Coordination

in

Hierarchical Control

Coordination in Hierarchical Control

Proefschrift

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Contents

Chapter 1

Introduction

Many processes, both technological and economic, consist of subprocesses that interact with each other and are each controlled by a different decision unit. Often there also exists a separate decision unit which influences every subprocess simultaneously. We call such a process a hierarchical system. Some examples of hierarchical systems are computer networks, some production processes, the ecological system, the European Union. In all these examples there exist several subprocesses that one tries to control separately while a central entity (in the above examples the main system, the production manager, the national government, the European council) tries to steer the whole system. This is formalized by the notion of coordination; the central entity (coordinator) steers the whole system by coordinating the actions of the individual decision units.

In this dissertation we discuss models of coordination in a hierarchical control framework. We will argue that in the existing models for the coordination process to be successful it is necessary that the individual decision units commit themselves to cooperate with the coordinator. However, in practice this commitment is not always a realistic assumption. By reacting strategically to the actions of the coordinator and the other decision units an individual decision unit might actually gain. The term "strategic" is used here to indicate that the individual decision units also use their actions to influence the behavior of the coordinator and the other decision units. In this dissertation we incorporate strategic behavior by the individual decision units into a new framework, making use of dynamic noncooperative game theory (see e.g. Başar and Olsder (1995)). This is done first in the context of two-player repeated games and later in the context of continuously repeated two-player games. The latter model requires the analysis of an infinite-horizon differential game. In this dissertation we provide tools for analyzing the stationary feedback Nash equilibria of such a game.

A prominent example of a hierarchical system, and one that has provided a background motivation for the work in this dissertation, is the European Union. In this case the lower level subsystems are formed by the member countries. On the upper level a major role is played by the Council of Ministers and the European Commission (see e.g. Weeren et al. (1993); Douven (1995)). The institutions on the upper level have the task to coordinate the policies on the individual countries, as can be concluded from (Delors, 1989, page 6):

> ... The integration process thus requires more intensive and effective policy coordination ... Decision-making authorities are subject to many pressures and even best efforts to take into account the international repercussions of their policies are likely to fail at certain times. While voluntary cooperation should be relied upon as much as possible to arrive at increasingly consistent national policies, thus taking account of divergent constitutional situations in member countries, there is also likely to be a need for more binding procedures. The success of the internal market programme hinges to a decisive extent on a much closer coordination of national economic policies, as well as on more effective Community policies.

In this excerpt from the Delors report on Economic and Monetary Union, we see that one of the key issues in the study of EMU concerns issues of coordination. The statement that "voluntary cooperation should be relied upon as much as possible" indicates that a cooperative mode of play is preferred by the upper level, but that it does not exclude the possibility that sometimes "rules" must be enforced. In particular this raises the question under which circumstances voluntary cooperation is likely to occur and how this can be influenced by a coordinator (in this case the European Commission and the Council of Ministers). Of course, if one wants to study the effects of coordination in the European Union, strategic behavior is only one aspect that plays a role. Other important issues are e.g. constitutional changes in the economic structure of the European Union involved with the transition to the final stage of EMU, the influences of the world economy on the European Union, as well as social and political influences. Clearly, building a model for coordination in the European Union, which takes the above-mentioned aspects into account, is a daunting task.

The models introduced in this dissertation are very simple models, based on two-player repeated games in continuous time. The main reason for concentrating on two-player games is that in the two-player case we do not have to concern ourselves with possible coalition forming between the players, because in the two-player case there are only two options, namely players can either cooperate or not. If we would allow for more than two players as would be suggested by the EU-example, players also have to decide on which of the other players they will cooperate with, which would add a huge amount of complexity to our analysis (see for instance the treatment of three-player bargaining situations in Houba (1994)). Needless to say that one should be very careful in concluding whether the approach towards hierarchical systems as taken in this dissertation is suited to study coordination within the European Union. Nevertheless, we believe that the approach as taken in this dissertation can be used as first step towards modelling coordination issues regarding the European Union.

Although coordination in the European Union is an important motivation for our research, we will not discuss the European Union any further in the remainder of this dissertation. As already noted, the models introduced in this dissertation are very simple, and not able to capture all the relevant issues in European policy coordination. Moreover, we believe that our models are suited for a broader range of hierarchical systems. Therefore, in this dissertation we will use a more general and abstract setup, and we will not go into detail regarding some specific applications.

1.1 Outline

The dissertation has the following outline:

Chapter 1: Introduction,

Chapter 2: Models of coordination,

Chapter 3: Repeated games,

Chapter 4: Nash equilibria of differential games,

Chapter 5: Continuously repeated games,

Chapter 6: Conclusions.

In chapter two, we start out by specifying the hierarchical control framework. This setup involves the control of a large-scale system, which can be divided into N subsystems, interacting with each other. For every subsystem a separate policymaker has to decide how to control that subsystem. In order to achieve some prespecified global control objective a coordinator is introduced. This coordinator exchanges information with the separate policymakers in order to achieve the global control objective. In section 2.2 we recall a model specifying such a coordination process, based on Mesarovic et al. (1970); Jamshidi

(1983); Singh (1980). We conclude (as in Weeren (1993)) that for the coordination process to be successful, it is necessary that all policymakers commit themselves to cooperate with the coordinator. As a consequence we note that in the framework as introduced in section 2.2, strategic behavior by the individual policymakers is not modelled, for strategic behavior implies that the individual policymakers should be allowed to deviate from the cooperative strategy in order to influence the coordinator and the other policymakers. This is our aim for the remainder of this dissertation: we will discuss how strategic behavior can be introduced into the hierarchical control framework.

In chapter three we construct a simplified model incorporating strategic behavior in a hierarchical control framework. We focus on repeated games in discrete time, which enables us to concentrate on strategic aspects of coordination without having to worry about other aspects like for instance informational issues. The main tool we use to specify and to analyze the model is dynamic noncooperative game theory (see Başar and Olsder (1995)). Therefore, we start chapter three by recalling some results from noncooperative game theory. After this recapitulation we briefly discuss strategic bargaining theory (see e.g. Houba (1994); Osborne and Rubinstein (1991)). This theory involves the specification of a bargaining procedure as a dynamic game and the analysis of its equilibria. Inspired by this idea we develop in section 3.4 a model for strategic behavior in a hierarchical framework. This model involves the construction of a difference game, the so-called controlled game, based on a static two-player game that is played repeatedly. Analysis of the controlled game shows that it is desirable to rephrase the model as a differential game over an infinite horizon. The specification of the continuous-time controlled game and its analysis is postponed to chapter five.

In chapter four we take a closer look at a special class of nonzero-sum differential games, namely nonzero-sum differential games of the linear-quadratic type. In the case of open-loop information, i.e. the case where every player knows at time t only the initial condition x_0, we derive necessary and sufficient conditions for the existence of a unique open-loop Nash equilibrium. Moreover, a sufficient condition is given under which the open-loop Nash equilibrium can be obtained in the usual manner, i.e. via the solution of a system of coupled Riccati differential equations. Furthermore, we show that, under some well-posedness assumptions, the open-loop Nash equilibrium converges to a unique solution when the horizon tends to infinity.

We will also show that the asymptotic behavior of the so-called feedback Nash equilibrium is more complicated. We give a detailed analysis for the simplest case, namely the case in which the dynamics are scalar. We show that for the feedback Nash equilibrium the associated system of Riccati differential equations can have different stable critical points. This implies in particular that the asymptotic behavior of feedback Nash equilibria depends critically on the weights put on the terminal values of the state $x(t_f)$.

The final subject in chapter four is the study of linear stationary feedback Nash equilibria for linear-quadratic differential games over an infinite horizon. In contrast with the generic uniqueness of the linear feedback Nash equilibrium for linear-quadratic differential games over a finite horizon, we find that in the infinite-horizon case nonuniqueness can be expected, even within the class of linear stationary feedback strategies. The explanation of this apparent contradiction lies in the critical dependence on the weights put on terminal values of the state in the finite-horizon case. Furthermore we show that the criterion of dynamic stability of the critical points is not sufficient to fully eliminate this nonuniqueness.

After the digression in chapter four on the Nash equilibria of LQ-games, in chapter five we will rephrase the model as introduced in chapter three in continuous time. Starting out with a two-player static game which is played repeatedly over time, we introduce a coordination mechanism and a decision rule for the coordinator, influencing the payoffs and strategies of the underlying static game. This leads to a nonlinear differential game with a one-dimensional state space, which we refer to as the controlled game. Unfortunately, it is in general impossible to handle nonlinear differential games analytically, but we will show how two-player nonlinear differential games with a one-dimensional state space can be handled numerically. As is well known, feedback Nash equilibria of differential games can be described by the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations. In chapter five we show how the HJBI equations describing stationary feedback Nash equilibria for infinite-horizon differential games can be handled numerically using recently developed methods for calculating solutions of differential-algebraic equations. In a worked example, where the underlying static game is a symmetric Cournot duopoly, we illustrate the numerical method and discuss the obtained stationary feedback Nash equilibria. In particular we will see that this example allows for uncountably many stationary feedback Nash equilibria. We conclude chapter five with the observation that the choice of coordination mechanism and the choice of decision rule for the coordinator can be viewed as a control problem; by choosing the appropriate mechanism and decision rule the coordinator can steer towards a global control objective.

Finally in chapter six we will review the results obtained in the previous chapters. We draw some conclusions and give some indications for future research. In particular we discuss in which directions the model described in chapter five might be extended.

Chapter 2

Models of coordination

2.1 Introduction

The main purpose of this chapter is to discuss the hierarchical control framework, and to study typical problems associated with such a framework. The setup can be visualized as in figure 2.1.

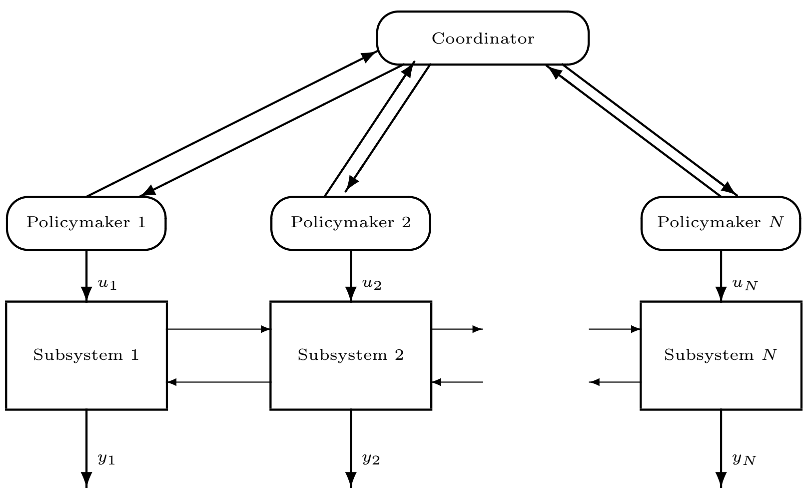


Figure 2.1: The hierarchical control problem.

This setup involves the control of a large-scale system, which can be divided in N subsystems, interacting with each other. We assume that for every subsystem i, $i = 1, \ldots, N$, there is a separate policymaker, who decides on the input for the i-th subsystem, u_i. One of the first references dealing with large-scale systems posessing a hierarchical structure is March and Simon (1958). Mesarovic et al. (1970) presented one of the earliest formal quantitative treatments of hierarchical (multilevel) systems. Since then a great deal of work has been done in the field (see e.g. Cohen (1977); Mahmoud et al. (1977); Singh (1980); Jamshidi (1983)). In Jamshidi (1983) it is noted that there is no uniquely or universally accepted set of properties associated with hierarchical systems. Nevertheless, Jamshidi (1983) notes the following key properties:

1. A hierarchical system consists of decision making components structured in a pyramid shape (as depicted in figure 2.1);
2. the system has an overall goal which may (or may not) be in harmony with all its individual components;
3. the various levels of hierarchy in the system exchange information (usually vertically) among themselves iteratively;
4. as the level of hierarchy goes up, the time horizon increases, i.e. the lower-level components are faster than the higher-level ones.

The main reason for studying this kind of systems in the seventies and early eighties has always been that the direct computation of optimal control solutions for large-scale systems was a rather cumbersome, time-consuming job. Hierarchical systems were mostly constructed by decomposition of large-scale systems, with the purpose of splitting up a large optimization problem into smaller subproblems, which could be handled more easily. In this sense the hierarchy was mostly artificial and mainly introduced just for computational convenience. The idea of decomposition was first treated theoretically in the context of large linear programming problems by Dantzig and Wolfe (1960). One often used method of decomposition for continuous time large-scale systems, is based on time scale separation (see e.g. Avramović (1979); Kokotović et al. (1980)).

Nowadays, in the presence of powerful computers, the motivation for studying hierarchical systems for computational reasons is less pressing, and so only little research in this direction has been done since the seventies and early eighties. However, in reality, many systems have a hierarchical structure by nature or by construction, e.g. computer networks, ecological systems, the economy of the European Community. One can imagine that controlling such systems involves some special problems. Clearly, in the process of finding an appropriate control law a policymaker depends on the other policymakers. To facilitate the choice of suitable control laws, a coordinator is introduced, with the task to steer the individual policy makers towards a situation in which a global control objective is established. As we will argue at the end of this chapter, one can make a distinction here between cooperative and noncooperative hierarchical control. In the cooperative setting we assume that all policymakers commit themselves to cooperate with the coordinator, in the sense that they will follow the directions of the coordinator exactly. As opposed to this we can study coordination in a noncooperative setting, where we do not assume such a commitment a priori. The model for coordination as treated in section 2.2, which goes back to the model of Mesarovic et al. (1970), is a typical example of cooperative hierarchical control. We will note that commitment to cooperate with the coordinator is a crucial assumption in this model. A first model using noncooperative ideas is proposed by Ito and de Zeeuw (1990). In section 2.3 we will briefly discuss this model. We will end this chapter with some concluding remarks in section 2.4.

2.2 Solution concepts in hierarchical optimal control

2.2.1 Introduction

This section is mainly based on Jamshidi (1983); Singh (1980); Weeren (1993). In this section we confine ourselves to open-loop information structures (see also chapter four), where every policymaker knows at time instant t the values of the initial states x_{0i} for all i. We will define a special class of hierarchical control problems. We assume that the dynamics of the subsystems are described by linear difference equations, with time-invariant parameters. The local control objectives we study involve the minimization of quadratic cost functionals. The global control objective we study is the minimization of an aggregate cost functional. Note that from a game theoretic point of view this can be related to the problem of finding a Pareto efficient solution, in case we view the hierarchical system as a dynamic game (see theorem 3.41). The restriction to a discrete-time linear-quadratic setup is for convenience only, the same approach as used in this section can be applied to more general systems.

2.2.2 Problem definition

As mentioned in the introduction, we study large-scale linear time-invariant systems, consisting of N smaller subsystems, interacting with each other. We can describe such a system in the following way :

$$x_i(t+1) = A_i x_i(t) + \sum_{j \neq i} A_{ij} x_j(t) + B_i u_i(t),\ x_i(0) = x_{i0}, \tag{2.1}$$

where:

$x_i(t)$ **:** an n_i-dimensional vector, representing the state of the i-th subsystem;

$u_i(t)$ **:** an m_i-dimensional vector, representing the control input of the i-th subsystem.

For each subsystem we can now define a cost functional in the following way:

$$J_i(u) = x_i(t_f)'Q_i x_i(t_f) + \sum_{t=0}^{t_f-1} \{x_i(t)'Q_i x_i(t) + u_i(t)'R_i u_i(t)\}, \tag{2.2}$$

where $Q_i \geqslant 0$ and $R_i > 0$ are weighting matrices.

For the whole interconnected system we define the aggregate cost functional:

$$J(u) := \sum_{i=1}^{N} J_i(u). \tag{2.3}$$

We can formulate the global minimization problem:

$$\begin{cases} \min\limits_u J(u) \\ \text{w.r.t.} \\ x_i(t+1) = A_i x_i(t) + \sum\limits_{j \neq i} A_{ij} x_j(t) + B_i u_i(t), \\ x_i(0) = x_{i0}. \end{cases} \tag{2.4}$$

In order to describe a process of coordination, we use a approach which goes back to Mesarovic et al. (1970). We first assume that an extra input, z_i is entering the i-th subsystem, being the interactions coming from the other $N-1$ subsystems. Now, in a sense, we cut the links between the subsystems and we suppose z_i acts as a variable which is manipulated by the i-th subsystem, just like x_i and u_i. Since z_i can be arbitrarily chosen by policymaker i, it is clear that in general $z_i \neq \sum\limits_{j \neq i} A_{ij} x_j$. In this way the minimization problem is completely decoupled into N subproblems, to be solved by the N policymakers. In order to make sure that the individual subproblems yield a solution to the original problem, it is necessary that the *interaction balance principle* is satisfied, i.e. the independently selected z_i's actually become equal to $\sum A_{ij} x_j$, for all i. Formally, this can be achieved by introducing a weighting parameter λ, which is called the *coordination variable* , and which penalizes the performance of the system when the interactions do not balance. Hence, to the cost functional (2.3) a penalty term is added:

$$L(x, u, z, \lambda) := \sum_{i=1}^{N} J_i(x_i, u_i, z_i) + \sum_{t=0}^{t_f-1} \lambda(t)'e(t), \tag{2.5}$$

where $e_i(t)$ is the error

$$e_i(t) := z_i(t) - \sum_{j \neq i} A_{ij} x_j(t). \tag{2.6}$$

The problem is now split up into two levels as follows. On the first level every policymaker minimizes for given λ

$$L_i(x_i, u_i, z_i, \lambda) := J_i(x_i, u_i) + \sum_{t=0}^{t_f - 1} \left(\lambda_i(t)' z_i(t) - \sum_{j \neq i} \lambda_j(t)' A_{ji} x_i(t) \right), \tag{2.7}$$

with respect to the dynamics

$$x_i(t+1) = A_i x_i(t) + B_i u_i(t) + z_i(t),\ x_i(0) = x_{i0}. \tag{2.8}$$

On the second level of the problem the coordinator manipulates the coordination variable λ in order to achieve an interaction error zero.

Remark 2.1 The interaction variable λ can be interpreted as the Lagrange parameter corresponding to the constraint $z_i = \sum_{j \neq i} A_{ij} x_j$.

2.2.3 The coordination process

In this section we solve the minimization problems, and we derive an algorithm, the interaction prediction algorithm, as a numerical example of how a coordination process can be implemented.

First we solve the first-level problems, i.e. we are going to minimize L_i for given λ, with respect to the constraint

$$x_i(t+1) = A_i x_i(t) + z_i(t) + B_i u_i(t),\ x_i(0) = x_{i0}.$$

We do this by using the maximum principle. (See for instance Başar and Olsder (1995)). The Hamiltonians are given by

$$H_i(\cdot) = x_i' Q_i x_i + u_i' R_i u_i + \lambda_i' z_i - \sum_{j \neq i} \lambda_j' A_{ji} x_i + p_i' \left[A_i x_i + z_i + B_i u_i \right], \tag{2.9}$$

and the adjoint states p_i satisfy

$$p_i(t) = Q_i x_i(t) - \sum_{j \neq i} A_{ji}' \lambda_j(t) + A_i' p_i(t+1), \tag{2.10}$$

with $p_i(t_f) = Q_i x_i(t_f)$.

The optimal u_i, obtained using $\frac{\partial H_i}{\partial u_i} = 0$, is given by

$$u_i(t) = -E_i(t+1)^{-1} B_i' \left[K_i(t+1)\left(A_i x_i(t) + z_i(t)\right) + g_i(t+1)\right]. \tag{2.11}$$

$K_i(t)$ satisfies the backward Riccati difference equation

$$\begin{aligned} K_i(t) &= Q_i + A_i' K_i(t+1) A_i - A_i' K_i(t+1) B_i E_i(t+1)^{-1} B_i' K_i(t+1) A_i, && (2.12)\\ K_i(t_f) &= Q_i, && (2.13) \end{aligned}$$

with

$$E_i(t+1) := R_i + B_i' K_i(t+1) B_i. \tag{2.14}$$

$g_i(t)$ satisfies the following backward difference equation:

$$\begin{aligned} g_i(t) &= A_i' G_i(t+1)' \left[K_i(t+1) z_i(t) + g_i(t+1)\right] - \sum_{j \neq i} A_{ji}' \lambda_j(t), && (2.15)\\ g_i(t_f) &= 0, && (2.16) \end{aligned}$$

where

$$G_i(t+1) := I - B_i E_i(t+1)^{-1} B_i' K_i(t+1). \tag{2.17}$$

Note that g_i is *not* independent of z_i.

Another necessary condition for optimality is $\frac{\partial H}{\partial z_i} = 0$, where $H = \sum_{i=1}^{N} H_i$. Note that $\frac{\partial H}{\partial z_i} = 0$ if and only if $\frac{\partial H_i}{\partial z_i} = 0$. From $\frac{\partial H_i}{\partial z_i} = 0$ we derive

$$\lambda_i(t) = -p_i(t+1). \tag{2.18}$$

Remember that we have chosen λ arbitrarily. In general our choice for λ will not satisfy (2.18). The coordinator's objective, in the second-level problem, is to find a better choice for λ, better in the sense that λ is closer to its optimal value and for all i z_i is closer to $\sum_{j \neq i} A_{ij} x_j$. In the first-level problem we have found (for fixed λ and z) optimal $(\hat{x}_i, \hat{u}_i, \hat{p}_i)$ belonging to the pairs (λ, z_i). Then we can use (2.18) to update λ and we can calculate $z_i = \sum A_{ij} \hat{x}_j$. For these new choices we can again solve the first-level problems, etc. We obtain the following algorithm:

Algorithm 2.2 (Interaction Prediction.)
Step 1 : $k := 1$. For all i and for all t initialize $\lambda_i^1(t) := 0$ and $z_i^1(t) := 0$.

