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On consistent solutions for strategic games*

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Abstract. Nash equilibria for strategic games were characterized by Peleg and Tijs (1996) as those solutions satisfying the properties of consistency, converse consistency and one-person rationality.

There are other solutions, like the ε-Nash equilibria, which enjoy nice properties and appear to be interesting substitutes for Nash equilibria when their existence cannot be guaranteed. They can be characterized using an appropriate substitute of one-person rationality. More generally, we introduce the class of "personalized" Nash equilibria and we prove that it contains all of the solutions characterized by consistency and converse consistency.

Key words: Consistency, axiomatization, strategic games, choice rules

1. Introduction

The starting point for this work was an attempt to see whether the axiomatizations for Nash equilibria given in Peleg and Tijs (1996) and in Peleg, Potters and Tijs (1996), could be adapted to get axiomatizations for ε-equilibria also.

In Peleg and Tijs (1996) it was proved that (OPR) (One Person Rationality), (CONS) (Consistency) and (COCONS) (Converse Consistency) can be used to provide an axiomatic characterization for Nash equilibria (briefly: NE). We refer to that paper for motivation and references to related work.

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Further references, in which the above mentioned results have been extended, are: Norde, Potters, Reijnierse and Vermeulen (1996), Peleg and Sudhölter (1994) and Shinotsuka (1994).

Looking for a characterization of ε-NE, it was clear that the key point was to replace (OPR) by (ε-OPR). By (OPR), in Peleg and Tijs (1996) it was named the requirement that for 1-person games the solution to be characterized coincides with payoff maximization. Clearly, (ε-OPR) should mean that for 1-person games we look for ε-maximizers instead of maximizers.

Being interested also in other kinds of approximate NE, like the (ε-k)-NE introduced in Lucchetti, Patrone and Tijs (1986), and studied also in Jurg and Tijs (1993) and Norde and Potters (1995), we looked at a unified way to achieve this kind of axiomatic characterizations.

The solution is simple: it is sufficient to introduce a (possibly personalized) "choice rule" p for all of the potential players, and to substitute a conveniently defined ($\mathscr{P}$-OPR) for (OPR). In this way, we get the axiomatic characterizations of NE and of the various kinds of approximate NE in which we were interested: the distinction between the various solution concepts is simply done by means of an appropriate choice of the choice rules p.

To be more detailed, by means of a set $\mathscr{P}$ of choice rules p we define $\mathscr{P}$-NE. Then, $\mathscr{P}$-NE is characterized by ($\mathscr{P}$-OPR), (CONS) and (COCONS). Furthermore, it is proved that every solution satisfying (CONS) and (COCONS) is in fact a $\mathscr{P}$-NE, for an appropriate family $\mathscr{P}$ of choice rules p: so, $\mathscr{P}$-NE appear to be all of the solutions which can be considered if we want to respect consistency (direct and converse).

Also a characterization of $\mathscr{P}$-NE is provided by means of ($\mathscr{P}$-OPR), (NEM) (non emptiness) and (CONS), along the lines of Peleg, Potters, and Tijs (1996).

2. $\mathscr{P}$-Nash equilibria

The main tool to define $\mathscr{P}$-NE is the "choice rule" that was mentioned in the introduction. Actually, it will be assumed that different players may have different choice rules: i.e., we are prepared to consider "personalized" choice rules, which is reflected in the name of $\mathscr{P}$-NE.

Definition 2.1. *A choice rule is a pair* $(\mathscr{U},p)$*, where:*

- $\mathscr{U}$ *is a nonempty set of real valued functions*
- p *is a map that to every* $u \in \mathscr{U}$, $u: A \to \mathbf{R}$, *associates a subset* $p(u)$ *of* A.

Remark 2.1: We (implicitly) assume that every function $u \in \mathscr{U}$ is defined on a nonempty set. We do *not* assume that $p(u) \neq \emptyset$. □

The definition says, in words, that the rule $(\mathscr{U},p)$ "chooses", for a given function u, a subset of A, the domain of u. Usually, we shall refer to the rule simply by p. The set $\mathscr{U}$ will be called the domain of the rule.

