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Autonomous and induced learning: an optimal control approach

Steffen Jørgensen* and Peter M. Kort†

*Department of Organization and Management, University of Southern Denmark, Odense University, Campusvej 55, DK-5230, Odense M, Denmark

†Department of Econometrics and Center, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

Abstract: The paper considers a production and process improvement problem of a firm in which the total manufacturing costs of a specific product depend on the output rate as well as the effects of autonomous and induced learning. Autonomous learning typically results from the repetition of a particular task, whereas induced learning is a result of explicit investments in production process improvements. The cumulative effects of the learning processes are represented by two stocks of knowledge. The stock of autonomous knowledge is built up at a rate corresponding to current output. Induced knowledge is built up by process improvement investments. Both stocks are subject to decay, due to obsolescence or forgetting of knowledge. For short and long term planning situations, we study the optimal evolution over time of the output rate, the process improvement expenditures rate and the two stocks of knowledge.

Keywords: production, autonomous and induced learning, cost reduction, optimal control.

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1 Introduction

The paper deals with productivity increases and cost reductions that result from learning processes. The concept of learning in production is well established in management theory and practice. It means that increased experience in the manufacturing of a product leads to higher productivity, better quality, lower unit production costs, lower set-up costs, or other performance improvements.

Following Rosenberg [1], who states that a sizeable portion of productivity growth takes the form of a slow accretion of individually small improvements and innovations, our focus will be on gradual changes in labour productivity, rather than infrequent adoptions of innovations to enhance productivity. We suppose that these gradual changes are the product of two sources of learning within the organization. *Autonomous* learning is a consequence of producing *per se* and hence requires little explicit managerial influence. *Induced* learning is the result of deliberate (management induced) investments in production process improvements. The manufacture of automobiles is an example of production processes that have benefited from the autonomous learning effects of mass production as well as induced learning (e.g. due to various 'Japanese' process improvement methods).

To measure quantitatively the benefits of learning processes, one uses a learning curve to provide a mathematical relationship between a specific performance measure and the firm's experience in manufacturing the product. Traditionally, cumulative output has been used as a proxy for autonomous learning. The assumption here is that labour productivity, in particular, increases as output accumulates. These productivity improvements translate into lower costs of manufacturing. To formalize the idea, let $Z(t)$ denote a firm's cumulative output of a product by time t and $c(t)$ the unit cost of production. The relationship between production experience and cost is given by a cost learning curve $c = h(Z)$, $h'(Z) < 0$. Thus, the unit manufacturing cost decreases as experience accumulates. Popular choices for the cost function have been the power and exponential laws, but learning curve theory often assumed that the parameters of these laws were exogenously fixed. Zangwill and Kantor observed that 'traditional learning curve theory offers no organized way for management to improve the slope of the learning curve so that learning ... occurs faster' ([2], p. 912). But even in the fixed parameter set-up, management has an opportunity to influence the rate of learning; one can deliberately choose to operate at high output levels to obtain a fast decrease of the unit cost of production.* This type of production policy has often been observed for new product innovations, e.g. in electronic products industries.

Apart from expanding output to generate the traditional learning by doing effects, management has other instruments to influence the accumulation of production experience. Here the distinction between autonomous and induced learning is useful [5]. As mentioned, autonomous learning typically results from the repetition of a certain task such that learning improves the ability to perform the task ('practice makes perfect'). These benefits can be reaped without very much explicit management action and their cost effects are modelled by the learning curve $c = h(Z)$. In addition to its descriptive purposes, the autonomous learning curve has been used in dynamic optimization problems in production and marketing [3,4,6].

Induced learning, on the other hand, should be viewed as a result of explicit managerial actions by which manufacturing processes are changed to augment the capabilities of the workers and enhance the efficiency of the production system. This type of learning has been considered in descriptive models [5] and in dynamic optimization problems [7–11]. Arrow [12] suggested to use cumulative investments in induced learning as a proxy for this type of knowledge. This approach is similar to the one used in human capital accumulation models [7,13,14].

With respect to prescription, there are only a few studies of the problem of *simultaneous* autonomous and induced learning [8,10]. Being particularly interested in quality, these authors consider two types of knowledge: productivity knowledge and quality knowledge. In this paper we look at learning in a different way and introduce two stocks of knowledge: knowledge which is built up by continued

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* Clearly, an output policy has to satisfy other objectives than the mere reduction of unit manufacturing costs. Thus, the policy must be coordinated with pricing, promotion, inventory, and distribution strategies, cf. Clarke *et al.* [3], Jørgensen *et al.* [4].

production (autonomous knowledge) and knowledge acquired through explicitly designed process improvement efforts (induced knowledge).

The paper proceeds as follows. In Section 2 we propose a dynamic optimization model in which management can influence the development of both types of learning through its choice of production volume and process improvement efforts and where the total manufacturing costs depend on output and the two stock levels. Using optimal control theory, Section 3 derives the solution of the problem. This section also provides the managerial prescriptions that can be inferred from the mathematical analysis. Section 4 concludes.

2 Dynamic optimization problem

The firm's efforts to improve the production of a particular product are reflected in the process improvement expenditure rate, denoted by $u(t)$. Let $K(t)$ denote the stock of induced knowledge accumulated by time t. Spence [7] used the dynamics $\dot{K}(t) = dK(t)/dt\ u(t)$, but here we wish to allow for decay of the stock $K(t)$. Experience in production can be forgotten or lost, due to factors such as employee turnover and technological obsolescence. Moreover, we wish to modify the assumption of constant marginal effects of process improvement expenditures that underlies the model $\dot{K}(t) = u(t)$. The dynamics for the stock $K(t)$ are given by (see also Harti [14])

$$\dot{K}(t) = f(u(t), K(t) - aK(t), K(0) = K_0 \geq 0 \tag{1}$$

in which $a = \text{const.} > 0$ is the decay rate of knowledge. Function f is twice continuously differentiable on the open set $\{u > 0,\ K > 0\}$ and satisfies

$$\begin{aligned} &f(u, \cdot) > 0\ \forall u > 0, f(0) = 0, f_u(0,) = +\infty \\ &f_u > 0, f_{uu} < 0, f_K < 0, f_{KK} < 0, f_{uK} < 0, f_{uu}f_{KK} - (f_{uK})^2 > 0 \end{aligned} \tag{2}$$

Here and in the sequel, a variable appearing as a subscript, denotes partial differentiation with respect to that variable.

Since we wish to study the impacts of process improvement expenditures on the firm's intertemporal development, situations in which management choose not to influence its stock of induced knowledge are less interesting. Hence, we impose the last condition in the first row of (2) to make $u(t) = 0$ suboptimal at any instant of time. In the second row of (2), the two first inequalities mean that (given any level of knowledge), process improvement expenditures increase the stock of knowledge, but subject to decreasing marginal effectiveness. The next two inequalities mean that (given the expenditure rate) the rate of change of the stock decreases (at an increasing speed) as the current stock of knowledge increases. Further, the marginal effectiveness of process improvement expenditures (f_u) is decreasing in K, i.e. the higher the stock of knowledge, the smaller the effect of an additional dollar spent on process improvements. Finally, f is a strictly concave function.

