, v)$, respectively.

A *solution concept* σ on the class Υ is a map that assigns to each game $(N, v) \in \Upsilon$ a (possibly empty) set of allocations $\sigma(N, v)$. Hence, $\sigma(N, v) \subseteq A(N, v)$ for all $(N, v) \in \Upsilon$.

This chapter concentrates on core concepts, i.e. concepts that rule out allocations that are in some sense unstable. We define two different core concepts.

Definition 15.5 The *core* $C(N, v)$ of a game (N, v) is the set of allocations upon which no coalition can improve:

$$\begin{aligned} C(N, v) \;=\; \{x \in A(N, v) \mid\ & \text{there exist no } S \in 2^N \setminus \{\emptyset\} \text{ and } y \in v(S) \text{ s.t.} \\ & \textstyle\sum_{i \in S} x^i_{|D} \leqq y_{|D} \quad \text{and} \\ & x^i_{|P} \leqq y_{|P} \quad \text{for each } i \in S \\ & \text{with at least one of the inequalities being strict } (\leq)\} \end{aligned}$$

◁

Definition 15.6 The *dominance outcome core* $DOC(N, v)$ of a game (N, v) is the set of imputations for which there is no coalition S and another imputation y such that y^i is better than x^i for each player $i \in S$ and such that the players in S can jointly guarantee themselves at least what they get according to the allocation y:

$$\begin{array}{rcll} DOC(N,v) & = & \{x \in I(N,v) \mid & \text{there exist no } S \in 2^N \setminus \{\emptyset\}, y \in I(N,v), \\ & & & \text{and } z \in v(S) \text{ s.t.} \\ & & & \begin{array}{lcll} y^i & \geq & x^i & \text{for each } i \in S, \\ \sum_{i\in S} y^i_{|D} & \leqq & z_{|D} & \text{and} \\ y^i_{|P} & \leqq & z_{|P} & \text{for each } i \in S \end{array} \quad \} \end{array}$$

◁

15.3 The core and dominance outcome core

In this section we prove that the core of a game is always included in the dominance outcome core. Moreover, we prove that the core equals the dominance outcome core under some mild conditions.

Proposition 15.7 *For each game $(N,v) \in \Upsilon$ it holds that $C(N,v) \subseteq DOC(N,v)$.*

Proof. Let $(N,v) \in \Upsilon$. If $C(N,v) = \emptyset$ we are done. So, assume $C(N,v) \neq \emptyset$ and let $x = (x^i)_{i\in N} \in C(N,v)$. Then x is an allocation upon which neither N nor individual players can improve, so $x \in I(N,v)$. Now suppose $x \notin DOC(N,v)$. Then let $S \in 2^N \setminus \{\emptyset\}, y \in I(N,v)$, and $z \in v(S)$ be such that

$$\begin{array}{rcll} y^i & \geq & x^i & \text{for each } i \in S; \\ \sum_{i\in S} y^i_{|D} & \leqq & z_{|D}; & \\ y^i_{|P} & \leqq & z_{|P} & \text{for each } i \in S. \end{array}$$

Hence, there exist an $S \in 2^N \setminus \{\emptyset\},\ z \in v(S)$ such that

$$\begin{array}{lclcll} \sum_{i\in S} x^i_{|D} & \leqq & \sum_{i\in S} y^i_{|D} & \leqq & z_{|D} & \\ x^i_{|P} & \leqq & y^i_{|P} & \leqq & z_{|P} & \text{for each } i \in S \end{array}$$

with at least one strict inequality since $y^i \geq x^i$ for all $i \in S$, i.e., S can improve upon x, contradicting $x \in C(N,v)$. □

In general, the core is not equal to the dominance outcome core. In the following example both cores do not coincide.

Example 15.8 Consider a three-player, bicriteria game (N, v) where the first criterion is divisible and the second public. Define $v(i) = \{(1, 10)\}$ for all $i \in N$ and $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2, 3\}) = \{(3, 10)\}$. Then

$$I(N, v) = DOC(N, v) = \{(x^1, x^2, x^3) \mid x^1 = x^2 = x^3 = (1, 10)\}.$$

However, $C(N, v) = \emptyset$, since every two-player coalition can improve upon the unique imputation. For instance, for $S = \{1, 2\}$ and $y = (3, 10) \in v(S)$: $x_1^1 + x_1^2 < 3 = y_1$ and $x_2^1 = x_2^2 = 10 = y_2$. ◁

Under some restrictions, however, the two cores coincide.

Proposition 15.9 *Let $(N, v) \in \Upsilon$ be a game for which the following four properties hold:*

1. *Comprehensiveness of $v(i)$ for each $i \in N$ and of $v(N)$:*

$$\begin{cases} \text{for all } i \in N \quad \text{and all } a \in v(i) & : \quad \{x \in \mathbb{R}^{D \cup P} \mid x \leqq a\} \subseteq v(i) \\ \qquad\qquad\qquad \text{for all } a \in v(N) & : \quad \{x \in \mathbb{R}^{D \cup P} \mid x \leqq a\} \subseteq v(N) \end{cases}$$

2. *Compactness conditions:*

$$\begin{cases} \text{for all } i \in N \quad \text{and all } a \in v(i) & : \quad (\{a\} + \mathbb{R}_+^{D \cup P}) \cap v(i) \quad \text{is compact} \\ \qquad\qquad\qquad \text{for all } a \in v(N) & : \quad (\{a\} + \mathbb{R}_+^{D \cup P}) \cap v(N) \quad \text{is compact} \end{cases}$$

3. *Nonlevelness of $v(i)$ for each $i \in N$ and of $v(N)$:*

$$\begin{cases} \text{for all } i \in N \quad \text{and all } a, b \in \mathrm{Par}(v(i)) & : \quad \text{if } a \geqq b, \quad \text{then } a = b \\ \qquad\qquad\qquad \text{for all } a, b \in \mathrm{Par}(v(N)) & : \quad \text{if } a \geqq b, \quad \text{then } a = b \end{cases}$$

4. *A superadditivity condition:*

   For each $S \in 2^N \setminus \{\emptyset\}, y \in v(S)$, and $z^i \in v(i)$ for each $i \in N \setminus S$ it holds that if $y_{|P} \geqq z^i_{|P}$ for all $i \in N \setminus S$, then $a \in v(N)$, where $a \in \mathbb{R}^{D \cup P}$ is defined as follows:

$$\begin{aligned} a_{|D} &= y_{|D} + \textstyle\sum_{i \in N \setminus S} z^i_{|D} \\ a_{|P} &= y_{|P} \end{aligned}$$

Then $C(N, v) = DOC(N, v)$.

Remark 15.10 The definition of nonlevel sets given above is a standard definition. In the proof of the proposition it is convenient to use the following equivalent formulation:

$$\begin{cases} \text{for all } i \in N, \quad \text{all } b \in \mathrm{Par}(v(i)), \quad \text{and all } a \in \mathbb{R}^{D\cup P} \ : \ \text{if } a \geq b, \ \text{ then } a \notin v(i) \\ \qquad\qquad\qquad \text{for all } b \in \mathrm{Par}(v(N)) \quad \text{and all } a \in \mathbb{R}^{D\cup P} \ : \ \text{if } a \geq b, \ \text{ then } a \notin v(N) \end{cases}$$

◁

Proof (Prop. 15.9). By Proposition 15.7: $C(N,v) \subseteq DOC(N,v)$. To prove that $DOC(N,v) \subseteq C(N,v)$, let $x = (x^i)_{i\in N} \in I(N,v)$ and assume that $x \notin C(N,v)$. Then there exist $S \in 2^N \setminus \{\emptyset, N\}$ and $y \in v(S)$ such that

$$\begin{cases} \sum_{i\in S} x^i_{|D} & \leqq & y_{|D} \\ x^i_{|P} & \leqq & y_{|P} \quad \text{for each } i \in S \end{cases} \tag{15.1}$$

where at least one inequality is strict ($\leq$). For each $i \in N \setminus S$, let $z^i \in v(i)$ be such that $z^i_{|P} \leqq y_{|P}$ and $z^i \in \mathrm{Par}(v(i))$, the Pareto edge of $v(i)$. Such z^i exist: let $i \in N \setminus S$ and $a \in v(i)$, which is possible by nonemptiness of $v(i)$. Either $a_{|P} \leqq y_{|P}$ or, using comprehensiveness of $v(i)$, one can lower the coordinates in $\{k \in P \mid a_k > y_k\}$ without leaving $v(i)$. So let $b \in v(i)$ be such that $b_{|P} \leqq y_{|P}$. By assumption the set $\{c \in v(i) \mid c \geqq b\}$ is compact and hence the set $\{c \in v(i) \mid c \geqq b, c_{|P} = b_{|P}\}$ is compact. Define $u \in \mathbb{R}^{D\cup P}$ such that $u_k = 1$ for $k \in D$ and $u_k = 0$ for $k \in P$. By nonemptiness and compactness of $\{c \in v(i) \mid c \geqq b, c_{|P} = b_{|P}\}$, we know that

$$\alpha^* := \max\{\alpha \in \mathbb{R}_+ \mid b + \alpha u \in \{c \in v(i) \mid c \geqq b, c_{|P} = b_{|P}\}\} \tag{15.2}$$

exists. We claim that $b + \alpha^* u \in \mathrm{Par}(v(i))$. Suppose to the contrary, that $b + \alpha^* u$ is not on the Pareto edge of $v(i)$. Then $d > b + \alpha^* u$ for some $d \in v(i)$. In particular, $d_k > b_k + \alpha^* u_k = b_k + \alpha^*$ for each $k \in D$. Take $\beta = \min\{d_k - b_k \mid k \in D\}$. Then $\beta > \alpha^*$ and $b + \beta u \leqq d$. By comprehensiveness of $v(i)$ it follows that $b + \beta u \in v(i)$. Also $b + \beta u \in \{c \in v(i) \mid c \geqq b, c_{|P} = b_{|P}\}$. Hence by (15.2), $\beta \leq \alpha^*$ must hold. This yields a contradiction. So $b + \alpha^* u \in \{c \in v(i) \mid c \geqq b, c_{|P} = b_{|P}\} \cap \mathrm{Par}(v(i))$. Since $b_{|P} \leqq y_{|P}$, we can now define the desired z^i by $z^i := b + \alpha^* u$.

By the superadditivity condition the vector $a \in \mathbb{R}^{D\cup P}$ with $a_{|D} = y_{|D} + \sum_{i\in N\setminus S} z^i_{|D}$ and $a_{|P} = y_{|P}$ is an element of $v(N)$. Using the comprehensiveness of $v(N)$ and the compactness assumption on $v(N)$, it follows in a similar manner as demonstrated above, that the set $\{c \in v(N) \mid c \geqq a, c_{|P} = a_{|P}\}$ contains an element b on the Pareto edge of $v(N)$. Take such a $b \in v(N)$. This b is used to construct an imputation $\hat{x}$ that dominates imputation x via coalition S. Define $\hat{x} = (\hat{x}^i)_{i\in N} \in (\mathbb{R}^{D\cup P})^N$ as follows:

$$\begin{aligned}
\hat{x}^i_{|P} &= b_{|P} && \text{for each } i \in N \\
\hat{x}^i_{|D} &= z^i_{|D} + \tfrac{1}{|N\setminus S|}(b - y - \textstyle\sum_{i\in N\setminus S} z^i)_{|D} && \text{for each } i \in N \setminus S \\
\hat{x}^i_{|D} &= x^i_{|D} + \tfrac{1}{|S|}(y - \textstyle\sum_{i\in S} x^i)_{|D} && \text{for each } i \in S
\end{aligned}$$

Notice that

- $\sum_{i\in N} \hat{x}^i_{|D} = b_{|D}$ and $\hat{x}^i_P = b_{|P}$ for all $i \in N$. Since $b \in v(N)$, it follows that $\hat{x}$ is an allocation;
- Since b is on the Pareto edge of $v(N)$, using the nonlevelness of $v(N)$ yields that the allocation $\hat{x}$ cannot be improved upon by the grand coalition N;
- Since $y_{|D} \geqq \sum_{i\in S} x^i_{|D}$ and $b_{|P} = a_{|P} = y_{|P} \geqq x^i_{|P}$ for all $i \in S$, we have that $\hat{x}^i \geqq x^i$ for each player $i \in S$. Also, $x \in I(N, v)$ by assumption. Hence, singleton coalitions $\{i\}$ with $i \in S$ cannot improve upon $\hat{x}$;
- Since $b \geqq a, a_{|D} = y_{|D} + \sum_{i\in N\setminus S} z^i_{|D}$, and $b_{|P} = a_{|P} = y_{|P} \geqq z^i_{|P}$ for each $i \in N \setminus S$, we have that $\hat{x}^i \geqq z^i$ for each $i \in N \setminus S$. Using the nonlevelness of $v(i)$ and the fact that z^i lies on the Pareto edge of $v(i)$, we derive that singleton coalitions $\{i\}$ with $i \in N \setminus S$ cannot improve upon $\hat{x}$.

From the four points above we deduce that $\hat{x} \in I(N, v)$. Moreover, $y \in v(S), \sum_{i\in S} \hat{x}^i_{|D} = y_{|D}$, and $\hat{x}^i_{|P} = b_{|P} = y_{|P}$ for each $i \in S$. Thus, by (15.1) and the construction of $\hat{x}$: $\hat{x}^i \geq x^i$ for each $i \in S$ (recall that $\hat{x}^i_{|P} = \hat{x}^j_{|P}$ for all players i, j). Conclude that $x \notin DOC(N, v)$. Hence, $DOC(N, v) \subseteq C(N, v)$, which completes the proof. □

15.4 Core axiomatizations with converse consistency

In this section we study some properties of the core and provide several axiomatizations, all based on the notions of consistency and converse consistency. The consistency principle essentially means that if the grand coalition of players reaches an agreement, then no subcoalition of players has an incentive to renegotiate within the subcoalition after giving the players outside it their part of the solution, because the proposed agreement is also in the solution of the reduced game played within the subcoalition. The converse consistency axiom requires that a proposed agreement must be in the solution of a game if for every subcoalition it holds that the restriction of this agreement to that subcoalition is in the solution of the reduced game. Hence, it provides information about the solution of a game, given information about the solution of its reduced games, justifying the name 'converse' consistency. The axiomatizations are similar to those of Peleg (1985, 1986, 1987).

Definition 15.11 Let $(N, v) \in \Upsilon, x \in A(N, v)$, and $S \in 2^N \setminus \{\emptyset, N\}$. The *reduced game* (S, v_S^x) of (N, v) with respect to allocation x and coalition S is the game defined by

$$\begin{aligned} v_S^x(S) &= v(N) - \tilde{x}(N \setminus S) \\ v_S^x(T) &= \cup_{Q \subseteq N \setminus S} \; (v(T \cup Q) - \tilde{x}(Q)) \quad \text{for all } T \in 2^S \setminus \{\emptyset, S\}, \end{aligned}$$

where $\tilde{x} = (\tilde{x}^i)_{i \in N} \in (\mathbb{R}^{D \cup P})^N$ is defined for all $i \in N$ by

$$\tilde{x}_k^i = \begin{cases} x_k^i & \text{if } \; k \in D \\ 0 & \text{if } \; k \in P. \end{cases}$$

◁

The interpretation of the reduced game is as follows. Suppose the group of all players initially agrees on an allocation x, and the players in $N \setminus S$ withdraw from the decision-making process taking their agreed-upon share of the private goods with them. Then, if the agents in S reconsider, they are facing the game v_S^x, because in their negotiations they take into account that they can cooperate with some of the players in $N \setminus S$ as long as those are given their shares of the private goods. Note that the players who leave the decision-making process are not guaranteed anything about the public criteria. Since these criteria are public, their level will ultimately be determined by the players who still take part in the decision-making process. Hence, players who leave this process take a risk, but if the solution concept is consistent, then the remaining players will not change their minds about the initially agreed-upon levels of the public criteria. This is similar to the treatment of public goods in van den Nouweland *et al.* (1998).

Let us consider some axioms that are used in the remainder of this section. A solution concept σ on Υ satisfies:

- **One-Person Efficiency (OPE)** if for each game $(N, v) \in \Upsilon$ with $|N| = 1$ it holds that $\sigma(N, v) = IR(N, v)$;

- **Individual Rationality (IR)** if for each game $(N, v) \in \Upsilon$ it holds that $\sigma(N, v) \subseteq IR(N, v)$;

- **Inclusion of Imputation Set for two-player Games (II$_2$)** if for every two-player game $(N, v) \in \Upsilon$ it holds that $\sigma(N, v) \supseteq I(N, v)$;

- **Restricted Nonemptiness (r-NEM)** if for each game $(N, v) \in \Upsilon$ it holds that if $C(N, v) \neq \emptyset$, then $\sigma(N, v) \neq \emptyset$;

- **Consistency (CONS)** if for each game $(N, v) \in \Upsilon$ it holds that $x \in \sigma(N, v)$ implies $x_S \in \sigma(S, v_S^x)$ for each coalition $S \in 2^N \setminus \{\emptyset, N\}$;

- **Converse Consistency (COCONS)** if for each game $(N,v) \in \Upsilon$ with $|N| \geqq 2$ and each allocation $x \in A(N,v)$ it holds that if $x_S \in \sigma(S, v_S^x)$ for each $S \in 2^N \setminus \{\emptyset, N\}$, then $x \in \sigma(N,v)$;
- **Converse Consistency for Two-Player Reductions (COCONS$_2$)** if for each game $(N,v) \in \Upsilon$ with $|N| \geqq 3$ and each allocation $x \in A(N,v)$ it holds that if $x_S \in \sigma(S, v_S^x)$ for each $S \in 2^N \setminus \{\emptyset, N\}$ with $|S| = 2$, then $x \in \sigma(N,v)$.

The next proposition states that the core satisfies all these axioms.

Proposition 15.12 *The core satisfies OPE, IR, II$_2$, r-NEM, CONS, COCONS, and COCONS$_2$.*

Proof. It is obvious that the core satisfies OPE, IR, II$_2$ and r-NEM.

To prove that the core satisfies CONS, let $(N,v) \in \Upsilon, x \in C(N,v)$, and $S \in 2^N \setminus \{\emptyset, N\}$. Suppose that $x_S \notin C(S, v_S^x)$. Then there exist a coalition $T \in 2^S \setminus \{\emptyset\}$ and a vector $z \in v_S^x(T)$ such that

$$\begin{array}{rcll} \sum_{i \in T} x^i_{|D} & \leqq & z_{|D} & \\ x^i_{|P} & \leqq & z_{|P} & \text{for all } i \in T \end{array}$$

with at least one strict inequality ($\leq$). Since $z \in v_S^x(T)$, there exist a $Q \subseteq N \setminus S$ and $y \in v(T \cup Q)$ such that $z = y - \tilde{x}(Q)$. Observe that by definition of the reduced game, $Q = N \setminus S$ if $T = S$. Now we have

$$\begin{array}{rcccccccl} \sum_{i \in T \cup Q} x^i_{|D} & = & \sum_{i \in T} x^i_{|D} + \sum_{i \in Q} \tilde{x}^i_{|D} & \leqq & (z + \tilde{x}(Q))_{|D} & = & y_{|D} & \\ x^i_{|P} & \leqq & z_{|P} & = & (y - \tilde{x}(Q))_{|P} & = & y_{|P} & \text{for all } i \in T \cup Q \end{array}$$

where at least one of the inequalities is strict ($\leq$). But then x cannot be in the core of (N,v), since $T \cup Q$ can improve upon it. Hence $x_S \in C(S, v_S^x)$ and the core satisfies CONS.

To prove that the core satisfies COCONS$_2$, Let $(N,v) \in \Upsilon$ with $|N| \geqq 3$ and $x \in A(N,v)$ such that $x_S \in C(S, v_S^x)$ for every two-player coalition $S \in 2^N \setminus \{\emptyset, N\}$. We will prove that no coalition of players can improve upon x, and hence $x \in C(N,v)$.

Suppose that N can improve upon x. Then, for some $y \in v(N)$:

$$\begin{array}{rcll} \sum_{i \in N} x^i_{|D} & \leqq & y_{|D} & \\ x^i_{|P} & \leqq & y_{|P} & \text{for all } i \in N \end{array}$$

where at least one of the inequalities is strict ($\leq$). Let $S \in 2^N \setminus \{\emptyset, N\}$ have two players. Then, for a y as mentioned above it holds that

$$\begin{array}{rcccll} \sum_{i \in S} x^i_{|D} & \leqq & y_{|D} - \sum_{i \in N \setminus S} x^i_{|D} & = & (y - \tilde{x}(N \setminus S))_{|D} & \\ x^i_{|P} & \leqq & y_{|P} & = & (y - \tilde{x}(N \setminus S))_{|P} & \text{for all } i \in S \end{array}$$

where at least one of the inequalities is strict ($\leq$). Since $y - \tilde{x}(N \setminus S) \in v_S^x(S)$, we find that S can improve upon x_S in (S, v_S^x). This contradicts $x_S \in C(S, v_S^x)$. We conclude that N cannot improve upon x in (N, v).

Now, let $T \in 2^N \setminus \{\emptyset, N\}$. To prove that T cannot improve upon x in (N, v), let $i \in T, j \in N \setminus T$, and $S := \{i, j\}$. Then $x_S \in C(S, v_S^x)$, so in particular $\{i\}$ cannot improve upon x_S in (S, v_S^x). Consequently, for $T \setminus \{i\} \subseteq N \setminus S$, there is no $z \in v((T \setminus \{i\}) \cup \{i\}) - \tilde{x}(T \setminus \{i\}) \subseteq v_S^x(i)$ such that $x^i_{|D} \leqq z_{|D}$ and $x^i_{|P} \leqq z_{|P}$, where at least one of the inequalities is strict ($\leq$). So there is no $y \in v(T)$ such that

$$\begin{array}{rcl} \sum_{k \in T} x^k_{|D} = x^i_{|D} + \sum_{k \in T \setminus \{i\}} x^k_{|D} & \leqq & y_{|D} \\[2ex] x^k_{|P} & \leqq & y_{|P} \quad \text{for all } k \in T \end{array}$$

where at least one of the inequalities is strict ($\leq$). Consequently, T cannot improve upon x on (N, v). We conclude that $x \in C(N, v)$ and that the core satisfies COCONS$_2$.