For ease of reference, we shall use also the notation $p(A,u)$, where A stands for the domain of the function u. Clearly superfluous, but useful to give "a name" to the domain of u.

Before providing some examples, let us point out that we could have considered choice rules based on preferences, instead of (utility) functions: however, for definiteness, we shall not pursue this point of view.

Example 2.1: In the cases (a)–(g), there is no particular restriction on the choice of the domain $\mathcal{U}$ for p. For definiteness, one could think of $\mathcal{U}$ as being $\mathcal{U}_F$, the set of all real-valued functions defined on all finite sets

(a) $p(A,u) = \text{argmax}_A\, u$
(b) $p(A,u) = A$
(c) $p(A,u) = \{a \in A: u(a) \geq \sup_A u - \varepsilon\}$, where $\varepsilon \geq 0$ is given, independent of u
(d) $p(A,u) = \{a \in A: u(a) \geq k\}$, where $k \in \mathbf{R}$ is given, independent of u
(e) $p(A,u) = \{a \in A: u(a) \geq \sup_A u - \varepsilon \text{ or } u(a) \geq k\}$, where ε and k are as before
(f) $p(A,u) = \{a \in A: u(a) = \max_A u$, if any; else $u(a) \geq \sup A\, u - \varepsilon$, if any; else $u(a) \geq k\}$, where ε and k are as before
Notice that, for $\varepsilon > 0$, in (e) and (f) we have $p(A,u) \neq \emptyset$ for every $u \in \mathcal{U}$ (also for $\mathcal{U} \neq \mathcal{U}_F$)
(g) $p(A,u) = \{a \in A: u(a)$ is an even integer$\}$
(h) Assume $\mathcal{U} = \mathcal{U}_F$:

$$p(A,u) = \{a \in A: u(a) \text{ is equal to the mean value of } u\}$$

(i) Assume $\mathcal{U}$ is the set of all real valued functions defined on subsets of $\mathbf{N}$:

$$p(A,u) = \{a \in A: a \text{ is even}\} \quad \square$$

First of all, we shall assume that it is given a set $\mathcal{N}$ (to be interpreted as the set of potential players), which will be kept fixed throughout all of the paper.

A game is $G = (N, A, u)$, where N is a finite subset of $\mathcal{N}$, $A = \prod_{i \in N} A_i$, where A_i are nonempty sets, and $u = (u_i)_{i \in N}$, with u_i: $A \to \mathbf{R}$.

We shall denote by $\mathcal{G}_{\mathcal{N}}$ the class of all of such games.

Assume now that for each (player) $i \in \mathcal{N}$ is given a choice rule $(\mathcal{U}_i, p_i)$: we shall denote by $\mathcal{P} = (\mathcal{U}_i, p_i)_{i \in \mathcal{N}}$ the profile of these choice rules.

Consider a game $G = (N, A, u) \in \mathcal{G}_{\mathcal{N}}$: this game is coherent with $\mathcal{P}$ if, for every $i \in N$ and for every $a \in A$, the function $\hat{u}_i^a = u_i(\cdot, a_{-i})$: $A_i \to \mathbf{R}$ belongs to $\mathcal{U}_i$. We shall denote by $\mathcal{G}_{\mathcal{P}}$ the class of all of such games.

Definition 2.2. *Let be given $G \in \mathcal{G}_{\mathcal{P}}$, and let $\bar{a} = (\bar{a}_i)_{i \in N} \in A$. We shall say that $\bar{a}$ is a $\mathcal{P}$-NE if $\bar{a}_i \in p_i(A_i, \hat{u}_i^{\bar{a}})$ for every $i \in N$.*

Example 2.2: Let G be a game. If the class $\mathcal{U}$ is conveniently chosen, then $\mathcal{P}$-NE corresponding to the case in which all of the players use the choice rule described in (a) of Example 2.1 are just the Nash equilibria, while the case in which all use the rule (c) gives rise to ε-NE. $\square$

Example 2.3: Let G be a semi-infinite bimatrix game. Assume that $\mathcal{U}$ is chosen as in the previous example. Then, the definition of weakly determined game, given in Lucchetti, Patrone and Tijs (1986), can be given using the rules described in cases (c) and (d) of Example 2.1. The results established in Jurg and Tijs (1993) and Norde and Potters (1995) can be rephrased saying that the games considered there have a $\mathscr{P}$-NE for every $\varepsilon > 0$ and $k \in \mathbf{R}$, where:

- player I uses the rule given in (a)
- player II uses the rule given in (e) □

Example 2.4: In the game given by the following table, (T, L) is the unique $\mathscr{P}$-NE (we consider only pure strategies), if both players adopt the rule described in Example 2.1, (g).