To model the accumulation of autonomous learning, denote by $x(t)$ the production rate at time t. Current production adds linearly to the stock of autonomous knowledge $Z(t)$. Allowing for decay of autonomous knowledge yields the following dynamics for the stock Z

$$\dot{Z}\ (t) = x(t) - \delta Z(t),\ Z(0) = Z_0 > 0 \tag{3}$$

in which δ is a positive and constant decay rate of the stock of autonomous knowledge.

Denote by $C(t)$ the *total* cost of production at time t. Following Spence [7] we suppose that induced learning affects this cost in an indirect way; process improvement expenditure $u(t)$ influences the stock $K(t)$ through (1) and $C(t)$ is a function of $K(t)$. The cost effects of autonomous learning are accounted for by letting $C(t)$ depend directly on autonomous knowledge $Z(t)$ [3,4]. Finally, $C(t)$ also depends on current output $x(t)$. Altogether, one obtains a general production cost function $C = g(K, Z, x)$. Not unexpectedly, this formulation is too general to admit analytical insights and we shall employ the following specialization:

$$C = g(K,\ Z,\ x) = G(Z,\ x) + F(K) \tag{4}$$

In (4) it is important to note that the cost effects of autonomous knowledge and are separated from those of induced learning. Thus, induced learning affects the total production cost by shifting the part of the cost curve (G) that depends on current output x and autonomous knowledge Z. As any other assumption, our hypothesis of separated effects of the two types of learning is open to critique. Thus, if process improvement efforts do interact with autonomous learning effects, the assumption will not be satisfied. Clarke *et al.* [3] argued that in some instances, process improvement efforts and the resulting induced knowledge is beneficial to the general organization of the production process and leads to a shift of the variable cost function.

Functions G and F are twice continuously differentiable on the open sets $\{Z > 0,\ x > 0\}$ and $\{K > 0\}$, respectively and satisfy

$$\begin{aligned} &G(Z, 0) = 0, G > 0 \text{ for } x > 0, G_z < 0, G_x > 0 \\ &G_{ZZ} < 0, G_{xx} > 0, G_{Zx} < 0, G_{ZZ}G_{xx} > (G_{Zx})^2 \\ &F > 0, F' < 0, F'' > 0 \end{aligned}$$

The assumptions in (5) imply that the cost component $G(Z, x)$ decreases as autonomous knowledge Z increases, whereas the component F decreases as induced knowledge K accumulates. Each type of learning is subject to decreasing marginal effectiveness. Functions G and F are strictly convex which makes C strictly convex.

Turning to the firm's revenue, we assume that all output is sold at a unit price p which depends on current output (= demand) $x(t)$. The demand function $p(x)$ is twice continuously differentiable on $\{x > 0\}$ and downward sloping, i.e. $p'(x) < 0$. The revenue function $R(x) = p(x)x$ is assumed to be strictly concave: $R''(x) < 0$. To make a zero output rate suboptimal, we introduce the assumption $R'(0) = p(0) > G_x(Z(t),\ 0)$ for all feasible Z. Figure 1 depicts the overall structure of the model.

If the firm plans for a finite period of time (i.e. T is finite), we impose the terminal constraints $K(T) = K_T = \text{const.} > K_0$, $Z(T) = Z_T = \text{const.} > Z_0$. These constraints reflect that the firm sets target levels for its stocks of knowledge; K_T, Z_T represent these targets. Note that we are only interested in situations in which the initial knowledge levels are below their target levels. If the horizon is infinite, the terminal stock constraints make no sense and are omitted. In this case we confine our interest

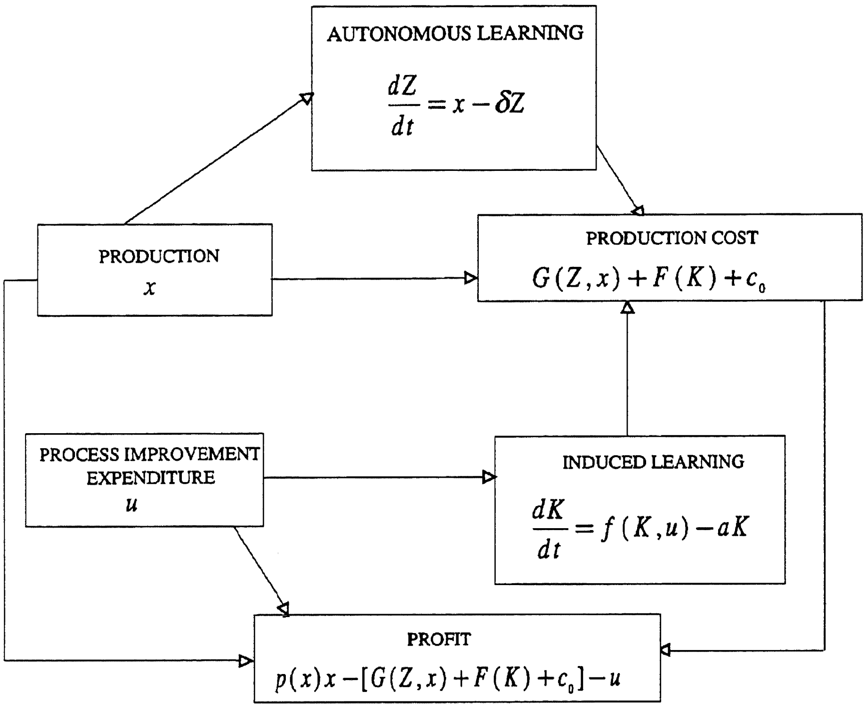

Figure 1 The dynamic structure of the model.

to situations in which initial knowledge levels are below their steady state levels. Thus, representing the steady states by $\hat{K}$, $\hat{Z}$, we only consider cases where $k_0 < \hat{K}$, $Z_0 < \hat{Z}$.

The firm's decision problem is to find controls $x(t)$ and $u(t)$, $t \in [0, T]$, that are piecewise continuous and maximize the objective

$$J = \int_0^T e^{-rt}\{R(x(t)) - [G(Z(t)) + F(K(t)) + c_0] - u(t)\}dt \tag{6}$$

subject to the dynamical constraints (1) and (3), the terminal constraints on K and Z (if the horizon is finite) and feasibility of state and control variables:

$$K(t) \sim 0,\ Z(t) \geq 0,\ u(t) \geq 0,\ x(t) \geq 0\ \forall t \in [0, T] \tag{7}$$

3 Results and managerial implications

This section starts out by stating the optimality conditions for the problem stated in Section 2. Section 3.2 characterizes the optimal output and process improvement expenditure policies.

Section 3.3 uses phase diagrams to study the intertemporal development of optimal output, process improvement expenditures and the stocks of autonomous and induced knowledge. Sensitivity analyses are presented in Section 3.4.

3.1 Optimality conditions

First we deal with feasibility, cf. Equation 7. By assumption, optimal process improvement expenditures and output remain strictly positive. Use Equations 1–3 to obtain

$$\dot{K}(t)|_{K(t)=0} = f(u(t), 0) > 0, \dot{Z}(t)|_{Z(t)=0} = x(t) > 0$$

Invoking the initial conditions $K(0) \geq 0$, $Z(0) > 0$ shows that the nonnegativity constraints $K \geq 0$, $Z \geq 0$ are satisfied.