Step 2 : Calculate for all i and t the solutions $K_i(t)$ of the backward Riccati difference equation (2.12).

Step 3 : Calculate for all i and t the solutions $g_i^k(t)$ of the backward difference equation (2.15).

Step 4 : Calculate u_i^k and x_i^k using (2.11) and (2.1). Calculate p_i^k using

$$p_i^k(t) = K_i(t)x_i^k(t) + g_i^k(t).$$

Step 5 : Check whether

$$\sum_{i,t} \|z_i^k(t) - \sum_{j \neq i} A_{ij}x_j^k(t)\|^2 < \varepsilon$$

for some small $\varepsilon > 0$ given a priori.
If not then $\lambda_i^{k+1}(t) := -p_i^k(t+1)$ and $z_i^{k+1}(t) := \sum_{j \neq i} A_{ij}x_j^k(t)$. Go to *step 3*.

Step 6 : Stop.

Remark 2.3 The name interaction prediction can be explained by the fact that in the $(k+1)$-th iteration we use $z_i^{k+1} := \sum_{j \neq i} A_{ij}x_j^k$, where x^k is the optimal x found in the k-th iteration, as a prediction for the interactions $\sum_{j \neq i} A_{ij}x_j$. Another approach could be that one only updates λ every step and calculates z_i^k in the first-level problems. Then the second-level problem can be solved using well-known iterative search methods. This is called the goal coordination approach (see Mesarovic et al. (1970)). However, the goal coordination method has some serious drawbacks (see e.g. Jamshidi (1983); Singh (1980)). Convergence is slower, but more important is that in order to use the goal coordination method, one has to include a term $z_i' S_i z_i$, $S_i > 0$, into the cost functionals J_i in order to avoid singularity problems.

Remark 2.4 In general convergence of the interaction prediction algorithm cannot be guaranteed. However, a sufficient condition for convergence, in terms of two self-adjoint operators can be obtained (see Cohen (1977)).

2.2.4 The noncooperative problem

In this section we again study the system given by (2.1). We have seen in subsection 2.2.3 that it is possible to find a coordination scheme in order to achieve minimization of the global cost functional J, as defined by (2.3). In this setting we suppose that for every subsystem i, $i = 1 \ldots N$, policymaker i decides what the control u_i will be. As one can imagine, in most practical situations every policymaker is primarily interested in minimizing his own cost functional J_i. In general this will not lead to minimization of the aggregate cost functional $J = \sum_{i=1}^{N} J_i$. In order to achieve minimization of the overall cost functional J, we introduced a coordinator. The task of the coordinator is to negotiate with all policymakers in order to balance the interactions between the subsystems.

From the nature of the hierarchical control problem, it is necessary that all policymakers are willing to cooperate in order to minimize the overall cost functional J. One can imagine that not all policymakers are willing to cooperate with a coordinator. Therefore, we will now study the problem where the policymakers act strictly noncooperative. For this problem we will formulate the Nash equilibrium concept, and we will investigate whether a Nash equilibrium exists and how a Nash equilibrium can be calculated. For a more extensive treatment on Nash equilibria we refer to chapters three and four of this dissertation.

Definition 2.5 Consider the interconnected system described by (2.1). We call a control function $\bar{u} = (\bar{u}_1', \ldots, \bar{u}_N')'$ a *Nash equilibrium* if

$$J_i(\bar{u}_1, \ldots, \bar{u}_i, \ldots, \bar{u}_N) = \min_{u_i} J_i(\bar{u}_1, \ldots, u_i, \ldots, \bar{u}_N) \tag{2.19}$$

for all $i = 1, \ldots, N$.

It can be proved that for the system (2.1) there exists a generically unique Nash equilibrium (see Başar and Olsder (1995), section 6.2.1[1]), due to the fact that for every i, $i = 1, \ldots, N$, J_i is strictly convex in u_i.

Lemma 2.6 *Suppose a control function $\bar{u}$ (with corresponding $\bar{x}$) satisfies:*

$$\bar{u}_i(t) = -E_i(t+1)^{-1} B_i' \left(K_i(t+1) \left(A_i \bar{x}_i(t) + \sum_{j \neq i} A_{ij} \bar{x}_j(t) \right) + \bar{g}_i(t+1) \right), \tag{2.20}$$

where $K_i(t)$ satisfies the backward difference Riccati equation (2.12). $E_i(t)$ is defined by (2.14), and $\bar{g}_i(t)$ satisfies the following backward difference equation:

$$\bar{g}_i(t) = A_i' G_i(t+1)' \left(K_i(t+1) \sum_{j \neq i} A_{ij} \bar{x}_j(t) + \bar{g}_i(t+1) \right), \tag{2.21}$$

$$\bar{g}_i(t_f) = 0. \tag{2.22}$$

[1] In this chapter we assume an open-loop information structure. Therefore we only consider open-loop Nash equilibria. In Başar and Olsder (1995), section 6.2.1, it is proved that in the linear-quadratic case, with $R_i > 0$ and $Q_i \geqslant 0$ for all i, there exists a unique open-loop Nash equilibrium, under some regularity condition.

Then $\bar{u}$ is a Nash equilibrium.

Proof : This follows immediately by minimizing J_i over u_i, for given $\bar{u}_j$, $j \neq i$. □

Lemma 2.7 *Suppose $\bar{u}$ is a Nash equilibrium. Then $\bar{u}$ is of the form (2.20).*

Proof : Suppose $\bar{u}$ is a Nash equilibrium. Then $\bar{u}_i$ minimizes J_i, $i = 1 \dots N$, for given $\bar{u}_j$, $j \neq i$. Because $R_i > 0$ and $Q_i \geqslant 0$, it follows that $\bar{u}_i$ is of the form (2.20). □

Combining both lemmata with the existence and generic uniqueness as established in Başar and Olsder (1995), we find the following corollary:

Corollary 2.8 *For the hierarchical control problem, there exists a generically unique open-loop Nash equilibrium $\bar{u}$ given by (2.20,2.12,2.14,2.17,2.21).*

Although we know there exists a generically unique Nash equilibrium for our system it is not yet a trivial matter to calculate the Nash equilibrium. Note that the equation (2.20) only gives an implicit characterization of the Nash equilibrium, and therefore an algorithm is needed in order to calculate the Nash equilibrium. Based on equations (2.20)–(2.21) we find the following algorithm:

Algorithm 2.9
***Step 1* :** For all i calculate $K_i(t)$ from (2.12).

***Step 2* :** k:=0. Choose suitable u_i^0, x_i^0, g_i^0, for all $i = 1, \dots, N$.

***Step 3* :** Calculate x_i^{k+1} using :

$$\begin{aligned} x_i^{k+1}(t+1) &= G_i(t+1)\left(A_i x_i^{k+1}(t) + \sum_{j \neq i} A_{ij} x_j^k(t)\right) \\ &\quad - B_i E_i(t+1)^{-1} B_i' g_i^k(t+1), \\ x_i^{k+1}(0) &= x_{i0}. \end{aligned} \tag{2.23}$$

***Step 4* :** Calculate g_i^{k+1} using (2.21). Then calculate u_i^{k+1} using (2.20).

***Step 5* :** If $\|u_i^{k+1} - u_i^k\| < \varepsilon$ then stop else $k := k+1$, go to step 3.

Remark 2.10 When one compares this algorithm with the interaction prediction algorithm 2.2, the main difference is the absence of the coordination variable λ, and related to that the fact that the adjoint state variables p_i are no longer needed in the calculations.

Lemma 2.11 *Suppose the algorithm 2.9 starts with* $x^0 = \bar{x}$, $u^0 = \bar{u}$, $g^0 = \bar{g}$. *Then* $x^1 = \bar{x}$, $u^1 = \bar{u}$ *and* $g^1 = \bar{g}$. *Hence* $(\bar{x}, \bar{u}, \bar{g})$ *is a fixed point for the algorithm.*

Proof : By equation (2.23) we have:

$$\begin{aligned} x_i^{k+1}(t+1) &= G_i(t+1)\left(A_i x_i^{k+1}(t) + \sum_{j\neq i} A_{ij} x_j^k(t)\right) - B_i E_i(t+1)^{-1} B_i' g_i^k(t+1), \\ x_i^{k+1}(0) &= x_{i0}. \end{aligned}$$

Now suppose $(x^0, u^0, g^0) = (\bar{x}, \bar{u}, \bar{g})$. Then x^1 satisfies:

$$\begin{aligned} x_i^1(t+1) &= G_i(t+1)\left(A_i x_i^1(t) + \sum_{j\neq i} A_{ij} \bar{x}_j(t)\right) - B_i E_i(t+1)^{-1} B_i' \bar{g}_i(t+1), \\ x_i^{k+1}(0) &= x_{i0} \;=\; \bar{x}_i(0). \end{aligned}$$

Necessarily $x^1 = \bar{x}$. Then it follows straightforwardly that $u^1 = \bar{u}$ and $g^1 = \bar{g}$. Hence $(\bar{x}, \bar{u}, \bar{g})$ is a fixed point for the algorithm 2.9. □

By corollary 2.8 we see that, provided that the algorithm 2.9 converges, $\bar{u} := \lim_{k\to\infty} u^k$ is the Nash equilibrium.

Finally we will study the question under what conditions the algorithm 2.9 converges. We define the following matrices:

$$E_{ii}^{(1)} := \begin{pmatrix} I & & & \\ -G_i(2)A_i & I & & \\ & \ddots & \ddots & \\ & & -G_i(t_f)A_i & I \end{pmatrix}, \tag{2.24}$$

$$E_{ii}^{(2)} := \begin{pmatrix} I & -A_i' G_i(2)' K_i(2) & & \\ & \ddots & \ddots & \\ & & I & -A_i' G_i(t_f)' K_i(t_f) \\ & & & I \end{pmatrix}, \tag{2.25}$$

$$E_{ii} := \left(\begin{array}{c|c} E_{ii}^{(1)} & \\ \hline & E_{ii}^{(2)} \end{array}\right), \tag{2.26}$$

$$E_{ij} := \left(\begin{array}{ccc|c} & 0 & & 0 \\ \hline -A_i' G_i(1)' K_i(1) A_{ij} & & & \\ & \ddots & & 0 \\ & & -A_i' G_i(t_f) K_i(t_f) A_{ij} & \end{array}\right), \tag{2.27}$$

$$F_{ii} := \left(\begin{array}{c|ccc} & -B_iE_i(1)^{-1}B_i' & & \\ 0 & & \ddots & \\ & & & -B_iE_i(t_f)^{-1}B_i' \\ \hline 0 & & 0 & \end{array}\right), \tag{2.28}$$

$$F_{ij} := \left(\begin{array}{ccc|c} 0 & \cdots & 0 & \\ G_i(2)A_{ij} & & & \\ & \ddots & & 0 \\ & & G_i(t_f)A_{ij} & \\ \hline & 0 & & 0 \end{array}\right), \tag{2.29}$$

$$H_i := \begin{pmatrix} G_i(1)A_{i1} & \cdots & G_i(1)A_i & \cdots & G_i(1)A_{iN} \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}, \tag{2.30}$$

$$E := \begin{pmatrix} E_{11} & \cdots & E_{1N} \\ \vdots & \ddots & \vdots \\ E_{N1} & \cdots & E_{NN} \end{pmatrix}, \tag{2.31}$$

$$F := \begin{pmatrix} F_{11} & \cdots & F_{1N} \\ \vdots & \ddots & \vdots \\ F_{N1} & \cdots & F_{NN} \end{pmatrix}, \tag{2.32}$$

$$H := \begin{pmatrix} H_1 \\ \vdots \\ H_N \end{pmatrix}. \tag{2.33}$$

We also define:

$$\xi_{i,1}(k) := \left(x_i^k(1)', \ldots, x_i^k(t_f)'\right)', \tag{2.34}$$

$$\xi_{i,2}(k) := \left(g_i^k(1)', \ldots, g_i^k(t_f-1)'\right)', \tag{2.35}$$

$$\xi_i(k) := (\xi_{i,1}(k)', \xi_{i,2}(k)')', \tag{2.36}$$

$$\xi(k) := (\xi_1(k)', \xi_2(k)', \ldots, \xi_N(k)')'. \tag{2.37}$$

The algorithm 2.9 is in fact nothing else than the iteration described by:

$$E\xi(k+1) = F\xi(k) + Hx_0, \tag{2.38}$$

$$\xi_{i,1}(0) = \left(x_i^0(1)', \ldots, x_i^0(t_f)'\right)', \tag{2.39}$$

$$\xi_{i,2}(0) = \left(g_i^0(1)', \ldots, g_i^0(t_f-1)'\right)'. \tag{2.40}$$

Remark 2.12 Note that the algorithm 2.9 for every $x_0 \in \mathbb{R}^{n_1+n_2+\cdots+n_N}$ and given initial choice $\xi(0)$ defines a unique sequence $(\xi(k))_{k\in\mathbb{N}}$. This implies that the system given by (2.38) is well defined.

Hence, a necessary and sufficient condition for the algorithm 2.9 to converge is that the system (2.38) is asymptotically stable. This gives us the following main result:

Theorem 2.13 *The algorithm 2.9 converges to the Nash equilibrium if and only if*

$$\forall_{|z|\geqslant 1} \det\left(E - \frac{1}{z}F\right) \neq 0.$$

Proof : We have already argued that the algorithm 2.9 is convergent if and only if the system (2.38) is asymptotically stable. The system (2.38) is asymptotically stable if and only if for all z with $|z| \geqslant 1$ we have that $\det(zE - F) \neq 0$, or equivalently if and only if

$$\forall_{|z|\geqslant 1} \det\left(E - \frac{1}{z}F\right) \neq 0.$$

□

Corollary 2.14 *A sufficient condition for convergence of the algorithm 2.9 is*

$$\|F\| < \sigma_{\min}(E),$$

where $\sigma_{\min}(E)$ is the minimal singular value of the matrix E.

Remark 2.15 Although theorem 2.13 gives a necessary and sufficient condition for global convergence of the algorithm 2.9, the criterion will in practice often be too complicated to be useful. The theorem can be used as a starting point to derive simpler conditions that are either necessary or sufficient.

2.3 The Ito model

In Ito and de Zeeuw (1990) an alternative framework for hierarchical control is proposed. The main purpose of this model, to be called the Ito model in the sequel, is to study integration issues in the European Community. In the Ito model the dynamics on the lower level, are assumed to be described by

$$x_i(t+1) = A_i x_i(t) + A_{0i} x_0(t) + B_i u_i(t) + D_i d_i(t), \; x_i(0) = x_{i0}, \tag{2.41}$$

and moreover for the coordinator, an aggregate model is introduced

$$x_0(t+1) = A_0 x_0(t) + B_0 u_0(t) + D_0 d_0(t), \; x_0(0) = x_{00}. \tag{2.42}$$

Here $x_i(t) \in \mathbb{R}^{n_i}$ is the state variable of the i-th subsystem, $u_i(t) \in \mathbb{R}^{m_i}$ is the control variable of the i-th subsystem and $d_i(t) \in \mathbb{R}^{\ell_i}$ is the vector of exogenous inputs entering the i-th subsystem, $i = 1, \ldots, N$. Moreover, the aggegrates x_0, u_0 and d_0 are obtained as

$$x_0(t) \;:= \sum_{i=1}^{N} W_i^x(t) x_i(t), \tag{2.43}$$

$$u_0(t) \;:= \sum_{i=1}^{N} W_i^u(t) u_i(t), \tag{2.44}$$

$$d_0(t) \;:= \sum_{i=1}^{N} W_i^d(t) d_i(t), \tag{2.45}$$

where for all t $W_i^x(t), W_i^u(t)$ and $W_i^d(t)$ are weighting matrices of appropriate dimensions, $i = 1, \ldots, N$. It is understood that (2.41) and (2.42) must be compatible with (2.43,2.44,2.45), i.e. the following assumption is implicitly made:

Assumption 2.16
There exist control functions u_i and exogenous inputs d_i such that the Ito model is consistent, i.e. there exist u_i and d_i such that for all $t = 0, \ldots, t_f - 1$

$$\begin{aligned} &\sum_{i=1}^{N} W_i^x(t+1) \left(A_i x_i(t) + A_{0i} \sum_{j=1}^{N} W_j^x(t) x_j(t) + B_i u_i(t) + D_i d_i(t) \right) = \\ &\sum_{i=1}^{N} \left(A_0 W_i^x(t) x_i(t) + B_0 W_i^u(t) u_i(t) + D_0 W_i^d(t) d_i(t) \right), \end{aligned} \tag{2.46}$$

for given consistent initial conditions, i.e. initial conditions such that

$$x_{00} = \sum_{i=1}^{N} W_i^x(0) x_{i0}.$$

In Ito and de Zeeuw (1990) the hierarchical control problem is split up in a central problem, to be solved by the coordinator and N local problems to be solved by the coordinator.

Definition 2.17 (Central problem) Define the cost functional J_0 by

$$J_0 := \sum_{t=0}^{t_f-1} \left\{ \|x_0(t) - x_0^*(t)\|_{Q_0}^2 + \|u_0(t) - u_0^*(t)\|_{R_0}^2 \right\}.$$

Then the *central problem* is the minimization problem

$$\min_{u_0} J_0(u_0)$$

with respect to the dynamics

$$x_0(t+1) = A_0 x_0(t) + B_0 u_0(t) + D_0 d_0(t),\ x_0 = x_{00}.$$

Here $R_0 > 0$ and $Q_0 \geqslant 0$ are weighting matrices and $\|x_0 - x_0^*\|^2_{Q_0}$ is a shorthand notation for $(x_0 - x_0^*)' Q_0 (x_0 - x_0^*)$. $x_0^*(t)$ and $u_0^*(t)$ are desired paths given a priori.

Denote by $\hat{x}_0$ the optimal solution of the central problem. The local problems are defined by

Definition 2.18 (Local problem) Define the cost functional J_i by

$$J_i := \sum_{t=0}^{t_f-1} \left\{ \|x_i(t) - x_i^*(t)\|^2_{Q_i} + \|u_i(t) - u_i^*(t)\|^2_{R_i} \right\}.$$

Then the *i-th local problem* is the minimization problem

$$\min_{u_i} J_i(u_i)$$

with respect to the dynamics

$$x_i(t+1) = A_i x_i(t) + A_{0i}\hat{x}_0(t) + B_i u_i(t) + D_i d_i(t),\ x_i = x_{i0}.$$

Here $R_i > 0$ and $Q_i \geqslant 0$ are weighting matrices. $x_i^*(t)$ and $u_i^*(t)$ are desired paths given a priori.

For determining the desired paths u_0^* and x_0^* in Ito and de Zeeuw (1990) two different approaches are proposed. In the first approach, the so-called *top-down* approach, the coordinator selects u_0^* and x_0^* himself, independently of the individual (local) policymakers. As opposed to this top-down approach, the *bottom-up* approach is introduced, in which

$$u_0^*(t) = \sum_{i=1}^{N} W_i^u(t) u_i^*(t),$$

and similarly

$$x_0^*(t) = \sum_{i=1}^{N} W_i^x(t) x_i^*(t),$$

for $t = 0, \ldots, t_f - 1$.

In Ito et al. (1991) an econometric model for the European Community is estimated, satisfying assumption 2.16. Using this model the central problem and the local problems are solved in a myopic fashion (as was also done in Ito and de Zeeuw (1990)). This means that at every time instant the optimization problems

$$\min_{u_0(t)} \left\{ \|x_0(t) - x_0^*(t)\|_{Q_0}^2 + \|u_0(t) - u_0^*(t)\|_{R_0}^2 \right\}$$

for the central problem, and

$$\min_{u_i(t)} \left\{ \|x_i(t) - x_i^*(t)\|_{Q_i}^2 + \|u_i(t) - u_i^*(t)\|_{R_i}^2 \right\},$$

on the lower level, are solved. In Weeren et al. (1993) the open-loop solutions to the central problem and the local problems are derived using the maximum principle, and have been used to simulate the model as proposed in Ito et al. (1991).

There are some drawbacks to the Ito model. The most severe drawbacks orginate from assumption 2.16. Although *a priori* this assumption is satisfied by construction, it is by no means guaranteed that the optimal solutions $\hat{x}_0$, $\hat{u}_0$ satisfy (2.43) and (2.44). In this sense assumption 2.16 is not necessarily satisfied *a posteriori*. In order to correct this, one should constrain the set of admissible solutions. However, in general one might expect that the set of admissible solutions, i.e. solutions $(\hat{x}_i, \hat{u}_i, d_i)$, $i = 1, \ldots, N$, satisfying (2.46), is too small to allow for interesting solutions, because the equation (2.46) imposes too heavy a restriction on the set of admissible solutions. In particular, once the central problem is solved (within the class of admissible solutions), the freedom to choose the individual $\hat{u}_i$'s is severely limited, leading to the conclusion that the individual policymakers do not have enough freedom left to choose their policies. This leads to the conclusion that for studying coordination issues, in this way the model is overspecified.