Notice that COCONS is not implied by COCONS$_2$, since COCONS$_2$ is not applicable to games $(N, v) \in \Upsilon$ with $|N| = 2$. The proof that the core satisfies COCONS, however, is similar to the proof that it satisfies COCONS$_2$ and is therefore omitted. □

Our next proposition lays the basis for the first axiomatization of the core.

Proposition 15.13 *Let ϕ and ψ be two solution concepts on Υ. If ϕ satisfies OPE and CONS and ψ satisfies OPE and COCONS, then $\phi(N, v) \subseteq \psi(N, v)$ for each game $(N, v) \in \Upsilon$.*

Proof. The proof is by induction on the number of players. First, let $(N, v) \in \Upsilon$ have only one player. Then $\phi(N, v) = \psi(N, v)$ by OPE. Next, assume that the claim holds for each game with at most $n \in \mathbb{N}$ players and let $(N, v) \in \Upsilon$ have $n + 1$ players. Let $x \in \phi(N, v)$. By CONS of ϕ: $x_S \in \phi(S, v_S^x)$ for every $S \in 2^N \setminus \{\emptyset, N\}$. By induction $\phi(S, v_S^x) \subseteq \psi(S, v_S^x)$ for every $S \in 2^N \setminus \{\emptyset, N\}$. Using COCONS of ψ we obtain $x \in \psi(N, v)$. □

Applying this proposition twice gives us the following axiomatization of the core.

Theorem 15.14 *A solution concept σ on Υ satisfies OPE, CONS, and COCONS, if and only if σ is the core.*

Proof. The core satisfies the three axioms according to Proposition 15.12. Let σ be a solution concept on Υ that also satisfies the axioms. Now apply Proposition 15.13. Since σ satisfies OPE and CONS and the core satisfies OPE and COCONS, we find that $\sigma(N, v) \subseteq C(N, v)$ for each $(N, v) \in \Upsilon$. Since the core satisfies OPE and CONS and σ satisfies OPE and COCONS, we find that $C(N, v) \subseteq \sigma(N, v)$ for each $(N, v) \in \Upsilon$. Hence, $\sigma(N, v) = C(N, v)$ for all $(N, v) \in \Upsilon$. □

According to our next result, if a solution concept σ on Υ satisfies individual rationality and consistency, then it is included in the core.

Proposition 15.15 *Let σ be a solution concept on Υ that satisfies IR and CONS. Then $\sigma(N,v) \subseteq C(N,v)$ for each game $(N,v) \in \Upsilon$.*

Proof. Let $(N,v) \in \Upsilon$. We discern three cases.

- If $|N| = 1$, then $\sigma(N,v) \subseteq IR(N,v) = C(N,v)$ by IR of σ;
- If $|N| = 2$, let $x \in \sigma(N,v)$. Individual players cannot improve upon x by IR of σ. It remains to show that N cannot improve upon x. Suppose to the contrary that N *can* improve upon x. Then there exists a vector $y \in v(N)$ such that

$$\begin{array}{lll} \sum_{i \in N} x^i_{|D} & \leqq & y_{|D} \\ x^i_{|P} & \leqq & y_{|P} \quad \text{for all } i \in N \end{array}$$

where at least one of the inequalities is strict ($\leq$). Let $i \in N$. Then

$$\begin{array}{lllll} x^i_{|D} & \leqq & y_{|D} - \sum_{j \in N \setminus \{i\}} x^j_{|D} & = & (y - \tilde{x}(N \setminus \{i\}))_{|D} \\ x^i_{|P} & \leqq & y_{|P} & = & (y - \tilde{x}(N \setminus \{i\}))_{|P} \end{array}$$

where at least one of the inequalities is strict ($\leq$). Since $y - \tilde{x}(N \setminus \{i\}) \in v(N) - \tilde{x}(N \setminus \{i\}) = v^x_{\{i\}}(i)$, it follows that $x^i \notin IR(\{i\}, v^x_{\{i\}})$. By IR of σ, $x^i \notin \sigma(\{i\}, v^x_{\{i\}})$. But $x \in \sigma(N,v)$ and CONS of σ imply that $x^i \in \sigma(\{i\}, v^x_{\{i\}})$, a contradiction. Hence, one has to conclude that N cannot improve upon x in (N,v).

  This leads to the conclusion that $\sigma(N,v) \subseteq C(N,v)$ for two-player games $(N,v) \in \Upsilon$;
- If $|N| \geqq 3$, let $x \in \sigma(N,v)$. By CONS of σ, $x_S \in \sigma(S, v^x_S)$ for each $S \in 2^N \setminus \{\emptyset\}$ with $|S| = 2$. By the previous step, $\sigma(S, v^x_S) \subseteq C(S, v^x_S)$ for such two-player coalitions S. Using COCONS$_2$ of the core, it follows that $x \in C(N,v)$. □

In the part of the proof of Proposition 15.15 where we indicate that the grand coalition N in a two-player game (N,v) cannot improve upon an allocation $x \in \sigma(N,v)$ the use of summation signs and notations like $N \setminus \{i\}$ seems unnecessarily complicated, since $N \setminus \{i\}$ consists of only one player. We adopt the more general notation, however, because with this notation it is easily seen that it also proves that the grand coalition cannot improve upon an allocation $x \in \sigma(N,v)$ in games with an arbitrary number of players.

Our next axiomatization applies the converse consistency axiom for two-player reductions.

Theorem 15.16 *A solution concept σ on Υ satisfies IR, II$_2$, CONS, and COCONS$_2$ if and only if σ is the core.*

Proof. The core satisfies the four axioms by Proposition 15.12. Let σ be a solution concept on Υ that also satisfies the axioms. Proposition 15.15 shows that $\sigma(N,v) \subseteq C(N,v)$ for every $(N,v) \in \Upsilon$.

It remains to show that $C(N,v) \subseteq \sigma(N,v)$ for each $(N,v) \in \Upsilon$. We consider three separate cases in which the game has one, two, or more than two players. First we consider two-player games, since this result is required for the argumentation in one-player games.

- If $|N| = 2$, we know that $\sigma(N,v) \subseteq C(N,v)$ from Proposition 15.15 and $C(N,v) = I(N,v) \subseteq \sigma(N,v)$ by II_2 of σ. So $C(N,v) = \sigma(N,v)$;

- Consider a one player game $(\{i\},v)$ and let $x^i \in C(\{i\},v)$. Consider $j \in U \setminus \{i\}$ and the game $(\{i,j\},w) \in \Upsilon$ defined by $w(i) = w(\{i,j\}) = v(i)$ and $w(j) = \{a\}$ with $a_{|D} = 0$ and $a_{|P} = x^i_{|P}$. Denote the allocation in $(\{i,j\},w) \in \Upsilon$ which gives x^i to player i and a to player j by (x^i,a). Then $(x^i,a) \in C(\{i,j\},w) = \sigma(\{i,j\},w)$. Also, $(\{i\}, w^{(x^i,a)}_{\{i\}}) = (\{i\},v)$, since $w^{(x^i,a)}_{\{i\}}(i) = w(\{i,j\}) - \tilde{a} = v(i)$. By CONS of σ, $x^i \in \sigma(\{i\}, w^{(x^i,a)}_{\{i\}}) = \sigma(\{i\},v)$. Hence, $C(N,v) \subseteq \sigma(N,v)$ if $|N| = 1$;

- If $|N| \geqq 3$, let $x \in C(N,v)$. By CONS of the core: $x_S \in C(S,v^x_S) = \sigma(S,v^x_S)$ whenever $|S| = 2$, hence $x \in \sigma(N,v)$ by COCONS_2 of σ.

We conclude that $\sigma(N,v) = C(N,v)$ for all games $(N,v) \in \Upsilon$. □

15.5 A core axiomatization with enlightening

In the proofs of Theorem 15.14 and Proposition 15.15, we showed that a solution concept σ on Υ satisfies $\sigma(N,v) \subseteq C(N,v)$ for each game $(N,v) \in \Upsilon$ by assuming that σ satisfies consistency and some form of individual rationality or one-person efficiency, i.e, an assumption that focuses on individual players. The other inclusion, $C(N,v) \subseteq \sigma(N,v)$ was harder to prove. In the previous section two notions of converse consistency were used to establish this part. In the article of Peleg (1985) on an axiomatization of the core of NTU games, it was shown that — given an infinite set of potential agents from which the finite player sets are drawn — the converse consistency axiom can be replaced by a (restricted) nonemptiness axiom to establish inclusion of the core in σ. The same is observed in axiomatizations of equilibria in noncooperative games (cf. Peleg and Tijs, 1996, and Norde *et al.*, 1996), where properties like restricted nonemptiness, individual rationality, consistency and converse consistency are studied in a different set-up (see also Chapter 11). Peleg and Tijs (1996) prove that if a solution concept on a set of noncooperative games satisfies consistency and a requirement on single player games, it is a subset of the Nash equilibrium set. If, in addition, a converse consistency property is imposed, the solution concept coincides with the set of Nash equilibria. Norde *et al.*

(1996) show that in mixed extensions of finite noncooperative games converse consistency can be replaced by nonemptiness.

In the current section we slightly modify the definition of reduced games of the previous section and show that the core can be axiomatized by means of restricted nonemptiness, consistency with respect to the new type of reduced games, and individual rationality. A similar definition of reduced games can be used to provide a new axiomatization of the core for games with transferable utility. This is done in Section 15.6.

The section concludes with an example showing that converse consistency cannot be replaced with restricted nonemptiness if the definition of reduced games from Section 15.4 is used.

Definition 15.17 Let $(N, v) \in \Upsilon, x \in A(N, v)$, and $S \in 2^N \setminus \{\emptyset, N\}$. The *reduced game* $(S, \overline{v}_S^x)$ of (N, v) with respect to allocation x and coalition S is the game defined by:

$$\begin{aligned} \overline{v}_S^x(S) &= v(N) - \tilde{x}(N \setminus S) \\ \overline{v}_S^x(T) &= \cup_{Q \subseteq N \setminus S, Q \neq \emptyset} (v(T \cup Q) - \tilde{x}(Q)) \quad \text{for all } T \in 2^S \setminus \{\emptyset, S\}. \end{aligned}$$

◁

The difference between this definition of a reduced game and the one in Definition 15.11 is that we require the set Q in the specification of $\overline{v}_S^x(T)$ to be nonempty. This reflects the intuition that, although attention is restricted to the players in S, the players in $N \setminus S$ do not *leave* the game, but strongly influence the game from behind the scenes. The remaining players don't ignore those in $N \setminus S$, but always cooperate with at least some of them.

With the reduction as given in Definition 15.17, we obtain a new consistency axiom $\overline{\text{CONS}}$. A solution concept σ on Υ satisfies:

- $\overline{\textbf{CONS}}$ if for each game $(N, v) \in \Upsilon$ it holds that $x \in \sigma(N, v)$ implies $x_S \in \sigma(S, \overline{v}_S^x)$ for each coalition $S \in 2^N \setminus \{\emptyset, N\}$.

The core satisfies $\overline{\text{CONS}}$. This is shown in the following proposition, along with other statements concerning the core and $\overline{\text{CONS}}$.

Proposition 15.18 *The following claims are true:*

1. *The core satisfies $\overline{CONS}$;*

2. *Consider a game $(N, v) \in \Upsilon$ with $|N| \geqq 3$. If $x \in IR(N, v)$ and $x_S \in C(S, \overline{v}_S^x)$ for each $S \in 2^N \setminus \{\emptyset, N\}$ with $|S| = 2$, then $x \in C(N, v)$;*

3. *Let σ be a solution concept on Υ that satisfies IR and $\overline{CONS}$. Then $\sigma(N, v) \subseteq C(N, v)$ for each $(N, v) \in \Upsilon$.*

Proof.

1. The proof that the core satisfies $\overline{\text{CONS}}$ is similar to the proof that the core satisfies consistency in Proposition 15.12;

2. Suppose $x \in IR(N,v)$ and $x_S \in C(S, \overline{v}_S^x)$ for each $S \in 2^N \setminus \{\emptyset, N\}$ with $|S| = 2$. Then individual players cannot improve upon x because $x \in IR(N,v)$. To show that N and other coalitions $T \in 2^N$ with $|T| \geqq 2$ cannot improve upon x, apply the arguments used in the proof that the core satisfies COCONS_2 in Proposition 15.12;

3. Let $(N,v) \in \Upsilon$. The proof that $\sigma(N,v) \subseteq C(N,v)$ if $|N| \in \{1,2\}$ is completely analogous to the corresponding part of the proof of Proposition 15.15. If $|N| \geqq 3$, let $x \in \sigma(N,v)$. By $\overline{\text{CONS}}$ of σ, $x_S \in \sigma(S, \overline{v}_S^x)$ for each $S \in 2^N \setminus \{\emptyset, N\}$ with $|S| = 2$. Hence, using the previous step of this proof, we find that $x_S \in C(S, \overline{v}_S^x)$ for each $S \in 2^N \setminus \{\emptyset, N\}$ with $|N| = 2$. By IR of σ, $x \in \sigma(N,v) \subseteq IR(N,v)$. Then, by part 2 of the current proposition, it follows that $x \in C(N,v)$. □

The main result of this section is the following axiomatization of the core.

Theorem 15.19 *A solution concept σ on Υ satisfies IR, $\overline{\text{CONS}}$, and r-NEM if and only if σ is the core.*

Proof. We have already seen that the core satisfies the three axioms. Let σ be a solution concept on Υ that also satisfies the three axioms. From Proposition 15.18, part 3, we know that $\sigma(N,v) \subseteq C(N,v)$ for each $(N,v) \in \Upsilon$. It remains to show that $C(N,v) \subseteq \sigma(N,v)$ for each $(N,v) \in \Upsilon$.

Let $(N,v) \in \Upsilon$. If $C(N,v) = \emptyset$ we are done, so assume $C(N,v) \neq \emptyset$, and let $x = (x^i)_{i \in N} \in C(N,v)$. Also, let $n \in U \setminus N$ and define a game $(N \cup \{n\}, w) \in \Upsilon$ as follows:

$$\begin{aligned}
w(n) &= \{y \in \mathbb{R}^{D \cup P} \mid \text{there exists a } k \in D \text{ s.t. } y_k < 0\} \\
&\cup \ \{y \in \mathbb{R}^{D \cup P} \mid \text{there exists a } k \in P \text{ s.t. } y_k < x_k^i\} \\
w(i) &= \{y \in \mathbb{R}^{D \cup P} \mid \text{there exists a } k \in D \cup P \text{ s.t. } y_k < x_k^i\} \text{ for } i \in N \\
w(S \cup \{n\}) &= v(S) \qquad \text{for } S \subseteq N, S \neq \emptyset \\
w(S) &= v(S) \qquad \text{for } S \subseteq N, |S| \geqq 2.
\end{aligned}$$

(Recall that for public criteria $k \in P$ one has that $x_k^i = x_k^j$ for all players $i, j \in N$. Consequently, it does not matter which player $i \in N$ is chosen in the definition of $w(n)$ above.)

We show that $C(N \cup \{n\}, w) = \{(x, d)\}$, where (x, d) is the allocation that gives $x^i \in \mathbb{R}^{D \cup P}$ to each player $i \in N$ and $d \in \mathbb{R}^{D \cup P}$ to player n, with $d_{|D} = 0$ and $d_{|P} = x^i_{|P}$ (for arbitrary $i \in N$, as above). Obviously, $(x, d) \in C(N \cup \{n\}, w)$. Now, let $(b^i)_{i \in N} \times \{b^n\} \in C(N \cup \{n\}, w)$. Using the definitions of $(w(j))_{j \in N \cup \{n\}}$, we see that it must hold that $b^i \geqq x^i$ for each player $i \in N$ and $b^n \geqq d$, to make sure that individual players in $N \cup \{n\}$ cannot improve upon $(b^i)_{i \in N} \times \{b^n\}$. If one or more of these inequalities are strict, then

$$\begin{array}{rclcll} \sum_{i \in N \cup \{n\}} b^i_{|D} & \geqq & \sum_{i \in N} x^i_{|D} + d_{|D} & = & \sum_{i \in N} x^i_{|D} & \\ b^i_{|P} & \geqq & x^i_{|P} & & & \text{for each player } i \in N \cup \{n\}, \end{array}$$

with at least one strict inequality. This would contradict $(x, d) \in C(N \cup \{n\}, w)$. Hence, $(b^i)_{i \in N} = (x^i)_{i \in N}$ and $b^n = d$ and this proves that (x, d) is the unique core element of $(N \cup \{n\}, w)$.

Also, we claim that $(N, \overline{w}_N^{(x,d)}) = (N, v)$. Namely,

$$\begin{array}{rclclcll} \overline{w}_N^{(x,d)}(N) & = & w(N \cup \{n\}) - \tilde{d} & = & w(N \cup \{n\}) - 0 & = & v(N) & \\ \overline{w}_N^{(x,d)}(S) & = & w(S \cup \{n\}) - \tilde{d} & = & w(S \cup \{n\}) - 0 & = & v(S) & \text{for } S \notin \{\emptyset, N\}. \end{array}$$

By r-NEM of σ we know that $\sigma(N \cup \{n\}, w) \neq \emptyset$ and we already saw that $\sigma(N \cup \{n\}) \subseteq C(N \cup \{n\}, w) = \{(x, d)\}$. So, $\sigma(N \cup \{n\}, w) = \{(x, d)\}$. Hence, by $\overline{\text{CONS}}$ of σ: $x = (x, d)_N \in \sigma(N, \overline{w}_N^{(x,d)}) = \sigma(N, v)$. This proves that $C(N, v) \subseteq \sigma(N, v)$. □

The main step in the proof, showing that $C(N, v) \subseteq \sigma(N, v)$ for each game $(N, v) \in \Upsilon$, proceeds by 'enlightening' core elements. In this procedure, one considers a game with a nonempty core and an arbitrary allocation in this core. Then, a game is constructed with a player set that strictly includes the players of the original game in such a way that this larger game has a unique core element and such that this new, enlarged, game and its unique core element reduced to the original player set are the original game and core element. Restricted nonemptiness is then used to derive the desired inclusion.

We conclude this section by showing that the analogon of Theorem 15.19 does not hold if we replace $\overline{\text{CONS}}$ by consistency with respect to the old definition of reduced games. In particular, we construct a solution concept σ on Υ that satisfies IR, CONS, and r-NEM, which is not equal to the core.

Let $\mathcal{T} \subset \Upsilon$ be the class of games with a nonempty core, one divisible criterion, and zero public criteria:

$$\mathcal{T} := \{(N, v) \in \Upsilon \mid C(N, v) \neq \emptyset, |D| = 1, P = \emptyset\}.$$

Since for each game $(N, v) \in \mathcal{T}$ the core is nonempty, there is only one criterion, and v takes nonempty values (see Section 15.2), we conclude that the function v is bounded from above. Hence, the function $\sup v$, where $\sup v(S)$ is the supremum of $v(S)$ for each $S \in 2^N \setminus \{\emptyset\}$, is well-defined. Define a solution concept σ on Υ as follows:

$$\sigma(N,v) = \begin{cases} C(N,v) & \text{if} \quad (N,v) \notin \mathcal{T} \\ \\ \{Nu(N, \sup v)\}, \text{ the nucleolus of } (N, \sup v) & \text{if} \quad (N,v) \in \mathcal{T} \end{cases}$$

If $(N,v) \in \mathcal{T}$, then $C(N,v) = C(N, \sup v)$. The game $(N, \sup v)$ is a TU-game. Recall (cf. Schmeidler, 1969) that the nucleolus of a TU-game with a nonempty core is always included in the core. The solution concept σ satisfies r-NEM because the nucleolus exists for TU-games.

To prove IR of σ, we distinguish between $(N,v) \in \mathcal{T}$ and $(N,v) \notin \mathcal{T}$. If $(N,v) \notin \mathcal{T}$, it is clear that $\sigma(N,v) = C(N,v) \subseteq IR(N,v)$ by IR of the core. If $(N,v) \in \mathcal{T}$, then $C(N,v) = C(N, \sup v)$. Consequently,

$$\sigma(N,v) = \{Nu(N, \sup v)\} \subseteq C(N, \sup v) = C(N,v) \subseteq IR(N,v)$$

by IR of the core and inclusion of the nucleolus in the core if the core of a TU-game is nonempty.

The solution concept σ also satisfies CONS. If $(N,v) \notin \mathcal{T}$, then $(S, v_S^x) \notin \mathcal{T}$ for each $x \in A(N,v)$ and $S \in 2^N \setminus \{\emptyset, N\}$ and hence it follows from consistency of the core that $x_S \in \sigma(S, v_S^x)$ for each $x \in \sigma(N,v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. So, suppose $(N,v) \in \mathcal{T}$, so that $\sigma(N,v) = \{Nu(N, \sup v)\}$. Let $S \in 2^N \setminus \{\emptyset, N\}$ and $x \in \sigma(N,v)$, i.e., $x = Nu(N, \sup v)$.

Notice, first of all, that the reduced game (S, v_S^x) is again an element of $\mathcal{T}$. It is clear that the reduced game has no public and exactly one private criterion. Also, $x \in \sigma(N,v) = \{Nu(N, \sup v)\} \subseteq C(N, \sup v) = C(N,v)$ and the core satisfies CONS. This shows that $x_S \in C(S, v_S^x)$ and, hence, $C(S, v_S^x) \neq \emptyset$.