If the planning horizon is finite, a necessary and sufficient optimality condition for the optimal control problem of Section 2 is as follows (cf. Feichtinger and Hartl [15], Theorems 2.2 and 2.4, or Seierstad and Sydsaeter [16], Theorem 5 (p. 107) and Theorem 14 (p. 236)). Define the current value Hamiltonian

$$\begin{aligned} H = \overline{w}[R(x) - (G(Z,x) + F(K) + c_0) - u] + \lambda(f(u,K) - aK) \\ + \pi(x - \delta Z) \end{aligned} \tag{8}$$

in which $\bar{w}$ is a nonnegative constant. Let $(K^*(t), Z^*(t), x^*(t)\ u^*(t))$ be a feasible quadruple. If there exists continuous and piecewise continuously differentiable costate variables $\lambda(t)$, $\pi(t)$ such that for all $t \in [0, T]$ the following condition is satisfied with $w > 0$

$$u^*(t) \text{ and } x^*(t) \text{ maximize } H(K^*(t), Z^*(t), u, x, \lambda(t), \pi(t)) \tag{9}$$

and, at all points of continuity of $u^*(t)$, $x^*(t)$, the costate equations

$$\dot{\lambda}(t) = [r + a - f_K(u^*(t), K^*(t))])\lambda(t) + \bar{w}F'(K^*(t)), \lambda(T) = 0 \tag{10}$$

$$\dot{\pi}(t) = (r + \delta)\pi(t) + \bar{w}G_z(Z^*(t), x^*(t)), \pi(T) = 0 \tag{11}$$

are satisfied and the maximized Hamiltonian is concave in (K, Z) for all t, then the quadruple $(K^*(t), Z^*(t), x^*(t)u^*(t))$ is optimal. If the horizon is infinite, the above conditions are necessary and sufficient for optimality in the catching-up sense if we replace $\lambda(T) = \pi(T) = 0$ in Equations 10 and 11 by the limiting transversality conditions

$$\begin{aligned} &\underline{\lim}_{T\to+\infty} e^{-T}\lambda(T)[K(T) - K^*(T)] \\ &\quad \geq 0, \underline{\lim}_{T\to+\infty} e^{-T}\pi(T)[Z(T) - Z^*(T)] \geq 0 \end{aligned} \tag{12}$$

in which $(K(\cdot), Z(\cdot))$ is an arbitrary, feasible state trajectory. In the appendix we prove that the constant $\overline{w}$ is positive and then it can, by normalization, be put equal to one. The appendix also proves that the maximized Hamiltonian is strictly concave in (K, Z). The remaining optimality conditions are dealt with in Sections 3.2–3.3.

3.2 Optimal output and process improvement policies

From now on we frequently omit the time-argument. The Hamiltonian maximization conditions in Equation 9 become

$$H_x = 0 \Leftrightarrow R'(x) + \pi = G_x(Z, x); H_u = 0 \Leftrightarrow -1 + \lambda f_u(u, K) = 0 \tag{13}$$

Note that $H_{xx} < 0$ and $H_{uu} < 0$ which implies that the optimal output and process improvement expenditure rates are unique and continuous in t. Since the maximized Hamiltonian is strictly concave in (Z, K), the optimal state trajectory (Z^*, K^*) is unique.

The first condition in Equation 13 is intuitive since $R'(x)$ is marginal revenue and π (t) is the shadow price of the stock of autonomous knowledge Z. The shadow price measures the marginal contribution to the optimal objective function of a one-unit addition to the stock Z. Hence, the optimality condition balances current as well as future benefits of producing an additional unit against the current cost of manufacturing that unit. This can also be illustrated by using Equation 11 and the first condition in Equation 13 from which it is straightforward to derive

$$G_x(Z(t), x(t)) + \int_t^\infty G_z(Z(s), x(s))e^{-(r+8)(s-t)}ds = '(x(t)) \ \forall t \in [0, T]$$

The first term on the left-hand side is the usual 'short-run' marginal production cost and the right-hand side is the usual 'short-run' marginal revenue. These terms reflect the immediate costs and benefits of producing and selling one more unit. However, producing the marginal unit also increases the stock of autonomous knowledge which implies that as of time t, production cost decreases (by G_Z) over the remaining planning period. The integral on the left-hand side measures this stream of cost reductions. Note that one must correct for decay of knowledge (multiplying by exp{$-\delta(s-t)$}) and discount to time t (multiplying by exp{$-r(s-t)$}). We obtain the usual marginal cost equals marginal revenue condition if $G_Z = 0$. In this case neither type of learning influences the marginal production cost.

The following proposition characterizes the optimal production rate (all proofs have been relegated to the appendix).

PROPOSITION 1. The first condition in Equation 13 determines implicitly the optimal production rate x^*as a continuously differentiable function of (Z, π). It holds that

$$\frac{\partial x^*}{\partial Z} = \frac{G_{xZ}}{R''(x) - G_{xx}} > 0, \frac{\partial x^*}{\partial \pi} = \frac{-1}{R''(x) - G_{xx}} > 0$$

The first result clearly is driven by the assumption $G_{xZ} < 0$. Thus, when the stock Z increases, the marginal cost of producing falls and it certainly pays to increase output. The intuition of the second result simply is that when the shadow price of Z increases (i.e. the stock Z becomes more valuable), the production rate should be increased.

The second condition in Equation 13 also has an intuitive interpretation. Consider a one-dollar increase in process improvement expenditures u. The term -1

is the marginal contribution to current profit (since the only effect of the increase in expenditure is to decrease the instantaneous profit by one dollar). The costate variable λ is the shadow price of the stock of induced knowledge K and the effect of a marginal increase in u on the stock K is measured by the derivative f_u. The product λf_u then is the marginal contribution of current expenditure to future objective function value, through the current expenditure's effect on the growth rate of the stock K. The next proposition characterizes the optimal process improvement expenditure rate.

PROPOSITION 2. The second condition in Equation 13 determines implicitly the optimal expenditure rate u^* as a continuously differentiable function of (K, A). It holds that

$$\frac{\partial u^*}{\partial K} = \frac{-f_{uK}}{f_{uu}} < 0, \frac{\partial u^*}{\partial \lambda} = \frac{-f_u}{\lambda f_{uu}} > 0$$

The first inequality states that the optimal expenditure rate is decreasing in the stock K. This result is mainly a consequence of the assumption that the higher the current stock of knowledge, the less efficient are the current expenditures in raising the stock ($f_{uK} < 0$). The second inequality has a similar interpretation as the one in Proposition 1.