$I \backslash II$	L	R
T	(0, −2)	(2,1)
B	(1,0)	(1,1)

□

Notice that in Example 2.3 the players use different rules. Another obvious instance of this is the case of a saddle point: it is a $\mathscr{P}$-NE for a game where both players look at the same payoff function, but player I maximizes and player II minimizes.

Let us conclude with a warning. We did not put any restriction on the choice rules p_i. For this reason, one could have obtained different strategy profiles as $\mathscr{P}$-NE simply defining them in a slightly different way: e.g., using $\bar{u}_i^{\bar{a}} = u_i|_{A_i \times \{\bar{a}_{-i}\}}$, instead of the $\hat{u}_i^{\bar{a}}$. To avoid such a kind of troubles, one can add the requirement that the choice rules satisfy the following "compatibility condition":

(CC) | given (A, u) and (B, v), s.t. u, $v \in \mathscr{U}$ and A, B are their respective domains, assume that there is a bijection $f: A \to B$ s.t. $v \circ f = u$: then, $p(v) = f(p(u))$.

3. Consistency and converse consistency

Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \mathscr{G}_{\mathscr{N}}$ be a game. Let $S \subseteq N$, $S \neq \emptyset$ and $\bar{x} = (\bar{x}_i)_{i \in N} \in A$. We shall say that $G^{S,\bar{x}} = (S, (A_i)_{i \in S}, (u_i^{\bar{x}})_{i \in S})$ is the *reduced game* of G, w.r.t. S and $\bar{x}$, if $u_i^{\bar{x}}: A_S = \prod_{i \in S} A_i \to \mathbf{R}$ is defined as follows:

$$u_i^{\bar{x}}(y_S) = u_i(y_S, \bar{x}_{N \setminus S})$$

Given a class $\mathscr{G}$ of games, we shall say that ϕ is a solution (on $\mathscr{G}$) if, for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \mathscr{G}$, it is $\phi(G) \subseteq A$.

Definition 3.1. *Let $\mathscr{G}$ be a class of games. We shall say that $\mathscr{G}$ is* **closed under**

reduction *if the following holds:*

$(CLOS)$	*for every* $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \mathcal{G}$, *for every* $\bar{x} \in A$ *and for every* $S \subseteq N$, $S \neq \emptyset$, *we have* $G^{S,\bar{x}} \in \mathcal{G}$.

Assume now that a solution ϕ on $\mathcal{G}$ is given. If the condition above is satisfied for every $\bar{x} \in \phi(G)$, instead of every $\bar{x} \in A$, we shall say that $\mathcal{G}$ is closed under reduction w.r.t. ϕ and we shall denote it by (CLOSϕ).

Definition 3.2. *Let be given a class* $\mathcal{G}$ *of games and a solution* ϕ *(on* $\mathcal{G}$*).*
We shall say that ϕ *satisfies* **consistency** *if:*

$(CONS)$	*for every* $G \in \mathcal{G}$, *for every* $\bar{x} \in A$ *and for every* $S \subseteq N$, $S \neq \emptyset$ *s.t.* $G^{S,\bar{x}} \in \mathcal{G}$: [*if* $\bar{x} \in \phi(G)$, *then* $\bar{x}_S \in \phi(G^{S,\bar{x}})$].

We say that a solution ϕ*, defined on a class* $\mathcal{G}$ *of games satisfying (CLOS), satisfies* **converse consistency** *if:*

$(COCONS)$	*for every* $G \in \mathcal{G}$, *and for every* $\bar{x} \in A$: [*if, for every* S *s.t.* $S \subseteq N$, $S \notin \{\emptyset, N\}$, $\bar{x}_S \in \phi(G^{S,\bar{x}})$, *then* $\bar{x} \in \phi(G)$].