We know by consistency of the nucleolus for TU-games (cf. Peleg, 1986) that x_S is the nucleolus of (S,w), where the reduced game w is defined by

$$\begin{aligned} w(S) &= (\sup v)(N) - x(N \setminus S) \\ w(T) &= \max\nolimits_{Q \subseteq N \setminus S} \{(\sup v)(T \cup Q) - x(Q)\} \quad \text{for } T \in 2^S \setminus \{\emptyset, S\}. \end{aligned}$$

Notice that

$$\begin{aligned} w(S) &= (\sup v)(N) - x(N \setminus S) \\ \\ &= \sup(v(N) - x(N \setminus S)) \\ \\ &= \sup v_S^x(S), \end{aligned}$$

and for $T \in 2^S \setminus \{\emptyset, S\}$:

$$w(T) = \max_{Q \subseteq N \setminus S} \{(\sup v)(T \cup Q) - x(Q)\}$$

$$= \max_{Q \subseteq N \setminus S} \{\sup(v(T \cup Q) - x(Q))\}$$

$$= \sup \cup_{Q \subseteq N \setminus S} \{v(T \cup Q) - x(Q)\}$$

$$= \sup v_S^x(T).$$

So $x_S = Nu(S, w) = Nu(S, \sup v_S^x) \in \sigma(S, v_S^x)$, completing our proof that σ satisfies CONS.

To show that $\sigma \neq C$, consider the two-player game $(\{1,2\}, v) \in \mathcal{T}$ with $v(1) = v(2) = \{0\}$ and $v(\{1,2\}) = \{1\}$. Then $\sigma(\{1,2\}, v) = \{Nu(\{1,2\}, v)\} = \{(\frac{1}{2}, \frac{1}{2})\} \neq C(\{1,2\}, v) = \{(x^1, x^2) \in \mathbb{R}^2 \mid x^1 \geqq 0, x^2 \geqq 0, x^1 + x^2 = 1\}$.

As an aside, notice that the solution concept σ also satisfies OPE. This follows from OPE of the core and $\sigma(N, v) = C(N, v)$ if $|N| = 1$. This implies that in Theorem 15.14 the converse consistency axiom cannot be replaced by restricted nonemptiness and individual rationality.

15.6 Application to TU-games

Applying the reduced games as defined in Section 15.5 to single-criterion games with transferable utility, one obtains an axiomatization of the core which differs from Peleg's (1986) axiomatization in the sense that it does not require Peleg's superadditivity axiom.

Again, let U be an infinite set. A *game with transferable utility*, or a (TU) game for ease of notation, is a tuple (N, v) with $N \subset U$ a finite set of players and $v : 2^N \to \mathbb{R}$ a map which assigns to each coalition $S \subseteq N$ of players a value $v(S) \in \mathbb{R}$. We assume that $v(\emptyset) = 0$. The set of all TU games is denoted by Υ_{TU}.

For a vector $x \in \mathbb{R}^N$ and a coalition $S \in 2^N$, we denote $x(S) := \sum_{i \in S} x_i$. The empty sum $x(\emptyset)$ is zero by definition. The vector x_S is the vector x restricted to the components in S.

A *payoff vector* in a game $(N, v) \in \Upsilon_{TU}$ is a vector $x \in \mathbb{R}^N$ such that $x(N) \leqq v(N)$. The set of all payoff vectors of (N, v) is denoted $P(N, v)$. A payoff vector is *individually rational* if $x_i \geqq v(i)$ for each player $i \in N$. The set of all individually rational payoff vectors is denoted $IR(N, v)$.

A *solution* σ on Υ_{TU} is a map that assigns to each game $(N, v) \in \Upsilon_{TU}$ a subset $\sigma(N, v) \subseteq P(N, v)$ of payoff vectors.

We study a particular solution. The *core* of a game (N, v) is the set of payoff vectors upon which no coalition can improve:

$$C(N, v) = \{x \in P(N, v) \mid x(S) \geqq v(S) \; \forall S \in 2^N\}.$$

Notice that $x(N) = v(N)$ for each $x \in C(N, v)$.

We provide a new axiomatization of the core of TU games based on consistency with respect to reduced games analogous to Definition 15.17, which differ slightly from those defined by Peleg (1986). This modification allows us to omit the superadditivity axiom and characterize the core with one axiom less than Peleg (1986).

We start by introducing the properties we use to characterize the core. A solution σ on Υ_{TU} satisfies

- **Restricted Nonemptiness (r-NEM)** if for each $(N, v) \in \Upsilon_{TU}$ with $C(N, v) \neq \emptyset$ we have that $\sigma(N, v) \neq \emptyset$;
- **Individual Rationality (IR)** if for each $(N, v) \in \Upsilon_{TU}$ we have that $\sigma(N, v) \subseteq IR(N, v)$.

In order to introduce consistency we need to define reduced games.

Definition 15.20 Let $(N, v) \in \Upsilon_{TU}, x \in P(N, v)$, and $S \in 2^N \setminus \{\emptyset, N\}$. The *reduced game* (S, v_S^x) of (N, v) with respect to payoff vector x and coalition S is the game in Υ_{TU} defined by:

$$
\begin{aligned}
v_S^x(\emptyset) &= 0 \\
v_S^x(S) &= v(N) - x(N \setminus S) \\
v_S^x(T) &= \max\nolimits_{Q \subseteq N \setminus S, Q \neq \emptyset} \{v(T \cup Q) - x(Q)\} \quad \forall T \in 2^S \setminus \{\emptyset, S\}
\end{aligned}
$$

$\triangleleft$

Consequently, in the reduced game (S, v_S^x), the players in $N \setminus S$ do not leave the game, they only leave the decision making process. They are paid according to $x_{N \setminus S}$ and no longer play against the players in S. But considering that they are still needed to distribute the remainder $v(N) - x(N \setminus S)$ of the value of the grand coalition, the remaining players are required to cooperate with at least some of them. A solution concept σ on Υ_{TU} satisfies

- **Consistency (CONS)** if for each game $(N, v) \in \Upsilon_{TU}$ and each coalition $S \in 2^N \setminus \{\emptyset, N\}$ we have that $x \in \sigma(N, v)$ implies $x_S \in \sigma(S, v_S^x)$.

Proposition 15.21 *The core satisfies r-NEM, IR, and CONS.*

Proof. It is obvious that the core satisfies r-NEM and IR. It remains to show that the core satisfies CONS. Let $(N, v) \in \Upsilon_{TU}, x \in C(N, v), S \in 2^N \setminus \{\emptyset, N\}$, and $T \in 2^S \setminus \{\emptyset\}$. If $T = S$, then $v_S^x(T) = v(N) - x(N \setminus T) = x(N) - x(N \setminus T) = x(T)$. If $T \neq S$, then

$$
v_S^x(T) \quad = \quad \max_{Q \subseteq N \setminus S, Q \neq \emptyset} \{v(T \cup Q) - x(Q)\}
$$

$$
\begin{aligned}
&\leqq \max_{Q \subseteq N \setminus S, Q \neq \emptyset} \{x(T \cup Q) - x(Q)\} \\
&= \max_{Q \subseteq N \setminus S, Q \neq \emptyset} x(T) \\
&= x(T),
\end{aligned}
$$

where the inequality follows from $x \in C(N, v)$. Thus, $x_S \in C(S, v_S^x)$. □

Proposition 15.22 *Let* $(N, v) \in \Upsilon_{TU}$ *with* $|N| \geqq 3$. *If* $x \in IR(N, v)$ *and* $x_S \in C(S, v_S^x)$ *for each* $S \in 2^N$ *with* $|S| = 2$, *then* $x \in C(N, v)$.

Proof.

- Since $x \in IR(N, v)$ we have that $x_i \geqq v(i)$ for each $i \in N$;
- Let $S \subset N$ have two players. Since $x_S \in C(S, v_S^x)$ we know that $x(S) = v_S^x(S) = v(N) - x(N \setminus S)$, so $x(N) = v(N)$;
- Now let $T \in 2^N \setminus \{N\}$ with $|T| > 1$. Take $i \in T,\ j \in N \setminus T,\ S = \{i, j\}$. Since $x_S \in C(S, v_S^x)$:

$$
\begin{aligned}
x_i &\geqq v_S^x(i) \\
&= \max_{Q \subseteq N \setminus S, Q \neq \emptyset} \{v(\{i\} \cup Q) - x(Q)\} \\
&\geqq v(\{i\} \cup T \setminus \{i\}) - x(T \setminus \{i\}) \\
&= v(T) - x(T \setminus \{i\}),
\end{aligned}
$$

  where the second inequality follows from the observation that $T \setminus \{i\} \subseteq N \setminus S$, and $T \setminus \{i\} \neq \emptyset$. So $x(T) \geqq v(T)$.

Consequently, $x(S) \geqq v(S)$ for each coalition $S \in 2^N$, so $x \in C(N, v)$. □

Proposition 15.23 *Let* σ *be a solution on* Υ_{TU} *that satisfies IR and CONS. Then* $x(N) = v(N)$ *for each* $x \in \sigma(N, v)$.

Proof. If $|N| = 1$, then $x(N) \leqq v(N)$ since x is a payoff vector and $x(N) \geqq v(N)$ by IR. So assume $|N| \geqq 2$. Let $x \in \sigma(N, v),\ i \in N$. By CONS: $x_i \in \sigma(\{i\}, v_{\{i\}}^x)$. By our previous step: $x_i = v_{\{i\}}^x(i) = v(N) - x(N \setminus \{i\})$, so $x(N) = v(N)$. □

Proposition 15.24 *Let* σ *be a solution on* Υ_{TU} *that satisfies IR and CONS. Then* $\sigma(N, v) \subseteq C(N, v)$ *for each* $(N, v) \in \Upsilon_{TU}$.

Proof. Let $(N, v) \in \Upsilon_{TU}$.

- If $|N| = 1$, then $\sigma(N, v) \subseteq IR(N, v) = C(N, v)$ by IR;
- If $|N| = 2$, then $\sigma(N, v) \subseteq C(N, v)$ by IR and Proposition 15.23;
- If $|N| \geqq 3$, let $x \in \sigma(N, v)$. By CONS, $x_S \in \sigma(S, v_S^x)$ for each two-player coalition $S \subset N$. By the previous step, $\sigma(S, v_S^x) \subseteq C(S, v_S^x)$ for each two-player coalition S. By IR we know that $x \in IR(N, v)$. So by Proposition 15.22: $x \in C(N, v)$. □

Theorem 15.25 *A solution σ on Υ_{TU} satisfies IR, r-NEM, and CONS if and only if σ is the core.*

Proof. We showed in Proposition 15.21 that the core indeed satisfies the three axioms. Now let σ be a solution on Υ_{TU} that also satisfies the three axioms. By Proposition 15.24 we have that $\sigma(N, v) \subseteq C(N, v)$ for each $(N, v) \in \Upsilon_{TU}$. Remains to show that $C(N, v) \subseteq \sigma(N, v)$ for each $(N, v) \in \Upsilon_{TU}$. Let $(N, v) \in \Upsilon_{TU}$. If $C(N, v) = \emptyset$ we are done, so assume that this is not the case and let $x \in C(N, v)$. Take $n \in U \setminus N$ and define a game $(N \cup \{n\}, w) \in \Upsilon_{TU}$ as follows:

$$\begin{aligned} w(i) &= x_i && \forall i \in N \\ w(n) &= 0 && \\ w(S) &= v(S) && \text{if } |S| \geqq 2 \text{ and } n \notin S \\ w(S) &= v(S \setminus \{n\}) && \text{if } |S| \geqq 2 \text{ and } n \in S. \end{aligned}$$

Obviously $C(N \cup \{n\}, w) = \{(x, 0)\}$, where $(x, 0) \in \mathbb{R}^{N \cup \{n\}}$ is the payoff vector that gives x_i to each player $i \in N$ and 0 to player n. By r-NEM and Proposition 15.24: $(x, 0) \in \sigma(N \cup \{n\}, w)$.

The reduced game $(N, w_N^{(x,0)})$ equals (N, v). Namely: $w_N^{(x,0)}(N) = w(N \cup \{n\}) - x_n = v(N) - 0 = v(N)$ and for an arbitrary coalition $S \in 2^N \setminus \{\emptyset, N\}$ we find that $w_N^{(x,0)}(S) = \max_{Q \subseteq (N \cup \{n\} \setminus N), Q \neq \emptyset} w(S \cup Q) - x(Q) = w(S \cup \{n\}) - x_n = v(S) - 0 = v(S)$. Hence, by CONS: $x = (x, 0)_N \in \sigma(N, w_N^{(x,0)}) = \sigma(N, v)$, which finishes our proof. □

Like the proof of Theorem 15.19, the main step of the proof above, showing that σ includes the core, proceeds by 'enlightening' core elements. Peleg (1985, 1986) also applies this procedure. It is easy to show that the three axioms are independent.

Example 15.26 Define σ^1 on Υ_{TU} by $\sigma^1(N, v) = IR(N, v)$ for each $(N, v) \in \Upsilon_{TU}$. Then σ^1 satisfies r-NEM and IR, but not CONS. Define σ^2 on Υ_{TU} by $\sigma^2(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$ for each $(N, v) \in \Upsilon_{TU}$. Then σ^2 satisfies r-NEM and CONS, but not IR. Define σ^3 on Υ_{TU} by $\sigma^3(N, v) = \emptyset$ for each $(N, v) \in \Upsilon_{TU}$. Then σ^3 satisfies IR and CONS, but not r-NEM. ◁

The only respect in which our definition of a reduced game differs from that of Peleg (1986) is that we require the players in the reduced game (S, v_S^x) to cooperate with at least one of the players in $N \setminus S$. This seems a plausible assumption if we take into account that the players in $N \setminus S$ do not *leave* the game. They are paid according to some payoff vector and no longer take part in the decision making process. However, considering that they are still needed to distribute the remainder of the value $v(N)$ of the grand coalition, the players in S are required to take account of the players in $N \setminus S$ and to cooperate with at least some of them.

For antimonotonic solutions our notion of consistency is implied by Peleg's notion of consistency. A solution σ on Υ_{TU} is called antimonotonic if for two games (N, v) and $(N, w) \in \Upsilon_{TU}$ with $v(N) = w(N)$ and $v(S) \geqq w(S)$ for each coalition S we have that $\sigma(N, v) \subseteq \sigma(N, w)$. This property is intuitive if a solution is based on objections: the more powerful the coalitions are, the more they can reject, so the smaller the solution of the game will be. Thus, the core satisfies consistency with respect to *our* definition of a reduced game because of its consistency with respect to *Peleg's* definition of a reduced game and its antimonotonicity: in our definition of reduced games, the maximum is taken over a smaller collection of coalitions.

Moreover, our axiomatization differs from Peleg's result by the absence of the superadditivity axiom.

Using a reduced game for NTU-games similar to the reduced game defined in this paper, we can also provide an axiomatization of the core of games with non-transferable utility in terms of individual rationality, restricted nonemptiness, and consistency. This result is similar to the result of Peleg (1985).

15.7 Claim games

In this section, based on Voorneveld and van den Nouweland (1999), the core allocations of a cooperative multicriteria game with public and private criteria are related with the equilibria of a noncooperative multicriteria game. For our analysis, a slightly extended definition of noncooperative or strategic form multicriteria games is required.

A *generalized noncooperative multicriteria game* is a tuple

$$G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N}, (\succeq_i)_{i \in N} \rangle,$$

where

- N is a finite set of players,
- X_i is the set of strategies of player $i \in N$,
- $u_i : \prod_{i \in N} X_i \twoheadrightarrow \mathbb{R}^r(i)$ is a payoff correspondence that maps each strategy profile $x = (x_i)_{i \in N} \in \prod_{i \in N} X_i$ to a nonempty subset $u_i(x)$ of $r(i)$-dimensional Euclidean

space, where $r(i) \in \mathbb{N}$ denotes the number of criteria taken into account by player $i \in N$,

- $\succeq_i$ is a binary relation on (a superset of) $\{u_i(x) \subseteq \mathbb{R}^{r(i)} \mid x \in \prod_{i \in N} X_i\}$, denoting the preferences of player $i \in N$ over his outcome sets.

This generalizes the noncooperative multicriteria games defined in Chapter 10 in two ways. First of all, set-valued payoff functions are admitted. Secondly, in the usual definition one omits the $\succeq_i$. Definitions of several solution concepts are as follows.

Definition 15.27 Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N}, (\succeq_i)_{i \in N} \rangle$ be a generalized noncooperative multicriteria game. A strategy profile $x \in \prod_{i \in N} X_i$ is

- an *equilibrium* if there does not exist a player $i \in N$ and a strategy $y_i \in X_i$ such that $u_i(y_i, x_{-i}) \succ_i u_i(x)$;

- an *undominated equilibrium* if it is an equilibrium and, moreover, there does not exists a strategy profile $y \in \prod_{i \in N} X_i$ such that

$$\begin{aligned} \forall j \in N &: \quad u_j(y) \quad \succeq_j \quad u_j(x), \\ \exists i \in N &: \quad u_i(y) \quad \succ_i \quad u_i(x). \end{aligned}$$

- a *strong equilibrium* if there does not exist a coalition $S \in 2^N \setminus \{\emptyset\}$ and profile $y_S \in \prod_{i \in S} X_i$ such that

$$\begin{aligned} \forall j \in S &: \quad u_j(y_S, x_{N \setminus S}) \quad \succeq_j \quad u_j(x), \\ \exists i \in S &: \quad u_i(y_S, x_{N \setminus S}) \quad \succ_i \quad u_i(x). \end{aligned}$$

The set of equilibria, undominated equilibria, and strong equilibria of the game G are denoted by $E(G)$, $UE(G)$, and $SE(G)$, respectively. $\triangleleft$

Let $(N, v) \in \Upsilon$ be a cooperative multicriteria game with public and private criteria. Define the *claim game* $G(N, v) = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N}, (\succeq_i)_{i \in N} \rangle$ as follows:

- The player set, as specified, equals N,

- Player $i \in N$ has strategy space $X_i := \{S \in 2^N \mid i \in S\} \times \mathbb{R}^{D \cup P}$,

- Player i's payoff correspondence $u_i : \prod_{j \in N} X_j \twoheadrightarrow \mathbb{R}^{D \cup P}$ is defined, for each $x =$

$(S^i, t^i)_{i\in N} \in \prod_{i\in N} X_i$ as follows:

$$u_i(x) = \begin{cases} \{t^i\} & \text{if} \quad \forall j \in S^i : S^j = S^i \text{ and} \\ & \quad\quad \forall j \in S^i : t^j_{|P} = t^i_{|P} \text{ and} \\ & \quad\quad \exists y \in v(S^i) \text{ s.t. } \sum_{j\in S^i} t^j_{|D} \leqq y_{|D}, t^j_{|P} \leqq y_{|P} \\ v(i) & \text{otherwise} \end{cases}$$

- For each player $i \in N$ the dominance relation $\succeq_i$ is the partial order on the subsets of $\mathbb{R}^{D\cup P}$ defined as follows. Let $A, B \subseteq \mathbb{R}^{D\cup P}$. Then

$$A \succ_i B \quad :\Leftrightarrow \quad \forall b \in B\ \exists a \in A \ :\ a \geq b, \text{ and}$$

$$A \succeq_i B \quad :\Leftrightarrow \quad [A \succ_i B \text{ or } A = B].$$

In a claim game, each player $i \in N$ states a coalition S^i of which he wants to be a member of and claims a payoff $t^i \in \mathbb{R}^{D\cup P}$ he wants to receive for joining. He receives his claim if this is feasible, i.e., if all other players in S^i also want to form this coalition, they agree on the public criteria, and there is a feasible payoff $y \in v(S^i)$ that can be used to finance the claims. In this case, the claimed coalition S^i is called fitting.

Formally, let $G(N, v)$ be a claim game and $x = (S^i, t^i)_{i\in N} \in \prod_{i\in N} X_i$ a strategy profile. The *fitting* F_x is the partition of the player set N defined as follows. For $S \in 2^N$ with $|S| \geqq 2$ we have that $S \in F_x$ if

$$\begin{array}{rrcll} & S^i & = & S & \text{for each } i \in S \\ & t^i_{|P} & = & t^j_{|P} & \text{for all } i, j \in S \\ \exists z \in v(S) \text{ s.t.} & \sum_{j\in S} t^j_{|D} & \leqq & z_{|D} & \text{and} \\ & t^i_{|P} & \leqq & z_{|P} & \text{for each } i \in S \end{array}$$

Further, for $i \in N$, we have that $\{i\} \in F_x$ if and only if

$$\{i\} \notin \bigcup \{S \in F_x : |S| \geqq 2\}.$$

Coalitions in the fitting are called *fitting coalitions.*

The first proposition shows that each imputation in a game (N, v) coincides with an equilibrium payoff in the claim game.

Proposition 15.28 *Let $(N, v) \in \Upsilon$ and $a \in I(N, v)$. There exists an $x \in E(G(N, v))$ such that $u_i(x) = \{a^i\}$ for all $i \in N$.*

Proof. Define $x = (N, a^i)_{i \in N}$. Obviously $u_i(x) = \{a^i\}$ for all $i \in N$. Suppose $x \notin E(G(N,v))$. Then some player $i \in N$ can profitably deviate to $y_i = (S^i, t^i)$. Since i had an individually rational payoff a^i, $\{i\}$ cannot be fitting in (y_i, x_{-i}). If it was, it would mean that $u_i(y_i, x_{-i})$ either equals $v(i)$ or $\{t^i\}$ if $t^i \leqq z$ for some $z \in v(i)$. This is clearly not a profitable deviation. So it must be the case that $S^i = N$ and there exists a $z \in v(N)$ such that

$$\begin{array}{rcccccc} & \sum_{j \in N} a^j_{|D} & \leq & \sum_{j \in N \setminus \{i\}} a^j_{|D} + t^i_{|D} & \leqq & z_{|D} \\ \forall j \in N: & t^i_{|P} & = & a^j_{|P} & \leqq & z_{|P}, \end{array}$$

where the inequality $\leq$ follows from the fact that $t^i \geq a^i$ in order for i to be better off after deviating. But then N can improve upon a, contradicting $a \in I(N,v)$. $\square$

The next proposition shows that each core element of a game (N,v) coincides with a strong equilibrium payoff in its claim game.