The effects of taking into account dynamic learning effects can be assessed by comparing the optimal decisions in Equation 13 with the optimal decisions of a myopic firm. The optimal output decision of a myopic firm is an x_m which satisfies $R'(x_m) = G_x(Z, x_m)$. Let x_d denote the optimal dynamic output rate in Equation 13. Using Equation 13 and the fact that the shadow price π is positive, shows that $x_d > x_m$. Thus, a myopic firm produces less than a far-sighted firm. The reason is that a myopic firm does not take into account the benefits from future learning, as represented by the shadow price $\pi > 0$. The optimal process improvement expenditure of a myopic firm clearly is $u_m = 0$: any positive expenditure would be an unnecessary cost because a myopic firm disregards the future impacts of its current expenditures. The conclusion is that a non-myopic firm takes into account both current and future benefits of autonomous and induced learning and produces more output and spends more on process improvement efforts than a myopic firm.

3.3 Phase diagram analysis

Since cost and revenue functions are not explicitly specified, it is impossible to get closed-form expressions for the optimal control and state trajectories. Instead we derive a series of qualitative results. In a decision support context, they should be seen as qualitative guidelines since they only can give the directions of change of control and state variables. If we had fully specified functional forms, it might be possible to get explicit solutions. On the other hand, it is sometimes seen that functional forms are chosen specifically to satisfy such an objective. This creates some arbitrariness whereas our cost function retains some generality and the results are to a lesser extent driven by specific assumptions.

Suppose that $x^*(t)$ and $u^*(t)$ are continuously differentiable with respect to t for all $t \in (0, T)$. Then the control variables must satisfy

$$\frac{d}{dt}(H_x) = \frac{d}{dt}(H_u) = 0 \text{ for } t \in (0, T). \tag{14}$$

Using Equations 10, 11 and 13 we eliminate the costate variables from Equation 14 to obtain

$$\dot{x} = \frac{1}{R''(x) - G_{xx}}[(r+\delta)(R'(x) - G_x(Z,x)) + (x - \delta Z)G_{xZ}(Z,x) - G_z(Z,x)] \tag{15}$$

$$\dot{u} = \frac{-1}{f_{uu}(K,u)}[F(K)(f_u(K,u))^2 + (r + a - f_K(K,u))f_u(K,u) + f_{uK}(K,u) \times (-aK + f(K,u))] \tag{16}$$

The system consisting of Equations 1, 3, 15, 16 is a 4-dimensional system in (K, Z, x, u) space and has a unique optimal solution. Mainly because of the separability of the cost function with respect to K and Z, this system can be decoupled in two independent systems. One system consists of Equation 15 and 3 and can be depicted in the (Z, x) plane. The other consists of Equation 16 and 1 and can be depicted in the (K, u) plane. To determine the steady states of the two systems we have the algebraic equations

$$R'(\hat{x}) + \hat{\pi} = G_x(\hat{Z}, \hat{x}), \hat{\pi} = G_Z(\hat{Z}, \hat{x})/(r+\delta), \hat{x} = \delta\hat{Z} \tag{17}$$

$$\hat{\lambda} f_u(\hat{K}, \hat{u}) = 1, \hat{\lambda} = -F'(\hat{K})/[r + a - f_K(\hat{u}, \hat{K})], \hat{K} = f(\hat{u}, \hat{K})/a \tag{18}$$

where Equation 17 uniquely determines the steady state triple $(\hat{Z}, \hat{x}, \hat{\pi})$ and Equation 18 uniquely determines $(\hat{K}, \hat{u}, \hat{\lambda})$. The first Equation in 17 and 18, respectively, is the Hamiltonian maximization condition. The second equation comes from costate Equations 10 and 11 and the third from state Equations 1 and 3. All six steady state values are positive.

For the $(7, x)$ differential equation system given by Equations 3 and 15 we have the following result.

PROPOSITION 3. (A) If

$$(r + 2\delta)G_{xZ}(\hat{Z}, \hat{x}) + G_{ZZ}(\hat{Z}, \hat{x}) > 0 \tag{19}$$

the steady state in the (Z, x) phase plane is a saddle point. The stable branch is the only trajectory $(Z(t), x(t))$ that converges to the steady state. This branch is downward sloping, at least in a neighbourhood of the steady state. (B) *If*

$$(r + 2\delta)G_{xZ}(\hat{Z}, \hat{x}) + G_{ZZ}(\hat{Z}, \hat{x}) = 0 \tag{20}$$

the steady state in the (Z, x) phase plane is a saddle point. The stable branch is *horizontal, at least in a neighbourhood of the steady state.* (C) *If the inequalities*

$$(r + 2\delta)G_{xZ}(\hat{Z}, \hat{x}) + G_{ZZ}(\hat{Z}, \hat{x}) < 0 \tag{21}$$

$$-\delta(\delta + r)[R''(\hat{x}) - G_{xx}(\hat{Z}, \hat{x})] + G_{ZZ}(\hat{Z}, \hat{x}) + G_{xZ}(\hat{Z}, \hat{x})(r + 2\delta) > 0 \tag{22}$$

are satisfied, the steady state in the (Z, x) *phase plane is a saddle point. The stable branch is upward sloping, at least in a neighbourhood of the steady state.*

The inequality in Equation 19 can be satisfied if the discount and decay rates are sufficiently small (since $G_{xZ} < 0$ and $G_{ZZ} > 0$). This happens if the firm is sufficiently far-sighted and the decay of autonomous knowledge is not too fast. The situation in Equation 20 is a hairline case which generates a rather extreme result: constant output rate at all instants of time. The inequality in Equation 21 holds for sufficiently large discount and decay rates, but here we need the additional assumption in Equation 22 to guarantee saddle point stability. Note that Equation 22 becomes harder (or even impossible) to satisfy if $|G_{xZ}|$ is very large. Then the learning effect is so significant that it is optimal to increase the output rate and the solution will not converge along a stable saddle point path to a steady state.

Remark. Our conclusions about the slope of the stable branch are valid only in a neighbourhood of the steady state. In the proof of Proposition 3 we show the existence of the stable branch on the time interval $[0, \infty)$, but this is not sufficient to guarantee the existence of a stable branch emanating from any feasible initial state Z_0. To assure this we must verify a global saddle point theorem (see, e.g. Feichtinger and Hartl [15]). In our model, the verification of such a theorem has not been successful. Notice, however, that our conclusions about the slopes of stable branches are guaranteed for initial states Z_0 that are sufficiently close to the steady state $\hat{Z}$.

In an *infinite horizon* problem there are only two kinds of optimal trajectories in (Z,x) space; the stock Z is either montonically increasing or decreasing over time. In any of the three phase diagrams of Figure 2, the solid curves leading to the steady state represent the stable path. On this path, the pair $(Z(t), x(t))$ converges to the steady state as time goes to infinity. The initial condition for Z is Z_0. To obtain the initial condition for x we start out with the steady state values $\hat{Z}, \hat{x}$ and, exploiting the differential Equations 3 and 15, we work backward in time to find the value $x(0)$ that corresponds to Z_0. Recall the assumption $Z0 < \hat{Z}$, that is, we are only interested in situations in which the initial stock of autonomous knowledge is below the steady state value.