Let be given a class $\mathcal{G} \subseteq \mathcal{G}_{\mathcal{P}}$. On such classes of games, we shall say that a solution ϕ satisfies personalized one-person rationality if the following holds:

($\mathcal{P}$-OPR)	for every $i \in \mathcal{N}$, and for every game $G = (\{i\}, A, u) \in \mathcal{G}$ we have that $\phi(G) = p_i(A, u)$

For the result we want to prove, we need the following:

Lemma 3.1. *Let* $\mathcal{G} \subseteq \mathcal{G}_{\mathcal{P}}$ *be a family of games satisfying (CLOS). Let* ϕ_1, ϕ_2 *be solutions on* $\mathcal{G}$ *s.t.* ϕ_1 *satisfies (*$\mathcal{P}$*-OPR) and (CONS), while* ϕ_2 *satisfies (*$\mathcal{P}$*-OPR) and (COCONS). Then,* $\phi_1 \subseteq \phi_2$.

Proof: By induction on the number of players. For card$(N) = 1$, it is guaranteed by ($\mathcal{P}$-OPR); we have even equality of the sets.

Assume that $\phi_1(H) \subseteq \phi_2(H)$ for every game H s.t. card$(N(H)) \leq k$. We shall prove that for every game G s.t. card$(N(G)) = k + 1$ we have $\phi_1(G) \subseteq \phi_2(G)$.

Let $\bar{x} \in \phi_1(G)$. By (CLOS), $G^{S,\bar{x}} \in \mathcal{G}$ for every $S \subseteq N$. By (CONS), $\bar{x}_S \in \phi_1(G^{S,\bar{x}})$. So, by the induction hypothesis, $\bar{x}_S \in \phi_1(G^{S,\bar{x}}) \subseteq \phi_2(G^{S,\bar{x}})$. But (COCONS) guarantees that $\bar{x} \in \phi_2(G)$. □

We can now state

Theorem 3.1. *Let $\mathscr{G} \subseteq \mathscr{G}_{\mathscr{P}}$ be a family of games satisfying (CLOS). Then, a solution ϕ on $\mathscr{G}$ is the $\mathscr{P}$-NE if and only if ϕ satisfies ($\mathscr{P}$-OPR), (CONS) and (COCONS).*

Proof: It is straightforward to verify that $\mathscr{P}$-NE satisfies ($\mathscr{P}$-OPR), (CONS) and (COCONS). Notice that, to achieve (CONS) and (COCONS), condition (CLOS) and the fact that $\mathscr{G} \subseteq \mathscr{G}_{\mathscr{P}}$ guarantee that the appropriate choice rules can be applied.

Since $\mathscr{P}$-NE satisfies all of the three properties, Lemma 3.1 gives us both $\phi \subseteq \mathscr{P}$-NE and $\mathscr{P}$-NE $\subseteq \phi$. Hence, the proof. □

The proofs of Lemma 3.1 and of Theorem 3.1 show that ($\mathscr{P}$-OPR) lies, so to say, in the background. It has some kind of parametric rôle. In other words, everything is pointing to the fact that a solution satisfying (CONS) and (COCONS) should be some kind of $\mathscr{P}$-NE.

Actually, the following result is an instance of the principle sketched above. It is stated in a special class to avoid too many technical details.

Theorem 3.2. *Let $\mathscr{G} \subseteq \mathscr{G}_{\mathscr{N}}$ be the class of* **all finite games**. *Let ϕ be a solution on $\mathscr{G}$, satisfying (CONS) and (COCONS).*

Then ϕ determines, by one-person games, a **unique** *family of choice rules $(p_i)_{i \in \mathscr{N}}$, each of whom is defined on $\mathscr{U}_F$. Moreover, ϕ is the $\mathscr{P}$-NE determined by these choice rules.*

Proof: Let $i \in \mathscr{N}$, A be a finite set and let $u: A \to \mathbf{R}$. Define $p_i(A, u) = \phi(\{i\}, A, u)$.