Even if one decides to allow for solutions that do not satisfy the conditions mentioned in assumption 2.16, there are some additional drawbacks to the model. Most notable is the absence of direct interactions between the subsystems in the Ito model.

To improve the Ito model some changes are necessary. First of all the restrictions imposed by the aggregate model should be relaxed. Moreover direct interactions between the subsystems should be added. In Douven (1995) the following alternative is proposed. Consider a model in which the dynamics are described by

$$x_i(t+1) = A_i x_i(t) + \sum_{j \neq i} A_{ij} x_j(t) + B_i u_i(t) + D_i d_i(t),\ x_i(0) = x_{i0}, \tag{2.47}$$

and cost functionals given by

$$J_i = \sum_{t=0}^{t_f-1} \left\{ \|x_i(t) - x_i^*(t)\|_{Q_i}^2 + \|u_i(t) - u_i^*(t)\|_{R_i}^2 \right\}. \tag{2.48}$$

For the coordinator the cost functional C is introduced[2], which is defined as

$$C = \sum_{i=1}^{N} C_i, \tag{2.49}$$

where

$$C_i = \sum_{t=0}^{t_f-1} \|x_i(t) - \bar{x}(t)\|_{M_i(t)}^2, \tag{2.50}$$

$$\bar{x}(t) = \sum_{i=1}^{N} W_i(t) x_i(t). \tag{2.51}$$

Then some equilibria of the difference game (see section 3.2 for the definition of difference games), with modified cost functionals

$$\tilde{J}_i := (1-\lambda)\, J_i + \lambda C_i, \tag{2.52}$$

for $\lambda \in (0,1)$ are studied (see Douven (1995)). Note that in this setup the idea of an aggregate model for the coordinator is abandoned.

2.4 Conclusions

We will conclude this chapter on coordination models with a discussion of the model as introduced in section 2.2 and its properties. In section 2.2 we discussed a model for hierarchical control in which the coordinator was given the task to steer the model towards a Pareto efficient solution. This was accomplished by the introduction of the coordination variable λ, manipulated by the coordinator in order to balance the interactions. The information flow for the coordination process, between the coordinator and an individual policy maker, can be depicted as in figure 2.2.

In every step of the coordination process every policymaker transmits his intended control function u_i and the desired value of the interaction input z_i to the coordinator, and in return

[2] In Douven (1995) this cost functional is called the convergence function.

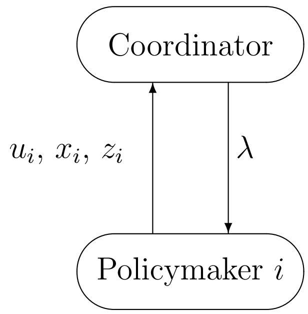


Figure 2.2: Information flow in the coordination process

he receives the coordination variable λ. The coordination process, as specified in section 2.2, takes place before the actual controls $\hat{u}_i$ are implemented. The information used by the policymakers to determine the control functions $\hat{u}_i$ is the initial value of the state x_0, which is assumed to be known to every policymaker, and the coordination variable λ. Therefore we can conclude that the system is controlled in an open-loop fashion. However, in most applications it is not reasonable to look for open-loop solutions. Moreover, it is more likely that coordination will take place while the system is controlled, instead of before the system is controlled. This would lead to a flow of information where at every time instant t a policymaker transmits the current value of his state $x_i(t)$, the intended control action $u_i(t)$ and his wish $z_i(t+1)$ to the coordinator, and in return he receives information $\lambda(t)$ from the coordinator. In this case the information available for policymaker i at time instant t is

$$\eta_i(t) = \{x_i(t), \lambda(t-1)\}, \tag{2.53}$$

and the information the coordinator has at t is

$$\eta^c(t) = \{x_i(t), u_i(t), z_i(t+1) \mid i = 1, \ldots, N\}. \tag{2.54}$$

So the first-level problem is now how to choose $u_i(t)$ and $z_i(t+1)$ based on $\eta_i(t)$. The second-level problem involves the choice of $\lambda(t)$ based on $\eta^c(t)$. The solution of these problems is not a trivial matter, as it is related to the issue of incomplete information in dynamic games, which is still an unsolved problem (except in some very special cases).

Apart from these informational considerations, we like to stress another problem regarding the model as described in section 2.2. We have noted in subsection 2.2.4 that for the coordination process to be successful, it is necessary that all policymakers commit themselves to cooperate with a coordinator. As a consequence strategic behavior by the individual policymakers is not incorporated in the framework as sketched in section 2.2. By strategic behavior we mean that the individual policymakers also use their controls u_i to influence the behavior of the coordinator and of the other policymakers. By the commitment to co-operate with a coordinator, individual policymakers cannot deviate from $\hat{u}_i$. This implies that the framework of section 2.2 is not very well suited to model hierarchical systems where policymakers have possibly conflicting interests, for in this case especially strategic considerations are of predominant importance. In the remainder of this dissertation, we will focus on some of the problems involved with the introduction of strategic behavior into the hierarchical control framework.

Chapter 3

Repeated games

3.1 Introduction

In this chapter, we will make a first effort towards the incorporation of strategic behavior into the hierarchical control framework. In chapter two, we noted that one of the shortcomings of the hierarchical control framework as described in Jamshidi (1983) and Singh (1980) is the absence of the possibility to react strategically to the coordinator's directions. As a start to model strategic behavior, we will concentrate on repeated games in discrete time. In this way, we obtain a stylized model in which we can concentrate on strategic aspects of coordination and neglect other aspects like e.g. the dynamics of the underlying system or imperfect information. Based on a static game which is repeatedly played, we construct a model describing strategic behavior by the individual players. This model is partly inspired by the theory of (strategic) bargaining (see e.g. Houba (1994); Osborne and Rubinstein (1991)); therefore we will also briefly discuss some elements of bargaining theory. A detailed analysis of the final model is postponed to chapter five, where we will consider the model in continuous time.

As the remainder of this dissertation relies heavily on solution concepts from noncooperative game theory, we start this chapter by recalling some standard results from noncooperative game theory (for a more extensive treatment see e.g. Fudenberg and Tirole (1991); Gibbons (1992)). In particular, we will introduce the Nash equilibrium concept and the concept of Pareto efficiency. Furthermore, we will define the concept of subgame perfectness and discuss the Folk theorem for infinitely repeated games in discrete time.

This chapter can roughly be divided into three parts. In section 3.2 we recall some preliminaries from noncooperative game theory, which are used in the remainder of this dissertation. In section 3.3 we will briefly consider the bargaining model as introduced by Rubinstein (see van Damme (1991); Fudenberg and Tirole (1991); Gibbons (1992); Osborne and Rubinstein (1991)). Then, in section 3.4 we will propose a model introducing strategic behavior in a hierarchical situation.

3.2 Preliminaries on game theory

We will use the following notation. Let G be an N-player game in strategic form. This means that G is characterized by sets Γ_i, $i = 1, \dots, N$, called the (pure[1]) strategy spaces of player i, and mappings $\pi_i : \Gamma_1 \times \cdots \times \Gamma_N \to \mathbb{R}$, denoting the payoffs of player i, $i = 1, \dots, N$. The objective of player i is to choose $\gamma_i \in \Gamma_i$ such that the payoff π_i is maximized. Throughout this dissertation we allow the strategy spaces Γ_i to be infinite. Denote by

$$\Gamma := \Gamma_1 \times \cdots \times \Gamma_N \tag{3.1}$$

the cartesian product of the individual strategy spaces and by γ_{-i} the $(N-1)$-tuple of strategies

$$\gamma_{-i} := \big(\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_N\big). \tag{3.2}$$

3.2.1 Nash equilibria

In this subsection we introduce the standard equilibrium concept in noncooperative game theory: the Nash equilibrium (see Nash (1951)). Following van Damme (1991) we note

> A solution of a noncooperative game is a set of recommendations which tell each player how to behave in every situation that may arise. This solution should be consistent, i.e. no player should have an incentive to deviate from his recommendation. Hence, a solution must be *self-enforcing*: As long as the others obey their recommendations, it should not be in my interest to deviate.

The Nash equilibrium does exactly this. It is constructed in such a way, that no player has the incentive to deviate unilaterally from the Nash equilibrium. Formally, the Nash equilibrium is introduced in the following way. We start by defining the concept of best replies.

[1]In this dissertation we only consider pure strategies. Because of the fact that in general we will use infinite strategy spaces, introduction of mixed strategies may lead to measure-theoretic difficulties.

Definition 3.1 $\hat{\gamma}_i$ is a *best reply against* γ_{-i} if

$$\pi_i(\gamma_1, \ldots, \hat{\gamma}_i, \ldots, \gamma_N) = \max_{\gamma_i \in \Gamma_i} \pi_i(\gamma_1, \ldots, \gamma_i, \ldots, \gamma_N).$$

Denote by $R_i(\gamma_{-i})$ the set of all best replies of player i against γ_{-i}.

Then the Nash equilibrium is defined in the following way:

Definition 3.2 The N-tuple of strategies $\bar{\gamma} \in \Gamma$ is a *Nash equilibrium* of G, if

$$\bar{\gamma}_i \in R_i(\bar{\gamma}_{-i})$$

for every $i \in \{1, \ldots, N\}$.

Note that the game G does not necessarily admit a Nash equilibrium. Moreover, if there exists a Nash equilibrium it is not necessarily unique. Rosen (1965) proved the following theorem (see also van Damme (1991); Fudenberg and Tirole (1991)):

Theorem 3.3 (Rosen,1965) *Let $G = (\Gamma_1, \ldots, \Gamma_N, \pi_1, \ldots, \pi_N)$ be a game in strategic form such that the following three conditions are satisfied for each $i \in \{1, \ldots, N\}$:*

1. *Γ_i is a nonempty, compact and convex subset of some finite-dimensional Euclidean space,*
2. *π_i is continuous,*
3. *π_i is concave in γ_i for all γ_{-i}.*

Then G possesses at least one Nash equilibrium.

In general G will have multiple Nash equilibria. One way of dealing with the problem of nonuniqueness is by looking at refinements of the Nash equilibrium concept (see van Damme (1991)). This is motivated by the observation that in case of multiple Nash equilibria, some of the equilibria are "less desirable" than others. The key idea is to select Nash equilibria which satisfy some additional nice properties like stability or perfection. One way to introduce a refinement of a Nash equilibrium is by requiring the equilibrium to belong to a subset S of Γ which has some nice properties (see for instance the definition of Markov perfect equilibria in Maskin and Tirole (1994)). A possible complication in using this approach is that the game G restricted to S might allow for a Nash equilibrium which is not a Nash equilibrium of the original game G. This can be avoided by demanding S to be closed under best replies.

Definition 3.4 Let $S \subset \Gamma$. Then S is called *closed under best replies* if

$$\forall_{i \in \{1,\ldots,N\}} R_i(\gamma_{-i}) \cap S_i \neq \emptyset$$

for all $\gamma \in S$. Here S_i is $S|_{\Gamma_i}$.

This definition basically says that if all other players choose their strategy in S, then I have a best reply against that strategy, which also lies in S. Now the following proposition is obvious.

Proposition 3.5 *Let $S \subset \Gamma$ and consider the restricted game $G|_S$, i.e. the game G with the strategy spaces Γ_i replaced by S_i. Suppose S is closed under best replies. Suppose $\bar{\gamma}$ is a Nash equilibrium of $G|_S$. Then $\bar{\gamma}$ is also a Nash equilibrium of G.*

Dynamic games

In the remainder of this subsection, we will discuss (deterministic) dynamic games in extensive form. The extensive-form representation of a dynamic game is the representation that explicitly displays the rules of the game, i.e. it specifies the following data (see e.g. Başar and Olsder (1995); Fudenberg and Tirole (1991); van Damme (1991)):

(i) the order of moves in the game,

(ii) for every decision point, which player has to move at that point,

(iii) the information a player has whenever it is his turn to move,

(iv) the choices available to a player when he has to move,

(v) the payoffs for all players.

Formally:

Definition 3.6 The *extensive form* of a N-person finite dynamic game is a five-tuple

$$\mathcal{G} = (T, P, \Upsilon, \mathcal{C}, \pi)\,,$$

where

- the game tree T is a finite tree with a distinguished node o, the origin of T; $\text{succ}(x)$ the immediate successors of x; the endpoints

$$Z := \{x \mid \text{succ}(x) = \emptyset\};$$

  and the decision points X, the complement of Z;

- the player partition P is a partition of X into N sets $P_1, \dots, P_N$, the set of decision points of player i,

- the information partition Υ is an N-tuple $(\Upsilon_1, \dots, \Upsilon_N)$, where Υ_i is a partition of P_i (into so called *information sets* of player i) such that for every information set every path intersects the information set at most once and all nodes in the information set have the same number of immediate successors; the information set $\eta \in \Upsilon_i$ which contains $x \in P_i$ represents the set of nodes player i cannot distinguish from x based on the information he has when he has to move at x;

- the choice partition $\mathcal{C}$ is a collection of partitions $C(\eta)$

$$\mathcal{C} = \left\{ C(\eta); \eta \in \bigcup_{i=1}^{N} \Upsilon_i \right\},$$

  where $C(\eta)$ is a partition of $\bigcup_{x \in \eta} \text{succ}(x)$ into so called *choices* at η, such that every choice contains exactly one element of $\text{succ}(x)$ for every $x \in \eta$,

- the payoff function π is an N-tuple $(\pi_i, \dots, \pi_N)$ where π_i is a real-valued function with domain the endpoints of T; if the endpoint z is reached, player i gets the payoff $\pi_i(z)$.

Definition 3.7 A (pure) strategy γ_i of player i is a mapping which assigns a choice $c \in C(\eta)$ to every information set $\eta \in \Upsilon_i$. Denote by Γ_i the set of all mappings γ_i from Υ_i to $\mathcal{C}$.

When an N-tuple of strategies γ is played, this will result in a unique path in T, connecting o to a certain $\tilde{z} \in Z$. Hence playing the N-tuple of strategies γ will result in the N-tuple of payoffs $\pi(\tilde{z})$. This gives rise to a mapping

$$\Pi : \Gamma \to \mathbb{R}, \ \gamma \mapsto \pi(\tilde{z}). \tag{3.3}$$

Note that (Γ, Π) defines a game in strategic form.

Definition 3.8 The game

$$\mathcal{S}(\mathcal{G}) := (\Gamma, \Pi)$$

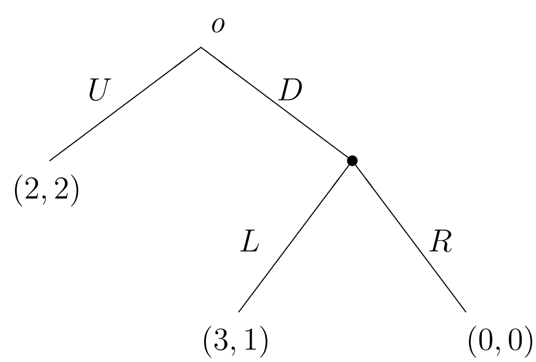
is called the *strategic form of* $\mathcal{G}$.

The definition of a Nash equilibrium of an extensive form game is now obvious.

Definition 3.9 Let $\mathcal{G} := (T, P, \Upsilon, \mathcal{C}, \pi)$ be a finite dynamic game in extensive form. Then γ is a Nash equilibrium of $\mathcal{G}$ if γ is a Nash equilibrium of $\mathcal{S}(\mathcal{G})$.

It is easy to show that for an N-tuple of stategies to be a Nash equilibrium it is only necessary that rational behavior is prescribed at every information set which is actually reached when the equilibrium is played; at every other information set the behavior may be arbitrary. Selten (in Selten (1965)) argues that for this reason some Nash equilibria are "more reasonable" then others. Consider the following example.

Example 3.10
Consider the following extensive form game $\mathcal{G}$:



The strategic form of $\mathcal{G}$ is given by:

	L	R
U	2,2	2,2
D	3,1	0,0

Inspection of the game reveals that there exist two Nash equilibria, i.e. (D, L) and (U, R). However, the Nash equilibrium (U, R) is not credible in the sense that it relies on the empty threat by player 2 to play R.

From this example it is clear that especially in the case of extensive form games there is a need for refinement of the Nash equilibrium concept to exclude "unreasonable" Nash equilibria. To this end one defines subgame perfect Nash equilibria. First we define subgames.

Definition 3.11 Let $x \in X$ and T_x be the subtree descending from x. If every information set of $\mathcal{G}$ either is contained in T_x or is disjoint from T_x, then the restriction of $\mathcal{G}$ to T_x constitutes a game of its own, to be called the *subgame* $\mathcal{G}_x$ *starting at* x. In this case every N-tuple of strategies decomposes into a pair (γ_{-x}, γ_x) where γ_x is an N-tuple of strategies in $\mathcal{G}_x$ and γ_{-x} is an N-tuple of strategies for the remaining part of the game (*the truncated game*). If it is known that γ_x will be played in $\mathcal{G}_x$ then, in order to analyze $\mathcal{G}$, it suffices to analyze the truncated game $\mathcal{G}_{-x}$, which results from $\mathcal{G}$ by replacing the subtree T_x by a single endpoint with payoff $\Pi_{i,x}(\gamma_x)$ for every player i.

Now we can introduce the subgame perfectness refinement.

Definition 3.12 Let $\bar{\gamma}$ be a Nash equilibrium of $\mathcal{G}$. Then $\bar{\gamma}$ is called a *subgame perfect* Nash equilibrium if for every subgame $\mathcal{G}_x$ of $\mathcal{G}$ the restriction $\bar{\gamma}_x$ of $\bar{\gamma}$ to $\mathcal{G}_x$ constitutes a Nash equilibrium of $\mathcal{G}_x$.

The following lemma can easily be proved:

Lemma 3.13 *If $\bar{\gamma}_x$ is a Nash equilibrium of the subgame $\mathcal{G}_x$ and $\bar{\gamma}_{-x}$ is a Nash equilibrium of the truncated game $\mathcal{G}_{-x}$, then $\bar{\gamma} := (\bar{\gamma}_{-x}, \bar{\gamma}_x)$ is a Nash equilibrium of $\mathcal{G}$.*

Proof : Because of the fact that $\bar{\gamma}_x$ is a Nash equilibrium for $\mathcal{G}_x$, no player has the incentive to deviate from $\bar{\gamma}$ once the node x is reached. Moreover, because $\bar{\gamma}_{-x}$ is a Nash equilibrium of $\mathcal{G}_{-x}$, the equilibrium path of $\mathcal{G}_{-x}$ ends in x and no player has the incentive to deviate from $\bar{\gamma}$ until x is reached. □

Remark 3.14 This lemma implies that a subgame perfect Nash equilibrium can be found by means of dynamic programming.

Differential and difference games

Infinite dynamic games, i.e. games in which the the choice sets of the players comprise a continuum of alternatives and players gain some dynamic information throughout the decision process, cannot be described by an extensive form consisting of a finite tree structure. Instead, the extensive form of an infinite dynamic game involves a difference (in discrete time) or a differential (in continuous time) equation, describing the evolution of the decision process. An extensive treatment on extensive forms for infinite dynamic games can be found in (Başar and Olsder, 1995, chapter 5).

Definition 3.15 A *difference game* of prescribed fixed duration involves

(i) An index set $\mathbb{T} = \{0, \ldots, t_f\}$, denoting the *stages* of the game,

(ii) a set X, called the *state space* of the game, to which the *state* $x(t)$ belongs for all $t \in \mathbb{T}$,

(iii) sets $U_i(t)$, defined for $t \in \{0, \ldots, t_f - 1\}$ and $i = 1, \ldots, N$, which are called the *control* sets of player i at time t; their elements are the permissible actions $u_i(t)$ at time t,

(iv) a function $f_t : X \times U_1(t) \times \cdots \times U_N(t) \to X$, defined for each $t \in \{0, \ldots, t_f - 1\}$, such that

$$x(t+1) = f_t(x(t), u_1(t), \cdots, u_N(t)),$$

for given $x_0 \in X$, x_0 is called the *initial state* of the game; this difference equation is called the *state equation* of the difference game,

(v) sets $Y_i(t)$, defined for each $t \in \{0, \ldots, t_f - 1\}$ and each $i = 1, \ldots, N$, which is called the *observation set* of player i at time t, to which the *observation* $y_i(t)$ belongs at time t,

(vi) a function $h_{i,t} : X \to Y_i(t)$, defined for each $t \in \{0, \ldots, t_f - 1\}$ and $i = 1, \ldots, N$, such that

$$y_i(t) = h_{i,t}(x(t)),$$

which is the *observation* equation of player i with respect to the state $x(t)$,

(vii) a finite collection $\eta_i(t)$, defined for each $t \in \{0, \ldots, t_f - 1\}$ and $i = 1, \ldots, N$, as a part of

$$\left(y_1(0), \ldots, y_1(t), u_1(0), \ldots, u_1(t-1), \ldots, y_N(0), \ldots, y_N(t), u_N(0), \ldots, u_N(t-1)\right),$$

which determines the information gained and recalled by player i at time t; specification of $\eta_i(t)$ for all t characterizes the *information structure* of player i, and the collection over $i = 1, \ldots, N$ characterizes the *information structure of the game*,

(viii) a set $\Upsilon_i(t)$ defined for all $t \in \{0, \ldots, t_f - 1\}$ and each $i = 1, \ldots, N$, as an appropriate subset of

$$\prod_{j=1,\ldots,N} Y_j(0) \times \cdots \times Y_j(t) \times U_j(0) \times \cdots \times U_j(t-1),$$

compatible with $\eta_i(t)$, called the *information space* of player i at time t, induced by its information $\eta_i(t)$,

(ix) a prescribed class $\Gamma_i(t)$ of mappings $\gamma_i(t) : \Upsilon_i(t) \to U_i(t)$ which are the admissible strategies of player i at time t, the aggregate mapping

$$\gamma_i = \{\gamma_i(0), \ldots, \gamma_i(t_f - 1)\}$$

is a *strategy* for player i in the game, and the class Γ_i of all such mappings γ_i is the *strategy space* of player i,

(x) functionals

$$\pi_i : (X \times U_1(0) \times \cdots \times U_N(0)) \times \cdots \times (X \times U_1(t_f - 1) \times \cdots \times U_N(t_f - 1)) \times X \to \mathbb{R},$$

called *the payoff*[2] of player i in the game of fixed duration.