Consider Figure 2A which corresponds to Case A of Proposition 3. The firm has a 'small' initial stock of autonomous knowledge Z_0 and the stock of knowledge increases steadily over time. The output rate is decreasing over time. The reason is that the inequality in Equation 19 is satisfied, which means that the decrease in marginal production cost due to autonomous learning, as measured by $|G_{xZ}|$, is relatively small. It does not outweigh the reduction of the negative effect of autonomous knowledge on production cost ($G_{ZZ} > 0$). Although the output rate decreases, output remains sufficiently large to compensate for the decay of knowledge. Hence the stock of knowledge increases. The optimal price increases over time (clearly due to the decrease of output).

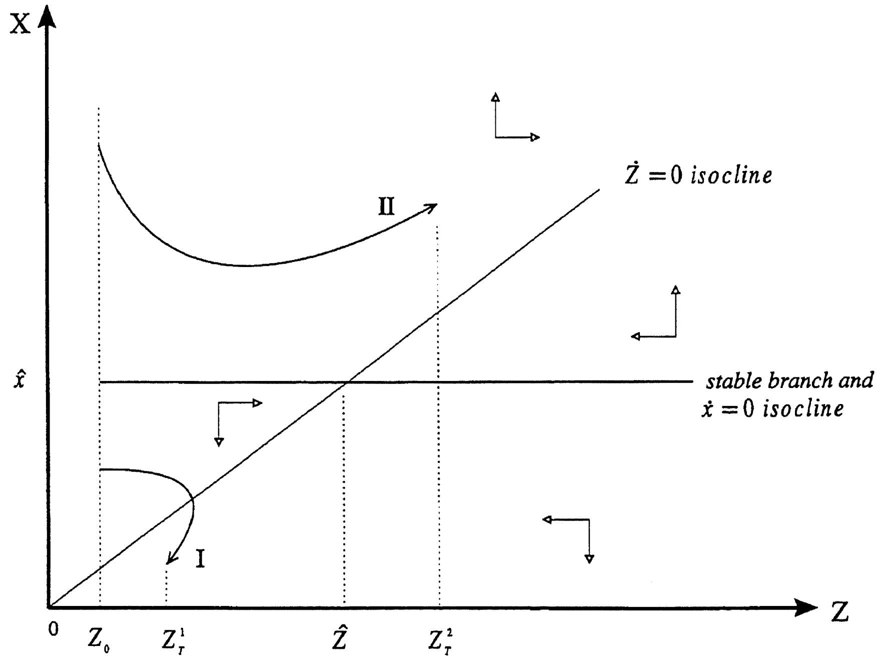

Figure 2A (Z, x) phase diagram – case A.

In Figure 2B, which depicts Case B of Proposition 3, the production rate is constant over time, equal to its steady state value $\hat{x}$. The case of a constant output rate over an infinite horizon is somewhat extreme and can be thought of as a hairline case.

Finally, Case C of Proposition 3 is illustrated in Figure 2C. Here, both the knowledge stock and the production rate increase over time. The reason is that the decreasing effect of learning on the marginal production cost ($G_{xZ} < 0$) is sufficiently large to warrant an increasing output policy. Convergence to the steady state is still optimal. This is because the additional assumption 22 implies that the total stabilising effects of concavity of the production function and convexity of the cost function outweigh the destabilizing effect induced by the negative sign of G_{xZ} (this effect is destabilizing since production increases knowledge which lowers production costs, making it more attractive to increase production). The stock increases because the production rate is sufficiently large, even initially, to compensate for the decay of knowledge. The policy of increasing output is accompanied by a falling price. Such policies have been noticed in many instances, perhaps most prominently in the electronic products industries; see, e.g. Philips *et al.* [17].

In the *finite horizon* problem, all optimal trajectories are *off* the stable path. To illustrate, Figure 2A depicts two possible trajectories, I and II. Which trajectory is optimal depends on the initial and terminal states, Z_0 and Z_T ($Z_0 < Z_T$) and the value of T.

$Z_0 < \hat{Z} < Z_T^2$: The stock Z is always increasing. The optimal trajectory is of Type II.

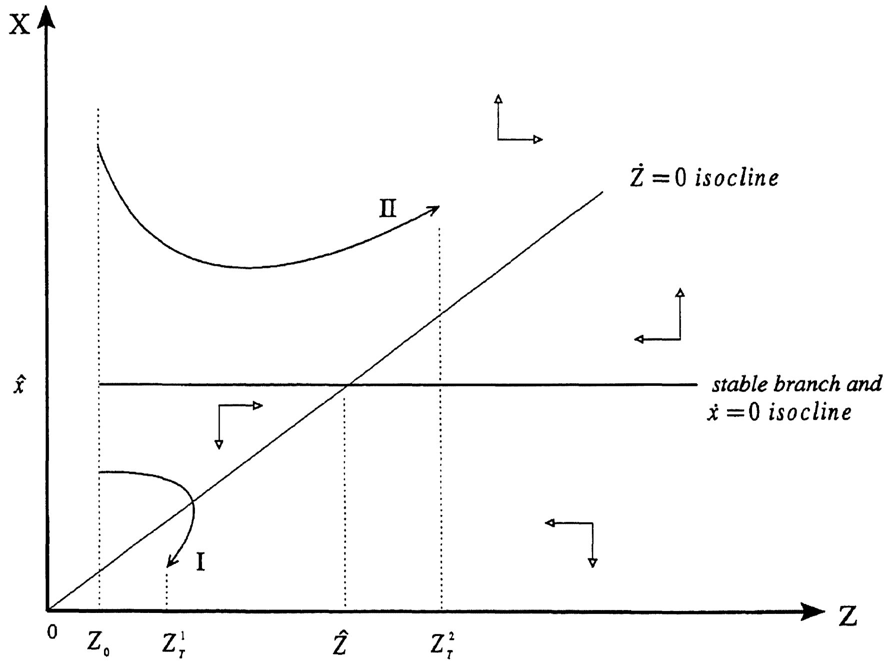

Figure 2B (Z, x) phase diagram – case B.

Depending on the initial stock, the production rate either is monotonically increasing over time, or it is first decreasing and then increasing.

$Z_0 < Z_T^1 < \hat{Z}$: The optimal trajectory is either of Type I or II. A Type II trajectory is as above. On a Type I trajectory, production always decreases.

Remark. The optimal output policy of the Li and Rajagopalan [10] model differs from the above. In the Li and Rajagopalan model, the optimal production rate is steadily increasing over time. This result may occur in the case where $Z_0 < \hat{Z} < Z_T^2$ but in our set-up other policies are also possible. The difference is due to our assumption $G_{xx} > 0$, that is, marginal production cost increases with the production rate. Hence it becomes more expensive to produce as output x increases. This output stabilizing effect is absent in the Li and Rajagopalan model because their production cost function is linear in x.

The next proposition deals with the (K, u) differential equations system in Equations 1 and 16, having the steady state in 18.