Proposition 15.29 *Let $(N,v) \in \Upsilon$ and $a \in C(N,v)$. There exists an $x \in SE(G(N,v))$ such that $u_i(x) = \{a^i\}$ for all $i \in N$.*

Proof. Define $x = (N, a^i)_{i \in N}$. Obviously $u_i(x) = \{a^i\}$ for all $i \in N$. Suppose $x \notin SE(G(N,v))$. Then there exist a coalition $S \in 2^N \setminus \{\emptyset\}$, a player $i \in S$, and a profile $y_S = (S^j, t^j)_{j \in S}$ such that

$$\left\{ \begin{array}{lcccccc} \forall j \in S & : & u_j(y_S, x_{N \setminus S}) & \succeq_j & u_j(x) & = & \{a^j\}, \\ \exists i \in S & : & u_i(y_S, x_{N \setminus S}) & \succ_i & u_i(x) & = & \{a^i\}. \end{array} \right. \tag{15.3}$$

As in Proposition 15.28, $\{i\}$ cannot be fitting in $(y_S, x_{N \setminus S})$ by individual rationality of a. Hence $S^i \in F_{(y_S, x_{N \setminus S})}$ and $|S^i| \geqq 2$. The fact that S^i is fitting implies that all its members receive their claimed payoff. Discern two cases.

Case I: $S^i \subseteq S$.
Since the claims $(t^j)_{j \in S^i}$ are feasible, there exists a $z \in v(S^i)$ such that

$$\begin{array}{ccll} \sum_{j \in S^i} t^j_{|D} & \leqq & z_{|D} & \\ t^j_{|P} & \leqq & z_{|P} & \text{for each } j \in S^i \end{array}$$

And by (15.3)

$$\begin{array}{ccll} \sum_{j \in S^i} a^j_{|D} & \leqq & \sum_{j \in S^i} t^j_{|D} & \\ a^j_{|P} & \leqq & t^j_{|P} & \text{for each } j \in S^i \end{array}$$

with at least one strict inequality ($\leq$). But then z dominates allocation a via coalition S^i, contradicting $a \in C(N,v)$.

CASE II: Not $S_i \subseteq S$.
Let $j \in (N \setminus S) \cap S^i$. Since $S^i \in F_{(y_S, x_{N \setminus S})}$: $S^j = S^i$. Since $j \in N \setminus S$: $S^j = N$. So $S^i = N$. By feasibility of the claims, there exists a $z \in v(N)$ such that

$$\begin{array}{rcll} \sum_{j \in S} t^j_{|D} + \sum_{j \in N \setminus S} a^j_{|D} & \leqq & z_{|D} & \\ t^j_{|P} & \leqq & z_{|P} & \text{for each } j \in S \\ a^j_{|P} & \leqq & z_{|P} & \text{for each } j \in N \setminus S \\ t^j_{|P} & = & a^k_{|P} & \text{for each } j \in S, k \in N \setminus S \end{array}$$

And by (15.3)

$$\begin{array}{rcll} \sum_{j \in N} a^j_{|D} & \leqq & \sum_{j \in S} t^j_{|D} + \sum_{j \in N \setminus S} a^j_{|D} & \\ a^j_{|P} & \leqq & t^j_{|P} & \text{for each } j \in S \end{array}$$

with at least one strict inequality ($\leq$). But then z dominates allocation a via coalition N, contradicting $a \in C(N,v)$.

Conclude that $(N, a^i)_{i \in N}$ is indeed a strong equilibrium of the claim game (N,v). □

Proposition 15.30 *Let $(N,v) \in \Upsilon$ and $a \in A(N,v)$. Allocation a is an element of $C(N,v)$ if and only if there exists an $x \in SE(G(N,v))$ such that $u_i(x) = \{a^i\}$ for all $i \in N$.*

Proof. The 'only if' part was shown in Proposition 15.29. To prove the converse, let $x \in SE(G(N,v))$ be such that $u_i(x) = \{a^i\}$ for all $i \in N$ and suppose that $a \notin C(N,v)$. Then there exists a coalition $S \in 2^N \setminus \{\emptyset\}$ and a $y \in v(S)$ such that

$$\begin{array}{rcll} \sum_{i \in S} a^i_{|D} & \leqq & y_{|D} & \text{and} \\ a^i_{|P} & \leqq & y_{|P} & \text{for each } i \in S \end{array}$$

with at least one of the inequalities being strict ($\leq$). Define $y_S = (S, t^j)_{j \in S}$ with for all $j \in S$:

$$\begin{array}{rcl} t^j_{|D} & = & a^j_{|D} + \frac{1}{|S|}(y_{|D} - \sum_{i \in S} a^i_{|D}), \\ t^j_{|P} & = & y_{|P}. \end{array}$$

Then $u_j(y_S, x_{N\setminus S}) = \{t^j\} \succ_j \{a^j\} = u_j(x)$ for all $j \in S$, contradicting the assumption that $x \in SE(G(N,v))$. □

What is it we showed in the previous proposition? We already knew that core elements induce strong equilibrium payoffs. The basic content of the proposition is that no coalition can profitably deviate from payoff vectors induced by strong equilibria in the claim game. Combining this with the assumption that these payoff vectors constitute an allocation in the game (N,v) gives rise to the conclusion that this allocation is in the core.

The assumption that the payoffs a constitute an allocation becomes obsolete if the game (N,v) is assumed to be superadditive, as is done in the following result. A game $(N,v) \in \Upsilon$ is *superadditive* if $\forall S, T \in 2^N \setminus \{\emptyset\}$ with $S \cap T = \emptyset$:

$$\forall s \in v(S)\ \ \forall t \in v(T)\ \ \exists u \in v(S \cup T)\ : \begin{cases} s_{|D} + t_{|D} & \leqq \quad u_{|D} \\ \\ \max\{s_k, t_k\} & \leqq \quad u_k \qquad \text{for each } k \in P. \end{cases}$$

Proposition 15.31 *Let $(N,v) \in \Upsilon$ be a superadditive game and $a = (a^i)_{i \in N} \in \prod_{i \in N} \mathbb{R}^{D \cup P}$. Then $a \in C(N,v)$ if and only if there exists an $x \in SE(G(N,v))$ such that $u_i(x) = \{a^i\}$ for all $i \in N$.*

Proof. The 'only if' part follows from Proposition 15.29. Conversely, assume there exists an $x \in SE(G(N,v))$ such that $u_i(x) = \{a^i\}$ for all $i \in N$. Proposition 15.30 implies that no coalition can improve upon $a = (a^i)_{i \in N}$. Remains to show that $a \in A(N,v)$. Consider the fitting $F_x = (N(1), \ldots, N(k))$ and $N(m) \in F_x$. Discern two cases.

Case I: $|N(m)| = 1$, say $N(m) = \{i\}$.
There are two possibilities. Either $u_i(x) = \{a^i\} = v(i)$, or $a^i \leqq z(m)$ for some $z(m) \in v(N(m))$. In either case, there exists a $z(m) \in v(N(m))$ such that $a^i \leqq z(m)$.

Case II: $|N(m)| \geqq 2$.
By definition there exists a $z(m) \in v(N(m))$ such that

$$\begin{array}{lll} \sum_{j \in N(m)} a^j_{|D} & \leqq & z(m)_{|D} \quad \text{and} \\ \\ a^j_{|P} & \leqq & z(m)_{|P} \quad \text{for each } j \in N(m) \end{array}$$

By superadditivity and the existence of the $z(m)$ as above, it follows that there is a $z \in v(N)$ such that

$$\sum_{i \in N} a^i_{|D} = \sum_{m=1}^{k} \sum_{i \in N(m)} a^i_{|D} \leqq \sum_{m=1}^{k} z(m)_{|D} \leqq z_{|D}$$

and for each $i \in N$ and each $l \in P$:

$$a_l^i \leqq \max_{m=1,\ldots,k} \{z(m)_l\} \leqq z_l.$$

So we have that

$$\begin{array}{lcll} \sum_{i\in N} a^i_{|D} & \leqq & z_{|D} & \text{and} \\ a^i_{|P} & \leqq & z_{|P} & \text{for each } i \in N. \end{array}$$

If any of these inequalities is strict, N can profitably deviate from a, contradicting the result that *no* coalition can profitably deviate. Hence the inequalities are all equalities, finishing the proof that $a \in A(N,v)$ and hence $a \in C(N,v)$. □

Similar to Proposition 15.31 one can prove:

Proposition 15.32 *Let $(N,v) \in \Upsilon$ be a superadditive game and $a = (a^i)_{i\in N} \in \prod_{i\in N} \mathbb{R}^{D\cup P}$. Then $a \in I(N,v)$ if and only if there exists an $x \in UE(G(N,v))$ such that $u_i(x) = \{a^i\}$ for all $i \in N$.*

This chapter is concluded with two simple examples. Example 15.33 indicates that even if the game is superadditive and has a nonempty core, there may be strong equilibria of the claim game in which the payoff to each player i is set-valued, i.e., equals $v(i)$. Example 15.34 indicates that in cooperative games with an empty core, the set of strong equilibria of its claim game need not be empty.

Example 15.33 Take $|D| = 1, P = \emptyset, N = \{1,2\}$, and $v(\{1\}) = v(\{2\}) = v(\{1,2\}) = (-\infty, 0]$. This is a superadditive game with a nonempty core: $(0,0) \in C(N,v)$. The strategy combination (x_1, x_2) with $x_i = (\{i\}, 1)$ in which each player wants to be on his own and claims payoff 1 yields $u_i(x) = v(i)$ for all $i \in N$ and is clearly a strong equilibrium of the claim game. ◁

Example 15.34 Take $|D| = 1, P = \emptyset, N = \{1,2\}$, and $v(\{1\}) = v(\{2\}) = v(\{1,2\}) = \{1\}$. Then $C(N,v) = \emptyset$. The strategy combination (x_1, x_2) with $x_i = (\{i\}, 1)$ yields $u_i(x) = \{1\}$ and is clearly a strong equilibrium of the claim game. ◁

Chapter 16

Best-Reply Matching in Ordinal Games

16.1 Introduction

In multicriteria games it is common that a good outcome in one criterion coincides with a bad outcome in another criterion. This can cause outcomes to be incomparable. Moreover, in Example 9.1 we saw that plausible decision making procedures may lead to nontransitive order relations. In both cases it is natural to consider ordinal games, games where the preferences of the players are not represented by real-valued utility functions, but by binary relations over the strategy space. This chapter, based on Droste, Kosfeld, and Voorneveld (1998a), considers such ordinal games in which — in addition — the common rationality assumptions are abandoned.

How rational should agents be? Or at least how rational should they be modelled? This question plays a substantial role in decision theory and game theory. Pioneering work in this field goes back to Simon in the mid 50's (Simon, 1957). Since then many different views and conceptions have been floating around, partly competing or contradicting each other. See Selten (1991) for a good description of the discussion upto the late 80's. The debate is still far from being settled. Recently, the topic of 'bounded rationality' has attracted a lot of interest again. Important research includes the work on psychology and economics, focusing on behavioral assumptions which can be based on psychological evidence (Camerer, 1997; Rabin, 1998). A related project is on evolution (Weibull, 1995; Vega-Redondo, 1996; Samuelson, 1997; Young, 1998) and learning (Fudenberg and Levine, 1998), where concepts from biological and social evolution are explored together with ideas on individual learning and adaptation.

Further research contains the work of Rubinstein (1998), who similar to the psychological literature argues that bounded rationality can not simply mean to assume players make mistakes but requires a new understanding of how players actually behave in decision making situations.

The present chapter follows these lines in proposing a new model for boundedly rational behavior of players in interactive situations that are captured by a noncooperative game. Our main idea focuses on the role of best replies, forming the set of actions a player can not improve upon given an action profile of his opponents. Roughly said, if players end up playing an action that was not a best reply to the actions of their opponents, they may feel regret of not having done the right thing. Consequently, the anticipation of regret may influence their decision making and determine their own behavior, i.e. their own mixed strategy. Our model studies a possible way of how this influence can work. The assumptions are as follows. Firstly, players focus on best replies only. This leads to an ordinal equilibrium concept, ignoring any cardinal issues as, e.g., actual payoff differences. Secondly, the anticipation of regret induces a player to compose his mixed strategy by matching the probability of playing an action to the (subjective) probability that this action is a best reply. The resulting behavior is called *best-reply matching*.

That regret may play an important role in situations of decision making is hardly a new point. Articles of Loomes and Sudgen (1982, 1987) and Bell (1982) have explored the possibility to incorporate regret considerations into the rational choice framework of standard decision theory under uncertainty. However, as far as interactive situations are concerned, upto now no analysis of behavior that is influenced by regret has been given. This chapter fills the gap. Yet, in contrast to the suggestions of Loomes and Sudgen (1982) or Bell (1982), where decision makers maximize a modified utility function, we do not propose a model of utility maximizing behavior. As explained above our model takes a bounded rationality approach, studying alternative procedures of how players actually behave. The main deviation we pursue is to assume 'matching'.

Over the past 25 to 30 years a mass of empirical evidence has been accumulated supporting the observation that individuals, both human beings and animals, produce behavior which obeys a pattern of so-called 'probability matching' or simply 'matching'. See, e.g., Davison and McCarthy (1988), Williams (1988), and Herrnstein (1997) for recent collections of these findings. In a general way matching says that an individual chooses an alternative from a given set of alternatives with a probability proportional to the value derived from that alternative. That is, if S is the set of alternatives and $v(s)$ denotes the 'value' of alternative $s \in S$, the probability of choosing s is equal to $v(s)/\sum_{s \in S} v(s)$. When we consider best-reply matching behavior, the individuals are the players of the game, the set of alternatives is given by the set of pure strategies of each player, and the value of each alternative is determined by the fact whether, ex post, the action is a best reply to the choice profile of the opponents or not.

Best-reply matching explicitly studies players that are clever enough to understand the strategic issues involved in a noncooperative game. That is, we do not consider players to ignore the fact that their payoff depends on the decisions of other players, as it is done, for example, in Osborne and Rubinstein (1998) and Rustichini (1998), where players simplify their situation by regarding it as a game against nature.

With respect to matching behavior in games, it should be noted that a prominent explanation of the matching phenomenon says that decision makers do not believe that the mechanism that causes the uncertainty is genuinely random and may therefore try to decipher the pattern (Cross, 1983, p.10). This explanation is particularly convincing in interactive situations, where players are confronted with other players making strategy choices rather than nature.

In order to prepare our general model we start with a simple example, which is in fact a game theoretic modification of a lottery given in Loomes and Sudgen (1982, p. 822). Consider the following situation. Player 1 faces a one-shot game against player 2 that is described by the payoff matrix in Figure 16.1. Payoffs given are in dollars and are to player 1, who chooses between A and B. In this example we ignore all strategic issues with respect to player 2, therefore payoffs to this player are neglected. Suppose that player 1 believes that player 2 first rolls a die and then chooses the respective column indicated by the roll of the die. Thus, beliefs of player 1 are such that all actions of his opponent have the same probability equal to $\frac{1}{6}$. Which row will player 1 choose, A or B?

	1	2	3	4	5	6
A	1	2	3	4	5	6
B	6	1	2	3	4	5

Figure 16.1: A First Example

Suppose first that player 1 is a neo-classical expected utility maximizer. Then, obviously, he is indifferent between A and B since both give the same expected payoff 3.5.

Suppose now that player 1 follows a different procedure. Comparing payoffs of actions A and B *for each individual choice* of player 2, he realizes that although both actions give indeed the same expected payoff, action A is a best reply for every action from 2 to 6, while action B is a best reply only in case player 2 chooses 1. Thus, in terms of player 1's beliefs the probability to play a best reply is five times higher when choosing action A than when choosing action B. Following this reasoning player 1 may choose action A with a higher probability than action B. Now suppose that the precise probability is determined as follows. For each of the individual choices of player 2, if the choice was known beforehand, player 1 would easily be able to decide between A and B. If player 2 was known to choose 1 he takes B, in all other cases he takes A. Now, of course player 1 does not know beforehand which action player 2 is going to choose. But using his beliefs he can calculate that the probability for being called to play B is $\frac{1}{6}$ and the probability for being called to play A is $\frac{5}{6}$. So this is what he does: he matches the probability of playing an action to the probability that this action is a best reply. With probability $\frac{1}{6}$ he plays B, with probability $\frac{5}{6}$ he plays A.

The example outlines the main essence of the behavioral approach we are going to study in this chapter. The remainder of this chapter is organized as follows. The next section specifies the class of games under consideration. Section 16.3 formally defines the notion of best-reply matching and best-reply matching equilibrium. We give a deeper motivation of the concept in Section 16.4. Section 16.5 proves existence of equilibria and analyzes the concept in more detail. Section 16.6 looks at the size and structure of the set of best-reply matching equilibria. Section 16.7 illustrates its properties by means of well-known examples. Finally, Section 16.8 concludes.

16.2 Preliminaries

We give the definition of some standard game theoretic notions which are used hereafter.

An *(ordinal noncooperative) game* is a tuple $G = \langle N, (X_i)_{i\in N}, (\succeq_i)_{i\in N}\rangle$, where $N = \{1, \ldots, n\}$ is a finite set of players; each player $i \in N$ has a finite set X_i of pure strategies, henceforth called *actions*, and a binary relation $\succeq_i$ over $\prod_{i\in N} X_i$, reflecting his preferences over the outcomes. The binary relation $\succeq_i$ is assumed to be reflexive and its asymmetric part $\succ_i$, defined for all $s, t \in \prod_{i\in N} X_i$ by

$$s \succ_i t \Leftrightarrow [s \succeq_i t \text{ and not } t \succeq_i s],$$

is assumed to be acyclic. In the following we also consider cases in which the preference relations $\succeq_i$ induce von Neumann-Morgenstern utility functions $u_i : \prod_{i\in N} X_i \to \mathbb{R}$ and denote the corresponding game by $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$.

Standard notation is used: $X = \prod_{i\in N} X_i$, $X_{-i} = \prod_{j\in N\setminus\{i\}} X_j$, etc. We denote by

$$\Delta_i := \{\sigma_i : X_i \to \mathbb{R} \mid \forall x_i \in X_i : \sigma_i(x_i) \geq 0, \sum_{x_i\in X_i} \sigma_i(x_i) = 1\}$$

the set of mixed strategies, henceforth called *strategies*, for player i. Analogously to the action case, we use notations $\Delta = \prod_{i\in N} \Delta_i$, $\Delta_{-i} = \prod_{j\in N\setminus\{i\}} \Delta_j$, $\sigma = (\sigma_i, \sigma_{-i})$. For a strategy profile $\sigma_{-i} \in \Delta_{-i}$, we write $\sigma_{-i}(x_{-i}) := \prod_{j\in N\setminus\{i\}} \sigma_j(x_j)$, the probability that the opponents of player i play the action profile $x_{-i} \in X_{-i}$. Thus, in particular we restrict attention to independent strategy profiles.

Consider a game $\langle N, (X_i)_{i\in N}, (\succeq_i)_{i\in N}\rangle$. Denote for each player $i \in N$ and each profile $x_{-i} \in X_{-i}$ of actions of his opponents the *set of pure best replies*, i.e., the actions that player i cannot improve upon, by $B_i(x_{-i})$:

$$B_i(x_{-i}) := \{x_i \in X_i \mid \nexists \tilde{x}_i \in X_i : (\tilde{x}_i, x_{-i}) \succ_i (x_i, x_{-i})\}.$$

Of course, for games $\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ with utility functions we have:

$$B_i(x_{-i}) := \{x_i \in X_i \mid \forall \tilde{x}_i \in X_i : u_i(x_i, x_{-i}) \geqq u_i(\tilde{x}_i, x_{-i})\}.$$

Since X_i is finite and $\succ_i$ is acyclic, $B_i(x_{-i})$ is nonempty. An action $x_i \in X_i$ in a game $G = \langle N, (X_i)_{i\in N}, (\succeq_i)_{i\in N}\rangle$ is a *never-best reply* if

$$\{x_{-i} \in X_{-i} \mid x_i \in B_i(x_{-i})\} = \emptyset.$$

For a game $\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ we have that $x_i \in X_i$ is a never-best reply if

$$u_i(x_i, x_{-i}) < \max_{\tilde{x}_i \in X_i} u_i(\tilde{x}_i, x_{-i})$$

for each $x_{-i} \in X_{-i}$. An action $x_i \in X_i$ is *weakly dominated* by a strategy $\sigma_i \in \Delta_i$ if

$$\forall x_{-i} \in X_{-i} : u_i(\sigma_i, x_{-i}) \geqq u_i(x_i, x_{-i})$$

with strict inequality for at least one x_{-i}, and *strictly dominated* if all inequalities are strict. A strictly dominated action is clearly a never-best reply. We next define best-reply matching behavior and best-reply matching equilibrium.

16.3 Definition

As the title of this chapter suggests, our approach focuses on two things: 'best reply' and 'matching'. Let us start with the first. Consider a game G and some player $i \in N$. Then, we assume that to every action $x_i \in X_i$ player i associates the set

$$\{x_{-i} \in X_{-i} \mid x_i \in B_i(x_{-i})\},$$

which gives all opponents' action profiles to which x_i is a best reply. The collection of these sets is obtained directly from the game G. It contains all relevant information concerning player i's best-reply structure. Note that in games with utility functions a lot of information may be ignored by focusing simply on the best-reply structure of the game. In particular, all cardinal issues do not enter a player's consideration. Best-reply matching is an ordinal concept.

The second expression, 'matching', captures how players use the information on their best-reply structure in order to determine their own behavior, i.e. the strategy being played. We assume that additional to the information on the game a player has beliefs about the opponents' behavior. The belief of player i is given by a strategy $\hat{\sigma}_{-i} \in \Delta_{-i}$ determining for each action profile $x_{-i} \in X_{-i}$ the probability $\hat{\sigma}_{-i}(x_{-i})$ with which player i believes that particular profile to occur. Now, our assumption says that a player builds his own strategy by matching his individual probabilities to play an action to the probabilities that these actions are a best reply. We obtain the following definition.