Remark 3.16 More general definitions, e.g. difference games of variable duration or difference games over an infinite horizon can be given along the same lines, see e.g. (Başar and Olsder, 1995, section 5.2).

Similarly, differential games can be introduced.

Definition 3.17 A *differential game* $D(\Gamma; \mathbb{T})$ of prescribed fixed duration involves

(i) A time interval $\mathbb{T} = [0, t_f]$, specified *a priori*, which denotes the duration of the evolution of the game,

(ii) a set S_0 with some topological structure, called the *trajectory space* of the game, its elements $x : \mathbb{T} \to S^0$ constitute the *admissible state trajectories* of the game, where $S^0 \subseteq \mathbb{R}^n$,

(iii) sets U_i with some topological structure, defined for $i = 1, \ldots, N$, which is called the *control space* of player i, whose elements are the *control functions* $u_i : \mathbb{T} \to S^i$ of player i, where $S^i \subseteq \mathbb{R}^{m_i}$

(iv) a differential equation,

$$\dot{x}(t) = f(t, x(t), u_1(t), \ldots, u_N(t)), \; x(0) = x_0$$

whose solution describes the state trajectory of the game corresponding to the N-tuple of control functions $(u_1 \ldots, u_N)$, for given initial state $x_0 \in S^0$,

[2]Usually difference games and differential games are defined in terms of costs instead of payoffs. Obviously, replacing costs by payoffs and minimization by maximization does not change any of the results in this dissertation.

(v) nondecreasing functions $\epsilon_i : \mathbb{T} \to \mathbb{T}$ such that for all $t \in \mathbb{T}$ $0 \leqslant \epsilon_i(t) \leqslant t$ and a function $\eta_i(t)$, defined for $i = 1, \ldots, N$, as

$$\eta_i(t) = x|_{[0,\epsilon_i(t)]},$$

$\eta_i(t)$ determines the state information gained and recalled by player i at time $t \in \mathbb{T}$; specification of $\eta_i(\cdot)$ characterizes the *information structure* of player i, and the collection over $i = 1, \ldots, N$ characterizes the *information structure of the game*,

(vi) the sigma field $\Upsilon_i(t)$ in S_0, generated by the cylinder sets

$$\{x \in S_0 \mid x(s) \in B(s), 0 \leqslant s \leqslant \epsilon_i(t)\}$$

where $B(s)$ is a Borel set in S^0 for all $s \in [0, \epsilon_i(t)]$; $\Upsilon_i(t)$ is called the *information field* of player i,

(vii) a prescribed class Γ_i of mappings $\gamma_i : \mathbb{T} \times S_0 \to S^i$, with the property that $\gamma_i(t, \cdot)$ is $\Upsilon_i(t)$ measurable for all $t \in \mathbb{T}$; Γ_i is called the *strategy space* of player i,

(viii) two functionals $q_i : S^0 \to \mathbb{R}$, $g_i : \mathbb{T} \times S^0 \times S^1 \times \cdots \times S^N \to \mathbb{R}$, defined for $i = 1, \ldots, N$, such that the composite functional

$$L_i(u_1, \ldots, u_N) := \int_0^{t_f} g_i(t, x(t), u_1(t), \ldots, u_N(t))dt + q_i(x(t_f))$$

is well defined[3], for every $u_j(t) = \gamma_j(t, x)$, $\gamma_j \in \Gamma_j$, and for all $i = 1, \ldots, N$; L_i is the *cost (or payoff) functional* of player i.

Remark 3.18 Note that a differential game, as formulated above is not yet well-defined, unless we impose some additional constraints on some of the terms introduced. In particular, conditions on Γ_i and f are needed, such that the differential equation admits a unique solution for every N-tuple of strategies $\gamma \in \Gamma$ (see Başar and Olsder (1995)).

Remark 3.19 We will often consider *memoryless perfect state* information, which is defined as $\eta_i(t) = (x_0, x(t))\,,\ t \in \mathbb{T}$. Note that strictly speaking this information structure is not covered by definition 3.17. However, (see theorem 5.1 in Başar and Olsder (1995)), it can be shown that for this information structure, when $S_0 = C^n(\mathbb{T} \to S^0)$, f continuous in t and uniformly Lipschitz in $x, u_1, \ldots, u_N$ and γ_i continuous in t and uniformly Lipschitz in x, the differential equation admits a unique solution for every $\gamma \in \Gamma$.

[3] See Başar and Olsder (1995).

Note that, for any difference game and well-defined differential game, every N-tuple of strategies $\gamma \in \Gamma$ defines a unique state trajectory, and hence induces a mapping

$$J : \Gamma \to \mathbb{R}, \gamma \mapsto L(\gamma_1(\eta_1), \ldots, \gamma_N(\eta_N)),$$

such that (Γ, J) defines a game in strategic form. Hence a Nash equilibrium of a difference (resp. differential) game is now easily defined as the Nash equilibrium of the associated strategic form game.

Consider the memoryless perfect state information structure for a difference or a differential game, $\eta_i(t) = (x_0, x(t))$. It can easily be shown, see e.g. chapter four of this dissertation, that in general there exist many Nash equilibria for dynamic games with this information structure. In this case it can again be argued that some Nash equilibria are more "reasonable" than others. Therefore we introduce the following refinement[4].

Definition 3.20 Let $\beta \in \Gamma$. The *truncated* differential game $D^{\beta}_{[s,t_f]}$ is defined by

$$D^{\beta}_{[s,t_f]} := D(\left\{\gamma \in \Gamma \mid \gamma_{[0,s)} = \beta_{[0,s)}, \gamma_{[s,t_f]} \in \Gamma_{[s,t_f]}\right\}, [0, t_f]).$$

A Nash equilibrium $\bar{\gamma} \in \Gamma$ is *strongly time consistent* if its truncation $\bar{\gamma}_{[s,t_f]}$ to the interval $[s, t_f]$ is a Nash equilibrium for the truncated differential game $D^{\beta}_{[s,t_f]}$, for every $\beta \in \Gamma$.

Remark 3.21 Strongly time consistent Nash equilibria under memoryless perfect state information are necessarily independent of the initial state (see (Başar and Olsder, 1995, page 257)). Therefore a strongly time consistent Nash equilibrium necessarily involves feedback strategies, i.e. they only depend on current values of the state. Furthermore, (see also lemma 3.13), strongly time consistent Nash equilibria can be found by means of dynamic programming. This justifies the following definition of feedback Nash equilibria (see also definition 6.6 in Başar and Olsder (1995)).

Definition 3.22 For a differential game under the memoryless perfect state information structure, an N-tuple of strategies $\bar{\gamma}$ constitutes a *feedback* Nash equilibrium, if there exist functions $V_i(\cdot, \cdot)$ defined on $\mathbb{T} \times S^0$, satisfying the following relations for each $i = 1, \ldots, N$:

$$\begin{aligned} V_i(t, x) &= \int_t^{t_f} g_i(\tau, \bar{x}(\tau), \bar{\gamma}_1(\tau, \eta_1(\tau)), \ldots, \bar{\gamma}_N(\tau, \eta_N(\tau)))d\tau + q_i(\bar{x}(t_f)) \\ &\leqslant \int_t^{t_f} g_i(\tau, x_i(\tau), \bar{\gamma}_1(\tau, \eta_1(\tau)), \ldots, \gamma_i(\tau, \eta_i(\tau)), \ldots, \bar{\gamma}_N(\tau, \eta_N(\tau)))d\tau + q_i(x_i(t_f)), \end{aligned}$$

[4] We introduce time-consistency and feedback Nash equilibria here for differential games. It is however obvious how to define the same concepts and to obtain similar results for difference games.

for all $t \in \mathbb{T}$, $\gamma_i \in \Gamma_i$, $x \in \mathbb{R}^n$, and where, on $\mathbb{T}$

$$\dot{x}_i(s) = f(t, x_i(s), \bar{\gamma}_1(s, \eta_1(s)), \ldots, \gamma_i(s, \eta_i(s)), \ldots, \bar{\gamma}_N(s, \eta_N(s))),\ x_i(t) = x,$$
$$\dot{\bar{x}}(s) = f(s, \bar{x}(s), \bar{\gamma}_1(s, \eta_1(s)), \ldots, \bar{\gamma}_N(s, \eta_N(s))),\ \bar{x}(t) = x.$$

Remark 3.23 Another way of introducing the feedback Nash equilibrium is by restriction of the class of admissible strategies to feedback strategies (see also the introduction of Markov perfect equilibria in Maskin and Tirole (1994)). It is easily shown that the subclass of feedback strategies is closed under best replies (see definition 3.4).

We have the following result:

Theorem 3.24 *Every feedback Nash equilibrium is strongly time consistent and vice versa, i.e. the class of strongly time consistent Nash equilibria coincides with the class of feedback Nash equilibria.*

Proof : From definition 3.22 it is immediately clear that any feedback Nash equilibrium is strongly time consistent. Now let $\bar{\gamma}$ be a strongly time consistent Nash equilibrium. Then it is easily verified that the functions V_i defined on $\mathbb{T} \times S^0$ by

$$V_i(t, x) := \int_0^{t_f} g_i(\tau, \bar{x}(\tau), \bar{\gamma}_1(\tau, \eta_1(\tau)), \ldots, \bar{\gamma}_N(\tau, \eta_N(\tau)))d\tau + q_i(\bar{x}(t_f)),$$

satisfy the inequalities as specified in definition 3.22. Hence $\bar{\gamma}$ is a feedback Nash equilibrium. □

3.2.2 The Folk theorem for infinitely repeated games

A special class of difference games is formed by repeated games. Let $G = (\Gamma, \pi)$ be a game in strategic form. Denote by $G(T)$ the game in which G is played T times. Furthermore, let $0 < \delta < 1$, and denote by $G(\infty, \delta)$ the game in which G is repeated infinitely and the payoffs are discounted by a factor δ. Denote by $h(1)$ initial information available to all players before $G(T)$ or $G(\infty, \delta)$ is played and for $t \geqslant 2$ define

$$h(t) := (h(1), \gamma(1), \gamma(2), \ldots, \gamma(t-1)), \tag{3.4}$$

the history of the game until time t, and denote by

$$H(t) := \{h(1)\} \times \Gamma^{t-1}, \tag{3.5}$$

the set of all possible histories until time t. Throughout this dissertation we will assume that all the players have perfect recall, i.e. at every time instant t every player knows $h(t)$ exactly.

Definition 3.25 A (pure) strategy s_i is a sequence of maps

$$s_i(t) : H(t) \to \Gamma_i$$

which maps possible histories $h(t) \in H(t)$ to (pure) strategies $\gamma_i \in \Gamma_i$ of G. Denote by S_i the set of all (pure) strategies of player i and define

$$S := S_1 \times \cdots \times S_N.$$

For the the finitely repeated game the payoff Π_i of player i is

$$\Pi_i(s) := \sum_{t=1}^{T} \pi_i(s(h_i(t)),$$

and, under the assumption that π_i is bounded for all i, for the infinitely repeated game

$$\Pi_i(s) := \sum_{t=1}^{\infty} \delta^{t-1}\pi_i(s(h_i(t)).$$

Denote by $\mathcal{S}(G(T))$ the strategic form game (S, Π), and similarly $\mathcal{S}(G(\infty, \delta))$.

We can now straightforwardly define a Nash equilibrium for the repeated game $G(T)$ or $G(\infty, \delta)$.

Definition 3.26 $\bar{s} \in S$ is a Nash equilibrium of the repeated game $G(T)$ (resp. $G(\infty, \delta)$) if it is a Nash equilibrium of $\mathcal{S}(G(T))$ (resp. $\mathcal{S}(G(\infty, \delta))$).

Similar to definition 3.11 we can define subgames of a repeated game $G(T)$ or $G(\infty, \delta)$.

Definition 3.27 Let $G(T)$ be a repeated game and let s be an N-tuple of strategies. Then the *subgame* $G(T)|_{h(t)}$ is the repeated game $G(T - t + 1)$ with $h(1)$ replaced by $h(t)$ and s restricted accordingly.
Let $G(\infty, \delta)$ be an infinitely repeated game and let s be an N-tuple of strategies. Then the *subgame* $G(\infty, \delta)|_{h(t)}$ is the repeated game $G(\infty, \delta)$ with $h(1)$ replaced by $h(t)$ and s restricted accordingly.

Then a subgame perfect Nash equilibrium is defined in the following (obvious) way.

Definition 3.28 Let $\bar{s}$ be a Nash equilibrium of $G(T)$ (resp. $G(\infty, \delta)$). Then $\bar{s}$ is *subgame perfect* if $\bar{s}$ is a Nash equilibrium for every subgame of $G(T)$ (resp. $G(\infty, \delta)$) when restricted to that subgame.

The following proposition is easily proved.

Proposition 3.29 *Let $\bar{\gamma}$ be a Nash equilibrium of G. Then the strategy $\bar{s}$ defined by*

$$\forall_{i,t}\ \bar{s}_i(h(t)) = \bar{\gamma}_i$$

is a subgame perfect Nash equilibrium both for $G(T)$ and for $G(\infty, \delta)$.

Proof : It is easily verified that $\bar{s}$ is a Nash equilibrium for $G(T)$ or for $G(\infty, \delta)$. Subgame perfectness follows directly by noting that $\bar{s}$ does not use any information at all, and therefore is a Nash equilibrium of any subgame. □

For the finitely repeated game $G(T)$ we also find the following proposition.

Proposition 3.30 *Suppose G has a unique Nash equilibrium $\bar{\gamma}$. Then the subgame perfect Nash equilibrium $\bar{s}$ in which $\bar{\gamma}$ is played at every time instant is the unique subgame perfect Nash equilibrium of $G(T)$.*

Proof : By dynamic programming. Consider the subgame $G(T)|_{h(T)}$. Independently of $h(T)$ this subgame has the unique Nash equilibrium $\bar{\gamma}$. Backward induction yields the advertized result. □

Remark 3.31 In the proof of this proposition we saw that the final stage becomes of predominant importance; the players know that, no matter what has happened before, they will always play the one-shot equilibrium in the last round. However, in most actual situations one cannot exclude the possibility of meeting the opponent once more, hence, a model in which the exact number of repetitions is known in advance is unrealistic. In this sense infinitely repeated games offer a better approximation to model long term competition.

We end this subsection with the Folk theorem for infinitely repeated games (see van Damme (1991); Friedman (1971); Fudenberg and Tirole (1991); Gibbons (1992)).

Theorem 3.32 (Folk Theorem) *Consider the infinitely repeated game $G(\infty, \delta)$. Let $(e_1, \ldots, e_N)$ be the payoffs of a Nash equilibrium of G, and let $(x_1, \ldots, x_N)$ be any other N-tuple of admissible payoffs of G. If for every $i = 1, \ldots, N$ $x_i > e_i$, and δ is close enough to one, then there exists a subgame perfect Nash equilibrium of $G(\infty, \delta)$, with average payoffs*[5] *$(x_1, \ldots, x_N)$.*

[5] The average payoffs of $G(\infty, \delta)$ are defined as $(1 - \delta)\,\Pi$.

Proof : Suppose $\bar{\gamma}$ is a Nash equilibrium of G with payoffs $(e_1, \ldots, e_N)$. Define γ^x to be the strategies of G with result in the payoffs $(x_1, \ldots, x_N)$. Now consider s^{trigger} (the so-called trigger strategy), for player i:

$$s_i^{\text{trigger}}(1) := \gamma_i^x,$$
$$s_i^{\text{trigger}}(t) := \begin{cases} \gamma_i^x & \text{if } h(t) \text{ only contains } \gamma^x, \\ \bar{\gamma}_i & \text{otherwise.} \end{cases}$$

It is easily verified that when all players adopt s^{trigger} as their strategy, then at every time-instant γ^x will be played and the average payoff equals $(x_1, \ldots, x_N)$.

We will show in two steps that s^{trigger} is a subgame perfect Nash equilibrium. First we show that s^{trigger} is a Nash equilibrium of $G(\infty, \delta)$ for δ close enough to one. Suppose all other players play s^{trigger}. Because $\bar{\gamma}$ is a Nash equilibrium of G and every other player j will play $\bar{\gamma}_j$ for ever if player i deviates in the first period, the best reply of player i against s_{-i}^{trigger} is to play $\bar{\gamma}_i$ forever once he has deviated. Denote by γ_i^d a best reply of player i against γ_{-i}^x resulting in the payoff $d_i = \pi_i(\gamma_i^d, \gamma_{-i}^x)$. Then $d_i \geqslant x_i > e_i$. Playing γ_i^d in the first period results in the payoff d_i in that first period but "triggers" the play of $\bar{\gamma}$ in all subsequent periods. Hence the maximal payoff obtained by deviating in the first period equals

$$d_i + \sum_{t=2}^{\infty} \delta^{t-1} e_i = d_i + \frac{\delta}{1-\delta} e_i.$$

In case player i plays γ_i^x in the first period, the choice between γ_i^d and γ_i^x is postponed to the second period. If playing γ_i^x in the first period is optimal for player i, then $\Pi_i = x_i + \delta \Pi_i$, and hence $\Pi_i = x_i/(1-\delta)$. However if playing γ_i^d in the first period is optimal, then $\Pi_i = d_i + \frac{\delta}{1-\delta} e_i$. Hence, player i has no incentive to deviate from γ^x in the first period if and only if

$$\frac{x_i}{1-\delta} \geqslant d_i + \frac{\delta}{1-\delta} e_i, \text{ i.e. iff } \delta \geqslant \frac{d_i - x_i}{d_i - e_i}.$$

Hence s^{trigger} is a Nash equilibrium if and only if $\delta \geqslant \max \frac{d_i - x_i}{d_i - e_i}$, which is strictly smaller than 1 because $d_i \geqslant x_i > e_i$ for every i.

It remains to be shown that s^{trigger} is subgame perfect. There exist two possible subgames of $G(\infty, \delta)$:
(i) subgames in which in all previous periods γ^x has been played, and
(ii) subgames in which some player has deviated.

Whenever all players play s^{trigger},
(i) in subgames of the first kind, the strategies s^{trigger} restricted to such a subgame again

equal s^{trigger}, and
(ii) in subgames of the second kind, the strategies s^{trigger} restricted to such a subgame prescribe $\bar{\gamma}$ to be played every period, which is also a Nash equilibrium of $G(\infty, \delta)$.

Hence s^{trigger} is a subgame perfect Nash equilibrium with average payoffs $(x_1, \ldots, x_N)$. □

3.2.3 Pareto efficiency

In the previous subsection we have considered Nash equilibria. From a noncooperative point of view this solution concept is reasonable because it is self-enforcing; no single player can gain by unilateral deviation. However, when binding agreements are possible, it is not necessary that a solution is self-enforcing. In a cooperative context, i.e. when binding agreements are allowed, a solution should be such that it can not be improved upon by all players simultaneously. This motivates the concept of Pareto efficiency.

Definition 3.33 Let G be a game in strategic form. The N-tuple of strategies $\hat{\gamma}$ is called *Pareto efficient* if the set of inequalities

$$\pi_i(\gamma) \geqslant \pi_i(\hat{\gamma}), \quad i = 1, \ldots, N,$$

where at least one of the inequalities is strict, does not allow for any solution $\gamma \in \Gamma$.

Definition 3.34 Let $f : \mathcal{D} \to \mathbb{R}$ be a function. Then the set arg max is defined by

$$\arg\max_{x \in \mathcal{D}} \{f(x)\} := \left\{ \hat{x} \in \mathcal{D}, f(\hat{x}) = \max_{x \in \mathcal{D}} f(x) \right\}.$$

We have the following lemma:

Lemma 3.35 *Let* $\alpha_i \in (0,1)$*, with* $\sum_{i=1}^{N} \alpha_i = 1$*. If* $\hat{\gamma} \in \Gamma$ *is such that*

$$\hat{\gamma} \in \arg\max_{\gamma \in \Gamma} \left\{ \sum_{i=1}^{N} \alpha_i \pi_i(\gamma) \right\},$$

then $\hat{\gamma}$ *is Pareto efficient.*

Proof : Let $\alpha_i \in (0,1)$, with $\sum_{i=1}^{N} \alpha_i = 1$, and

$$\hat{\gamma} \in \arg\max_{\gamma \in \Gamma} \left\{ \sum_{i=1}^{N} \alpha_i \pi_i(\gamma) \right\}.$$

Assume $\hat{\gamma}$ is not Pareto efficient. Then, there exists an N-tuple of strategies $\tilde{\gamma}$ such that

$$\pi_i(\tilde{\gamma}) \geqslant \pi_i(\hat{\gamma}),\ i = 1, \ldots, N,$$

where at least one of the inequalities is strict. But then

$$\sum_{i=1}^{N} \alpha_i \pi_i(\tilde{\gamma}_i) > \sum_{i=1}^{N} \alpha_i \pi_i(\hat{\gamma}_i),$$

which contradicts the fact that $\hat{\gamma}$ is maximizing. □

Remark 3.36 Note that in this lemma we neither use any concavity conditions on the π_i's nor any convexity assumptions regarding the Γ_i's.