PROPOSITION 4. The steady state in the (K, u) *phase plane is a saddle point with a downward sloping stable branch.*

The negative slope of the stable branch implies that process improvement efforts decrease with the stock of induced knowledge (see also Proposition 2). This is because it is more difficult to build up knowledge if the stock is already large ($f_{Ku} < 0$) and additional knowledge has less effect on production costs if the level of

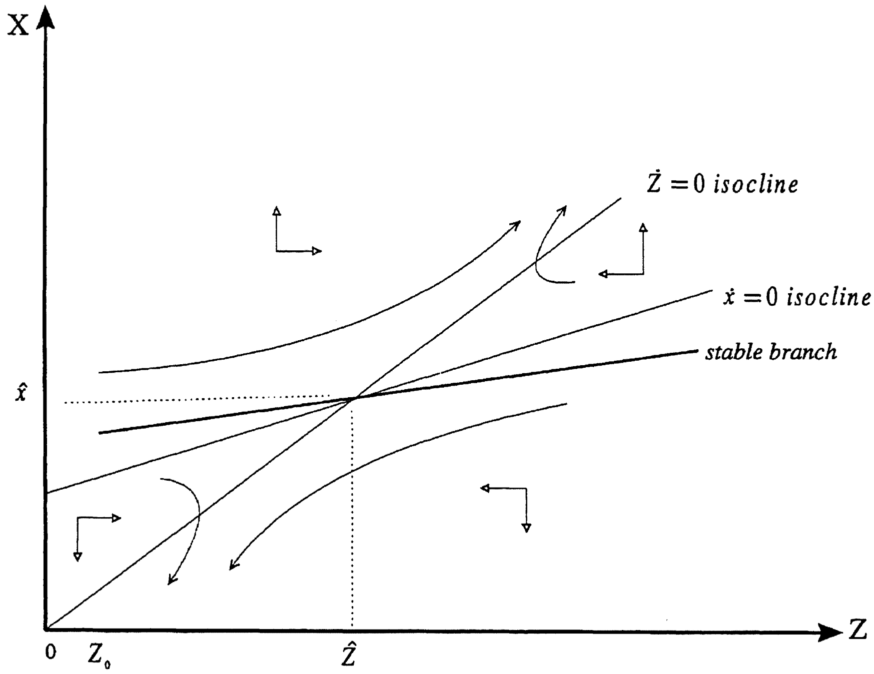

Figure 2C (Z, x) phase diagram – case C.

knowledge is already high $(F''(K) < 0)$. The phase diagram for the (K, u) differential equations system is depicted in Figure 3.

In the infinite horizon problem, the firm starts out in a point on the stable path (corresponding to the initial condition) and the trajectory $(K(t), u(t))$ converges to the steady state as time goes to infinity. It suffices to note that for $K_0 < \hat{K}$, process improvement expenditures decrease over time. They remain sufficiently large to compensate for the decay of learning such that the stock of induced knowledge increases over time. In the finite horizon case, similar remarks as those given in conjunction with Figure 2 apply.

3.4 Sensitivity analysis

To make a fully dynamic sensitivity analysis one needs closed-form expressions for the optimal trajectories or global information about implicit relationships. Here, optimal trajectories are only qualitatively characterized and we have only local information (i.e. in neighbourhoods of the steady state) on the variables Z and x. Hence, for this system we can only investigate the sensitivity of the steady state values.

By Equation 17 the steady state values of Z and x are implicitly given as functions of the model parameters δ and r and we have

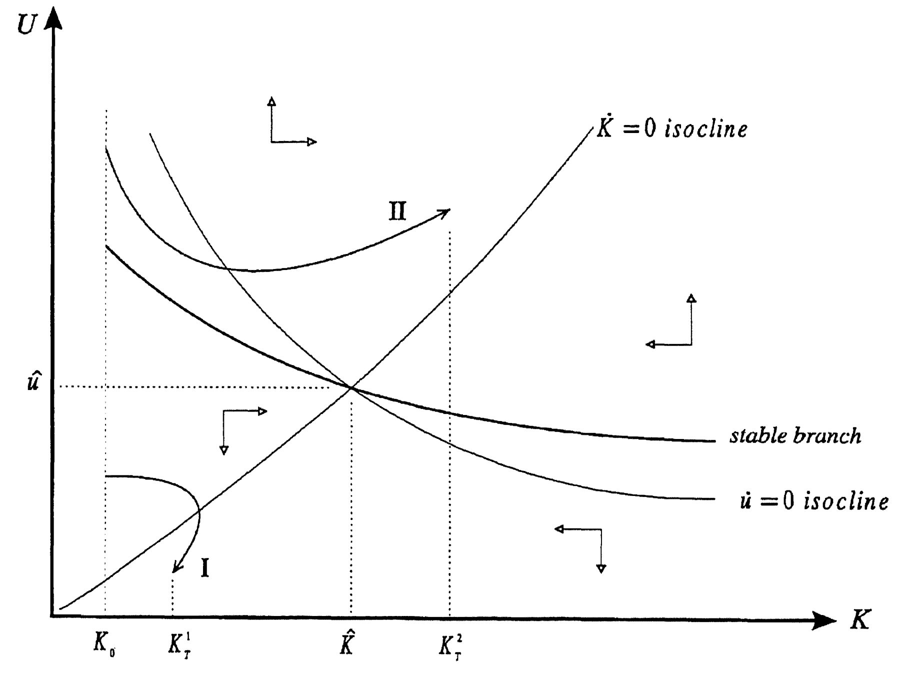

Figure 3 (K, U) phase diagram.

PROPOSITION 5. (A) The higher the rate of interest r, the lower the steady state output $\hat{x}$ and the stock of autonomous knowledge $\hat{Z}$.

(B) If $-\hat{Z}\, G_{xZ} < -\hat{Z}(r+\delta)(R'' - G_{xx}) + \pi$, the steady state stock of autonomous knowledge $\hat{Z}$ decreases with the decay rate δ.

The results in Part (A) are plausible, recognising that a high rate of discount means that the firm is myopic and puts less emphasis on future streams of profits. A far-sighted firm has a low discount rate which leads to higher steady state levels of the stocks and the control variables. The steady state output rate is decreasing in the discount rate which confirms our discussion about myopic behaviour where we concluded that a myopic firm produces less than a far-sighted one at any instant of time.

Part (B) shows that the intuitive result – the steady state stock decreases with the decay rate – is only valid with a disclaimer. Its interpretation is that the numerical value of the mixed partial derivative G_{xZ} must not be too large. If $|G_{xZ}|$ is large the stock of autonomous knowledge has a high impact on production cost. Moreover, if the decay rate δ is large, the steady state production level is large which implies high production costs. To reduce these costs one needs to have a higher steady state level of autonomous knowledge; the effect of this is sizeable when $|G_{xZ}|$ is large.

For the (K, u) system it is possible to make an analysis of the steady state values as well as a dynamic sensitivity analysis for the finite horizon problem:

PROPOSITION 6. (A) The higher the value of the decay rate δ, the lower the steady state stock of induced knowledge $\hat{K}$. The higher the rate of interest r, the lower the process improvement expenditures $\hat{u}$ and the stock of induced knowledge $\hat{K}$.

(B) In the finite horizon case it holds that the higher the rate of interest r, the lower the optimal stock of induced knowledge $K(t,r)$ for all $t \in [0,T)$ and the lower the optimal process improvement expenditures $u(t,r)$, at least on a time interval $[0, t_1)$, $t_1 < T$.