Definition 16.1 Let $G = \langle N, (X_i)_{i\in N}, (\succeq_i)_{i\in N}\rangle$ be a game. Consider a player $i \in N$. Denote by $\hat{\sigma}_{-i} \in \Delta_{-i}$ the strategy profile player i believes his opponents to play. Player i *matches best replies* if for every $x_i \in X_i$:

$$\sigma_i(x_i) = \sum_{\{x_{-i}\in X_{-i} \mid x_i \in B_i(x_{-i})\}} \frac{1}{|B_i(x_{-i})|}\hat{\sigma}_{-i}(x_{-i}). \qquad (16.1)$$

$\triangleleft$

As we have to take care of multiple best replies, dividing by $|B_i(x_{-i})|$ in (16.1) ensures that σ_i is well-defined, i.e. that probabilities sum up to one. Thereby we implicitly assume that all multiple best replies are weighted equally. However, it should be clear that any other weighting rule would be fine, too, though changing, of course, the probabilities that are assigned to actions. If best replies are unique the weighting rule is obviously irrelevant.

In a best-reply matching equilibrium every player matches best replies and beliefs are correct, i.e. for all $i \in N$, $\hat{\sigma}_{-i} = \sigma_{-i}$.

Definition 16.2 Let $G = \langle N, (X_i)_{i \in N}, (\succeq_i)_{i \in N} \rangle$ be a game. A mixed strategy profile $\sigma \in \Delta$ is a *best-reply matching (BRM) equilibrium* if for every player $i \in N$ and for every $x_i \in X_i$:

$$\sigma_i(x_i) = \sum_{\{x_{-i} \in X_{-i} | x_i \in B_i(x_{-i})\}} \frac{1}{|B_i(x_{-i})|} \sigma_{-i}(x_{-i}). \tag{16.2}$$

The set of BRM equilibria of a game G is denoted by *BRM(G)*. $\triangleleft$

Having defined the notion of best-reply matching and the equilibrium concept based on this notion, we now provide two interpretations of the new concept.

16.4 Motivation and interpretation

We give two different interpretations of best-reply matching behavior and BRM equilibrium. One interpretation of best-reply matching is based on the idea that a player wants to feel content after having played the game. That is, after the game is over a player would like to be able to say that 'he has done the right thing' in the sense that he has chosen 'the right action'. In a game the only reasonable ex post criterion for 'a right action' is the criterion of a best reply. Once the game is over and strategies are realized, the action profile of the opponents is fixed and can no longer be changed. Yet given a fixed action profile of the other players the best a player can do is indeed to play a best reply. They form the reference value to which any action has to be compared after the game is over.

Therefore intuitively our approach can be seen as saying that, in order to feel no regret after the game is over, players have an ex ante aspiration level of playing a best reply. Note that this assumption does not necessarily disagree with the basic idea of rational utility maximizing behavior. In fact, on the action level, i.e. whenever a player has deterministic beliefs, best-reply matching behavior is rational.

So suppose that beliefs of a player i are mixed, i.e., that several of his opponents' action profiles x_{-i} (possibly all) are believed to be played with positive probability. In this situation our behavioral assumption differs from rationality. The idea is the following. Since a player knows what to do *in each of the single cases* x_{-i} that can occur, namely play a best reply to the profile x_{-i}, he refers to these situations as the basis for his behavior. Firstly, he determines his mixed strategy by restricting its support to actions x_i that would be played in any of these single cases. Secondly, he weighs these individual actions by the probability with which they would be played if only the realized profile of the opponents would be known beforehand. The result is matching. Each action x_i is played with exactly the probability that this action would be played, i.e. that it is a best reply, given the beliefs of player i.

The reason why we assume the behavior of a player to be non-rational in case of uncertainty is that we presume it to be too complicated for a player to maximize over the set of mixed strategies, which is a continuum of alternatives. Although, or even because he is able to maximize over the set of actions, he fails to do so in case he must determine a mixed strategy. Similarly, best-reply matching is not based on expected payoffs, since these can never be observed in a one-shot setting. In particular, we assume that players do not aggregate an uncertain situation by summing up products of probabilities and payoffs. Instead, we propose that a player views an uncertain situation as a combination of several possible situations, in each of which he would know precisely what to do. He then weighs the reactions to each separate situation with the probability he assigns to the occurrence of this situation.

The notion of equilibrium in this setting is based on the usual static approach. Every player behaves according to the best-reply matching assumption and individual beliefs are correct. In other words, the interaction between players forms a fixed point.

An alternative motivation for the equilibrium concept is based on a repeated situation of play, where each single player myopically adapts his strategy from one period to the other. The description of this second motivation is brief; the reader is referred to Droste, Kosfeld, and Voorneveld (1998b) for details.

Suppose the game G is repeated infinitely often in discrete time. At any time t a player is assumed to play an action drawn from the distribution given by his mixed strategy σ_i^t at that time. Players behave myopically and update their strategy each period based only on the realization of play in that period. As in the one-shot setting the updating is based on best replies. However, contrary to above the updating procedure does not involve any matching but, as we will see, will imply matching behavior in every steady state, i.e. in equilibrium.

Consider a given period t and suppose that in this period action profile $x \in X$ was realized. Let players $i \in N$ adjust their strategy as follows. For each $x_i \in X_i$,

$$\sigma_i^{t+1}(x_i) = \begin{cases} (1-\theta)\,\sigma_i^t(x_i) + \frac{\theta}{|B_i(x_{-i})|} & \text{if } x_i \in B_i(x_{-i}), \\ (1-\theta)\,\sigma_i^t(x_i) & \text{otherwise,} \end{cases} \tag{16.3}$$

where $0 < \theta < 1$ is a parameter that is exogenously fixed.

Intuitively, the adjustment procedure specified in (16.3) says that a player $i \in N$ after the t-th repetition of the game adjusts his strategy by first proportionally decreasing all probabilities by a fraction θ. This then leaves the player with a probability θ that is reallocated to the actions that are best replies to the action profile of his opponents in the t-th repetition of the game. Hence, similar to the one-shot setting a player focuses on optimal behavior in terms of best replies. After each round he shifts his behavior towards strategies that are best replies to the last observation, where the degree of adjustment is determined by $0 < \theta < 1$. Note that, while the action currently played by player i does not directly influence how he adjusts his strategy, the action does influence the updating process of all of his opponents. Consequently, the action currently played does influence the future strategy profiles of his opponents and therefore it will influence his own adjustment process indirectly.

Alternatively, the adjustment process (16.3) can be motivated by a multipopulation model (cf. Weibull, 1995, Chapter 5). Suppose there exist $n = |N|$ large (technically infinite) populations of agents. Each agent in population $i \in N$ is programmed to an action in X_i. The share of agents in population $i \in N$ programmed to action $x_i \in X_i$ at time t is given by $\sigma_i^t(x_i)$. In each period t a fraction $\theta \in (0,1)$ of the agents in each population is randomly drawn to play the game. The agents who are called to play the game are randomly matched in n-tuples such that each agent is matched with exactly one agent from every other population. After all n-tuples of agents have played the game, the participating agents leave the system and are replaced by new agents who learn something about the prevailing states of the n populations. Suppose, in particular, that a randomly selected outcome $x \in X$ of one of the games played in period t is publicly announced. In other words, the probability of sampling a profile $x \in X$ equals its share in the current populations.

Now consider a new agent, replacing an agent that leaves population $i \in N$. This agent once and for all commits to an action $x_i \in X_i$, which he does by adopting a best reply against the publicly announced action profile x_{-i}. In case of multiple best replies, the new agent is assumed to adopt each of the best replies with equal probability. Obviously, the fraction $\sigma_i^{t+1}(x_i)$ of agents in population i programmed to action $x_i \in X_i$ in period $t+1$ then equals $(1-\theta)\sigma_i^t(x_i) + \frac{\theta}{|B_i(x_{-i})|}$ if $x_i \in B_i(x_{-i})$ and $(1-\theta)\sigma_i^t(x_i)$ otherwise, which is exactly the adjustment rule (16.3).

For a given initial random variable $\sigma^0 \in \Delta$ and a given parameter θ, the adjustment rule in (16.3) defines a discrete time Markov process $(\sigma^t)_{t=0}^{\infty}$ on the state space Δ. Without going into details, some key results from Droste, Kosfeld, and Voorneveld (1998b) are mentioned. If the Markov process is approximated by a dynamical system in continuous time, which follows the expected movement of the original stochastic process, then its steady states are exactly the best-reply matching equilibria of the underlying game G. Every pure-strategy best-reply matching equilibrium is an absorbing state of the stochastic process $(\sigma^t)_{t=0}^{\infty}$; every absorbing state of $(\sigma^t)_{t=0}^{\infty}$ is a best-reply matching equilibrium, possibly in mixed strategies. Thus, with repeated play matching turns out to be a stationarity property of the adjustment process of the players. For every game it holds that whenever the adjustment process settles down, players must match best replies.

16.5 Analysis

A fundamental question with respect to any equilibrium concept concerns its existence. The first proposition shows that BRM equilibria exist for every game.

Proposition 16.3 *Let $G = \langle N, (X_i)_{i\in N}, (\succeq_i)_{i\in N}\rangle$ be a game. Then $BRM(G) \neq \emptyset$.*

Proof. Let $i \in N, \sigma \in \Delta, x_i \in X_i$. Define

$$r_i(x_i, \sigma_{-i}) := \sum_{\{x_{-i}\in X_{-i} | x_i \in B_i(x_{-i})\}} \frac{1}{|B_i(x_{-i})|}\sigma_{-i}(x_{-i}).$$

Then

$$\begin{aligned}
\sum_{x_i\in X_i} r_i(x_i,\sigma_{-i}) &= \sum_{x_i\in X_i} \sum_{\{x_{-i}\in X_{-i}|x_i\in B_i(x_{-i})\}} \frac{1}{|B_i(x_{-i})|}\sigma_{-i}(x_{-i}) \\
&= \sum_{x_{-i}\in X_{-i}} \sum_{x_i\in B_i(x_{-i})} \frac{1}{|B_i(x_{-i})|}\sigma_{-i}(x_{-i}) \\
&= \sum_{x_{-i}\in X_{-i}} \sigma_{-i}(x_{-i}) \\
&= 1.
\end{aligned}$$

Hence the mapping

$$\begin{aligned}
r:\ & \Delta \to \Delta \\
& \sigma \mapsto r(\sigma)
\end{aligned}$$

with $r(\sigma)_i(x_i) = r_i(x_i, \sigma_{-i})$ is well-defined. In the definition of the function r_i neither the index set in the summation sign nor the number $|B_i(x_{-i})|$ of pure best replies depends on the strategy combination σ. Hence, this mapping is obviously continuous. Application of the Brouwer fixed-point theorem yields the existence of a strategy profile $\sigma \in \Delta$ such that $\sigma = r(\sigma)$, which is a BRM equilibrium. □

Remark 16.4 It follows from the proof of Proposition 16.3 that $\sum_{x_i\in X_i}\sigma_i(x_i) = 1 = \sum_{x_i\in X_i} r_i(x_i,\sigma_{-i})$ for each $\sigma \in \Delta, i \in N$. As a consequence, when computing BRM equilibria, one of the conditions $\sigma_i(x_i) = r_i(x_i,\sigma_{-i})$ of player i is redundant. ◁

A game H is said to be *obtained by iterated elimination of never-best replies* from a game $G = \langle N, (X_i)_{i\in N}, (\succeq_i)_{i\in N}\rangle$ if there exists a number $k \in \mathbb{N}$ of elimination rounds and for each player $i \in N$ a collection of sets $X_i^0, X_i^1, \ldots, X_i^k$ and a sequence $\succeq_i^0, \succeq_i^1, \ldots, \succeq_i^k$ of relations such that:

1. For each player $i \in N : X_i = X_i^0 \supseteq X_i^1 \supseteq \cdots \supseteq X_i^k$;

2. For each player $i \in N$ and each $l = 0, 1, \dots, k$: $\succeq_i^l$ is the preference relation $\succeq_i$ of the game G restricted to $\prod_{j \in N} X_j^l$;

3. For each $l = 0, 1, \dots, k-1$ and each player $i \in N$ the set $X_i^l \setminus X_i^{l+1}$ contains only never-best replies of player i in the game $\langle N, (X_i^l)_{i \in N}, (\succeq_i^l)_{i \in N} \rangle$ and there exists at least one player $i \in N$ for which $X_i^l \setminus X_i^{l+1}$ is nonempty;

4. H is the game $\langle N, (X_i^k)_{i \in N}, (\succeq_i^k)_{i \in N} \rangle$;

5. In the game H, no player $i \in N$ has never-best replies.

The behavior of the BRM equilibrium concept with respect to dominated actions and elimination thereof is summarized in the next result.

Proposition 16.5 *The following results hold:*

(i) In a BRM equilibrium σ^* of a game $\langle N, (X_i)_{i \in N}, (\succeq_i)_{i \in N} \rangle$ never-best replies are played with zero probability.

Moreover,

(ii) the set of BRM equilibria of a game $G = \langle N, (X_i)_{i \in N}, (\succeq_i)_{i \in N} \rangle$ equals — up to zero probability assigned to eliminated actions — the set of BRM equilibria of a game that is obtained by iterated elimination of never-best replies.

Finally,

(iii) let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game with von Neumann-Morgenstern utility functions and let σ^* be a BRM equilibrium of G. If player i's action x_i is weakly dominated by the strategy σ_i, then:

$$\textit{for all } \overline{x}_i \in X_i : \ \textit{if } \sigma_i(\overline{x}_i) > 0, \ \textit{then } \sigma_i^*(\overline{x}_i) \geq \sigma_i^*(x_i).$$

Proof. The proof of (i) is easy: if $x_i \in X_i$ is a never-best reply, then the set $\{x_{-i} \in X_{-i} \mid x_i \in B_i(x_{-i})\}$ is empty and hence according to (16.2): $\sigma_i^*(x_i) = r_i(x_i, \sigma_{-i}^*) = 0$.

To prove (ii), it suffices to prove that the first round of eliminations does not change the equilibrium set, since the proof can then be repeated for the additional rounds. Assume for simplicity that in the first elimination round we eliminate all the never-best replies

$$NB_i := \{x_i \in X_i \mid x_i \text{ is a never-best reply of player } i \text{ in } G\}$$

of each player $i \in N$, thus obtaining a smaller game G'. The equilibrium conditions in the game G are that for each $i \in N$ and each $x_i \in X_i$:

$$\sigma_i(x_i) \quad = \quad r_i(x_i, \sigma_{-i})$$

$$
\begin{aligned}
&= \sum_{\{x_{-i}\in X_{-i}|x_i\in B_i(x_{-i})\}} \frac{1}{|B_i(x_{-i})|}\sigma_{-i}(x_{-i}) \\
&= \sum_{\{x_{-i}\in X_{-i}|x_i\in B_i(x_{-i}) \text{ and } \forall j\in N\setminus\{i\}:\, x_j\notin NB_j\}} \frac{1}{|B_i(x_{-i})|}\sigma_{-i}(x_{-i}) \\
&+ \sum_{\{x_{-i}\in X_{-i}|x_i\in B_i(x_{-i}) \text{ and } \exists j\in N\setminus\{i\}:\, x_j\in NB_j\}} \frac{1}{|B_i(x_{-i})|}\sigma_{-i}(x_{-i}) \\
&= \sum_{\{x_{-i}\in\prod_{j\in N\setminus\{i\}} X_j\setminus NB_j|x_i\in B_i(x_{-i})\}} \frac{1}{|B_i(x_{-i})|}\sigma_{-i}(x_{-i}) \\
&+ \sum_{\{x_{-i}\in X_{-i}|x_i\in B_i(x_{-i}) \text{ and } \exists j\in N\setminus\{i\}:\, x_j\in NB_j\}} \frac{1}{|B_i(x_{-i})|}\sigma_{-i}(x_{-i})
\end{aligned}
$$

By (i), actions $x_j \in NB_j$ are played with zero probability in a BRM equilibrium. Hence the second sum in the last equality above equals zero. What remains, for each player $i \in N$ and each action $x_i \in X_i \setminus NB_i$, are exactly the equilibrium conditions for the game G'.

To prove (iii), let $x_{-i} \in X_{-i}$ and assume that $x_i \in B_i(x_{-i})$. Since σ_i weakly dominates x_i and $x_i \in B_i(x_{-i})$, for every $\overline{x}_i \in X_i$ such that $\sigma_i(\overline{x}_i) > 0$ we must have that $\overline{x}_i \in B_i(x_{-i})$. Hence for every $\overline{x}_i \in X_i$ with $\sigma_i(\overline{x}_i) > 0$:

$$\{x_{-i} \in X_{-i} \mid x_i \in B_i(x_{-i})\} \subseteq \{x_{-i} \in X_{-i} \mid \overline{x}_i \in B_i(x_{-i})\},$$

which together with the definition of $r_i(\cdot, \sigma^*_{-i})$ implies the result:

$$
\begin{aligned}
\sigma_i(\overline{x}_i) > 0 \Rightarrow \sigma_i^*(\overline{x}_i) &= r_i(\overline{x}_i, \sigma^*_{-i}) \\
&= \textstyle\sum_{\{x_{-i}\in X_{-i}|\overline{x}_i\in B_i(x_{-i})\}} \frac{1}{|B_i(x_{-i})|}\sigma^*_{-i}(x_{-i}) \\
&\geq \textstyle\sum_{\{x_{-i}\in X_{-i}|x_i\in B_i(x_{-i})\}} \frac{1}{|B_i(x_{-i})|}\sigma^*_{-i}(x_{-i}) \\
&= r_i(x_i, \sigma^*_{-i}) \\
&= \sigma_i^*(x_i).
\end{aligned}
$$

□

Notice that the result above does not rule out that weakly dominated actions are played with positive, even quite large probability.

Example 16.6 Consider the game in Figure 16.2. T weakly dominates B and L strictly dominates R. Both T and B are a best reply against L, and T is a unique best reply against R. Hence in equilibrium we have the condition that

$$\sigma_1(T) = \frac{1}{2}\sigma_2(L) + \sigma_2(R).$$

	L	R
T	1, 1	1, 0
B	1, 1	0, 0

Figure 16.2: The game from Example 16.6

The condition for $\sigma_1(B)$ is redundant, since the probabilities have to add up to one. Similarly, for player 2 we see that L is a unique best reply to both T and B, so that his equilibrium condition becomes

$$\sigma_2(L) = \sigma_1(T) + \sigma_1(B) = 1.$$

Solving these equations and taking into account that $(\sigma_1, \sigma_2) \in \Delta$ we find that the unique BRM equilibrium equals $((\frac{1}{2}, \frac{1}{2}), (1, 0))$. Observe that the weakly dominated action is not only played with positive probability, but that there is not even an alternative action with a higher probability. $\triangleleft$

Despite the relatively prudent behavior with respect to (weakly) dominated actions as expressed in Proposition 16.5, the set of BRM equilibria and Nash equilibria have no obvious relation. In the game of Figure 16.3, for instance, the unique Nash equilibrium equals $((\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3}))$ while the unique BRM equals $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.

	L	R
T	0, 2	2, 0
B	1, 0	0, 1

Figure 16.3: The Nash and BRM equilibrium concept differ

We can, however, indicate a relation with the notion of *strict equilibria* introduced by Harsanyi (1973) as those strategy profiles σ satisfying the condition that each player plays his unique best reply to the strategies of the opponent:

$$\forall i \in N : \{\sigma_i\} = \{\tau_i \in \Delta_i \mid \nexists \tilde{\tau}_i : u_i(\tilde{\tau}_i, \sigma_{-i}) > u_i(\tau_i, \sigma_{-i})\}.$$

It is clear that a strict Nash equilibrium is always a pure-strategy Nash equilibrium and (consequently) that strict Nash equilibria do not always exist. However, if they exist, they are exactly the pure-strategy BRM equilibria of the game.

Proposition 16.7 *The set of strict Nash equilibria of a game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ coincides with the set of pure strategy BRM equilibria.*

The proof is straightforward and left to the reader.

The results with respect to the iterated elimination of never-best replies in Proposition 16.5 call to mind the notion of rationalizability introduced in Bernheim (1984) and

Pearce (1984). Without going into the formal definitions, it follows immediately from Proposition 16.5 and Bernheim (1984, pp. 1015-1016) that every action that is played with positive probability in a BRM equilibrium is rationalizable. However, in a BRM equilibrium σ, the mixed strategies σ_i themselves need not be rationalizable. This shows again that the bounded rationality concept of best-reply matching agrees with standard rationality on the action level but disagrees with respect to mixed strategies.

16.6 Size and structure

The size of an equilibrium set can be seen as a measure of the cutting power of an equilibrium concept: if an equilibrium set contains many candidates, it can be seen as a weak concept, not ruling out many strategy profiles. With respect to the size of the set of BRM equilibria of a game $\langle N, (X_i)_{i\in N}, (\succeq_i)_{i\in N}\rangle$, remark that it is always a relatively small subset of Δ. A strategy tuple $\sigma_{-i} \in \Delta_{-i}$ completely determines $r_i(\cdot, \sigma_{-i})$ and hence in an n-player game it suffices to know only $n-1$ components of a BRM equilibrium to compute the equilibrium strategy for the remaining n-th player. This implies that $BRM(G)$ is always of lower dimension than Δ. In particular, it is impossible that $BRM(G) = \Delta$.

The structure of the set of Nash equilibria has been studied by several authors, including Winkels (1979) and Jansen (1981), who show that in two-person games the set of Nash equilibria has a nice decomposition into a finite number of polytopes. Concerning the structure of the set of best-reply matching equilibria, we see that if the game G has only two players, then $BRM(G)$ is a polytope, since the set of BRM equilibria is then determined by finitely many linear equations and linear weak inequalities in the variables $(\sigma_i(x_i))_{i\in N, x_i\in X_i}$. If the game has at least three players, its set of BRM equilibria is determined by a set of polynomial equations over a Cartesian product of simplices. This leads to the observations that — analogous to the set of Nash equilibria — the set of BRM equilibria may be curved or disconnected. The following two examples indicate that both possibilities may indeed occur.