Before we can formulate a lemma stating in a sense the converse of the above lemma, we first need the following separation theorem:

Theorem 3.37 *Let X be a nonempty convex set in $\mathbb{R}^n$. Furthermore, let $x_0 \in \mathbb{R}^n$, such that $x_0 \notin X$. Then there exists a $p \in \mathbb{R}^n$, $p \neq 0$, $|p| < \infty$, such that for all $x \in X$ $p'x \geqslant p'x_0$.*

Proof : See e.g. (Takayama, 1985, page 44). □

Remark 3.38 Note that usually this lemma is stated with the additional assumption that X is a closed subset of $\mathbb{R}^n$. The proof in Takayama (1985) however, shows that this condition is superfluous. Note that when we demand X to be closed, the inequality is strict.

Now we can formulate a converse to lemma 3.35.

Lemma 3.39 *Assume that the strategy spaces Γ_i, $i = 1, \ldots, N$ are convex. Moreover, assume that the payoffs π_i are concave. Then, if $\hat{\gamma}$ is Pareto efficient, there exist $\alpha_i \in [0, 1]$, $i = 1, \ldots, N$, with $\sum_{i=1}^{N} \alpha_i = 1$, such that for all $\gamma \in \Gamma$*

$$\sum_{i=1}^{N} \alpha_i \pi_i(\gamma) \leqslant \sum_{i=1}^{N} \alpha_i \pi_i(\hat{\gamma}).$$

Proof : Define for all $\gamma \in \Gamma$ the set $Z_\gamma \subset \mathbb{R}^N$ by

$$Z_\gamma := \left\{ z \in \mathbb{R}^N \mid z_i < \pi_i(\gamma) - \pi_i(\hat{\gamma}),\ i = 1, \ldots, N \right\},$$

and define Z by

$$Z := \bigcup_{\gamma \in \Gamma} Z_\gamma .$$

Then, because $\hat{\gamma}$ is Pareto efficient, $0 \notin Z$. Moreover, Z is convex. For, if $z \in Z_\gamma$, $\tilde{z} \in Z_{\tilde{\gamma}}$ and $\lambda \in [0,1]$, then

$$\begin{aligned} \lambda z_i + (1-\lambda)\,\tilde{z}_i &< \lambda \pi_i(\gamma) + (1-\lambda)\,\pi_i(\tilde{\gamma}) - \pi_i(\hat{\gamma}) \\ &\leqslant \pi_i(\lambda\gamma + (1-\lambda)\,\tilde{\gamma}) - \pi_i(\hat{\gamma}), \end{aligned}$$

and hence, $\lambda z + (1-\lambda)\,\tilde{z} \in Z_{\lambda\gamma + (1-\lambda)\tilde{\gamma}} \subset Z$.

By the separation theorem 3.37, we know that there exists a $\tilde{p} \neq 0$, such that $\tilde{p}'z \geqslant 0$ for all $z \in Z$. Because, for every $i = 1, \ldots, N$, we can choose $-z_i$ arbitrarily large, it is obvious that $\tilde{p}_i \leqslant 0$. Define $p := -\tilde{p}$. Then, for all $z \in Z$, $p'z \leqslant 0$, and $p_i \geqslant 0$, $i = 1, \ldots, N$.

Let $z \in Z$. Then there exists a $\gamma \in \Gamma$, and $\varepsilon \in \mathbb{R}^N$, with $\varepsilon_i > 0$, $i = 1, \ldots, N$, such that

$$z_i = \pi_i(\gamma) - \pi_i(\hat{\gamma}) - \varepsilon_i, \; i = 1, \ldots, N.$$

Moreover, by varying $\gamma \in \Gamma$ and $\varepsilon_i > 0$, $i = 1, \ldots, N$, we obtain all $z \in Z$. Hence, for all $\gamma \in \Gamma$ and for all $\varepsilon_i > 0$ we have

$$\sum_{i=1}^{N} p_i \left(\pi_i(\gamma) - \pi_i(\hat{\gamma}) - \varepsilon_i \right) \leqslant 0,$$

and therefore for all $\gamma \in \Gamma$

$$\sum_{i=1}^{N} p_i \pi_i(\gamma) \leqslant \sum_{i=1}^{N} p_i \pi_i(\hat{\gamma}).$$

If we define

$$\alpha_i := \frac{p_i}{\sum_{j=1}^{N} p_j},$$

we find that for all $\gamma \in \Gamma$

$$\sum_{i=1}^{N} \alpha_i \pi_i(\gamma) \leqslant \sum_{i=1}^{N} \alpha_i \pi_i(\hat{\gamma}).$$

□

Remark 3.40 This lemma is due to K. Fan, I. Glicksberg and A.J. Hoffman, and appeared as such in Fan et al. (1957). The proof as given above is essentially taken from Takayama (1985).

By combining the results from lemma 3.35 and lemma 3.39, we find the following theorem.

Theorem 3.41 *Let $\alpha_i > 0$, $i = 1, \ldots N$, satisfying $\sum_{i=1}^{N} \alpha_i = 1$. If $\hat{\gamma} \in \Gamma$ is such that*

$$\hat{\gamma} \in \arg\max_{\gamma \in \Gamma} \left\{ \sum_{i=1}^{N} \alpha_i \pi_i(\gamma) \right\},$$

then $\hat{\gamma}$ is Pareto efficient.
Moreover, if Γ_i is convex and π_i is concave for all $i = 1, \ldots N$, then for all Pareto efficient $\hat{\gamma}$ there exist $\alpha_i \geqslant 0$, $i = 1, \ldots N$, with $\sum_{i=1}^{N} \alpha_i = 1$, such that

$$\hat{\gamma} \in \arg\max_{\gamma \in \Gamma} \left\{ \sum_{i=1}^{N} \alpha_i \pi_i(\gamma) \right\}.$$

Remark 3.42 Verkama gives in Verkama (1994) an elegant and short proof of this theorem, in case the strategy spaces Γ_i are compact and convex subspaces of some Euclidean space $\mathbb{R}^{n_i}$, and the payoffs π_i are pseudoconcave.

3.3 Bargaining models

In this section we will elaborate on the theory of bargaining. This theory has its origin in two papers by John Nash (Nash (1950) and Nash (1953)). In these papers a bargaining problem is defined as a situation in which two (or more) individuals or organizations have to agree on the choice of one specific alternative from a set of alternatives available to them, while having conflicting interests over this set of alternatives. Note that these alternatives can be the choice of which strategy to play in a game in strategic form; this case is generally referred to as policy bargaining. For references on bargaining theory, we refer to van Damme (1991); Fudenberg and Tirole (1991); Houba (1994); Osborne and Rubinstein (1991). Nash (in Nash (1953)) proposes two different approaches to the bargaining problem, namely the *axiomatic* and the *strategic* approach. The axiomatic approach lists a number of desirable properties the solution must have, called the *axioms*. The strategic approach on the other hand, sets out a particular bargaining procedure and asks what outcomes would result from rational behavior by the individual players. Specifically, a dynamic game in extensive form is specified, of which e.g. the subgame perfect Nash equilibria are studied. In this section we will concentrate on this strategic approach. Furthermore, as in the remainder of this dissertation we will restrict ourselves to the two-player case. For bargaining models involving more than two players and the difficulties involved in treating these models, see e.g. Houba (1994).

A classic model in the context of strategic bargaining is Rubinstein's alternating offer model, in which two players must agree on how to share a pie of size 1 (see Rubinstein (1982)). In periods $0, 2, 4$, etc., player 1 proposes a sharing rule $(x, 1-x)$ that player 2 can either accept or reject. If player 2 accepts any offer, the game ends. However, if player 2 rejects player 1's offer in period $2k$, then in period $2k+1$ he can propose a sharing rule $(x, 1-x)$ that player 1 can accept or reject. If player 1 accepts one of player 2's offers the game ends, otherwise he can make an offer in the subsequent period, and so on. If $(x, 1-x)$ is accepted at time t, the payoffs are defined by $\pi := (\delta_1^t x, \delta_2^t (1-x))$. A first observation we make is that this game allows for many Nash equilibria. The strategy in which player 1 always demands $x = 1$, and rejects all smaller offers and in which player 2 always offers $x = 1$ and accepts any offer, is a Nash equilibrium. However, this Nash equilibrium is not subgame perfect, for if player 2 rejects player 1's first offer, and offers player 1 a share $x > \delta_1$, then player 1 should accept, because the best possible outcome if he rejects is to receive the entire pie tomorrow, which is worth only δ_1.

Define the continuation payoffs of a pair of strategies in a subgame starting at t to be the payoffs in time-t units of the outcome induced by that strategies. We find the following proposition.

Proposition 3.43 *The Rubinstein bargaining model has a subgame perfect Nash equilibrium. The Nash equilibrium, in which player i always demands a share $(1-\delta_j)/(1-\delta_i\delta_j)$ when it is his turn to make an offer, and he accepts any share which is greater than or equal to $\delta_i(1-\delta_j)/(1-\delta_i\delta_j)$, is subgame perfect. Moreover, subgame perfect equilibrium continuation payoffs are unique.*

Proof : (Adapted from Fudenberg and Tirole (1991)).
Define $\underline{v}_i$ and $\overline{v}_i$ to be the infimum resp. supremum of player i's continuation payoffs of any subgame in which he is allowed to make the first offer. Similarly, define $\underline{w}_i$ and $\overline{w}_i$ to be the infimum resp. supremum of player i's continuation payoffs in subgames where the other player makes the first offer. We will prove that $\underline{v}_i = \overline{v}_i$ and $\underline{w}_i = \overline{w}_i$.

When player i makes an offer x, the other player will accept any x such that his share $(1-x)$ exceeds $\delta_j\overline{v}_j$, since he cannot expect more than $\overline{v}_j$ in the continuation game following his refusal. Hence, $\underline{v}_i \leqslant 1 - \delta_j\overline{v}_j$.

Also, since player i will never offer player j more than $\delta_j\overline{v}_j$, $\overline{w}_j \leqslant \delta_j\overline{v}_j$.

Since player j can obtain at least $\underline{v}_j$ in the continuation game by rejecting player i's offer, he will reject any offer $1 - x$ such that $1 - x \leqslant \delta_j \underline{v}_j$. Therefore, $\overline{v}_i$ satisfies

$$\overline{v}_i \leqslant \max\left\{1 - \delta_j \underline{v}_j, \delta_i \overline{w}_i\right\} \leqslant \max\left\{1 - \delta_j \underline{v}_j, \delta_i^2 \overline{v}_i\right\} = 1 - \delta_j \underline{v}_j.$$

Combining all inequalities yields

$$\overline{v}_i \leqslant 1 - \delta_j \underline{v}_j \leqslant 1 - \delta_j \left(1 - \delta_i \overline{v}_i\right)$$

and

$$\underline{v}_i \geqslant 1 - \delta_j \overline{v}_j \geqslant 1 - \delta_j \left(1 - \delta_i \underline{v}_i\right).$$

Hence,

$$\frac{1 - \delta_j}{1 - \delta_i \delta_j} \leqslant \underline{v}_i \leqslant \overline{v}_i \leqslant \frac{1 - \delta_j}{1 - \delta_i \delta_j}.$$

Similarly, $\underline{w}_i = \overline{w}_i = \frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}$. This shows that the subgame perfect equilibrium continuation payoffs are unique.

By the above arguments player i must offer exactly $\underline{v}_i$. Player j is indifferent between accepting and rejecting this offer. If he always accepts this leads to the subgame perfect Nash equilibrium as stated above. □

Remark 3.44 It can be shown that in the subgame perfect equilibrium given above the players reach an immediate agreement on a Pareto efficient point. It can even be proved that this point is the so-called Nash bargaining solution. (See van Damme (1991)).

3.4 Coordination in repeated games

3.4.1 Introduction

In this section we provide a general model to study the role of a coordinator in reaching a cooperative equilibrium for a repeated game. The model allows the individual players to react in a strategic fashion to the behavior of the coordinator, and so the natural question arises whether there is a way in which a coordinator can encourage cooperative play. In Klompstra (1992), in the context of linear-quadratic differential games, it is shown that if players are allowed to switch in time between cooperative and noncooperative behavior, such switches do indeed occur. With this idea in mind, we can conclude that the decision whether or not to cooperate has a dynamical flavor, i.e. willingness to cooperate should be modelled in a dynamic way. The theory of (strategic) bargaining, briefly introduced in the previous section, shows that using threats to play noncooperatively (i.e. to reject), individual players can influence the final outcome of a game. This idea is also used in our model, in the sense that individual players can influence the behavior of the coordinator by deviating from a cooperative strategy. The coordinator can then be interpreted as an institution appointed to promote a prespecified mode of play. In this way our model can be interpreted as a first step in introducing strategic behavior into the hierarchical control model. In order to concentrate on the strategic aspects, our model is constructed in such a way that it ignores the dynamical and informational aspects of the hierarchical control problem. This is accomplished by building the model on a two-player game, which is played repeatedly, rather than on a system with a non-trivial state space (as was done in chapter two). Regarding this repeated game we introduce the notion of coordination and discuss the resulting difference game.

3.4.2 General model formulation

Consider the following situation. Two players repeatedly play a nonzero-sum game G. Assume now that the game G depends in some way (through the payoffs that the players receive, or through the strategy spaces that are available to them) on a parameter $\alpha \in [0,1]$ that may vary in time. The value of α is determined by a 'coordinator' through some decision rule that takes the actions of the players into account. In this way the decisions of the players can influence their future payoffs, and a difference game arises which we shall refer to as the 'controlled game'. Comparing the equilibria of the controlled game to the possible modes of play in the original game G, we can see whether the decision rule chosen by the coordinator is effective in establishing cooperation between the players.

We formalize this idea as follows. Consider a two-player static game G in strategic form, with strategy spaces Γ_i and payoff functions π_i. From this game G we construct a new game, $G(\alpha)$, for every α in $[0,1]$, where α is the variable that is manipulated by the coordinator. Denote by $\Gamma_i(\alpha)$ the strategy spaces of $G(\alpha)$ and by $\nu_i(\alpha, \gamma_1(\alpha), \gamma_2(\alpha))$ the payoffs. Now assume that the coordinator can observe the strategies $\gamma_i(\alpha)$ chosen by the individual players, and uses a decision rule

$$\alpha(t+1) = f(\alpha(t), \gamma_1(\alpha(t)), \gamma_2(\alpha(t)))$$

to determine the future values of α. Finally, by choosing as a criterion

$$\mathcal{L}_i = \sum_{t=0}^{t_f-1} \nu_i(\alpha(t), \gamma_1(\alpha(t)), \gamma_2(\alpha(t))),$$

a difference game is specified, which we refer to as the controlled game. So the construction of a controlled game from a static game G is done in the following steps:

Step 1: construction of a coordination mechanism $G \rightarrowtail G(\alpha)$,

Step 2: specification of a decision rule

$$\alpha(t+1) = f(\alpha(t), \gamma_1(\alpha(t)), \gamma_2(\alpha(t)))$$

for the coordinator.

3.4.3 Construction of a controlled game

In this subsection we will construct a class of controlled games we will use in the remainder of this chapter. First we make some assumptions on the underlying static game G.

Assumption 3.45
The strategy spaces $\Gamma_i \subseteq \mathbb{R}^k$ are convex.
The payoff functions $\pi_i : \Gamma_1 \times \Gamma_2 \to \mathbb{R}$, $i = 1, 2$, are twice differentiable and strictly concave, i.e. $\begin{pmatrix} \frac{\partial^2 \pi_i}{\partial \gamma_1^2} & \frac{\partial^2 \pi_i}{\partial \gamma_1 \partial \gamma_2} \\ \frac{\partial^2 \pi_i}{\partial \gamma_1 \partial \gamma_2} & \frac{\partial^2 \pi_i}{\partial \gamma_2^2} \end{pmatrix} < 0.$

By $\bar{\gamma} = \left(\bar{\gamma}_1, \bar{\gamma}_2\right) \in \Gamma_1 \times \Gamma_2$ we denote a Nash equilibrium of the game G.

Denote by $\gamma_i^*(\alpha)$ the cooperative strategy for player i, to be played when the coordinator selects α. Furthermore, denote by $\gamma_i^a(\alpha)$ an alternative strategy, that player i would play when playing noncooperatively. We have to make a choice for the alternative strategy γ^a. The issue on how to choose such an alternative strategy is closely related to the issue of choosing threatpoints or disagreement strategies in bargaining theory (see e.g. Houba (1994); Osborne and Rubinstein (1991)). A possible choice of alternative strategy is a Nash equilibrium for the underlying game G. Especially in the case that G has a unique Nash equilibrium this seems a good choice, for the Nash equilibrium is the standard equilibrium concept in noncooperative situations (see section 3.2).

We introduce $c_i(t)$, which is a parameter reflecting the willingness of player i to play cooperatively at time instant t. If $c_i(t) = 0$ then player i chooses to play the alternative strategy $\gamma_i^a(\alpha(t))$ and if $c_i(t) = 1$ then player i chooses to play the strategy $\gamma_i^*(\alpha(t))$. We allow the players to hesitate between cooperative and noncooperative play by allowing the parameter $c_i(t)$ to take values between 0 and 1. For given c_i, the strategy played by player i is given by $u_i(c_i) := c_i \gamma_i^*(\alpha) + (1 - c_i)\, \gamma_i^a(\alpha)$.

Now we assume that the coordinator, by observing the actions of both players at time-instant t, can determine the values of $c_i(t)$. Using this information the coordinator adjusts the value of $\alpha(t)$. The process of coordination can be described by a decision rule

$$\alpha(t+1) = f(\alpha(t), c_1(t), c_2(t)). \tag{3.6}$$

This decision rule has to satisfy some properties:

1. f is sufficiently smooth, i.e. f is at least twice differentiable w.r.t. c_i, and at least differentiable w.r.t. α,
2. $\forall_{c_1,c_2} 0 \leqslant f(\alpha, c_1, c_2) \leqslant 1$,
3. $\frac{\partial^2 f}{\partial c_i \partial c_j} = 0$, $\frac{\partial f}{\partial c_i} \neq 0$.

The smoothness condition is imposed in order to prevent some technical difficulties in the sequel of this chapter. Clearly this condition might be weakened at the expense of some technical complications. The second condition is crucial, in the sense that it guarantees that $\alpha(t)$ remains in $[0, 1]$ for all t. Finally, the third condition is sufficient to guarantee that the optimization problems we will encounter are strictly concave, and that the mechanism is not trivial. Obviously also this condition might be weakened, and in this case a more delicate analysis would be required. An example of a coordination rule satisfying properties 1 to 3 is

$$f(\alpha, c_1, c_2) = \alpha + \beta\alpha(1-\alpha)(c_2 - c_1),$$

where $\beta \in (0, 1)$ is an arbitrary constant. This decision rule reflects the intuition that whenever one of the players shows less willingness to cooperate, the coordinator might try to convince this player to play more cooperatively in the future by choosing a new α, which is more favorable for that particular player. When β is chosen in $(-1, 0)$, the decision rule is such that the coordinator punishes any player who is not playing cooperatively.

A further assumption we make is that both players exactly know the mechanism f the coordinator is using. It is this assumption that creates the possibility for strategic behavior by both players. By choosing c_1 and c_2 the players can influence the behavior of the coordinator. A nonlinear difference game emerges, where α is the state variable, c_1 and c_2 are the controls, and with the payoff functionals

$$L_i = \sum_{t=0}^{t_f-1} \nu_i(\alpha(t), u_1(c_1(t)), u_2(c_2(t))), \tag{3.7}$$

in which $u_i(c_i(t)) = c_i(t)\gamma_i^*(\alpha(t)) + (1-c_i(t))\,\gamma_i^a(\alpha(t))$. We refer to this newly defined difference game as the controlled game.

Note that by introducing $u_i(c_i) = c_i\gamma_i^*(\alpha) + (1 - c_i)\,\gamma_i^a(\alpha)$ the payoff for player i at time instant t is given by $\nu_i(\alpha(t), u_1(c_1(t)), u_2(c_2(t)))$, which we will sometimes write with some abuse of notation as $\nu_i(\alpha(t), c_1(t), c_2(t))$. In the sequel of this chapter we will assume that ν_1 and ν_2 are strictly concave in $\big(c_1, c_2\big)$.

3.4.4 Equilibria of the controlled game

A natural solution concept to consider for the controlled game is the feedback Nash equilibrium. We find the following proposition.