The results in Part A are intuitive: their interpretation is similar to that in Proposition 5. In particular, the optimal process improvement expenditure is decreasing in the discount rate. This confirms our discussion about myopic behaviour where we concluded that a myopic firm spends nothing on process improvements. The last result in Part B confirms a finding of Li and Rajagopalan [10].

4 Concluding remarks

The paper has considered the dynamic effects of two types of learning in production. Autonomous learning reflects the well known 'learning by doing' phenomenon. Besides contributing to sales revenue, an important positive benefit of producing more now is that the firm automatically gains production experience that benefits future production, in terms of a lower unit manufacturing cost in the future. This learning effect alone often leads to a production policy where output increases over time. But our model also takes into account some counterbalancing effects: variable manufacturing costs increase progressively with output and autonomous learning has more effect on production cost when production experience is low than when it is high. In the infinite horizon case, these effects stabilize the output policy such that the production rate does not increase infinitely, but converges to a steady state. An important contribution of the paper is the demonstration that the relative magnitude of the 'increased production now makes future production cheaper' effect is a key factor in the determination of the firm's production behaviour.

The second type of learning results from efforts that are purposefully aimed at enhancing the skills of the work force and improve the production process with respect to equipment and organization. Process improvement efforts creates a stock of induced knowledge (human capital) and our results show that optimal process improvement efforts are negatively related to the size of the stock of induced knowledge. The hypotheses driving this result are that it is more difficult to increase knowledge if knowledge is already high and production costs are reduced the most by process improvement efforts if knowledge is not too high.

For the agenda of future research we propose two items. Firstly, a generalisation of the demand function to include more than just current output (demand). Current demand, particularly for durable goods, is often observed to depend on the level of cumulative sales. In marketing literature one speaks about positive [negative] demand diffusion effects if consumers increase [decrease] their current demand with the level of cumulative sales. Positive demand diffusion effects is a factor that, in marketing theory and in practice, has been used as a rationale for policies of increasing output and decreasing price, in conjunction with unit cost reductions obtained by increasing cumulative output (autonomous learning). Secondly, it could be worthwhile to try to relax the assumption that the two stocks do not interact in the cost function, to see the impacts of 'spillover' of learning. Most likely, however, both suggestions will result in optimization problems that are too complex to be analytically tractable and one will have to resort to numerical simulations.

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References

1 Rosenberg, N. (1982) *Inside the Black Box: Technology and Economics*, Cambridge University Press, New York.

2 Zangwill, W.I. and Kantor, P.B. (1988) 'Toward a Theory of Continuous Improvement and the Learning Curve', *Management Science*, Vol. 44, pp. 910–920.

3 Clarke, F.H., Darrough, M.N. and Heineke, J.M. (1982) 'Optimal Pricing Policy in the Presence of Experience Effects', *Journal of Business*, Vol. 55, pp. 517–530.

4 Jørgensen, S., Kort, P.M. and Zaccour, G. (1999) 'Production, Inventory and Pricing Under Cost and Demand Learning Effects', *European Journal of Operational Research*, Vol. 117, pp. 382–395.

5 Adler, P.S. and Clark, K.B. (1991) 'Behind the Learning Curve: A Sketch of the Learning Process', *Management Science*, Vol. 37, pp. 267–281.

6 Spence, M. (1981) 'The Learning Curve and Competition', *Bell Journal of Economics and Management Science*, Vol. 12, pp. 49–70.

7 Spence, M. (1984) 'Cost Reduction, Competition and Industry Performance', *Econometrica*, Vol. 52, pp. 101–121.

8 Fine, C.H. (1986) 'Quality Improvement and Learning in Productive Systems', *Management Science*, Vol. 32, pp. 1301–1315.

9 Fine, C.H. and Porteus, E.L. (1989) 'Dynamic Process Improvement', *Operations Research*, Vol. 37, pp. 580–591.

10 Li, G. and Rajagopalan, S. (1998) 'Process Improvement, Quality and Learning Effects', *Management Science*, Vol. 44, pp. 1517–1532.

11 Li, G. and Rajagopalan, S. (1998) 'A Learning Curve Model with Knowledge Depreciation', *European Journal of Operational Research*, Vol. 105, pp. 143–154.

12 Arrow, K.J. (1962) 'The Economic Implications of Learning by Doing', *Review of Economic Studies*, Vol. 29, pp. 166–170.

13 Ritzen, J.R. and Winkler, D.R. (1979) 'On the Optimal Allocation of Resources in the Production of Human Capital', *Journal of the Operational Research Society*, Vol. 30, pp. 33–41.

14 Hartl, R.F. (1983) 'Optimal Allocation of Resources in the Production of Human Capital,' *Journal of the Operational Research Society*, Vol. 34, pp. 599–606.

15 Feichtinger, G. and Hard, R.F. (1986) *Optimale Kontrolle ökonomischer Prozesse*, Walter deGruyter, Berlin.

16 Seierstad, A. and Sydsaeter, K. (1987), *Optimal Control Theory with Economic Applications*, North-Holland, Amsterdam.

17 Philips, L.W., Chang, D.R. and Buzzel, R.D. (1985) 'Product Quality, Cost Position and Business Performance: A Test of Some Key Hypotheses', *Journal of Marketing*, Vol. 39.

Appendix

PROOF OF $\bar{w} > 0$. By contradiction. Assume that $\bar{w} = 0$. From (8) and (9) we have $H_u = -\bar{w} = +\lambda f_u(u, K) = 0$ which for $\bar{w} = 0$ becomes $H - u = \lambda f_u(u, K) = 0$. Since $f_u > 0$, it must hold that λ is zero for all t. Using 1 and 6, the optimal objective J^* is given by

$$\int_0^T e^{-rt} \left\{ R(x^*(t)) - [c_0 + G(Z^*(t), x^*(t)) + F(K_0) + \int_0^t (f(u^*(s), K^*(s)) - aK^*(s))ds)] - u^*(t) \right\} dt$$

so that $\partial J^*/\partial K_0 = F'(K^*) > 0$. It is known that $\partial J^*/\partial K_0 = \lambda(0)$ and hence $\lambda(0) > 0$. This contradicts $\lambda(t) = 0$ for all t and hence $\bar{w} = 0$ cannot be true. It follows that $\bar{w} > 0$.