By definition of p_i, clearly ϕ satisfies ($\mathscr{P}$-OPR) w.r.t. this family of choice rules.

Since by assumption ϕ satisfies also (CONS) and (COCONS), thanks to Theorem 3.1 we have that $\phi = \mathscr{P}$-NE. □

Let us notice at this point that, since perfect NE do not satisfy (CONS), they cannot be $\mathscr{P}$-NE. The same is true for other refinements of NE which do not satisfy (CONS): see Example 2.4 of Peleg and Tijs (1996).

One word on the independence of the properties ($\mathscr{P}$-OPR), (CONS) and (COCONS). Again, examples 2.16 and 2.17 from Peleg and Tijs (1996) show that neither (CONS) nor (COCONS) are implied by the other two. For what concerns ($\mathscr{P}$-OPR), the previous Theorem gives an answer. Let us notice that for the solution ϕ provided in Peleg and Tijs to show that (OPR) is not implied by (CONS) and (COCONS), we have actually that $\phi = \mathscr{P}$-NE, for p given as in Example 2.1 (b).

Referring to Peleg and Tijs (1996) once more, we point out that it is possible to extend to this setting also their results on the extensive form games: in particular, defining $\mathscr{P}$-SPE (that is, $\mathscr{P}$-subgame perfect equilibrium).

We end this section providing an example of a solution which is not a $\mathscr{P}$-NE. This solution could be seen as a refinement of the (ε-k)-NE which correspond to the $\mathscr{P}$-NE induced by the choice rule described in Example 2.1 (e). In this way it is shown that the phenomenon by which refinements of NE do not satisfy (CONS), extends also to "refinements" of $\mathscr{P}$-NE.

Example 3.1: Let be given $\varepsilon \geq 0$ and $k \in \mathbf{N}$, $k > 1$. Given $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$, we define

$$\phi(G) = \begin{cases} \{\bar{a}\text{: for every } i \in N, \\ \quad u_i(\bar{a}) \geq sup_{A_i} u_i(a_i, \bar{a}_{-i}) - \varepsilon\} & \text{if nonempty,} \\ \{\bar{a}\text{: for every } i \in N, \\ \quad (u_i(\bar{a}) \geq sup_{A_i} u_i(a_i, \bar{a}_{-i}) - \varepsilon \text{ or } u_i(\bar{a}) \geq k)\} & \text{otherwise.} \end{cases}$$

We provide a counterexample to show that ϕ does not satisfy (CONS).

Let $N = \{I, II, III\}$; $A = \{T, B\} \times \mathbf{N} \times \mathbf{N}$; $u_I(x, y, z) = \begin{cases} 1 & \text{if } x = T \text{ and } y = 1 \\ 0 & \text{otherwise} \end{cases}$; $u_{II}(x, y, z) = \begin{cases} 1 & \text{if } x = T \\ y & \text{if } x = B \end{cases}$; $u_{III}(x, y, z) = z$.

Clearly $\bar{x} = (B, k, k)$ is in $\phi(G)$ (notice that the payoff of III prevents the existence of ε-NE).

However, for the reduced game $G^{S, \bar{x}}$, with $S = \{I, II\}$, $(B, k) \notin \phi(G^{S, \bar{x}})$, because $G^{S, \bar{x}}$ has a NE (the point $(T, 1)$). □

4. The ancestor property

Given a class $\mathscr{G}$ of games and a solution ϕ on $\mathscr{G}$, Peleg, Potters and Tijs (1996) have introduced a directed graph, denoted by Graph($\mathscr{G}, \phi$), whose vertices are couples (G, x) with $G \in \mathscr{G}$ and $x \in \phi(G)$.

Two nodes (H, y) and (K, z) in the graph are connected if K is a reduced game of H, given y. That is:

$$N(K) \subseteq N(H), N(K) \notin \{\emptyset, N(H)\}$$

$$K = H^{N(K), y}$$

$$z = y|_{N(K)}$$

Also the following property was introduced in Peleg, Potters and Tijs (1996).