Proposition 3.46 *Let $\bar{\gamma}$ be a Nash equilibrium of G. Suppose there exist for all $\alpha \in (0,1)$ numbers $\lambda_i(\alpha) \in \mathbb{R}$ and $\mu_i(\alpha) \in (0,\infty)$, $i = 1,2$, such that*

$$\nu_i(\alpha, u_1(c_1), u_2(c_2)) = \lambda_i(\alpha) + \mu_i(\alpha)\pi_i(u_1(c_1), u_2(c_2)),$$

and moreover, suppose that the alternative strategies γ^a are such that for all $\alpha \in (0,1)$ the system of equations

$$\begin{aligned} \bar{c}_1(\alpha)\gamma_1^*(\alpha) + (1 - \bar{c}_1(\alpha))\,\gamma_1^a(\alpha) &= \bar{\gamma}_1, \\ \bar{c}_2(\alpha)\gamma_2^*(\alpha) + (1 - \bar{c}_2(\alpha))\,\gamma_2^a(\alpha) &= \bar{\gamma}_2, \end{aligned}$$

has a solution $(\bar{c}_1(\alpha), \bar{c}_2(\alpha)) \in [0,1] \times [0,1]$. Then $(\bar{c}_1(\alpha), \bar{c}_2(\alpha))$ constitutes a feedback Nash equilibrium for the controlled game. Moreover, the actions $(u_1(\bar{c}_1(\alpha(t))), u_2(\bar{c}_2(\alpha(t)))$ played at every time instant t are equal to the Nash equilibrium $\bar{\gamma}$ of G.

Proof : By dynamic programming. In the last stage the Nash equilibrium of the game with payoffs $\nu_i(\alpha, u_1(c_1), u_2(c_2))$ has to be determined. Because

$$\nu_i(\alpha, u_1(c_1), u_2(c_2)) = \lambda_i(\alpha) + \mu_i(\alpha)\pi_i(u_1(c_1), u_2(c_2)),$$

and because there exist $(\bar{c}_1(\alpha), \bar{c}_2(\alpha))$ such that

$$\begin{aligned} \bar{c}_1(\alpha)\gamma_1^*(\alpha) + (1 - \bar{c}_1(\alpha))\,\gamma_1^a(\alpha) &= \bar{\gamma}_1, \\ \bar{c}_2(\alpha)\gamma_2^*(\alpha) + (1 - \bar{c}_2(\alpha))\,\gamma_2^a(\alpha) &= \bar{\gamma}_2, \end{aligned}$$

we find

$$\nu_i(\alpha, u_1(\bar{c}_1(\alpha)), u_2(\bar{c}_2(\alpha))) = \lambda_i(\alpha) + \mu_i(\alpha)\pi_i(\bar{\gamma}_1, \bar{\gamma}_2),$$

and hence $(\bar{c}_1(\alpha), \bar{c}_2(\alpha))$ is a Nash equilibrium for the game played at the last time instant. This argument can be repeated at all other time instants, completing the proof. □

The following corollary is straightforwardly proved.

Corollary 3.47 *Suppose $\bar{\gamma}$ is the* unique *Nash equilibrium of G. Suppose there exist for all $\alpha \in (0,1)$ numbers $\lambda_i(\alpha) \in \mathbb{R}$ and $\mu_i(\alpha) \in (0,\infty)$, $i = 1,2$, such that*

$$\nu_i(\alpha, u_1(c_1), u_2(c_2)) = \lambda_i(\alpha) + \mu_i(\alpha)\pi_i(u_1(c_1), u_2(c_2)),$$

and moreover, suppose that the alternative strategies γ^a are such that for all $\alpha \in (0,1)$ the system of equations

$$\begin{aligned} \bar{c}_1(\alpha)\gamma_1^*(\alpha) + (1 - \bar{c}_1(\alpha))\,\gamma_1^a(\alpha) &= \bar{\gamma}_1, \\ \bar{c}_2(\alpha)\gamma_2^*(\alpha) + (1 - \bar{c}_2(\alpha))\,\gamma_2^a(\alpha) &= \bar{\gamma}_2, \end{aligned}$$

has a unique *solution $(\bar{c}_1(\alpha), \bar{c}_2(\alpha))$. Then $(\bar{c}_1(\alpha), \bar{c}_2(\alpha))$ is the* unique *feedback Nash equilibrium for the controlled game. The actions $(u_1(\bar{c}_1(\alpha(t))), u_2(\bar{c}_2(\alpha(t))))$ played at every time instant t are equal to the Nash equilibrium $\bar{\gamma}$ of G.*

Remark 3.48 Comparing this corollary with proposition 3.30, we observe again (as in remark 3.31) that the last stage becomes of predominant importance. Exactly as in remark 3.31 we might conclude that a model in which the exact number of repetitions is known in advance is often unrealistic, and that therefore we should consider the controlled game over an infinite time horizon.

3.4.5 A redistribution mechanism

As previously noted, there are several ways in which the coordination parameter α may affect the underlying static game G. In this subsection we consider the case in which the payoffs depend on α and the strategy spaces do not.

We make the following assumptions about the underlying static game G.

Assumption 3.49
(i) *The game G is symmetric, i.e. $\Gamma_1 = \Gamma_2$ and $\pi_1(\gamma_1, \gamma_2) = \pi_2(\gamma_2, \gamma_1)$,*

(ii) *G has a unique Nash equilibrium $(\bar{\gamma}_1, \bar{\gamma}_2)$, with equilibrium payoffs $(\bar{\pi}_1, \bar{\pi}_2)$,*

(iii) *the unique Nash equilibrium of G is not Pareto efficient.*

The symmetry suggests restricting our attention to Pareto efficient strategies $\hat{\gamma}(\frac{1}{2})$ corresponding to $\alpha_i = \frac{1}{2}$ (see theorem 3.41). So for the cooperative strategy we choose $\gamma_i^*(\alpha) = \hat{\gamma}_i(\frac{1}{2})$. The second assumption, that G has a unique Nash equilibrium, justifies the choice of this Nash equilibrium as the alternative strategy, i.e. $\gamma_i^a(\alpha) = \bar{\gamma}_i$. Note that both the cooperative strategies γ^* and the alternative strategies γ^a do not depend on α in this case. The extra payoffs from playing $u_i(c_i) = c_i\hat{\gamma}_i(\frac{1}{2}) + (1-c_i)\,\bar{\gamma}_i$ are given by

$$\pi^*(c_1,c_2) := \pi_1(u_1(c_1),u_2(c_2)) + \pi_2(u_1(c_1),u_2(c_2)) - \pi_1(\bar{\gamma}_1,\bar{\gamma}_2) - \pi_2(\bar{\gamma}_1,\bar{\gamma}_2). \tag{3.8}$$

Now suppose that these extra payoffs are redistributed over the players by the coordinator, according to the rule

$$\nu_1(\alpha,c_1,c_2) \;:=\; \alpha\pi^*(c_1,c_2), \tag{3.9}$$
$$\nu_2(\alpha,c_1,c_2) \;:=\; (1-\alpha)\,\pi^*(c_1,c_2). \tag{3.10}$$

Then we find the following result.

Proposition 3.50 *The unique feedback Nash equilibrium for the redistribution controlled game is given by* $\bar{c}_i \equiv 1$.

Proof : The symmetry of the game G implies $\bar{\gamma}_1 = \bar{\gamma}_2 =: \bar{\gamma}$ and $\hat{\gamma}_1(\frac{1}{2}) = \hat{\gamma}_2(\frac{1}{2}) =: \hat{\gamma}$. Then elementary calculus shows that

$$\begin{pmatrix} \frac{\partial^2\pi^*}{\partial c_1^2} & \frac{\partial^2\pi^*}{\partial c_1\partial c_2} \\ \frac{\partial^2\pi^*}{\partial c_1\partial c_2} & \frac{\partial^2\pi^*}{\partial c_2^2} \end{pmatrix} = (\hat{\gamma}-\bar{\gamma})^2 \left(\begin{pmatrix} \frac{\partial^2\pi_1}{\partial \gamma_1^2} & \frac{\partial^2\pi_1}{\partial \gamma_1\partial \gamma_2} \\ \frac{\partial^2\pi_1}{\partial \gamma_1\partial \gamma_2} & \frac{\partial^2\pi_1}{\partial \gamma_2^2} \end{pmatrix} + \begin{pmatrix} \frac{\partial^2\pi_2}{\partial \gamma_1^2} & \frac{\partial^2\pi_2}{\partial \gamma_1\partial \gamma_2} \\ \frac{\partial^2\pi_2}{\partial \gamma_1\partial \gamma_2} & \frac{\partial^2\pi_2}{\partial \gamma_2^2} \end{pmatrix} \right).$$

From the strict concavity of π_1 and π_2 it follows that

$$\begin{pmatrix} \frac{\partial^2\pi_i}{\partial \gamma_1^2} & \frac{\partial^2\pi_i}{\partial \gamma_1\partial \gamma_2} \\ \frac{\partial^2\pi_i}{\partial \gamma_1\partial \gamma_2} & \frac{\partial^2\pi_i}{\partial \gamma_2^2} \end{pmatrix} < 0,$$

$i = 1, 2$, and hence

$$\begin{pmatrix} \frac{\partial^2\pi^*}{\partial c_1^2} & \frac{\partial^2\pi^*}{\partial c_1\partial c_2} \\ \frac{\partial^2\pi^*}{\partial c_1\partial c_2} & \frac{\partial^2\pi^*}{\partial c_2^2} \end{pmatrix} < 0,$$

since $\hat{\gamma} \neq \bar{\gamma}$, by assumption 3.49 (iii).

Now, using this strict concavity of π^*, it is easily verified, again using dynamic programming and using similar arguments as in the proof of proposition 3.46, that at every time instant t, a necessary and sufficient condition for $\bar{c}_i$ to be a Nash equilibrium, is that in $(\bar{c}_1,\bar{c}_2)$

$$\frac{\partial\pi^*}{\partial c_1} = \frac{\partial\pi^*}{\partial c_2} = 0.$$

Again elementary calculus shows that $\frac{\partial\pi^*}{\partial c_1} = \frac{\partial\pi^*}{\partial c_2} = 0$, if and only if

$$\frac{\partial\pi_1}{\partial\gamma_1}(u_1(\bar{c}_1),u_2(\bar{c}_2)) + \frac{\partial\pi_2}{\partial\gamma_1}(u_1(\bar{c}_1),u_2(\bar{c}_2)) \;=\; 0,$$
$$\frac{\partial\pi_1}{\partial\gamma_2}(u_1(\bar{c}_1),u_2(\bar{c}_2)) + \frac{\partial\pi_2}{\partial\gamma_2}(u_1(\bar{c}_1),u_2(\bar{c}_2)) \;=\; 0,$$

which are exactly the first order conditions characterizing $\hat{\gamma}$ (see theorem 3.41). Hence, $\bar{c}_i = 1$. □

Remark 3.51 Also here remark 3.31 and remark 3.48 can be stated again. Also in this case it might be more appropriate to study the controlled game over an infinite time horizon.

3.5 Conclusions

In the first part of this chapter we have introduced solution concepts from noncooperative game theory. In particular we have introduced Nash equilibria for games in strategic form, for finite dynamic games in extensive form and for difference and differential games. Especially in the case of dynamic games, we saw that there is a need for refinements of the Nash equilibrium concept, to avoid "unreasonable" outcomes. In the case of finite dynamic games and repeated games this led to the subgame perfectness refinement, and for difference and differential games to feedback Nash equilibria. Finally, in subsection 3.2.3, we have introduced the concept of Pareto efficiency for cooperative situations, and we gave a characterization of Pareto efficient outcomes in theorem 3.41.

In section 3.3 we have briefly introduced (strategic) bargaining theory. By means of Rubinsteins alternating offer model we have seen an example of how bargaining can be modelled, by prescribing explicitly a bargaining procedure. Moreover it was shown how in such a model the outcomes can be determined. Specifically, the bargaining procedure was formulated as a dynamic game, of which the subgame perfect Nash equilibria are studied. Inspired by this idea we have developed in 3.4 a model incorporating strategic behavior in a hierarchical framework. This model involves the specification of a difference game, the so-called controlled game. Analysis of this model, i.e. determination of the feedback Nash equilibria of the controlled game, showed (see remarks 3.48 and 3.51) that it is desirable to rephrase the model over an infinite horizon. The analysis of infinite-horizon feedback Nash equilibria of difference games involves the solution of functional equations of the form

$$\begin{aligned} V_1(x) &= \max_{u_1} \left\{ rV_1(f(x,u_1,u_2)) + \pi_1(x,u_1,u_2) \right\}, \\ V_2(x) &= \max_{u_2} \left\{ rV_2(f(x,u_1,u_2)) + \pi_2(x,u_1,u_2) \right\}, \end{aligned}$$

which appears to be a nasty problem. In the continuous-time case, i.e. in the case of infinite-horizon differential games, we will show in chapter five (and appendix A) that these functional equations are replaced by differential-algebraic equations which we can solve numerically. Therefore, in chapter five we will redefine the controlled game as a differential game over an infinite horizon, and analyze it in more detail. Before doing so, we will consider in chapter four the effects of introducing an infinite horizon into a special class of differential games (namely linear-quadratic differential games) and also the asymptotics of finite-horizon linear-quadratic differential games when the horizon tends to infinity.

Chapter 4

Nash equilibria in differential games

4.1 Introduction

Differential games were first introduced in Isaacs (1956), within the framework of two-person zero-sum games. Recently, the theory of zero-sum differential games has successfully been used in the area of H_∞ control theory, see e.g. Başar and Bernard (1991); Stoorvogel (1992). Nonzero-sum differential games were introduced in the papers Starr and Ho (1969b,a). A good survey of the area of noncooperative dynamic games is provided in the book Başar and Olsder (1995). This chapter is mainly based on the papers Engwerda and Weeren (1995b); Weeren et al. (1994).

In this chapter we look at a special class of nonzero-sum differential games, namely nonzero-sum differential games of the linear-quadratic type. The dynamics are described by a linear differential equation,

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t),\ x(0) = x_0, \tag{4.1}$$

and for each player a quadratic cost functional is given:

$$\begin{aligned} L_1(u_1,u_2) &:= x(t_f)'K_{1f}x(t_f) + \\ &\int_0^{t_f} \{x(t)'Q_1x(t) + u_1(t)'R_{11}u_1(t) + u_2(t)'R_{12}u_2(t)\}\,dt, \end{aligned} \tag{4.2}$$

$$\begin{aligned} L_2(u_1,u_2) &:= x(t_f)'K_{2f}x(t_f) + \\ &\int_0^{t_f} \{x(t)'Q_2x(t) + u_1(t)'R_{21}u_1(t) + u_2(t)'R_{22}u_2(t)\}\,dt, \end{aligned} \tag{4.3}$$

in which all matrices are symmetric, and moreover $Q_i \geqslant 0$ and $R_{ii} > 0$.

The objective of the game for each player is the minimization of his own cost functional by choosing appropriate inputs for the underlying linear dynamical system.

For given information sets $\eta_i(t)$ and any pair of strategies $\big(\gamma_1, \gamma_2\big)$, the actions of the players are completely determined by the relations $\big(u_1, u_2\big) = \big(\gamma_1(\eta_1), \gamma_2(\eta_2)\big)$. Substitution of the pair $\big(u_1, u_2\big)$ in (4.2–4.3), together with the corresponding unique state trajectory, yields a pair of numbers $\big(L_1(u_1, u_2), L_2(u_1, u_2)\big)$. Therefore we have a mapping for each fixed initial state vector x_0, defined by

$$J_i : \Gamma_1 \times \Gamma_2 \to \mathbb{R}, \big(\gamma_1, \gamma_2\big) \mapsto L_i(u_1, u_2), \tag{4.4}$$

which we call the cost functional of player i for the game in strategic form (see section 3.2).

In Nash (1951) the Nash equilibrium concept was introduced, which was argued to be a natural concept in a noncooperative context. The Nash equilibrium is defined in the following way (see section 3.2):

Definition 4.1 A pair of strategies $(\gamma_1^*, \gamma_2^*) \in \Gamma_1 \times \Gamma_2$ is a *Nash equilibrium* for the differential game, if for all $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$ the following inequalities hold:

$$J_1(\gamma_1^*, \gamma_2^*) \leqslant J_1(\gamma_1, \gamma_2^*), \tag{4.5}$$
$$J_2(\gamma_1^*, \gamma_2^*) \leqslant J_2(\gamma_1^*, \gamma_2). \tag{4.6}$$

The Nash equilibrium is defined such that it has the property that there is no incentive for any unilateral deviation by any one of the players. A possible problem when dealing with Nash equilibria, is that in general one cannot expect to have a unique Nash equilibrium. Already in the paper Starr and Ho (1969a), for nonzero-sum differential games, non-uniqueness problems regarding Nash equilibria were discussed.

In almost all papers on linear-quadratic differential games, the games are studied over a fixed time period $[0, t_f]$. In the case of open-loop information, where every player knows at time $t \in [0, t_f]$ the initial state x_0 (denoted by $\eta_i(t) = x_0$), conditions for the existence of a unique Nash equilibrium can be given (see Engwerda and Weeren (1995b) and section 4.2). In the case of closed-loop perfect state information, where every player knows at time $t \in [0, t_f]$ the complete history of the state (denoted by $\eta_i(t) = x|_{[0,t]}$), one can show that there exist many Nash equilibria. In this case it is possible to define a refinement of the Nash equilibrium concept towards (linear) feedback Nash equilibria, which have the nice property of strong time consistency (see theorem 3.24). In the finite (fixed) horizon case, generic uniqueness of the linear feedback Nash equilibrium can be shown (see e.g. Başar and Olsder (1995)).

Only a few authors have studied the game over an infinite time horizon, or the asymptotic behavior of Nash equilibria for $t_f \to \infty$. In Abou-Kandil et al. (1993), the asymptotic behavior of open-loop Nash equilibria is studied. For the feedback Nash equilibrium, in Papavassilopoulos et al. (1979), an initial study is made of stationary feedback Nash equilibria for the differential game over an infinite horizon. However, in the paper Papavassilopoulos et al. (1979) the asymptotic behavior of feedback Nash equilibria is not studied. Also, the problem of existence of stationary feedback Nash equilibria for infinite-horizon differential games is not addressed. Instead, some sufficient solvability conditions for the coupled algebraic Riccati equations are derived, using Brouwer's fixed point theorem.

In section 4.2 of this chapter, we discuss the asymptotic analysis of Nash equilibria in the open-loop case, based on the fact that in the open-loop case the Nash equilibrium can be related to a linear differential system (see papers Abou-Kandil and Bertrand (1986); Abou-Kandil et al. (1993); Engwerda and Weeren (1994b)). In section 4.3, we will show that for feedback Nash equilibria it is not possible to follow a similar approach. Instead we will give a detailed asymptotic analysis for the special case that all system parameters are scalar. Finally, in section 4.4, we study linear stationary feedback Nash equilibria for the differential game over an infinite horizon, and use the results from section 4.3 to show that linear stationary feedback Nash equilibria are not unique.

4.2 Open-loop Nash equilibria

In this section we study the open-loop information structure , i.e. $\eta_i(t) = x_0$, $t \in [0, t_f]$.

4.2.1 Introduction

A well known problem studied in the literature on dynamic games is the existence of a unique open-loop Nash equilibrium. It is often stated (see e.g. Starr and Ho (1969b); Simaan and Cruz Jr. (1973); Abou-Kandil and Bertrand (1986); Abou-Kandil et al. (1993)) that the open-loop Nash equilibrium is given by

$$u_1^*(t) = -R_{11}^{-1} B_1' K_1(t) \Phi(t) x_0 \tag{4.7}$$

$$u_2^*(t) = -R_{22}^{-1} B_2' K_2(t) \Phi(t) x_0 \tag{4.8}$$

provided that the set of coupled asymmetric Riccati-type differential equations

$$\dot{K}_1 = -A'K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2, \ K_1(t_f) = K_{1f}, \tag{4.9}$$

$$\dot{K}_2 = -A'K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1, \ K_2(t_f) = K_{2f}, \tag{4.10}$$

has a solution $(K_1(t), K_2(t))$ on $[0, t_f]$. Here $\Phi(t)$ satisfies the transition equation

$$\dot{\Phi}(t) = (A - S_1K_1(t) - S_2K_2(t))\Phi(t),\ \Phi(0) = I \tag{4.11}$$

and $S_i = B_iR_{ii}^{-1}B_i', i = 1, 2$.

We will show by means of an example that, stated this way, this assertion is in general not correct. As correctly stated by Başar and Olsder in (Başar and Olsder, 1995, theorem 6.12) existence of a solution to the above mentioned Riccati differential equations is just a sufficient condition to conclude that there exists an open-loop Nash equilibrium for the game. Unfortunately, Başar and Olsder make an assumption in their proof on the costate variable (that it can be obtained by letting a differentiable matrix act on the state variable), which is flawed. For, we will show that in case this assumption does hold the existence of a solution to the Riccati equations is both a necessary and a sufficient condition for the existence of an open-loop Nash equilibrium, and we show by an example that this is not true. Therefore we present a correct proof of theorem 6.12 as stated in Başar and Olsder (1995).

We will analyze the open-loop Nash equilibrium from its roots: the corresponding Hamiltonian equations. In subsection 4.2.2 we will show how both necessary and sufficient conditions for the existence of a unique open-loop Nash equilibrium from the Hamiltonian equations can be derived, in terms of the invertibility of a certain matrix $H(t_f)$. In subsection 4.2.3 we give a correct proof of the theorem stated by Başar and Olsder. Moreover, we present a sufficient condition which guarantees the existence of the set of Riccati differential equations (4.9–4.10).