PROOF OF CONCAVITY OF THE MAXIMIZED HAMILTONIAN. We write the maximized Hamiltonian as

$$\begin{aligned} H^* &= \{R(x^*) - G(Z, x^*) + \pi(x^* - \delta Z)\} + \{-F(K) - u^* + \lambda(f(u^*, K) - aK)\} \\ &\cong \phi(Z, x^*) + \psi(K, u^*) \end{aligned}$$

in which $x^*(Z, \pi)$ and $u^*(K, \lambda)$ are the optimal controls stated in Propositions 1 and 2, respectively. If function ϕ is concave in Z for any given π and function ψ is concave in K for any given λ, then the maximized Hamiltonian is concave in (K, Z) for any given (π, λ). Differentiate function ϕ twice with respect to Z and use Equation 13 and Proposition 1 to obtain

$$\frac{\partial^2 \phi}{\partial Z^2} = -G_{zz}(Z, x^*) - \frac{(G_{xZ}(Z, x^*))^2}{R''(x^*) - G_{xx}(Z, x^*)} \tag{A1}$$

If the right-hand side of (A.1) is negative, function ϕ is strictly concave. To prove this, use the assumptions $G_{zz}G_{xx} > G_{Zx}^2$, $R'' < 0$, and $G_{zz} > 0$ to obtain $-G_{zz}R'' + G_{zz}G_{xx} > G_{Zx}^2$. It is readily seen that the latter inequality implies that the right-hand side of (A.1) is negative. Next differentiate function ψ twice with respect to K and use Equation 13 and Proposition 2 to obtain

$$\frac{\partial^2 \Psi}{\partial K^2} = -F''(K) + \frac{\lambda}{f_{uu}(K, u^*)} [f_{KK}(K, u^*) f_{uu}(K, u^*) - f_{uK}(K, u^*))^2] \tag{A2}$$

Exploiting the concavity of functions f and F and that $\lambda = 1/f_u$, shows that the right-hand side of A2 is negative. Hence function ψ is strictly concave.

PROOF OF PROPOSITIONS 1 AND 2. In Equation 13 define $h(x, Z, \pi) = R'(x) + \pi - G_x(Z, x) = 0$ and note that h is continuously differentiable by the assumptions on demand and cost functions. Let (x^0, Z, π) denote the unique point at which $h = 0$ for each admissible pair (Z, π). This point exists since $R''(x) < 0$ and $G_{xx} > 0$. Moreover, $h_x(x^0, Z, \pi) < 0$. In a neighbourhood of (x^0, Z, π), x is implicitly given as a continuously differentiable function of (Z, π) for which the implicit function rule yields the derivatives stated in Proposition 1. Proposition 2 is proved in a similar way.

PROOF OF PROPOSITION 3. Define the Jacobian matrix $\Im$ of the differential equation system given by Equations 3 and 15 and evaluate the elements of this matrix at the steady state $(\hat{Z}, \hat{x})$:

$$\Im = \begin{pmatrix} \frac{\partial \dot{Z}}{\partial Z} & \frac{\partial \dot{Z}}{\partial x} \\ \frac{\partial \dot{x}}{\partial Z} & \frac{\partial \dot{x}}{\partial x} \end{pmatrix} = \begin{pmatrix} -\delta & 1 \\ -\frac{G_{zz}(\hat{Z}, \hat{x}) + (r + 2\delta) G_{xZ}(\hat{Z}, \hat{x})]}{R''(\hat{x}) - G_{xx}(\hat{Z}, \hat{x})} & r + \delta \end{pmatrix} \tag{A3}$$

The steady state is a saddle if and only if $\det \Im < 0$, which by using A3, is equivalent to

$$-\delta(\delta + r)[R''(\hat{x}) - G_{xx}(\hat{Z}, \hat{x})] + G_{ZZ}(\hat{Z}, \hat{x}) + G_{xZ}(\hat{Z}, \hat{x})(r + 2\delta) > 0 \tag{A4}$$

The inequality A4 is the same as Equation 22. Calculating the slopes of the isoclines yields

$$\begin{aligned} \frac{dx}{dZ}|_{Z=0} &= \frac{-\partial \hat{Z}/\partial Z}{\partial \hat{Z}/\partial x} = \delta, \frac{dx}{dZ}|_{\dot{x}=0} = \frac{-\partial \dot{x}/\partial Z}{\partial \dot{x}/\partial x} \\ &= \frac{G_Z Z(G, x) + (r + 2\delta) G_{xZ}(Z, x)}{(r + \delta)[R''(x) - G_{xx}(Z, x)]} \end{aligned} \tag{A5}$$

The $\dot{Z} = 0$ isocline has a positive slope but the slope of the $\dot{x} = 0$ isocline depends on the sign of the numerator in the last expression in A5, that is,

$$sgn\left(\frac{dx}{dZ}|_{\dot{x}=0}\right) = -sgn(G_{ZZ}(Z, x) + (r + 2\delta) G_{xZ}, (Z, x))$$

The results of the proposition now follow. *Case* (*A*): Equation 19 implies A4 and the $\dot{x} = 0$ isocline is downward sloping, cf. A5, at least in a neighbourhood of the steady state. *Case* (*B*): obviously A4 is satisfied. When Equation 20 holds, A5 shows that the slope of the $\dot{x} = 0$ isocline is zero, at least in a neighbourhood of the steady state. The stable branch coincides with the $\dot{x} = 0$ isocline. *Case* (*C*): When Equation 21 holds, A4 must be satisfied in itself. Using A5 and Equation 24 shows that the $\dot{x} = 0$ isocline has a positive slope, at least in a neighbourhood of the steady state.

PROOF OF PROPOSITION 4. Consider the Jacobian matrix $\Im$ of the system Equations 1 and 16 and evaluate the elements of this matrix at the steady state $(\hat{K}, \hat{u})$:

$$\Im = \begin{pmatrix} \frac{\partial \dot{K}}{\partial K} & \frac{\partial \dot{K}}{\partial u} \\ \frac{\partial \dot{u}}{\partial K} & \frac{\partial \dot{u}}{\partial u} \end{pmatrix} = \begin{pmatrix} -a + f_k & f_u \\ f_{uu}[(r - 2(f_k - a))f_{uK} + (f_u)^2(-F'' + \frac{f_{KK}}{f_u})] & - F'f_u \end{pmatrix} \quad \text{(A6)}$$

Using A6 and the assumptions on functions f and F shows that $\det\Im < 0$ and the steady state is a saddle. For the slopes of the isoclines we have, using A6

$$\frac{du}{dK}|_{\dot{K}=0} = \frac{-\partial \dot{K}/\partial K}{\partial \dot{K}/\partial u} > 0, \frac{du}{dK}|_{\dot{u}=0} = \frac{-\partial \dot{u}/\partial K}{\partial \dot{u}/\partial u} > 0$$

The $\dot{K} = 0$ isocline is upward sloping while the $\dot{u} = 0$ isocline is downward sloping. With these results at hand, the phase diagram in Figure 3 follows.

The following two remarks apply to both Proposition 3 and Proposition 4. On the stable branches, the x, u, Z and K trajectories converge and hence the integrand of the optimal objective function is a bounded function. As the discount rate is positive, we conclude that the objective integral is finite. The trajectories $(Z(t), x(t))$ and $(K(t), u(t))$ in Figures 2 and 3 converge to steady states as time goes to infinity. Using this fact we have verified the limiting transversality conditions in Equation 12 because these conditions are satisfied when we have feasible and bounded state trajectories Z and K that converge to steady states. Optimality is implied.

PROOF OF PROPOSITION 5. The results of the proposition follow by straightforward differentiations in Equation 20.

PROOF OF PROPOSITION 6. Part A follows by partial differentiations in Equation 21. For Part B it suffices to note that in the Jacobian matrix given by A3, all elements are positive, except the one in the north-western corner which is negative. The result of Part B then follows by applying Theorem 4.10 in Feichtinger and Hartl [15].