Definition 4.1. *The graph Graph($\mathscr{G}, \phi$) has the ancestor property if:*

(*AP*) | *for every $(G, x) \in$ Graph $(\mathscr{G}, \phi)$, there is a game $H \in \mathscr{G}$ s.t. for every $y \in \phi(H)$ the vertex (H, y) is connected with (G, x).*

It is easy to notice that Theorem 1 of Peleg, Potters and Tijs (1996) holds also if we drop any reference to (OPR). That is, we can prove the following theorem. Before that, a piece of notation: if a solution ϕ on a class $\mathscr{G}$ is non-empty valued, we shall say that ϕ satisfies (NEM).

Theorem 4.1. *Let $\mathscr{G}$ be a class of games, and let ϕ be a solution on $\mathscr{G}$, satisfying (NEM) and (CONS).*

If ϕ is a solution on $\mathscr{G}$ which satisfies (AP), then ϕ is minimal (on $\mathscr{G}$, w.r.t. inclusion) in the class of solutions on $\mathscr{G}$ which satisfy (NEM) and (CONS).

Proof: Let $\bar{\phi}$ be given, which satisfies (NEM) and (CONS), s.t. $\bar{\phi} \subseteq \phi$. We shall prove that $\bar{\phi} = \phi$.

Let $G \in \mathscr{G}$ and $x \in \phi(G)$. Consider H, whose existence is guaranteed by (AP). We have, for (NEM), that $\bar{\phi}(H) \neq \emptyset$. So, let $\bar{y} \in \bar{\phi}(H)$. Now $\bar{y}|_{N(G)} \in \bar{\phi}(H^{N(G),\bar{y}})$ by (CONS) applied to $\bar{\phi}$. But $\bar{y} \in \bar{\phi}(H)$, hence $\bar{y} \in \phi(H)$. So, $\bar{y}|_{N(G)} = x$ by (AP).

Hence, we have proved that $x \in \bar{\phi}(H^{N(G),\bar{y}})$, but $H^{N(G),\bar{y}} = G$ by the definition of Graph$(\mathscr{G}, \phi)$. □

Now, we can prove the following result, which extends to $\mathscr{P}$-NE the considerations made by Peleg, Potters and Tijs (1996) after their Theorem 1.

Theorem 4.2. *Let $\mathscr{G} \subseteq \mathscr{G}_{\mathscr{P}}$ be a class of games s.t. $\mathscr{P}$-NE satisfies (NEM) and (AP) on $\mathscr{G}$.*

Let $\hat{\phi}$ be a solution on $\mathscr{G}$ that satisfies (NEM), (CONS) and ($\mathscr{P}$-OPR). If the class $\mathscr{G}$ satisfies (CLOS $\hat{\phi}$), then $\hat{\phi} = \mathscr{P}$-NE.

Proof: Since $\hat{\phi}$ satisfies ($\mathscr{P}$-OPR) and (CONS), while $\mathscr{P}$-NE satisfies ($\mathscr{P}$-OPR) and (COCONS), we have that $\hat{\phi} \subseteq \mathscr{P}$-NE by Lemma 3.1 (notice that in the proof is actually needed only (CLOS$\hat{\phi}$), not (CLOS)).

But $\mathscr{P}$-NE on $\mathscr{G}$ satisfies (CONS) and (NEM). Since we assumed also that $\mathscr{P}$-NE satisfies (AP) on $\mathscr{G}$, thanks to Theorem 4.1 we can conclude that $\mathscr{P}$-NE is minimal on $\mathscr{G}$ w.r.t. (NEM) and (CONS). So, $\mathscr{P}$-NE $= \hat{\phi}$. □

We shall now see how the same idea used in the proof of Theorem 3 of Peleg, Potters and Tijs (1996) can be used to characterize ε-NE, in a way similar to their characterization for NE. We shall assume that every player in $\mathscr{N}$ has its choice rule with domain $\mathscr{U}_F$. We shall, furthermore, assume that all of these rules satisfy (CC). We need a further restriction on these rules.