One area where games of this type are widely used is in policy coordination problems (see e.g. van Aarle et al. (1995); Dockner et al. (1985); Fershtman and Kamien (1987); Petit (1989)). In many economic policy coordination problems an interesting question is to analyze the effect of an expanding planning horizon on the resulting equilibria. Therefore we consider this effect if one expands the horizon t_f to infinity in a separate section. One nice property is that the equilibrium solution becomes much easier to calculate and implement than in the finite horizon case. Before we present the results on this subject in subsection 4.2.5, we first consider the algebraic Riccati equations associated with (4.9–4.10) and their solutions. In subsection 4.2.4 we show how all solutions of these equations can be determined from the eigenstructure of the matrix M, and that the eigenvalues of the associated closed-loop system, obtained by applying the control functions $u_i^*(t) = -R_{ii}^{-1}B_i'K_i\Phi(t)x_0$ to (4.1), are completely determined by the eigenvalues of matrix M.

Finally, in subsection 4.2.6 we will study the scalar case which is of particular interest for many economic applications. We will show that in the scalar case the above mentioned invertibility condition is always satisfied, that as a consequence the equilibrium solution is given by (4.7,4.8), and that the solutions converge to a stabilizing control if the planning horizon expands.

4.2.2 Existence conditions for an open-loop Nash equilibrium

In this subsection, we consider the existence of a unique open-loop Nash equilibrium of the differential game in detail. Due to the stated assumptions, both cost functionals $J_i, i = 1, 2$, are strictly convex functions of γ_i for all admissible control functions $u_j, j \neq i$ and for all x_0. This implies that the conditions following from the minimum principle are both necessary and sufficient (see e.g. (Başar and Olsder, 1995, section 6.5)).

Minimization of the Hamiltonian

$$H_i = (x'Q_ix + u_1'R_{i1}u_1 + u_2'R_{i2}u_2) + \psi_i'(Ax + B_1u_1 + B_2u_2),\ i = 1, 2, \tag{4.12}$$

with respect to u_i yields the optimality conditions (see e.g. Başar and Olsder (1995) or Abou-Kandil and Bertrand (1986)):

$$u_1^*(t) = -R_{11}^{-1}B_1'\psi_1(t), \tag{4.13}$$
$$u_2^*(t) = -R_{22}^{-1}B_2'\psi_2(t), \tag{4.14}$$

where the n-dimensional vectors $\psi_1(t)$ and $\psi_2(t)$ satisfy

$$\dot{\psi}_1(t) = -Q_1x(t) - A'\psi_1(t),\ \text{with } \psi_1(t_f) = K_{1f}x(t_f), \tag{4.15}$$
$$\dot{\psi}_2(t) = -Q_2x(t) - A'\psi_2(t),\ \text{with } \psi_2(t_f) = K_{2f}x(t_f), \tag{4.16}$$

and

$$\dot{x}(t) = Ax(t) - S_1\psi_1(t) - S_2\psi_2(t),\ x(0) = x_0. \tag{4.17}$$

In other words, the problem has a unique open-loop Nash equilibrium if and only if the differential equation

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix} = -\begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A' & 0 \\ Q_2 & 0 & A' \end{pmatrix}\begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix}, \tag{4.18}$$

with boundary conditions

$$\begin{aligned} x(0) &= x_0, \\ \psi_1(t_f) - K_{1f}x(t_f) &= 0, \\ \psi_2(t_f) - K_{2f}x(t_f) &= 0, \end{aligned}$$

has a unique solution. Denoting the state variable $(x'(t)\ \psi_1'(t)\ \psi_2'(t))'$ by $y(t)$, we can rewrite this two-point boundary value problem in the standard form

$$\dot{y}(t) = -My(t), \text{ with } Py(0) + Qy(t_f) = (x_0'\ 0\ 0)', \tag{4.19}$$

where

$$M = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A' & 0 \\ Q_2 & 0 & A' \end{pmatrix}, \tag{4.20}$$

$$P = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.21}$$

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ -K_{1f} & I & 0 \\ -K_{2f} & 0 & I \end{pmatrix}. \tag{4.22}$$

From (4.19) we have immediately that there exists a unique open-loop Nash equilibrium if and only if the equation

$$(P + Qe^{-Mt_f})y(0) = (x_0'\ 0\ 0)', \tag{4.23}$$

or equivalently

$$(Pe^{Mt_f} + Q)e^{-Mt_f}y(0) = (x_0'\ 0\ 0)', \tag{4.24}$$

is uniquely solvable for every x_0. Elementary matrix analysis shows then that

Theorem 4.2 *The two-player linear quadratic differential game has a unique open-loop Nash equilibrium for every initial state x_0 if and only if the matrix $H(t_f)$ is invertible, where*

$$H(t) := \begin{pmatrix} I & 0 & 0 \end{pmatrix} e^{Mt} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}. \tag{4.25}$$

Moreover, the open-loop Nash equilibrium together with the associated state trajectory can be calculated from the linear two-point boundary value problem (4.19).

4.2.3 Sufficient conditions for existence of an open-loop Nash equilibrium

In this subsection, we will reconsider the usual approach to the problem in terms of the Riccati equations (4.9–4.10) in more detail. First we show that whenever the set of Riccati equations (4.9–4.10) has a solution, there exists an open-loop Nash equilibrium.

Theorem 4.3 *If the set of Riccati equations (4.9–4.10) has a solution then there exists an open-loop Nash equilibrium.*

Proof : Let $K_i(t)$ satisfy the set of Riccati equations (4.9–4.10). Assume that the control functions $u_i(t) = -R_{ii}^{-1}B_i'K_i(t)\Phi(t)x_0 = -R_{ii}^{-1}B_i'K_i(t)x(t)$ are used to control the system given by (4.1).
Now, define $\psi_i(t) := K_i(t)x(t)$. Then, obviously $\dot{\psi}_i(t) = \dot{K}_i(t)x(t) + K_i(t)\dot{x}(t)$.
Substitution of $\dot{K}_i$ from (4.9–4.10) and $\dot{x}$ from (4.1) yields

$$\dot{\psi}_i = (-A'K_i - Q_i)x = -A'\psi_i - Q_i x.$$

From this we conclude that the two-point boundary value problem (4.19) has a solution, which proves the claim. □

Now, under the assumption that the open-loop problem has a solution, it follows immediately from theorem 4.2 and (4.24) that

$$y_0 = e^{Mt_f}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix} H^{-1}(t_f)x_0. \tag{4.26}$$

Since $y(t) = e^{-Mt}y_0$, it follows that the entries of $y(t)$ can be rewritten as

$$x(t) = (I\ 0\ 0)e^{M(t_f-t)}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix} H^{-1}(t_f)x_0, \tag{4.27}$$

$$\psi_1(t) = (0\ I\ 0)e^{M(t_f-t)}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix} H^{-1}(t_f)x_0, \tag{4.28}$$

$$\psi_2(t) = (0\ 0\ I)e^{M(t_f-t)}\begin{pmatrix} I \\ K_{1f} \\ K_{2f}\end{pmatrix} H^{-1}(t_f)x_0. \tag{4.29}$$

Using these formulas the following lemma can easily be proved.

Lemma 4.4 *If $H(t)$, as defined in (4.25) is invertible for all $t \in [0, t_f]$, then*

$$\psi_1(t) = K_1(t)x(t) \text{ and } \psi_2(t) = K_2(t)x(t)$$

for some continuously differentiable matrix functions $K_1(t)$ and $K_2(t)$.

Proof : From (4.27) we have that

$$x(t) = H(t_f - t)H^{-1}(t_f)x_0.$$

Since by assumption the matrix $H(t)$ is invertible for all t, it follows that

$$H^{-1}(t_f)x_0 = H^{-1}(t_f - t)x(t).$$

Substitution of this expression into the equations for ψ_i, $i = 1, 2$, in (4.28,4.29) yields:

$$\psi_1(t) = G_1(t_f - t)H^{-1}(t_f - t)x(t), \quad (4.30)$$
$$\psi_2(t) = G_2(t_f - t)H^{-1}(t_f - t)x(t), \quad (4.31)$$

for some continuously differentiable matrix functions G_i, $i = 1, 2$. Since also $H^{-1}(\cdot)$ is a continuously differentiable matrix function the advertized result is obvious now. □

We like to stress here that the condition as stated in lemma 4.4 is just a sufficient condition for the adjoint state variables ψ_i, $i = 1, 2$ to be written as the product of a differentiable matrix and the state variable. Given the fact that such a representation is possible, the next corollary shows that then the open-loop Nash equilibrium can be obtained by solving the set of Riccati differential equations. This shows the flaw in the proof given by Başar and Olsder. For it implies in particular (see the result of theorem 4.3) that whenever this representation is possible a unique open-loop Nash equilibrium exists *if and only if* the set of Riccati differential equations (4.9–4.10) has a solution, whereas it will be shown below (see example 4.6) that such an equivalence does not hold.

Corollary 4.5 *If $H(t)$ is invertible for every $t \in [0, t_f]$, then the unique open-loop Nash equilibrium is given by (4.7–4.10).*

Proof : From (4.13–4.14) we know that $\psi_1(t)$ and $\psi_2(t)$ satisfy

$$\dot{\psi}_1(t) = -Q_1x(t) - A'\psi_1(t),\ \psi_1(t_f) = K_{1f}x(t_f),$$
$$\dot{\psi}_2(t) = -Q_2x(t) - A'\psi_2(t),\ \psi_2(t_f) = K_{2f}x(t_f),$$

and

$$\dot{x}(t) = Ax(t) - S_1\psi_1(t) - S_2\psi_2(t),\ x(0) = x_0.$$

According to lemma 4.4, under the above mentioned condition, $\psi_1(t)$ and $\psi_2(t)$ can be written as $K_1(t)x(t)$ and $K_2(t)x(t)$ for some continuously differentiable matrix functions $K_1(t)$ and $K_2(t)$. So, in particular we have that

$$\dot{\psi}_i = \dot{K}_ix + K_i\dot{x},\ i = 1, 2.$$

Substitution of $\dot{\psi}_i$ and ψ_i, $i = 1, 2$ into the above formulas yields

$$\left(\dot{K}_1 + A'K_1 + K_1A + Q_1 - K_1S_1K_1 - K_1S_2K_2\right) e^{Mt}x_0 = 0,$$

with

$$(K_1(t_f) - K_{1f})\, e^{Mt_f}x_0 = 0,$$

and

$$\left(\dot{K}_2 + A'K_2 + K_2A + Q_2 - K_2S_2K_2 - K_2S_1K_1\right) e^{Mt}x_0 = 0,$$

with

$$(K_2(t_f) - K_{2f})\, e^{Mt_f}x_0 = 0,$$

for arbitrarily chosen x_0. From this the stated result is obvious. □

Note that this result in particular implies that under the above mentioned invertibility condition the existence of a solution to the set of Riccati equations is guaranteed. So verification of the solvability condition becomes superfluous.

The next example shows that there exist situations where the set of Riccati differential equations (4.9–4.10) does not have a solution, whereas there exists an open-loop Nash equilibrium for the game.

Example 4.6
Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -0.9 \end{pmatrix}, B_1 = B_2 = Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$R_{11} = \begin{pmatrix} 500 & -200 \\ -200 & 100 \end{pmatrix}^{-1}, R_{22} = \begin{pmatrix} 1000 & 200 \\ 200 & 50 \end{pmatrix}^{-1}, Q_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Now, choose $t_f = 0.1$. Then, numerical calculations show

$$\begin{aligned} H(0.1) &= \begin{pmatrix} 35.0323 & 9.3217 \\ -1.8604 & 0.4729 \end{pmatrix} + \begin{pmatrix} 366.4330 & -142.9873 \\ -36.5968 & 16.4049 \end{pmatrix} K_{1f} \\ &\quad + \begin{pmatrix} 850.3143 & 172.4050 \\ -22.0161 & -3.5423 \end{pmatrix} K_{2f} \\ &=: V \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}. \end{aligned}$$

Now, choose

$$K_{1f} = \begin{pmatrix} -\frac{1}{V_{23}} & \frac{1-V_{21}}{V_{24}} \\ \frac{1-V_{21}}{V_{24}} & 2 \end{pmatrix}, \text{ and } K_{2f} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{V_{22}+V_{23}K_{1f}(1,2)+2V_{24}}{V_{26}} \end{pmatrix}.$$

Then $H(0.1) = \begin{pmatrix} 20.11 & 1096.54 \\ 0 & 0 \end{pmatrix}$ is not invertible.

So, according to theorem 4.2 there exists no open-loop Nash equilibrium, and therefore (see theorem 4.3) the corresponding set of Riccati differential equations has no solution on $[0, 0.1]$.

Next consider $H(0.11)$. Numerical calculations show that with the system parameters as chosen above, $H(0.11)$ is invertible. So, according to theorem 4.2 again, the game does have an open-loop Nash equilibrium for $t_f = 0.11$. However, since the set of Riccati differential equations can be rewritten as one autonomous vector differential equation, whose solutions are known to be shift invariant, it is clear that the corresponding set of Riccati differential equations can not have a solution on $[0, 0.11]$, since it had no solution on $[0, 0.1]$.

4.2.4 The solutions for the algebraic Riccati equation

To study the asymptotic behavior of the open-loop Nash equilibrium, we first consider in this subsection the set of solutions of the algebraic Riccati equations corresponding with (4.9–4.10)

$$-A'K_1 - K_1A - Q_1 + K_1S_1K_1 + K_1S_2K_2 = 0, \quad (4.32)$$
$$-A'K_2 - K_2A - Q_2 + K_2S_2K_2 + K_2S_1K_1 = 0. \quad (4.33)$$

MacFarlane (1963) and Potter (1966) discovered independently that there exists a relationship between the stabilizing solution of the algebraic Riccati equation and the eigenvectors of a related Hamiltonian matrix for linear-quadratic control problems. We will follow their approach here and formulate similar results for our case. In fact Abou-Kandil et al. (1993) already pointed out the existence of a similar relationship. One of their results is that if the horizon t_f tends to infinity, under some technical conditions on the matrix M, the solution of the Riccati differential equations (4.9–4.10) converges to a solution of the algebraic Riccati equations (4.32–4.33) which can be calculated from the eigenspaces of matrix M.

In this subsection we will elaborate the relationship between solutions of the algebraic Riccati equations (4.32–4.33) and the matrix M in detail (see also Engwerda and Weeren (1995a)). We will present both necessary and sufficient conditions in terms of the matrix

M under which the algebraic Riccati equations (4.32–4.33) have (a) real solution(s). In particular we will see that all solutions can be calculated from the invariant subspaces of M and that the eigenvalues of the associated closed-loop system, obtained by applying the control functions $u_i^*(t) = -R_{ii}^{-1} B_i' \Phi(t) x_0$, are completely determined by the eigenvalues of matrix M. As a corollary of these results we obtain both necessary and sufficient conditions for the existence of a stabilizing control of this type, a result which will be used in the next subsection.

In our analysis, the set of all M-invariant subspaces will play a crucial role. Therefore we introduce a separate notation for this set:

$$\mathcal{M}^{\text{inv}} := \{\mathcal{T} \mid M\mathcal{T} \subseteq \mathcal{T}\} \tag{4.34}$$

It is well known (see e.g. Lancaster and Tismenetsky (1985)) that this set contains only a finite number of (distinct) elements if and only if all eigenvalues of M have geometric multiplicity one.

The set of possible solutions for the algebraic Riccati equation can, as will be shown in the next theorem, directly be calculated from the following collection of M-invariant subspaces:

$$\mathcal{K}^{\text{pos}} := \left\{ \mathcal{K} \in \mathcal{M}^{\text{inv}} \mid \mathcal{K} \oplus \text{im} \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = \mathbb{R}^{3n} \right\}. \tag{4.35}$$

Remark 4.7 Elements in the set $\mathcal{K}^{\text{pos}}$ can be calculated using the set of matrices

$$K^{\text{pos}} := \left\{ K \in \mathbb{R}^{3n \times n} \mid K = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \ X \text{ invertible} \right\}. \tag{4.36}$$

The exact result on how all solutions of the algebraic Riccati equations (4.32–4.33) can be calculated reads as follows:

Theorem 4.8 *The pair (K_1, K_2) is a real solution to the algebraic Riccati equations (4.32–4.33) if and only if $K_1 = YX^{-1}$ and $K_2 = ZX^{-1}$ for some*

$$\mathcal{K} = \text{im} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

such that

$$\mathcal{K} \in \mathcal{K}^{\text{pos}}.$$

Moreover, if the control functions $u_i^*(t) = -R_{ii}^{-1}B_i'K_i\Phi(t)x_0$ are used to control the system (4.1), the spectrum of the matrix $-A + S_1K_1 + S_2K_2$ coincides with $\sigma(M|_{\mathcal{K}})$.

Proof : " ⇒ " Assume $\big(K_1, K_2\big)$ solve (4.32–4.33). Then simple calculations show that

$$M \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} -A + S_1K_1 + S_2K_2 \\ Q_1 + A'K_1 \\ Q_2 + A'K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} (-A + S_1K_1 + S_2K_2)\,,$$

which completes this part of the proof.

" ⇐ " Let $\mathcal{K} \in \mathcal{K}^{\text{pos}}$. Then there exist K_1 and K_2 such that $\mathcal{K} = \text{im} \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix}$, and a matrix J such that

$$M \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} J.$$

Spelling out the left hand side of this equation gives

$$\begin{pmatrix} -A + S_1K_1 + S_2K_2 \\ Q_1 + A'K_1 \\ Q_2 + A'K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} J,$$

which immediately yields that $J = -A + S_1K_1 + S_2K_2$. Substitution of this equality into the right hand side of the equality shows then that

$$\begin{aligned} Q_1 + A'K_1 &= K_1\,(-A + S_1K_1 + S_2K_2)\,, \\ Q_2 + A'K_2 &= K_2\,(-A + S_1K_1 + S_2K_2)\,, \end{aligned}$$

or stated differently, $\big(K_1, K_2\big)$ satisfies (4.32–4.33). This proves the second part of the theorem.

The last statement of the theorem concerning the spectrum of the matrix $-A+S_1K_1+S_2K_2$ is a well-known fact. □

From theorem 4.8, a number of interesting properties concerning the solvability of the algebraic Riccati equations (4.32–4.33) follow. First of all, we observe that every element of $\mathcal{K}^{\text{pos}}$ defines exactly one solution of (4.32–4.33). Furthermore, this set contains a finite number of elements if the geometric multiplicities of all eigenvalues of M is one. So, in that case we immediately conclude that the algebraic Riccati equations will have at most a finite number of solutions and that the algebraic Riccati equations will have no real solution if $\mathcal{K}^{\text{pos}}$ is empty. Another conclusion which immediately follows from theorem 4.8 is that

Corollary 4.9 *The algebraic Riccati equations (4.32–4.33) will have a set of solutions* $\big(K_1, K_2\big)$ *stabilizing* $A - S_1K_1 - S_2K_2$ *if and only if there exists an* M*-invariant subspace* $\mathcal{K}$ *in* $\mathcal{K}^{\text{pos}}$ *such that* $\operatorname{Re}\lambda > 0$ *for all* $\lambda \in \sigma(M|_{\mathcal{K}})$.

To illustrate some of the above mentioned properties, reconsider example 4.6.

Example 4.10 (Continued from example 4.6)
Numerical calculations show that the eigenvalues of M are

$$\sigma(M) = \{-42.1181, -0.8866, -0.3441 \pm 4.6285i, -0.3168, 42.1096\},$$

and the corresponding eigenspaces

$$\begin{aligned}
\mathcal{T}_1 &= \operatorname{span}\big\{(-0.9968\ 0.0549\ 0.0471\ 0.0229\ 0.0242\ -0.0013)'\big\},\\
\mathcal{T}_2 &= \operatorname{span}\big\{(-0.0178\ 0.0108\ -0.2191\ -0.5272\ -0.1570\ 0.8056)'\big\},\\
\mathcal{T}_3 &= \operatorname{span}\big\{(-0.0439\ 0.1636\ -0.085\ -0.1382\ 0.0519\ -0.1906)',\\
&\qquad (0.2512\ -0.9146\ -0.0284\ -0.0425\ 0.0168\ -0.0582)'\big\},\\
\mathcal{T}_4 &= \operatorname{span}\big\{(0.1145\ -0.4047\ -0.2570\ -0.4975\ 0.1676\ -0.6939)'\big\},\\
\mathcal{T}_5 &= \operatorname{span}\big\{(-0.9970\ 0.0545\ -0.0450\ -0.0219\ -0.0231\ 0.0013)'\big\}.
\end{aligned}$$

It is easily verified that there exist $\begin{pmatrix} 4 \\ 2 \end{pmatrix} + 1 = 7$ two-dimensional M-invariant subspaces, i.e. $\mathcal{K}^{\text{pos}}$ has maximally 7 elements. Then, according to theorem 4.8, the algebraic Riccati equations can have at most 7 real solutions. Furthermore, there is no solution which stabilizes the closed-loop system matrix $A - S_1K_1 - S_2K_2$.
As an example consider

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} := \big(\mathcal{T}_5\ \mathcal{T}_4\big).$$

This yields the solution

$$\begin{aligned}
K_1 = YX^{-1} &= \begin{pmatrix} 0.0450 & -0.2570 \\ 0.0219 & -0.4975 \end{pmatrix} \begin{pmatrix} 0.9970 & 0.1145 \\ -0.0545 & -0.4047 \end{pmatrix}^{-1}\\
&= \begin{pmatrix} 0.0811 & 0.6581 \\ 0.0905 & 1.2549 \end{pmatrix},\\
K_2 = ZX^{-1} &= \begin{pmatrix} 0.0231 & 0.1676 \\ -0.0013 & -0.6939 \end{pmatrix} \begin{pmatrix} 0.9970 & 0.1145 \\ -0.0545 & -0.4047 \end{pmatrix}^{-1}\\
&= \begin{pmatrix} 0.0006 & -0.4141 \\ 0.0939 & 1.7411 \end{pmatrix}.
\end{aligned}$$

The eigenvalues of the "closed-loop" system matrix are

$$\sigma(A - S_1K_1 - S_2K_2) = \{-42.1096, 0.3168\}.$$

It is easily verified that the rank of the first two rows of every other candidate solution is also two, so we conclude that the system of algebraic Riccati equations has seven solutions, none of which is stabilizing.