Example 16.8 Consider the three-player game in Figure 16.4. Here we denote by $p, q, r \in [0,1]$ the probability with which player 1 chooses his first row, player 2 chooses his first column, and player 3 chooses his first matrix, respectively. Considering Remark

$$\begin{array}{c|c|c|}
 & q & 1-q \\
\hline
p & 1,1,1 & 1,0,0 \\
\hline
1-p & 0,1,1 & 0,0,0 \\
\hline
\multicolumn{3}{c}{r}
\end{array}
\qquad\qquad
\begin{array}{c|c|c|}
 & q & 1-q \\
\hline
p & 1,0,0 & 0,1,1 \\
\hline
1-p & 0,0,0 & 1,1,1 \\
\hline
\multicolumn{3}{c}{1-r}
\end{array}$$

Figure 16.4: A game with a curved set of BRM equilibria

16.4, it suffices to determine an equilibrium constraint only for p, q, and r, since those for

$1-p, 1-q, 1-r$ will follow immediately. The first strategy (the top row) of player 1 is a unique best response to three combinations of pure strategies of his opponents, namely to those in which player 2 chooses either his first or his second column and player 3 chooses the first matrix, which occurs with probability $qr+(1-q)r$, and to the strategy in which player 2 chooses his first column and player 3 chooses the second matrix, which occurs with probability $q(1-r)$. Together with the constraints for the other two players, we find that the conditions for a BRM equilibrium are

$$\left\{\begin{array}{lclcl} p & = & qr+(1-q)r+q(1-r) & = & q+(1-q)r \\ q & = & pr+(1-p)r & = & r \\ r & = & pq+(1-p)q & = & q \\ p,q,r & \in & [0,1] & & \end{array}\right.$$

Consequently, the set of BRM equilibria equals

$$\{((p,1-p),(q,1-q),(r,1-r)) \mid p=q(2-q), r=q, q\in[0,1]\},$$

which is a curved equilibrium set. ◁

Example 16.9 Consider the three-player game in Figure 16.5. The conditions for a

$$\begin{array}{c|c|c|} & q & 1-q \\ \hline p & 1,1,1 & 0,0,0 \\ \hline 1-p & 0,0,1 & 1,1,0 \\ \hline \end{array} \qquad \begin{array}{c|c|c|} & q & 1-q \\ \hline p & 0,0,0 & 1,1,1 \\ \hline 1-p & 1,1,0 & 0,0,1 \\ \hline \end{array}$$

$$r \qquad\qquad\qquad\qquad 1-r$$

Figure 16.5: A game with a disconnected set of BRM equilibria

BRM equilibrium are

$$\left\{\begin{array}{lclcl} p & = & qr+(1-q)(1-r) & = & 2qr-q-r+1 \\ q & = & pr+(1-p)(1-r) & = & 2pr-p-r+1 \\ r & = & pq+(1-p)q & = & q \\ p,q,r & \in & [0,1] & & \end{array}\right.$$

This is equivalent with (after substitution of $r=q$):

$$\left\{\begin{array}{lcl} p & = & 2q^2-2q+1 \\ q & = & 2pq-p-q+1 \\ r & = & q \\ p,q,r & \in & [0,1] \end{array}\right.$$

Subtracting the first equality from the second, we find:

$$\left\{\begin{array}{lcl} p & = & 2q^2-2q+1 \\ pq & = & q^2 \\ r & = & q \\ p,q,r & \in & [0,1] \end{array}\right.$$

Hence

$$\left\{\begin{array}{ccc} p & = & 2q^2-2q+1 \\ q & = & 0 \\ r & = & q \\ p,q,r & \in & [0,1] \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{ccc} p & = & 2q^2-2q+1 \\ p & = & q \\ r & = & q \\ p,q,r & \in & [0,1] \end{array}\right.$$

Consequently, the set of BRM equilibria equals

$$\{((1,0),(0,1),(0,1))\} \cup \{((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))\} \cup \{((1,0),(1,0),(1,0))\},$$

consisting of three components. ◁

16.7 Examples

In this section we apply the concept of a BRM equilibrium to several classes of games, including two-person coordination games and a class of Hawk-Dove games. Moreover, one can apply the concept of a BRM equilibrium to the reduced strategic form of extensive form games. We present one brief example and one more elaborate case, in which we solve a T-choice centipede game.

Example 16.10 Consider the Rock, Scissors, Paper game in Figure 16.6, where R, S, P, have the obvious meaning and the corresponding probabilities with which these strategies are played are denoted by p_i, q_i.

	R, q_1	S, q_2	$P, 1-q_1-q_2$
R, p_1	$0,0$	$1,-1$	$-1,1$
S, p_2	$-1,1$	$0,0$	$1,-1$
$P, 1-p_1-p_2$	$1,-1$	$-1,1$	$0,0$

Figure 16.6: Rock, Scissors, Paper

The conditions for a BRM equilibrium are

$$\left\{\begin{array}{ccc} p_1 & = & q_2 \\ p_2 & = & 1-q_1-q_2 \\ q_1 & = & p_2 \\ q_2 & = & 1-p_1-p_2 \\ p_1,p_2,q_1,q_2 & \in & [0,1] \\ p_1+p_2 & \leqq & 1 \\ q_1+q_2 & \leqq & 1 \end{array}\right.$$

Simple calculus leads to the conclusion that the unique BRM equilibrium equals the unique Nash equilibrium in which both players choose each of their pure strategies with probability $\frac{1}{3}$. ◁

Example 16.11 A two-player game is a coordination game if both players have the same set of actions and the unique best reply to an action of the opponent is to play the same action. An example of a coordination game is the Battle of the Sexes game given in Figure 16.7.

	boxing	ballet
boxing	$3,2$	$0,0$
ballet	$0,0$	$2,3$

Figure 16.7: Battle of the Sexes; a coordination game

From the definition of a coordination game it is clear that a profile of strategies is a BRM equilibrium if and only if both players play the same mixed strategy. This illustrates an important difference from the Nash equilibrium concept: The pure Nash equilibria of a coordination game are the combinations of pure strategies in which the players indeed coordinate (choose the same pure strategy). Since these Nash equilibria are strict, they are also BRM equilibria. However, there is a mixed strategy Nash equilibrium in which players do not coordinate exactly. In the example above, the mixed strategy Nash equilibrium is $((\frac{3}{5}, \frac{2}{5}), (\frac{2}{5}, \frac{3}{5}))$. This equilibrium is not a BRM equilibrium, since it is not symmetric. For example, player 1 puts more probability on 'boxing' than he believes player 2 does, which is not in accordance with matching. Intuitively, in a BRM equilibrium in order to avoid miscoordination players want to do exactly the same as their opponent. If the opponent goes 'boxing' with probability p they will go 'boxing' with the same probability p, too. ◁

Example 16.12 In this example we consider a class of Hawk-Dove games with the structure of the payoff matrix given in Figure 16.8. Here V and W are real numbers satisfying the condition $W < V$. We consider several cases.

	q	$1-q$
p	V, V	$0, 2V$
$1-p$	$2V, 0$	W, W

Figure 16.8: A class of Hawk-Dove games

1. If $W > 0$, we have a Prisoner's dilemma; the second action of both players strictly dominates the first, so the unique BRM equilibrium equals $((0,1),(0,1))$.
2. If $W = 0$, the conditions for a BRM equilibrium are

$$\begin{cases} p &= \frac{1}{2}(1-q) \\ q &= \frac{1}{2}(1-p) \\ p,q &\in [0,1] \end{cases}$$

   The BRM equilibrium is $((\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{2}{3}))$.

3. If $W < 0 < V$, we have a Chicken game. The conditions for a BRM equilibrium are
$$\begin{cases} p & = & 1-q \\ q & = & 1-p \\ p,q & \in & [0,1] \end{cases}$$
The set of BRM equilibria is $\{((p, 1-p), (1-p, p)) \mid p \in [0,1]\}$.

4. If $V = 0$, the conditions for a BRM equilibrium are
$$\begin{cases} p & = & \frac{1}{2}q + (1-q) \\ q & = & \frac{1}{2}p + (1-p) \\ p,q & \in & [0,1] \end{cases}$$
The BRM equilibrium is $((\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}))$.

5. If $V < 0$, the first action of both players strictly dominates the second, so the unique BRM equilibrium is $((1,0),(1,0))$.

◁

Example 16.13 Consider the extensive form game in Figure 16.9. In this game, player 1 is given the choice to stop (S) or continue (C). If he continues, player 2 is given the same choice. The game ends if either player decides to stop or both decide to continue. Assume that $c \succ_1 a$ and $c \succ_2 b$. Consequently, we have that the outcome c is the unique

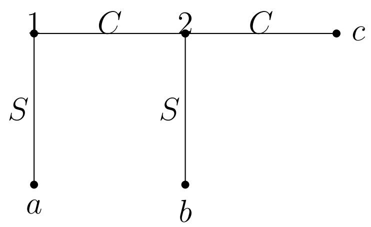


Figure 16.9: An extensive form game

subgame perfect equilibrium of the game. Denote by p the probability that player 1 chooses to stop and by q the probability that player 2 chooses to stop. By Remark 16.4, it suffices to determine the equilibrium conditions for p and q. Player 2's choice to stop is not a best reply to player 1's strategy to continue. If player 1 stops, it is of no concern what player 2 chooses: either strategy is a best reply. Hence the equilibrium condition for player 2 is:
$$q = \frac{1}{2}p, q \in [0,1].$$
The equilibrium condition for player 1 is either $p = q$ or $p = \frac{1}{2}q$ or $p = 0$, depending on whether he finds a better than, equivalent to, or worse than outcome b. In the first

two cases, i.e., if $a \succeq_1 b$, there is a Nash equilibrium yielding outcome a which is never played in a BRM equilibrium. No matter what preferences player 1 has over a and b, the unique BRM equilibrium is that both players continue with probability one. $\lhd$

Example 16.14 In the T-choice centipede game, introduced by Rosenthal (1981), players 1 and 2 alternately move. In any of the $2T$ periods, the player whose turn it is to move can decide to stop the game (S) or to continue (C). Consequently, both players have $T+1$ actions: stopping at any one of the T opportunities, or continue all the time. The game ends if one of the players decides to stop or if neither player has decided to do so after each of them has had T opportunities. For each player, the outcome when he stops the game in period t is better than that in which the other player stops the game in period $t+1$ (or the game ends), but worse than any outcome that is reached if in period $t+1$ the other player passes the move to him. Therefore:

Player 2's action to stop at his k-th opportunity is a best reply to the following actions of player 1:

- *player 1 stops immediately; then all of player 2's $T+1$ actions are a best reply;*
- *if $k = T$ the unique best reply to player 1's choice to continue always is to stop at the final stage;*
- *player 1 decides to stop at opportunity $k+1$.*

Player 1's action to stop at his k-th opportunity is a best reply to exactly one action of player 2:

- *player 2 decides to stop in the next period, at his k-th opportunity.*

An example of a 3-choice centipede game is given in Figure 16.10. Denote by $p_i[q_i]$

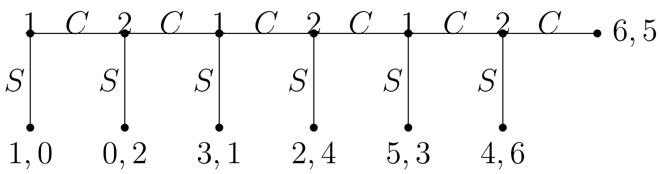


Figure 16.10: A 3-choice centipede game

the probability of player 1[2] to stop at his i-th opportunity, once this opportunity is reached ($i = 1, \ldots, T$). Thus, our computations are in behavioral, rather than in mixed strategies. We show that the unique BRM equilibrium satisfies for each number $T \in \mathbb{N}$ of choices and each $k \in \{0, \ldots, T-1\}$:

$$p_{T-k} = q_{T-k} = \frac{2}{k+3}. \tag{16.4}$$

In particular, if the number of choices T approaches infinity, the probability for each player to stop at the first (and by the same argument at any finite) opportunity, converges to zero. Osborne and Rubinstein (1998), who obtain a similar result, conclude that their equilibrium notion makes sense only if both players fail to understand the structure of the game. In our equilibrium notion, the equilibrium conditions form an almost immediate translation of the structure of the game, where it is a unique best reply to stop exactly one period ahead of your opponent's intent to do so. Still, we find a potential resolution of the paradox posed by the centipede game: The players play the unique Nash equilibrium of stopping immediately with positive probability, but the solution in (16.4) indicates that there is a strong urge to continue playing, thus providing the possibility to achieve more preferable outcomes.

The solution in (16.4) also indicates that players continue with positive probability at every node, but the probability to stop increases as players reach further nodes in the game. This feature is mentioned as the most obvious and consistent pattern in the experimental study of McKelvey and Palfrey (1992), who remark that 'any model to explain the data should capture this basic feature' (McKelvey and Palfrey, 1992, p.809). Thus, while no standard game theoretic solution concept can predict this outcome, the best-reply matching equilibrium concept does a good job. Moreover, the surprising result that a player continues with positive probability at his final node, even though this action is strictly dominated by stopping at that node, is observed in the experimental sessions of McKelvey and Palfrey as well.

We now show that (16.4) holds. The description of the T-choice centipede game in terms of best replies (emphasized above) immediately gives rise to the following equilibrium conditions for player 1:

$$p_1 = q_1 \tag{I.1}$$

$$(1-p_1)p_2 = (1-q_1)q_2 \tag{I.2}$$

$$\cdots$$

$$(1-p_1)(1-p_2)\cdots(1-p_{T-1})p_T = (1-q_1)(1-q_2)\cdots(1-q_{T-1})q_T, \tag{I.T}$$

and for player 2:

$$q_1 = \frac{p_1}{T+1} + (1-p_1)p_2 \tag{II.1}$$

$$(1-q_1)q_2 = \frac{p_1}{T+1} + (1-p_1)(1-p_2)p_3 \tag{II.2}$$

$$\cdots$$

$$(1-q_1)(1-q_2)\cdots(1-q_{T-2})q_{T-1} = \frac{p_1}{T+1} + (1-p_1)(1-p_2)\cdots(1-p_{T-1})p_T \tag{II.T − 1}$$

$$(1-q_1)(1-q_2)\cdots(1-q_{T-1})q_T = \frac{p_1}{T+1} + (1-p_1)(1-p_2)\cdots(1-p_T). \tag{II.T}$$

The conditions that arise from always continuing are redundant (see Remark 16.4).

We prove first of all, that in the T-choice centipede game we have for each $i = 1, \ldots, T-1$:

$$p_i = q_i, p_i \notin \{0,1\}. \tag{16.5}$$

We know from condition (I.1) that $p_1 = q_1$. Suppose $p_1 = 1$. Substitution in (II.1) yields $1 = \frac{1}{T+1}$, a contradiction. Suppose $p_1 = 0$. Then $p_2 = q_2$ by (I.2) and $p_2 = 0$ by (II.1). Hence $p_3 = q_3$ by (I.3) and $p_3 = 0$ by (II.2). Proceeding in this fashion yields that $p_k = q_k = 0$ for all $k = 1, \ldots, T$, which contradicts (II.T). Hence $p_1 = q_1, p_1 \notin \{0,1\}$. Now assume that we have shown for some $k \in \{1, \ldots, T-2\}$:

$$\forall n \leqq k : p_n = q_n, p_n \notin \{0,1\}.$$

We proceed to show that the same holds for $k+1$. First of all, we have from (I.k+1) that $p_{k+1} = q_{k+1}$. Consider condition (II.k+1):

$$(1-q_1)(1-q_2)\cdots(1-q_k)q_{k+1} = \frac{p_1}{T+1} + \underbrace{(1-p_1)(1-p_2)\cdots(1-p_{k+1})p_{k+2}}_{\geq 0}.$$

If $q_{k+1} = 0$, then its left hand side equals zero, which would imply that $p_1 \leqq 0$, whereas we know from the above that $p_1 > 0$. If $p_{k+1} = q_{k+1} = 1$, condition (II.k+2) reduces to

$$0 = \frac{p_1}{T+1},$$

a contradiction. This finishes the proof of (16.5). This part was necessary to avoid division by zero in the following solution of the game.

Substitute the left-hand side of player 1's conditions in the left-hand side of player 2's conditions. This yields

$$\begin{aligned}
p_1 &= \frac{p_1}{T+1} + (1-p_1)p_2 \\
(1-p_1)p_2 &= \frac{p_1}{T+1} + (1-p_1)(1-p_2)p_3 \\
&\cdots \\
(1-p_1)(1-p_2)\cdots(1-p_{T-2})p_{T-1} &= \frac{p_1}{T+1} + (1-p_1)(1-p_2)\cdots(1-p_{T-1})p_T \\
(1-p_1)(1-p_2)\cdots(1-p_{T-1})p_T &= \frac{p_1}{T+1} + (1-p_1)(1-p_2)\cdots(1-p_T)
\end{aligned}$$

The first equation is equivalent to

$$\frac{T}{T+1}p_1 = (1-p_1)p_2.$$

Using this equality to replace the left-hand side of the second equation leads to

$$\frac{T}{T+1}p_1 = \frac{p_1}{T+1} + (1-p_1)(1-p_2)p_3,$$

which is equivalent to

$$\frac{T-1}{T+1}p_1 = (1-p_1)(1-p_2)p_3.$$

Use this equation, again, to replace the left-hand side of the third equation. This leads to

$$\frac{T-2}{T+1}p_1 = (1-p_1)(1-p_2)(1-p_3)p_4.$$

Continuing in this way, we get the following equivalent system of T equations:

$$\begin{aligned}
\frac{T}{T+1}p_1 &= (1-p_1)p_2 \\
\frac{T-1}{T+1}p_1 &= (1-p_1)(1-p_2)p_3 \\
&\cdots \\
\frac{2}{T+1}p_1 &= (1-p_1)(1-p_2)\cdots(1-p_{T-1})p_T \\
\frac{1}{T+1}p_1 &= (1-p_1)(1-p_2)\cdots(1-p_{T-1})(1-p_T).
\end{aligned}$$

The final step is to roll it up backwards again. Add the last and the second to last equation to get

$$\frac{3}{T+1}p_1 = (1-p_1)(1-p_2)\cdots(1-p_{T-1}). \tag{16.6}$$

In combination with the second to last equation and (16.5), which assures that we do not divide by zero, this immediately leads to

$$p_T = \frac{2}{3}.$$

Now start with equation (16.6) and first, add the third to last equation, and second, divide in a similar way. This yields first,

$$\frac{6}{T+1}p_1 = (1-p_1)(1-p_2)\cdots(1-p_{T-2}), \tag{16.7}$$

and second,

$$p_{T-1} = \frac{3}{6} = \frac{1}{2}.$$

Now do this again with equation (16.7) in combination with the fourth to last equation and get

$$\frac{10}{T+1}p_1 = (1-p_1)(1-p_2)\cdots(1-p_{T-3}),$$

and so

$$p_{T-2} = \frac{4}{10} = \frac{2}{5}.$$

This procedure stops when reaching the first equation, thereby generating the following sequence of probabilities:

$$\frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \frac{1}{4}, \ldots$$

It is easy to see that

$$\forall T \in \mathbb{N}, \forall k \in \{0, \ldots, T-1\} : p_{T-k} = q_{T-k} = \frac{2}{k+3}.$$

$\triangleleft$

16.8 Conclusion

Empirical evidence shows that matching behavior occurs in many decision theoretic situations with uncertainty. See, e.g., Davison and McCarthy (1988), Williams (1988), and Herrnstein (1997). In decision theoretic situations the uncertainty is caused by 'nature'. In a game theoretic framework players face uncertainty from a different source: the strategic behavior of their opponents.

A common explanation of matching behavior says that people do not believe that the mechanism, which produces the uncertainty, is genuinely random and therefore they may try to decipher the pattern. This explanation is particularly convincing in an interactive situation, where players are confronted with other players rather than nature. Consequently, matching behavior may play an even more important role in games. In this chapter we have analyzed an equilibrium concept for boundedly rational players which deals with matching behavior in interactive situations.

A main issue for future work would be to gather empirical evidence to determine the extent to which matching behavior actually occurs in strategic games. Another direction for future research is to make the concept applicable directly to games in extensive form, without seeking recourse to the reduced normal-form game, as was done in Section 16.7. Finally, in a slightly different set-up, assuming the existence of payoff functions, it is of interest to investigate if best-reply matching equilibria can be extended to a cardinal equilibrium concept that takes payoff differences into account.

Chapter 17

Random Games

17.1 Introduction

A common feature of interactive decisions with multiple criteria is that they are based on an aggregate of conflicts: conflicts between the players, but also between the criteria a specific player takes into account, i.e., the relevant characteristics by which a decision maker evaluates his strategic possibilities. This final chapter looks at an aggregate of games that arises through the uncertainty of players about the game being played.

Noncooperative games in which players have incomplete information about the characteristics of the participating players are commonly modelled using the Bayesian games of Harsanyi (1967-1968). The private information players have about the uncertainties is captured by possible different types of each player. A prior random move by nature is assumed to select each player's type. Underlying assumptions are that at least the number of players is commonly known — even though there may be uncertainty about their characteristics (types) — and that the action space of a player is the same, irrespective of the state of nature.

Although useful for modelling many practical conflict situations with incomplete information, there are interactive real-life situations in which players have to decide on their course of action before the exact number of players and/or their action spaces are known. Consider the following examples:

1. Suppose several animal species share a common area like a forrest. On a given day, it is not clear which animals and how many of them will be out in the field and for what purpose (moving to a different shelter, foraging/hunting, mating).

2. Each commuter going from home to work in the morning is uncertain about the number of other people participating in traffic and about their opportunities to travel to work, due to possible unannounced strikes in public transportation, suddenly defective cars, and road blocks caused by accidents.