Definition 4.2. *A choice rule $(\mathscr{U}, p)$ is said to separate points if:*

(S) | *there exist $\beta, \gamma \in \mathbf{R}$ s.t. for every function $u: \Omega \to \mathbf{R} (u \in \mathscr{U})$ s.t. $u(\Omega) = \{\beta, \gamma\}$, it is $p(\Omega, u) = \{\omega \in \Omega: u(\omega) = \gamma\}$*

The idea behind this definition is clear. Let us add that (S) is not very useful without (CC). Referring to Example 2.1, notice that on $\mathscr{U}_F$ rules (a), (c),(d), (e), (f) and (g) satisfy (S), while the rules (b) and (h) do not. All of them satisfy (CC). For rule (c), when $\varepsilon > 0$, it is sufficient to take $\beta = 0$ and $\gamma = 2\varepsilon$. Notice that the functions u considered in (S) fail to be continuous on "nice" topological spaces (e.g., an interval of the reals), so that this road is barred if one is interested in extending the result that we shall give to a context like those studied in Peleg and Sudhölter (1994) or in Norde, Potters, Reijnierse and Vermeulen (1996).

Let us prove our result:

Theorem 4.3. *Let be given an infinite set $\mathscr{N}$ of potential players. Assume that for every $i \in \mathscr{N}$ we are given a choice rule $(\mathscr{U}_i, p_i)$ s.t. $\mathscr{U}_i = \mathscr{U}_F$ and satisfying (CC) and (S).*

Let $\mathscr{G} = \mathscr{G}_F^{\mathscr{P}\text{-}NE}$ be the class of all finite games, with players in $\mathscr{N}$, which have $\mathscr{P}$-NE.

Then, $\mathscr{P}$-NE satisfies (AP) on $\mathscr{G}$.

Proof: Let $(G, \bar{x}) \in \mathscr{G}$. We shall essentially repeat the construction in Peleg, Potters and Tijs (1996) of a game H to prove (AP).

We add one player to $N(G)$. Take a player $j \in \mathscr{N}$ s.t. $j \notin N$. So, $N(H) = N(G) \cup \{j\}$. The strategy spaces of H are: for $i \in N(G), A_i$ is as in G. For player j, $A_j = \{T, B\}$.

The payoff functions are defined as follows:

for player j:

$U_j(\bar{x}, B) = \gamma_j$
$U_j(\bar{x}, T) = \beta_j$
$U_j(x, B) = \beta_j$ for $x \in A$ s.t. $x \neq \bar{x}$
$U_j(x, T) = \gamma_j$ for $x \in A$ s.t. $x \neq \bar{x}$

for player i:

$U_i(x, B) = u_i(x)$
$U_i(x, T) = \beta_i$ if $x_i \neq \bar{x}_i$
$U_i(x, T) = \gamma_i$ if $x_i = \bar{x}_i$

It can be verified that $(\bar{x}, B)$ is the *unique* $\mathscr{P}$-NE for H (the arguments are as in Peleg, Potters and Tijs (1996)). So, $H \in \mathscr{G}$. It is straightforward to see that $(\bar{x}, B)_{N(G)} = \bar{x}$ and that $H^{N(G),(\bar{x},B)} = G$. □

Corollary 4.1. *Let $\mathscr{N}$ be infinite. Consider the profile $\mathscr{P}_\varepsilon$ that is obtained, given $\varepsilon \geq 0$, when every $i \in \mathscr{N}$ uses the choice rule of Example 2.1 (c) (that is: we are looking at ε-NE).*

Let $\mathscr{G} = \mathscr{G}_F^{\mathscr{P}_\varepsilon\text{-}NE}$ be the class of all finite games, with players in $\mathscr{N}$, which have $\mathscr{P}_\varepsilon$-NE.

Let ϕ be a solution on $\mathscr{G}$ that satisfies (NEM), ($\mathscr{P}_\varepsilon$-OPR), (CLOSϕ) and (CONS). Then, $\phi = \mathscr{P}_\varepsilon$-NE.

The proof of this corollary is a simple adaptation of the proof of Theorem 3 in Peleg, Potters and Tijs (1996), taking into account Theorems 4.3 and 4.2, and the fact that the choice rule used in $\mathscr{P}_\varepsilon$ satisfies (S) and (CC), as can easily be checked.

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