3. The (partial) deregulation of markets causes uncertainty among incumbent firms

about the identity and number of potential market entrants. Moreover, although the market has been opened for other firms, there will typically still be some government interference. Firms will not be certain about the specific form this interference will take.

4. A generation of students has to choose an education, thus selecting certain career opportunities and ruling out others. None of the students knows in advance what the labor market at the moment of their graduation will look like: which type of jobs will be available and with whom they will compete for jobs.

5. A coach has to prepare a sports team for its next match. He has to decide on matters like training for defensive and offensive tactics, surprise maneuvers, and which players to employ in the match. Moreover, he has to do this *before* knowing the line-up of the opposing team and whether or not players in his team with special skills that are of crucial importance to specific types of play will be without injury at the time of the match.

This chapter introduces a model of 'random games' in which the actual game being played — its player set, the action sets of the involved players, as well as their preferences — is determined by a stochastic state of nature. It is assumed that all potential players have beliefs about these states of nature.

After introducing some notation and preliminary results in Section 17.2, the formal definition of random games is given in Section 17.3. Play of a random game proceeds as follows: a state of nature is realized and all potential players, unaware of this state of nature and consequently of the exact strategic game that corresponds with this state of nature, simultaneously and independently choose an action. The actual players that are selected to play the game that corresponds with the realized state of nature implement their action choices. In case of infeasible action choices, the game ends in an outcome that is not explicitly modelled. The random game and all these rules are common knowledge among the potential players.

Many problems in economics and operations research involve a planning stage, where decision makers have to plan their behavior under the uncertainty whether unforeseen contingencies will make their choices impossible to implement. This is the main feature that random games are meant to capture: players know they may be involved in the game corresponding to the stochastic state of nature and will have to take a single action in this randomly selected game. What, then, is the appeal at the planning stage of such profiles of single actions?

The potential players have no additional information on which to condition their action choice. Still, a certain profile of actions x may be better than another profile y in the sense that the probability that a state of nature arises in which x (restricted to the appropriate set of selected players) yields a Nash equilibrium is larger than the probability that a state of nature arises in which y yields equilibrium play. This leads us in Section 17.4 to define *maximum likelihood equilibria*, action profiles x which are such that the probability of ending up in a state where x restricted to the set of selected players yields a Nash equilibrium, is maximal. For this equilibrium concept an existence result is provided.

Borm, Cao, and García-Jurado (1995) consider a special class of Bayesian games, where the private information of the players is always the same, irrespective of the state of nature; equivalently, this can be seen as an incomplete information game in which the players have no private information. Instead of searching for its Bayesian equilibria, they introduce maximum likelihood equilibria in a similar vein to this chapter. Two significant differences are, firstly, that in the paper of Borm *et al.* (1995) there was no apparent need to introduce a new equilibrium concept, since they study a class of Bayesian games, whereas in the random games introduced in the present chapter it was motivated by players seeking to avoid possibly infeasible action choices, and, secondly, that the games of Borm *et al.* (1995) form only a small subclass of the random games in this chapter: they keep both the set of selected players and their action sets fixed.

17.2 Preliminaries

For clarity and easy reference, this section contains definitions of some standard notions which are used in the remainder of the chapter and lists some preliminary results.

Let A be a nonempty set. A *preference relation* on A is a complete, reflexive, and transitive binary relation $\succeq$ on A. If $a \succeq b$ and not $b \succeq a$, write $a \succ b$. We write $a \preceq b$ if $b \succeq a$. Assuming a given topology on A, a preference relation $\succeq$ is *continuous* if the existence of two sequences $(a^k)_{k=1}^{\infty}$ and $(b^k)_{k=1}^{\infty}$ in A with $\lim_{k\to\infty} a^k = a$, $\lim_{k\to\infty} b^k = b$, and $a^k \succeq b^k$ for all $k \in \mathbb{N}$ implies $a \succeq b$.

A *strategic game* is a tuple $\langle N, (A_i)_{i\in N}, (\succeq_i)_{i\in N}\rangle$, where

- N is a finite set of players;
- each player $i \in N$ has a set A_i of actions;
- each player $i \in N$ has a preference relation $\succeq_i$ over the set $\times_{j\in N} A_j$ of action profiles.

Let $a = (a_j)_{j\in N} \in \times_{j\in N} A_j$ be a profile of actions, $i \in N$, $b_i \in A_i$, and $S \subseteq N$. We sometimes write:

- $a_{-i} = (a_j)_{j\in N\setminus\{i\}}$ to indicate the action profile of i's opponents;
- (b_i, a_{-i}) to indicate the strategy profile in which player i chooses action b_i and his opponents $j \in N \setminus \{i\}$ action a_j;
- $a_{|S} = (a_j)_{j\in S}$ to indicate the action profile a restricted to the players in S.

An action profile $a = (a_j)_{j \in N} \in \times_{j \in N} A_j$ is a *Nash equilibrium* of the strategic game $\langle N, (A_i)_{i \in N}, (\succeq_i)_{i \in N} \rangle$ if

$$\forall i \in N, \forall b_i \in A_i : \quad a \succeq_i (b_i, a_{-i}),$$

i.e., if no player can achieve a better outcome by unilateral deviation.

Recall that $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers. For a set A, $2^A = \{B \mid B \subseteq A\}$ denotes the collection of all subsets of A, $\overline{A}$ denotes the closure of A, and A^c denotes the complement of A.

A set A contained in a finite dimensional Euclidean space, say $A \subseteq \mathbb{R}^n$, is *separable* if it contains a countable subset C such that $\overline{C} = A$. In this case, it is said that A is *separable through* C. Notice that a separable set is closed. For instance, $\mathbb{R}$ is separable through $\mathbb{Q}$, but $\mathbb{R} \setminus \mathbb{Q}$ is not separable.

Lemma 17.1 *Let $\langle N, (A_i)_{i \in N}, (\succeq_i)_{i \in N} \rangle$ be a strategic game. Assume that for each $i \in N$: $\succeq_i$ is continuous and $A_i \subseteq \mathbb{R}^{n(i)}$ is separable through C_i. Then:*

(a) The set of Nash equilibria is closed.

(b) For each $i \in N, a_{-i} \in \times_{j \in N \setminus \{i\}} A_j$, and $a_i \in A_i$:

$$\forall b_i \in A_i : \; (b_i, a_{-i}) \preceq_i (a_i, a_{-i}) \quad \Leftrightarrow \quad \forall b_i \in C_i : \; (b_i, a_{-i}) \preceq_i (a_i, a_{-i})$$

Proof.
(a) If the set of Nash equilibria is empty, it is closed by definition. If Nash equilibria exist, let $(a^k)_{k=1}^{\infty}$ be a sequence of Nash equilibria converging to a strategy combination $a \in \times_{i \in N} A_i$. Then a has to be a Nash equilibrium: Let $i \in N$ and $b_i \in A_i$. For each $k \in \mathbb{N}$, a^k is a Nash equilibrium, so $a^k \succeq_i (b_i, a^k_{-i})$. By continuity of $\succeq_i$: $a \succeq_i (b_i, a_{-i})$.

(b) (⇒) Trivial, since $C_i \subseteq A_i$.
(b) (⇐) Assume $(b_i, a_{-i}) \preceq_i (a_i, a_{-i})$ for all $b_i \in C_i$. Let $b_i \in A_i$. Since $\overline{C_i} = A_i$, there is a sequence $(b_i^k)_{k=1}^{\infty}$ in C_i converging to b_i. This is a sequence in the set $\{c_i \in A_i \mid (c_i, a_{-i}) \preceq_i (a_i, a_{-i})\}$, which is closed by continuity of $\preceq_i$. Hence its limit b_i is in this set: $(b_i, a_{-i}) \preceq_i (a_i, a_{-i})$. □

17.3 Random games

In this section, random games are formally introduced. In a random game, the actual strategic game that is played — its player set, the action sets of the involved players, as well as their preferences — is determined by a stochastic state of nature; the potential players are assumed to have common beliefs about these states of nature.

Formally, in a random game a stochastic variable taking values in a nonempty set Ω of 'states of nature' determines the actual game that is played. In a given play of the random game, a certain state of nature $\omega \in \Omega$ is realized. This state determines a strategic game, i.e., a set of players, a set of actions available to each of the players, and the preferences of the players over the action profiles. Let us discuss each of these in turn.

A finite, nonempty set U specifies the potential players. The players in U have common beliefs over the states of nature, described by a probability space (Ω, Σ, p), where Σ is a σ-algebra on Ω and p is a probability measure w.r.t. this σ-algebra. A function $N : \Omega \to 2^U$ determines the set of players. If $\omega \in \Omega$ is the state of nature, the set of players that is selected to play the game is $N(\omega) \subseteq U$.

A nonempty set A specifies the potential actions. If $\omega \in \Omega$ is the state of nature, each selected player $i \in N(\omega)$ has a set $A_i(\omega) \subseteq A$ of actions.

Given state ω, each player $i \in N(\omega)$ has preferences over the set of action profiles $\times_{j \in N(\omega)} A_j(\omega)$. But player $i \in N(\omega)$ has no precise information about the exact state of the world. To incorporate preferences over such uncertain situations, it is assumed that each player $i \in U$ has preferences over lotteries (probability distributions) on the outcomes and states in which he is selected to play. Formally, let $i \in U$ and let $\Omega(i) \subseteq \Omega$ denote the set of states in which player i participates in the game:

$$\forall i \in U : \quad \Omega(i) := \{\omega \in \Omega \mid i \in N(\omega)\}.$$

Then each player $i \in U$ is characterized by a continuous preference relation $\succeq_i$ over lotteries on $\bigcup_{\omega \in \Omega(i)} \Big(\big(\times_{j \in N(\omega)} A_j(\omega) \big) \times \{\omega\} \Big)$. If $\omega \in \Omega$ is the state of nature, then the preference relation $\succeq_{i,\omega}$ of player $i \in N(\omega)$ over the action space $\times_{j \in N(\omega)} A_j(\omega)$ is just his preference relation $\succeq_i$ restricted to the set $(\times_{j \in N(\omega)} A_j(\omega)) \times \{\omega\}$:

$$\forall \omega \in \Omega, \forall a, b \in \times_{j \in N(\omega)} A_j(\omega) : \quad a \ \succeq_{i,\omega} \ b \ \Leftrightarrow \ (a, \omega) \ \succeq_i \ (b, \omega).$$

The function that maps each state of nature $\omega \in \Omega$ to its associated strategic game, is called G:

$$G : \omega \mapsto \langle N(\omega), (A_i(\omega))_{i \in N(\omega)}, (\succeq_{i,\omega})_{i \in N(\omega)} \rangle.$$

The above is summarized in the following definition.

Definition 17.2 A *random game* consists of

- a nonempty set Ω of states;
- a probability space (Ω, Σ, p), where Σ is a σ-algebra on Ω and p a probability measure w.r.t. this σ-algebra, specifying the beliefs of the players over the states of nature;
- a nonempty, finite set U of potential players;

- a nonempty set A of potential actions;
- for each player $i \in U$ a continuous preference relation $\succeq_i$ over lotteries on

$$\bigcup_{\omega \in \Omega(i)} \Big(\big(\times_{j \in N(\omega)} A_j(\omega) \big) \times \{\omega\} \Big)$$

- a function G on Ω which maps each state $\omega \in \Omega$ to a game

$$\langle N(\omega), (A_j(\omega))_{j \in N(\omega)}, (\succeq_{j,\omega})_{j \in N(\omega)} \rangle,$$

  where $N(\omega) \subseteq U$ is nonempty, and for each $i \in N(\omega)$ the set $A_i(\omega) \subseteq A$ is nonempty and $\succeq_{i,\omega}$ equals $\succeq_i$ restricted to $(\times_{j \in N(\omega)} A_j(\omega)) \times \{\omega\}$:

$$\forall a, b \in \times_{j \in N(\omega)} A_j(\omega) : \quad a \succeq_{i,\omega} b \Leftrightarrow (a, \omega) \succeq_i (b, \omega).$$

A random game is denoted $\Gamma = \langle \Omega, \Sigma, p, U, A, (\succeq_j)_{j \in U}, G \rangle$. $\triangleleft$

Play of a random game $\Gamma = \langle \Omega, \Sigma, p, U, A, (\succeq_j)_{j \in U}, G \rangle$ proceeds as follows: a state of nature $\omega \in \Omega$ is realized and all players $i \in U$, unaware of this state of nature, simultaneously and independently choose an action from A. The players in $N(\omega)$ implement their action choices in the game $G(\omega)$. In case of infeasible action choices, the game ends in an outcome that is not explicitly modelled. The random game and all these rules are common knowledge.

We make the additional assumption that the set U of potential players contains no superfluous players. This is an innocent assumption: players that are not involved in any of the games $(G(\omega))_{\omega \in \Omega}$ are not taken into account. Formally this means that

$$\forall i \in U : \quad \Omega(i) = \{\omega \in \Omega \mid i \in N(\omega)\} \neq \emptyset,$$

or equivalently:

Assumption 1 $U = \cup_{\omega \in \Omega} N(\omega)$.

Secondly, we assume that there are only finitely many action sets that a potential player can be offered. Formally, let $i \in U$ and let

$$\mathcal{A}(i) := \{A_i(\omega) \mid \omega \in \Omega\}$$

denote the collection of action sets player i can be offered.

Assumption 2 For each $i \in U$ the set $\mathcal{A}(i)$ is finite.

17.4 Maximum likelihood equilibria

Although the random game and its rules of play are assumed to be common knowledge, no additional information is provided to the potential players. In particular, they are not informed about their action sets. This captures a common problem in decision making, namely the situation in which players or decision makers have to plan their course of action while still uncertain about contingencies that may make their choices impossible to implement.

Without being informed about actual action sets, potential players cannot condition their strategic choices on such information. Therefore, a strategy simply selects an action.

Definition 17.3 Let $\Gamma = \langle \Omega, \Sigma, p, U, A, (\succeq_j)_{j \in U}, G \rangle$ be a random game. A *strategy* of player $i \in U$ is an element $a_i \in A$. $\triangleleft$

This definition of a strategy is in line with the intuition: players know they may be involved in the game corresponding to the stochastic state of nature and in that case will have to take a single action in this randomly selected game. In their planning stage, players will ask themselves how to evaluate profiles of such single actions. Not every action of a given player is feasible in each of the games in which he is selected to play. It is therefore natural to consider strategy profiles which are *more likely* than others to yield equilibrium play in the random game. Maximum likelihood equilibria are the topic of this section.

Informally, a *maximum likelihood equilibrium* of a random game is a strategy profile $a = (a_j)_{j \in U} \in \times_{j \in U} A$ that maximizes the probability that a state of nature occurs in which profile a, restricted to the set of players selected to play the game, is a Nash equilibrium: it is a strategy profile that is most likely to yield equilibrium play.

Definition 17.4 Let $\Gamma = \langle \Omega, \Sigma, p, U, A, (\succeq_j)_{j \in U}, G \rangle$ be a random game and $a = (a_j)_{j \in U} \in \times_{j \in U} A$ a strategy profile. The *Nash equilibrium indicator* of a is the function $NI_a : \Omega \to \{0, 1\}$ with

$$NI_a(\omega) = \begin{cases} 1 & \text{if } a_{|N(\omega)} \text{ is a Nash equilibrium of } G(\omega), \\ 0 & \text{otherwise.} \end{cases}$$

(Recall that $a_{|N(\omega)}$ is the strategy profile a restricted to the players in $N(\omega)$.) $\triangleleft$

Notice that a strategy profile $a \in \times_{j \in U} A$ may fail to be a Nash equilibrium of $G(\omega)$ for two reasons:

1. $a_{|N(\omega)}$ is feasible in $G(\omega)$, but there is a player $i \in N(\omega)$ with a profitable deviation,
2. one of the players $i \in N(\omega)$ plays an infeasible strategy: $a_i \notin A_i(\omega)$.

Let $a \in \times_{j \in U} A$ be a strategy profile. An event of interest is $\{\omega \in \Omega \mid NI_a(\omega) = 1\}$, the set of states in which the strategy profile a yields a Nash equilibrium. To make sure that this is a measurable set, i.e., an element of the σ-algebra Σ, we make some further assumptions.

We assume that A contains no redundant elements. Just like Assumption 1, this entails no loss of generality.

Assumption 3 $A = \cup_{j \in U} \cup_{A_j \in \mathcal{A}(j)} A_j$.

Each possible action set is required to be separable. This is formulated in a slightly different, yet equivalent way.

Assumption 4 $A \subseteq \mathbb{R}^n$ is separable through $C \subseteq A$ and for each $i \in U$ and each $A_i \in \mathcal{A}(i)$: A_i is separable through $A_i \cap C$.

Proposition 17.5 *Under Assumptions 2 and 3, Assumption 4 is equivalent with the following statement:*

$$\text{For each } i \in U \text{ and each } A_i \in \mathcal{A}(i): \quad A_i \text{ is separable.} \tag{17.1}$$

Proof. Assumption 4 implies (17.1). To see that (17.1) implies assumption 4, recall that each $\mathcal{A}(i)$ is finite by Assumption 2. Write $\mathcal{A}(i) = \{A_{i1}, A_{i2}, \ldots, A_{im(i)}\}$, where $m(i) \in \mathbb{N}$ is the number of elements of $\mathcal{A}(i)$. Assume that A_{ik} is separable through $C_{ik} \subseteq A_{ik}$. Then $\cup_{i \in U} \cup_{k=1}^{m(i)} C_{ik}$ is a finite union of countable sets, so countable, a subset of A, and

$$\overline{\cup_{i \in U} \cup_{k=1}^{m(i)} C_{ik}} = \cup_{i \in U} \cup_{k=1}^{m(i)} \overline{C_{ik}} = \cup_{i \in U} \cup_{k=1}^{m(i)} A_{ik} = A,$$

finishing the proof. □

The fifth assumption concerns the measurability of the functions N, $(A_i)_{i \in U}$, and $(\succeq_i)_{i \in U}$.

Assumption 5 The following measurability conditions hold:

(a; on N) $\forall S \subseteq U: \quad \{\omega \in \Omega \mid N(\omega) = S\} \in \Sigma$.

(b; on A_i) $\forall i \in U, \forall b_i \in A: \quad \{\omega \in \Omega \mid i \in N(\omega), b_i \in A_i(\omega)\} \in \Sigma$.

(c; on $\succeq_i$) $\forall i \in U, \forall b_i \in A, \forall a \in \times_{j \in U} A:$

$$\{\omega \in \Omega \mid a_{|N(\omega)} \in \times_{j \in N(\omega)} A_j(\omega),\ i \in N(\omega),\ b_i \in A_i(\omega),\ a_{|N(\omega)} \succeq_{i,\omega} (b_i, a_{-i})_{|N(\omega)}\} \in \Sigma.$$

Lemma 17.6 *Let $i \in U, b_i \in A, a \in \times_{j \in U} A$. Define three sets:*

$$\begin{aligned} X(i) &= \{\omega \in \Omega \mid i \notin N(\omega)\} \\ Y(i, b_i) &= \{\omega \in \Omega \mid i \in N(\omega), b_i \notin A_i(\omega)\} \\ Z(i, b_i, a) &= \end{aligned}$$

$$\{\omega \in \Omega \mid a_{|N(\omega)} \in \times_{j \in N(\omega)} A_j(\omega),\ i \in N(\omega),\ b_i \in A_i(\omega),\ a_{|N(\omega)} \succeq_{i,\omega} (b_i, a_{-i})_{|N(\omega)}\}$$

The following claims hold:

(a) *Assumption 5a is equivalent with the assumption that* $\forall i \in U : \{\omega \in \Omega \mid i \in N(\omega)\} \in \Sigma$*. Therefore* $X(i) \in \Sigma$*.*

(b) $Y(i, b_i) \in \Sigma$*.*

(c) $Z(i, b_i, a) \in \Sigma$*.*

Proof.

(a) Let $i \in U$. To see that assumption 5a implies $\{\omega \in \Omega \mid i \in N(\omega)\} \in \Sigma$, rewrite

$$\{\omega \in \Omega \mid i \in N(\omega)\} = \cup_{S \subseteq U : i \in S} \{\omega \in \Omega \mid N(\omega) = S\}.$$

This is a finite union of measurable sets, hence measurable. To see the converse, let $S \subseteq U$ and write

$$\{\omega \in \Omega \mid N(\omega) = S\} = [\cap_{i \in S} \{\omega \in \Omega \mid i \in N(\omega)\}] \ \bigcap \ [\cap_{i \in U \setminus S} \{\omega \in \Omega \mid i \notin N(\omega)\}].$$

This is a finite intersection of measurable sets, hence measurable.

That $X(i) \in \Sigma$ follows from the first part of the proof.

(b) Write

$$Y(i, b_i) = [\{\omega \in \Omega \mid i \notin N(\omega)\} \ \cup \ \{\omega \in \Omega \mid i \in N(\omega), b_i \in A_i(\omega)\}]^c.$$

As the complement of the finite union of measurable sets, this is a measurable set.

(c) This is simply Assumption 5c. □

Proposition 17.7 *Let* $\Gamma = \langle \Omega, \Sigma, p, U, A, (\succeq_j)_{j \in U}, G \rangle$ *be a random game that satisfies Assumptions 1, 2, 3, 4, and 5. Let* $a \in \times_{j \in U} A$ *be a strategy profile. Then the set* $\{\omega \in \Omega \mid NI_a(\omega) = 1\}$ *is an element of the* σ*-algebra* Σ*.*

Proof. Define the sets $X(i)$, $Y(i, b_i)$, and $Z(i, b_i, a)$ as in Lemma 17.6 and let C be as in Assumption 4. Then

$$\begin{aligned}
\{\omega \in \Omega \mid NI_a(\omega) = 1\} \quad &= \\
\{\omega \in \Omega \mid a_{|N(\omega)} \in \times_{j \in N(\omega)} A_j(\omega) \text{ and} \\
\forall i \in N(\omega), \forall b_i \in A_i(\omega) : \ a_{|N(\omega)} \ \succeq_{i,\omega} \ (b_i, a_{-i})_{|N(\omega)}\} \quad &= \\
\{\omega \in \Omega \mid a_{|N(\omega)} \in \times_{j \in N(\omega)} A_j(\omega) \text{ and} \\
\forall i \in N(\omega), \forall b_i \in A_i(\omega) \cap C : \ a_{|N(\omega)} \ \succeq_{i,\omega} \ (b_i, a_{-i})_{|N(\omega)}\} \quad &= \qquad (17.2) \\
\cap_{i \in U} \cap_{b_i \in C} \left[X(i) \cup Y(i, b_i) \cup Z(i, b_i, a) \right]. & \qquad (17.3)
\end{aligned}$$

The first equality follows from the definition of a Nash equilibrium, the second equality follows from Assumption 4 and Lemma 17.1. Remains to show the third equality.

Let ω be in the set in (17.2), $i \in U$, and $b_i \in C$. There are three possibilities:

1. If $i \notin N(\omega)$, then $\omega \in X(i)$;

2. If $i \in N(\omega)$ and $b_i \notin A_i(\omega)$, then $\omega \in Y(i, b_i)$;

3. If $i \in N(\omega)$ and $b_i \in A_i(\omega)$, then $a_{|N(\omega)} \in \times_{j \in N(\omega)} A_j(\omega)$ and $a_{|N(\omega)} \ \succeq_{i,\omega} \ (b_i, a_{-i})_{|N(\omega)}$ by assumption, so $\omega \in Z(i, b_i, a)$.

Hence $\omega \in X(i) \cup Y(i, b_i) \cup Z(i, b_i, a)$. Since $i \in U$ and $b_i \in C$ were taken arbitrarily, this proves that the set in (17.2) is contained in the set in (17.3).

Now let $\omega \in \cap_{i \in U} \cap_{b_i \in C} [X(i) \cup Y(i, b_i) \cup Z(i, b_i, a)]$, $i \in N(\omega)$, and $b_i \in A_i(\omega) \cap C$. Since $\omega \in X(i) \cup Y(i, b_i) \cup Z(i, b_i, a)$ and $\omega \notin X(i) \cup Y(i, b_i)$, it follows that $\omega \in Z(i, b_i, a)$, which implies that $a_{|N(\omega)} \in \times_{j \in N(\omega)} A_j(\omega)$ and $a_{|N(\omega)} \ \succeq_{i,\omega} \ (b_i, a_{-i})_{|N(\omega)}$. Since $i \in N(\omega)$, and $b_i \in A_i(\omega) \cap C$ were taken arbitrarily, this proves that the set in (17.3) is contained in the set in (17.2). This establishes the third equality.

Since each of the three sets $X(i)$, $Y(i, b_i)$, and $Z(i, b_i, a)$ is measurable by Lemma 17.6, their union is measurable. Since U is finite and C is countable, the set in (17.3) is a countable intersection of measurable sets and hence measurable. □

Definition 17.8 Let $\Gamma = \langle \Omega, \Sigma, p, U, A, (\succeq_j)_{j \in U}, G \rangle$ be a random game that satisfies assumptions 1, 2, 3, 4, and 5. Define the *likelihood function* $\mathcal{L} : \times_{j \in U} A \to [0, 1]$ by $\mathcal{L}(a) = p(\{\omega \in \Omega \mid NI_a(\omega) = 1\})$. A strategy profile $a \in \times_{j \in U} A$ is a *maximum likelihood equilibrium* of the random game if it maximizes $\mathcal{L}$:

$$\forall b \in \times_{j \in U} A : \quad \mathcal{L}(a) \geqq \mathcal{L}(b).$$

The set of maximum likelihood equilibria of Γ is denoted $MLE(\Gamma)$. ◁

Proposition 17.7 shows that the function $\mathcal{L}$ is well-defined.

Proposition 17.9 *Let $\Gamma = \langle \Omega, \Sigma, p, U, A, (\succeq_j)_{j \in U}, G \rangle$ be a random game that satisfies assumptions 1, 2, 3, 4, and 5. Let $(a^n)_{n=1}^\infty$ be a sequence in $\times_{j \in U} A$ with limit $a^0 \in \times_{j \in U} A$. If the sequence $(\mathcal{L}(a^n))_{n=1}^\infty$ converges, then $\mathcal{L}(a^0) \geqq \lim_{n \to \infty} \mathcal{L}(a^n)$.*

Proof. Define for each $n \in \mathbb{N}_0$:

$$B^n := \{\omega \in \Omega \mid NI_{a^n}(\omega) = 0\}.$$

The complement of B^n is $\{\omega \in \Omega \mid NI_{a^n}(\omega) = 1\}$ and was shown to be measurable in Proposition 17.7, so B^n is measurable.

If $B^0 = \emptyset$, the statement in the proposition is trivial, since then $\mathcal{L}(a^0) = p(\Omega) = 1 \geqq \lim_{n \to \infty} \mathcal{L}(a^n)$. So take $\omega \in B^0$. By definition, a^0 is not a Nash equilibrium of

$G(\omega)$. The set of Nash equilibria of $G(\omega)$ is closed by Lemma 17.1. Hence, there exists a neighborhood $\mathcal{O}$ of a^0 such that

$$\forall a \in \mathcal{O}: \quad NI_a(\omega) = 0. \tag{17.4}$$

Convergence of $(a^n)_{n=1}^{\infty}$ to a^0 implies

$$\exists k \in \mathbb{N} \text{ such that } \forall n \geqq k: \quad a^n \in \mathcal{O}. \tag{17.5}$$

Combining (17.4) and (17.5):

$$\exists k \in \mathbb{N} \text{ such that } \forall n \geqq k: \quad NI_{a^n}(\omega) = 0.$$

Since $\omega \in B^0$ was arbitrary, it follows that

$$\forall \omega \in B^0 \ \ \exists k(\omega) \in \mathbb{N} \text{ such that } \forall n \geqq k(\omega): \ \ \omega \in B^n.$$

This implies that

$$B^0 \subseteq \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} B^n = \liminf_{n\to\infty} B^n.$$

Hence

$$1 - \mathcal{L}(a^0) = p(B^0) \leqq p(\liminf_{n\to\infty} B^n) = \liminf_{n\to\infty} p(B^n) = 1 - \lim_{n\to\infty} \mathcal{L}(a^n),$$

so $\mathcal{L}(a^0) \geqq \lim_{n\to\infty} \mathcal{L}(a^n)$. □

This completes the preliminary work for the existence theorem of maximum likelihood equilibria. One additional assumption is made.

Assumption 6 A is a compact set.

Theorem 17.10 *Let $\Gamma = \langle \Omega, \Sigma, p, U, A, (\succeq_j)_{j\in U}, G\rangle$ be a random game that satisfies assumptions 1, 2, 3, 4, 5, and 6. Then Γ has a maximum likelihood equilibrium: $MLE(\Gamma) \neq \emptyset$.*

Proof. The strategy space $\times_{j\in U} A$ is compact in the product topology. The set $\{\mathcal{L}(a) \mid a \in \times_{j\in U} A\}$ is nonempty and bounded above by one, so it has a supremum M. Hence, there is a sequence $(a^n)_{n=1}^{\infty}$ in $\times_{j\in U} A$ with $\lim_{n\to\infty} \mathcal{L}(a^n) = M$. By compactness of $\times_{j\in U} A$, the sequence has a convergent subsequence (a^{n_k}) with limit a^0. From Proposition 17.9 it follows that

$$\mathcal{L}(a^0) \geqq \lim_{k\to\infty} \mathcal{L}(a^{n_k}) = M$$

Then a^0 is a maximum likelihood equilibrium of Γ. □

A random game as in Theorem 17.10 satisfies assumptions 1 through 6. Let us have a closer look at the assumptions and the role they play in the existence result for maximum likelihood equilibria.

As observed before, assumptions 1 and 3 are made without loss of generality: they just get rid of unnecessary ingredients.

Assumptions 4 and 6, concerning separability and compactness of A are topological conditions on the set of potential actions. Compactness is a standard assumption. Separability is only a weak condition. Typical examples of action sets that come to mind are strategy simplices (probability distributions over finitely many pure strategies), an interval $[0, \infty)$ of prices, or a subset of $\mathbb{N}$ denoting possible quantities (for instance of production). All these sets are separable, so Proposition 17.5 would imply separability of A.

Assumption 2, concerning the finiteness of the sets $\mathcal{A}(i)$ was used in the proof of Proposition 17.5.

Assumption 5 concerns the measurability of certain sets and was needed to guarantee the measurability of the set of states on which a certain strategy profile yielded a Nash equilibrium.

Remark 17.11 Random games were defined such that its players have a common prior p. This prior was used to define a likelihood function $\mathcal{L}$ that measures for all players simultaneously how likely it is, according to this prior, that a selected strategy profile yields equilibrium play in the random game. Without imposing the restriction that the players have a common prior, each player would have his own likelihood function, not necessarily coinciding with that of the others. In such a case, there will typically not be a strategy profile that simultaneously maximizes the likelihood function of all players, thereby making it difficult to single out a desirable strategy profile. $\triangleleft$

Borm *et al.* (1995) consider random games $\Gamma = \langle \Omega, \Sigma, p, U, A, (\succeq_j)_{j \in U}, G \rangle$ that satisfy the following structural assumptions:

- The player set is fixed: $\forall \omega \in \Omega : \; N(\omega) = U$.
- The action sets are fixed: $\forall i \in U \; \exists A_i \subseteq A \; \forall \omega \in \Omega : \;\; A_i(\omega) = A_i$.

This clearly implies that assumptions 1 and 2 are satisfied. Moreover, they require each A_i to be separable and $\times_{i \in U} A_i$ to be compact. Notice that this coincides with our assumptions 3, 4, and 6. Assumptions 5a and 5b are automatically fulfilled. Borm *et al.* (1995) also capture assumption 5c by requiring that for each $i \in U$, each $a \in \times_{j \in U} A_j$, and each $b_i \in A_i$ the set $\{\omega \in \Omega \mid a \succeq_{i,\omega} (b_i, a_{-i})\}$ is measurable.

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Samenvatting

Speltheorie is een wiskundige theorie die methodes ontwikkelt en gebruikt voor het bestuderen van de interactie tussen beslissers. Dit proefschrift behandelt een tweetal onderwerpen uit de speltheorie. In het eerste deel wordt gekeken naar potentiaalspelen; het tweede deel van het proefschrift handelt over multicriteriaspelen.

Potentiaalspelen zijn niet-coöperatieve spelen — spelen waarin de spelers onderling geen bindende afspraken kunnen maken — waar de informatie over de strategische mogelijkheden van alle betrokken spelers tegelijk kan worden samengevat in een enkele reëelwaardige functie op de strategieënruimte van het spel. Deze strategische informatie bestaat eruit hoe de uitbetaling van een speler verandert wanneer deze afwijkt van zijn huidige strategie, hierbij de keuzes van de overige spelers constant veronderstellend. Een hogere uitbetaling gaat gepaard met een hogere waarde van de potentiaalfunctie. Strategieëncombinaties waar de potentiaalfunctie een maximum aanneemt, zijn evenwichten van het spel: geen enkele speler kan door afwijken een hogere uitbetaling realiseren, omdat dit gepaard zou gaan met een toename in de potentiaal, die echter bij aanname maximaal is. Een belangrijke eigenschap van potentiaalspelen is dus, dat potentiaalspelen waarin er maar eindig veel zuivere strategieëncombinaties zijn, zuivere evenwichten hebben: evenwichten waarin elke speler een eenvoudige zuivere strategie kan spelen en niet zijn toevlucht hoeft te zoeken tot het gebruik van gerandomizeerde strategieën.

Er zijn diverse typen potentiaalspelen. Een belangrijke vraag is naar de structuur van deze spelen: wat zijn noodzakelijke en voldoende voorwaarden voor het bestaan van een bepaald type potentiaalfunctie? Dit onderwerp komt aan bod in hoofdstukken 2, 5, 7 en 9. De relatie tussen deze hoofdstukken is dat in alle gevallen een conditie op cykels in de strategieënruimte van het spel van centraal belang is. De vorm van deze conditie is misschien het best te begrijpen aan de hand van de litho *Klimmen en dalen* uit 1960 van Maurits C. Escher. In deze litho kunnen we een groep monniken op een trap volgen, die bij iedere stap een traptrede omhoog gaan, maar desondanks na verloop van tijd terugkeren naar een punt waar ze al eerder zijn geweest. Het uitsluiten van soortgelijke 'stijgende cykels' is nodig om het bestaan van een potentiaal in een niet-coöperatief spel te garanderen.

In eindige potentiaalspelen leveren maxima van een potentiaalfunctie evenwichten in zuivere strategieën. Zuivere evenwichten hoeven niet te bestaan in potentiaalspelen waarin spelers oneindig veel strategieën hebben. Er kunnen evenwel situaties zijn waarin spelers een hoge uitbetaling krijgen, waarmee ze tevreden zijn, of door af te wijken van hun keuze er maar in geringe mate op vooruit kunnen gaan. Het bestaan van zulke bijna-evenwichten wordt bestudeerd in hoofdstuk 8, waar blijkt dat de aanwezigheid van hooguit één speler met een oneindige verzameling zuivere strategieën voldoende is om het bestaan van bijna-evenwichten te garanderen.

Toepassingen van potentiaalspelen komen aan bod in hoofdstukken 2, 3, 4 en 6. In hoofdstukken 2 en 3 worden congestiemodellen bestudeerd, waar spelers gebruik maken van verschillende faciliteiten en het nut dat ze aan dit gebruik ontlenen afhangt van het aantal andere gebruikers. Hoofdstuk 4 bestudeert processen waarin produktie van goederen in verschillende stappen plaatsvindt en de kosten van een produktie-afdeling uitsluitend afhangen van de gekozen produktietechnieken van afdelingen die bij het produktieproces in voorgaande stappen of dezelfde stap betrokken zijn. Hoofdstuk 6 behandelt een methode voor het financieren van publieke goederen met de eigenschap dat spelers, die individuele bijdragen leveren aan de financiering van de publieke goederen, door het nastreven van hun eigen belang tegelijkertijd handelen in het belang van de sociale welvaart, de welvaart van de gehele groep betrokkenen. Bovendien legt dit hoofdstuk relaties tussen het niet-coöperatieve probleem en een coöperatief probleem, waar spelers in samenwerking een bijdrage leveren aan door hen te financieren en te kiezen publieke goederen.

Bij het nemen van beslissingen evalueert een beslisser situaties over het algemeen aan de hand van verschillende criteria. Deze criteria kunnen moeilijk met elkaar te vergelijken zijn en het kan voorkomen dat een situatie die aantrekkelijk is volgens één criterium onaantrekkelijk is volgens een ander criterium. Het tweede deel van het proefschrift handelt over de hiermee samenhangende multicriteriaspelen, waarin spelers meerdere criteria tegelijkertijd hanteren. Een speler in een multicriteriaspel kan bijvoorbeeld worden gezien als een organisatie met verschillende leden, ieder met een eigen doelfunctie. Gegeven deze interpretatie van een speler als een organisatie ontstaat er een aggregatie van conflicten, enerzijds *tussen* de verschillende organisaties, anderzijds *binnen* een organisatie, waar de leden gezamenlijk moeten beslissen over een strategieënkeuze. Dit is een gemeenschappelijk kenmerk van alle spelen in het tweede deel van het proefschrift: ze zijn gebaseerd op een opeenstapeling van conflicten.

Het evenwichtsconcept voor niet-coöperatieve spelen vereist dat spelers een beste antwoord spelen op de strategieën van de tegenstanders. Als een speler één reëelwaardige criteriumfunctie heeft, is een beste antwoord ondubbelzinnig gedefinieerd als een strategie die zodanig is dat een afwijking daarvan geen hoger nut kan opleveren. Maar als een speler meerdere criteria tegelijkertijd hanteert, is het niet zo duidelijk wat een beste antwoord is. Verschillende antwoorden op deze vraag leveren verschillende evenwichtsconcepten op voor multicriteriaspelen. Dit onderwerp komt aan bod in hoofdstukken 11 en 13. In de Pareto-evenwichten in hoofdstuk 11 zijn beste antwoorden gedefinieerd als strategieën die een Pareto-optimale uitbetaling leveren. In dit hoofdstuk worden eigenschappen van het Pareto-evenwicht beschreven en axiomatiseringen van het Pareto-evenwichtsconcept gegeven. Hoofdstuk 12 bestudeert de structuur van de verzameling Pareto-evenwichten in twee-persoons multicriteriaspelen. Hoofdstuk 13 introduceert een drietal evenwichtsconcepten, waarbij aspecten aan bod komen uit de multicriteria optimalisering, de niet-coöperatieve en de coöperatieve speltheorie. In een compromis-evenwicht probeert elke speler een uitkomst te realiseren die zo dicht mogelijk ligt bij een ideale uitkomst. Dit concept is nauw gerelateerd aan de compromiswaarden uit de literatuur over multicriteria optimalisering. In Nash bargaining evenwichten, gerelateerd aan de speltheoretische literatuur over bargaining, proberen spelers juist een onderhandelingsoplossing te genereren die ver van een onaangename oplossing verwijderd is. Perfecte evenwichten, geïnspireerd door de literatuur over evenwichtsverfijningen, tenslotte, vormen een verfijning van het Pareto-evenwichtsconcept en houden rekening met het feit dat spelers fouten kunnen maken bij het uitvoeren van hun keuzes.

Hoofdstuk 14 bekijkt Pareto-optimal security strategies in twee-persoons nulsomspelen met meerdere criteria. Pareto-optimal security strategies zijn 'veilige' strategieën in de zin dat een speler voor elk van zijn strategieën nagaat wat het ergste is wat hem kan overkomen in ieder van zijn criteria afzonderlijk. Zo kent een speler aan elke strategie een 'security vector' toe die het worst-case scenario beschrijft als deze strategie wordt gekozen. Een Pareto-optimal security strategy is een strategie waarvoor dit worst-case scenario het minst onaangenaam is. Verschillende karakteriseringen van Pareto-optimal security strategies worden gegeven. In het bijzonder wordt aangetoond dat ze samenvallen met minimax strategieën in een standaard twee-persoons matrixspel, waar elk van de spelers maar één criterium heeft.

Coöperatieve multicriteriaspelen worden bestudeerd in hoofdstuk 15. Er wordt onderscheid gemaakt tussen ondeelbare, publieke criteria, die voor elke speler binnen een coalitie dezelfde waarde aannemen, en deelbare, private criteria, waarvan de waarde over de leden van een coalitie verdeeld kan worden. De nadruk wordt gelegd op een core concept voor coöperatieve multicriteriaspelen, bestaande uit allocaties voor de afzonderlijke spelers met de eigenschap dat geen enkele coalitie van spelers een incentive heeft om de voorgestelde allocatie naast zich neer te leggen, omdat ze zelf geen betere uitkomst kunnen garanderen. Dit concept wordt geaxiomatiseerd en additionele motivatie voor het concept wordt gegeven door aan te tonen dat core elementen op natuurlijke wijze samenvallen met sterke evenwichten in gerelateerde niet-coöperatieve claim spelen, waar de spelers onafhankelijk een coalitie noemen die ze willen vormen en een uitbetaling die ze willen.

Het feit dat een speler meerdere criteria hanteert bij het evalueren van uitkomsten, impliceert dat hij vaak alleen een partiële ordening op de uitkomsten kan aanbrengen. Een niet-coöperatief spel waarin elke speler een partiële ordening heeft over de strategieënruimte wordt een ordinaal spel genoemd. Hoofdstuk 16 introduceert een nieuw model voor beperkt rationeel gedrag in ordinale spelen. Het model benadrukt de rol van beste antwoorden. Als een speler na afloop van het spel vaststelt dat hij geen beste antwoord speelde op de keuzes van de tegenstanders, kan hij spijt hebben van het maken van een onjuiste keuze. De anticipatie van spijt kan het beslissingsproces en de daaruit voortvloeiende keuze van een speler beïnvloeden. Hoofdstuk 16 stelt matching voor als een mogelijke manier waarop deze invloed gestalte kan krijgen. Matching wordt waargenomen in talloze experimenten over beslissen onder onzekerheid en komt er in essentie op neer dat een alternatief wordt gekozen met een kans die proportioneel is aan de waarde van dat alternatief. In ordinale spelen is best-reply matching gebaseerd op twee elementen. Ten eerste, elke speler is uitsluitend geïnteresseerd in de beste-antwoord structuur van het spel. Ten tweede, elke speler speelt een zuivere strategie met de kans dat deze strategie een beste antwoord zal zijn. De kansen komen voort uit de beliefs van een speler met betrekking tot het gedrag van zijn tegenstanders. Een evenwichtsconcept op basis van best-reply matching wordt gedefinieerd en er wordt aangetoond dat elk eindig ordinaal spel een best-reply matching evenwicht heeft. Enige eigenschappen van het nieuwe evenwichtsconcept worden onderzocht en het concept wordt toegelicht aan de hand van bekende voorbeelden. Een opvallend resultaat wordt gevonden in het Centipede spel. In het unieke best-reply matching evenwicht van dit spel zetten de spelers het spel voort met positieve kans, waarbij deze kans groter is naarmate het spel langer is.

In veel problemen in de economie en operations research moeten individuen hun gedrag plannen onder de onzekerheid of bepaalde omstandigheden het implementeren van hun keuze onmogelijk maken. Hoofdstuk 17 introduceert dit probleem in een speltheoretische context door het formuleren van random games. Random games laten onzekerheid toe over alle ingrediënten van het spel: de verzameling spelers, de acties en de voorkeuren van de betrokken spelers. Het voorgestelde evenwichtsconcept voor random games, maximum likelihood evenwichten, selecteert die acties, die het meest waarschijnlijk zijn om in het uiteindelijk gespeelde spel een goede uitkomst op te leveren.