4.2.5 Convergence results

As argued in the introduction of this section, it is for a number of reasons interesting to see how the open-loop Nash equilibrium changes when the horizon t_f tends to infinity. To study convergence properties, it seems reasonable to require the existence of a unique open-loop Nash-equilibrium for every finite horizon t_f. Therefore we will make in this section the following well-posedness assumption (see theorem 4.2)

$$H(t_f) \text{ is invertible for all } t_f < \infty. \tag{4.37}$$

Furthermore, we will see that general convergence results can only be derived if the eigenstructure of matrix M satisfies an additional property. Therefore we define this property first.

Definition 4.11 M is called *dichotomically separable* if there exist subspaces V_1 and V_2 such that $MV_i \subseteq V_i$, $i = 1, 2$, $V_1 \oplus V_2 = \mathbb{R}^{3n}$, $\dim V_1 = n$, $\dim V_2 = 2n$, and moreover $\operatorname{Re}\lambda > \operatorname{Re}\mu$ for all $\lambda \in \sigma(M|_{V_1})$ and $\mu \in \sigma(M|_{V_2})$.

Using corollary 4.5 we have now immediately from (4.37) that to study the convergence of the open-loop Nash equilibrium we can restrict ourselves to the study of the set of solutions to the Riccati differential equations (4.9–4.10) at time 0. We will denote the corresponding solutions of (4.9–4.10) by $K_i(0, t_f)$. So the question is under which conditions the solutions of this set of differential equations will converge if t_f increases. Note that $K_i(0, t_f)$ can be viewed as the solution $k(t)$ of an autonomous vector differential equation $\dot{k} = f(k)$, with $k(0) = k_0$ for some fixed k_0, and where f is a smooth function. Elementary analysis shows then that $K_i(0, t_f)$ can only converge to a limit $\bar{k}$ if the limit $\bar{k}$ satisfies $f(\bar{k}) = 0$. Therefore, we immediately deduce from theorem 4.8 the following necessary condition for convergence.

Lemma 4.12 *$K_i(0, t_f)$ can only converge to a limit $\bar{K}_i(0)$ if the set $\mathcal{K}^{\text{pos}}$ is nonempty.*

Note that dichotomic separability of M implies that $\mathcal{K}^{\text{pos}}$ is nonempty. On the other hand it is not difficult to construct an example where $\mathcal{K}^{\text{pos}}$ is nonempty, whereas M is not dichotomically separable.

To study the convergence of $K_i(0,t_f)$ we reconsider (4.30) and (4.31) in lemma 4.4. From these formulas we have that

$$K_1(0,t_f) = (0\ I\ 0)e^{Mt_f}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}\left((I\ 0\ 0)e^{Mt_f}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}\right)^{-1}, \tag{4.38}$$

$$K_2(0,t_f) = (0\ 0\ I)e^{Mt_f}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}\left((I\ 0\ 0)e^{Mt_f}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}\right)^{-1}. \tag{4.39}$$

We are now able to give an elementary proof of the following result (see also (Abou-Kandil et al., 1993, section 4))

Theorem 4.13 *Assume that the well-posedness assumption (4.37) holds. Then, if M is dichotomically separable and* $\text{span}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \oplus V_2 = \mathbb{R}^{3n}$,

$$K_1(0,t_f) \rightarrow Y_0X_0^{-1},$$
$$K_2(0,t_f) \rightarrow Z_0X_0^{-1},$$

where X_0, Y_0, Z_0 are defined by (using the notation of definition 4.11)

$$V_1 =: \text{span}\begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}.$$

Proof : Choose $\begin{pmatrix} I & 0 & 0 \\ K_{1f} & I & 0 \\ K_{2f} & 0 & I \end{pmatrix}$ as a basis for $\mathbb{R}^{3n}$. Then, because

$$\text{span}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \oplus V_2 = \mathbb{R}^{3n},$$

there exists an invertible matrix $V_{22} \in \mathbb{R}^{2n \times 2n}$ such that $V_2 = \text{span}\begin{pmatrix} 0 \\ V_{22} \end{pmatrix}$. Moreover, because M is dichotomically separable, there exist matrices J_1, J_2 such that

$$M = V \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} V^{-1},$$

where

$$V = \begin{pmatrix} X_0 & 0 \\ \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} & V_{22} \end{pmatrix},$$

and $\sigma(J_i) = \sigma(M|_{V_i})$, $i = 1, 2$.

Using this, we can rewrite $K_i(0, t_f)$, $i = 1, 2$, in (4.38,4.39) as $\tilde{G}_i(t_f)\tilde{H}^{-1}(t_f)$, $i = 1, 2$, where

$$\begin{aligned} \tilde{G}_1(t_f) &= (0\ I\ 0)Ve^{-\lambda_n t_f}\begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}, \\ \tilde{G}_2(t_f) &= (0\ 0\ I)Ve^{-\lambda_n t_f}\begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}, \\ \tilde{H}(t_f) &= (I\ 0\ 0)Ve^{-\lambda_n t_f}\begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}. \end{aligned}$$

Here λ_n is the element of $\sigma(M|_{V_1})$ which has the smallest real part.

Next, consider $\tilde{G}_1(t_f) - Y_0 X_0^{-1}\tilde{H}(t_f)$.
Simple calculations show that this matrix can be rewritten as

$$e^{-\lambda_n t_f}(-Y_0 X_0^{-1}\ I\ 0)V \begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}, \tag{4.40}$$

Since $(-Y_0 X_0^{-1}\ I\ 0)\,(X_0'\ Y_0'\ Z_0')' = 0$, (4.40) equals

$$e^{-\lambda_n t_f} \begin{pmatrix} I & 0 \end{pmatrix} V_{22} e^{J_2 t_f} V_{22}^{-1} \begin{pmatrix} K_{1f} - Y_0 X_0^{-1} \\ K_{2f} - Z_0 X_0^{-1} \end{pmatrix}.$$

As $e^{-\lambda_n t_f} e^{J_2 t_f}$ converges to zero for $t_f \to \infty$, it is obvious now that $\tilde{G}_1(t_f) - Y_0 X_0^{-1}\tilde{H}(t_f)$ converges to zero for $t_f \to \infty$. Similarly it can be shown that also $\tilde{G}_2(t_f) - Z_0 X_0^{-1}\tilde{H}(t_f)$

converges to zero for $t_f \to \infty$. To conclude from this that $K_1(0,t_f) \to Y_0 X_0^{-1}$, and $K_2(0,t_f) \to Z_0 X_0^{-1}$, it suffices to show that $\tilde{H}^{-1}(t_f)$ remains bounded for $t_f \to \infty$. This follows, however, directly by spelling out $\tilde{H}(t_f)$ as

$$\tilde{H}(t_f) = e^{-\lambda_n t_f} X_0 e^{J_1 t_f} X_0^{-1}.$$

□

Combination of the results from theorem 4.13 and corollary 4.9 yields then

Corollary 4.14 *If the horizon t_f in the linear quadratic differential game tends to infinity, the unique open-loop Nash equilibrium $u_i^*(t,t_f) = -R_{ii}^{-1} B_i' K_i(t,t_f)\Phi(t)x_0$ converges to $u_i^*(t) = -R_{ii}^{-1} B_i' K_i e^{t(A - S_1K_1 - S_2K_2)} x_0$, $i = 1,2$, which stabilizes the associated "closed-loop" system, if the following conditions are satisfied:*

1. *all conditions mentioned in theorem 4.13,*
2. $\operatorname{Re}\lambda > 0$ *for all* $\lambda \in \sigma(M|_{V_1})$.

Moreover, the constant matrices K_i can be calculated from the eigenspaces of the matrix M (see theorem 4.13).

4.2.6 The scalar case

We start this subsection by showing that the invertibility condition mentioned in corollary 4.5 is always satisfied if the dimensions of both the state space and the input spaces for the system (4.1) equal one. This implies that for this kind of systems the usually stated assertion that the open-loop Nash equilibrium is given by (4.7–4.10) is correct and, moreover, that the Riccati equations (4.9–4.10) yield the appropriate solution. To prove this result we first calculate the exponential of the matrix M. To stress the fact that in this section we are dealing with the scalar case, we will put the system parameters in lower case, so e.g. a instead of A.

Lemma 4.15 *Consider the matrix M in (4.19) and assume $s_1q_1 + s_2q_2 > 0$. The exponential of the matrix M, e^{Ms}, is given by*

$$V \begin{pmatrix} e^{-\mu s} & 0 & 0 \\ 0 & e^{as} & 0 \\ 0 & 0 & e^{\mu s} \end{pmatrix} V^{-1}, \tag{4.41}$$

where

$$V = \begin{pmatrix} a+\mu & 0 & a-\mu \\ -q_1 & -s_2 & -q_1 \\ -q_2 & s_1 & -q_2 \end{pmatrix},$$

and

$$V^{-1} = \frac{1}{\det V} \begin{pmatrix} (s_1q_1+s_2q_2) & s_1(a-\mu) & s_2(a-\mu) \\ 0 & -2q_2\mu & 2q_1\mu \\ -(s_1q_1+s_2q_2) & -s_1(a+\mu) & -s_2(a+\mu) \end{pmatrix},$$

where the determinant of V *is given by* $\det V = 2\mu\big(s_1q_1+s_2q_2\big)$, *and* $\mu = \sqrt{a^2+s_1q_1+s_2q_2}$.

Proof : Straightforward calculations. □

Next consider the matrix $H(s)$ from lemma 4.4 for an arbitrarily chosen $s \in [0, t_f]$. Obviously,

$$H(s) = (1\ 0\ 0)e^{Ms} \begin{pmatrix} 1 \\ k_{1f} \\ k_{2f} \end{pmatrix}.$$

Using the expressions in lemma 4.15 for V and V^{-1} we find

$$H(s) = 2\mu\left[(\mu-a)\,e^{\mu s} + (\mu+a)\,e^{-\mu s} + \left(e^{\mu s} - e^{-\mu s}\right)(s_1k_{1f}+s_2k_{2f})\right].$$

Clearly, $H(s)$ is positive for every $s \geqslant 0$. This implies in particular that $H(s)$ differs from zero for every $s \in [0, \infty)$. So from corollary 4.5 we now immediately have the following conclusion.

Theorem 4.16 *Assume* $s_1q_1+s_2q_2 > 0$. *Then the scalar linear-quadratic differential game has a unique open-loop Nash equilibrium:*

$$\gamma_1(x_0) = u_1^*(t) = -\frac{1}{r_{11}}b_1k_1(t)\phi(t)x_0,$$
$$\gamma_2(x_0) = u_2^*(t) = -\frac{1}{r_{22}}b_2k_2(t)\phi(t)x_0,$$

where $k_1(t)$ *and* $k_2(t)$ *are the solutions of the coupled asymmetric Riccati differential equations*

$$\dot{k}_1 = -2ak_1 - q_1 + s_1k_1^2 + s_2k_1k_2,\ k_1(t_f) = k_{1f},$$
$$\dot{k}_2 = -2ak_2 - q_2 + s_2k_2^2 + s_1k_1k_2,\ k_2(t_f) = k_{2f},$$

and

$$\dot{\phi}(t) = (a - s_1k_1(t) - s_2k_2(t))\,\phi(t),\ \phi(0) = 1.$$

Here $s_i = \frac{1}{r_{ii}}b_i^2$, $i = 1, 2$.

We conclude this subsection by considering the convergence properties of the open-loop equilibrium solution mentioned above. It turns out that in the scalar case we can prove that this solution always converges.

Theorem 4.17 *Assume that $s_1q_1 + s_2q_2 > 0$. Then, the open-loop Nash equilibrium as given in theorem 4.16 converges to the strategies:*

$$\begin{aligned}\gamma_1(x_0) &= u_1^*(t) = -\frac{1}{r_{11}}b_1k_1e^{-\mu t}x_0,\\ \gamma_2(x_0) &= u_2^*(t) = -\frac{1}{r_{22}}b_2k_2e^{-\mu t}x_0,\end{aligned}$$

where $k_1 = \frac{(a+\mu)q_1}{s_1q_1+s_2q_2}$ and $k_2 = \frac{(a+\mu)q_2}{s_1q_1+s_2q_2}$. Moreover, these strategies stabilize the system (4.1).

Proof : Since $s_1q_1 + s_2q_2 > 0$, it is clear from (4.41) that M is dichotomically separable. Furthermore we showed above that the well-posedness assumption is always satisfied in the scalar case. Note that $\mu > 0$, so according to corollary 4.14 the open-loop Nash equilibrium converges whenever k_{if}, $i = 1, 2$, are such that

$$s_1q_1 + s_2q_2 + s_1(a-\mu)k_{1f} + s_2(a-\mu)k_{2f} \neq 0.$$

Now consider the case that

$$s_1q_1 + s_2q_2 + s_1(a-\mu)k_{1f} + s_2(a-\mu)k_{2f} = 0.$$

To study this case, reconsider (4.38) and (4.39) for $t_f \to \infty$. Elementary spelling out of these formulas, using (4.41), shows that also in this case both $k_1(0, t_f)$ and $k_2(0, t_f)$ converge to the limit as advertized above, which concludes the proof. □

4.2.7 Concluding remarks

In this section, we reconsidered the existence and asymptotic behavior of a unique open-loop Nash equilibrium in the two-player linear-quadratic game. We analyzed the problem starting from its basics: the Hamiltonian equations. We derived necessary and sufficient conditions for the existence of a unique open-loop Nash equilibrium in terms of a full rank condition on a modified fundamental matrix. An open problem remains to find general conditions on the system matrices which guarantee that the rank condition is satisfied. Furthermore we showed by means of an example that in general a solution to the system of differential Riccati equations may fail to exist whereas an open-loop Nash equilibrium does exist. A sufficient condition is given under which the open-loop Nash equilibrium can be obtained via the solutions of the Riccati differential equations (4.9–4.10).

To study convergence of the open-loop Nash equilibrium if the horizon t_f tends to infinity, we argued that for well-posedness reasons we can restrict ourselves to study the asymptotic behavior of the Riccati differential equations. To that end we first considered the existence of real solutions for the corresponding algebraic Riccati equations. We showed how every real solution to the algebraic Riccati equations can be calculated from the invariant subspaces of the matrix M. Furthermore, we showed how the eigenvalues of the "closed-loop" system matrix if the open-loop strategies are used to control (4.1) correspond to the eigenvalues of the matrix M.

In particular this approach makes it possible to conclude whether the algebraic Riccati equations have a real solution, and if so, how many solutions there are (there are always only a finite number of solutions if the geometric multiplicity of all eigenvalues of M equals one) and which of these solutions give rise to control strategies that stabilize the "closed-loop" system. We noted that in general the algebraic Riccati equations will allow for more than one stabilizing solution. We like to note that it is not difficult to show by means of an example that this property is independent of the fact whether the matrix M is dichotomically separable or not.

These results raise a number of open interesting questions, for instance, is it possible to say a priori something on the relationship between the eigenstructure of matrix M (in particular the structure which guarantees the existence of stabilizing solutions to the algebraic Riccati equations, and more in particular the structure which generically implies convergence of the solutions of the Riccati differential equations) and geometric properties of the system parameters in (4.1). A first attempt to answer the question under which conditions on the system matrices there may exist a stabilizing solution was addressed in Engwerda and Weeren (1994a), where for a number of particular situations it is shown that matrix M always has at least n eigenvalues (counted with their algebraic multiplicities) with a positive real part. On the other hand it is shown there by means of an example that this property does not always hold.

The results on the existence of real solutions to the algebraic Riccati equations were used to show that if the dimension of the direct sum of the invariant subspaces corresponding to the n largest eigenvalues (counted again with algebraic multiplicities) equals n, then generically the solution to the Riccati differential equations converges to a solution which can be directly calculated from this direct sum.

Since there are a number of applications which just involve scalar systems we concluded this section with a detailed analysis of that case. We showed that for those systems the unique open-loop Nash equilibrium can always be found by solving the Riccati differential equations, and that this solution converges to a strategy which stabilizes the system when the planning horizon tends to infinity.

Finally we note that the obtained results can straightforwardly be generalized to the N-player game.

4.3 Feedback Nash equilibria

4.3.1 Introduction

In this section we study memoryless perfect state (MPS) information, i.e.

$$\eta_i(t) = (x_0, x(t)), \; t \in [0, t_f].$$

For this information structure, the following theorem is well known (see Başar and Olsder (1995); Starr and Ho (1969b)):

Theorem 4.18 *Let the strategies* (γ_1^*, γ_2^*) *be such that there exist solutions* (ψ_1, ψ_2) *to the differential equations*

$$\begin{aligned} \dot{\psi}_1' = &-\frac{\partial H_1}{\partial x}(x^*, \gamma_1^*(t, x_0, x^*), \gamma_2^*(t, x_0, x^*), \psi_1) \\ &- \frac{\partial H_1}{\partial u_2}(x^*, \gamma_1^*(t, x_0, x^*), \gamma_2^*(t, x_0, x^*), \psi_1) \cdot \frac{\partial \gamma_2^*}{\partial x}(t, x_0, x^*) \end{aligned} \tag{4.42}$$

$$\begin{aligned} \dot{\psi}_2' = &-\frac{\partial H_2}{\partial x}(x^*, \gamma_1^*(t, x_0, x^*), \gamma_2^*(t, x_0, x^*), \psi_2) \\ &- \frac{\partial H_2}{\partial u_1}(x^*, \gamma_1^*(t, x_0, x^*), \gamma_2^*(t, x_0, x^*), \psi_2) \cdot \frac{\partial \gamma_1^*}{\partial x}(t, x_0, x^*) \end{aligned} \tag{4.43}$$

in which, for $i = 1, 2$,

$$H_i(x, u_1, u_2, \psi_i) := x'Q_ix + u_1'R_{i1}u_1 + u_2'R_{i2}u_2 + \psi_i'(Ax + B_1u_1 + B_2u_2) \tag{4.44}$$

with terminal conditions, for $i = 1, 2$,

$$\psi_i(t_f) = K_{if}x^*(t_f), \tag{4.45}$$

such that for $i = 1, 2$,

$$\frac{\partial H_i}{\partial u_i}(x^*, \gamma_1^*(t, x_0, x^*), \gamma_2^*(t, x_0, x^*), \psi_i) = 0 \tag{4.46}$$

and x^* satisfies

$$\dot{x}^*(t) = Ax^*(t) + B_1\gamma_1^*(t, x_0, x^*(t)) + B_2\gamma_2^*(t, x_0, x^*(t)),\ x^*(0) = x_0. \tag{4.47}$$

Then (γ_1^*, γ_2^*) is a Nash equilibrium with respect to the MPS information structure, and the following equalities hold:

$$u_i^*(t) = \gamma_i^*(t, x_0, x^*(t)) = -R_{ii}^{-1}B_i'\psi_i(t).$$

Remark 4.19 Let $(\gamma_1^{ol}, \gamma_2^{ol})$ be an open-loop Nash equilibrium. Then it is easily seen that $(\gamma_1^{ol}, \gamma_2^{ol})$ is also a Nash equilibrium with respect to the MPS information structure, because $\frac{\partial}{\partial x}\gamma_i^{ol} = 0$.

When we restrict the admissible strategies to the class of (possibly time-varying) *linear* feedback strategies, i.e. $\Gamma_i^{fb} := \{\gamma_i \in \Gamma_i \mid \gamma_i(x,t) = F_i(t)x\}$ (see section 3.2), then there exists a generically unique feedback Nash equilibrium (see e.g. (Başar and Olsder, 1995, section 6.5.2)). The following theorem can be found in Başar and Olsder (1995); Starr and Ho (1969b).

Theorem 4.20 *Suppose (K_1, K_2) satisfy the coupled Riccati equations, given by*

$$\begin{aligned}
\dot{K}_1 &= -A'K_1 - K_1A - Q_1 + K_1S_1K_1 + K_1S_2K_2 + K_2S_2K_1 - K_2S_{02}K_2 && (4.48)\\
\dot{K}_2 &= -A'K_2 - K_2A - Q_2 + K_2S_2K_2 + K_2S_1K_1 + K_1S_1K_2 - K_1S_{01}K_1 && (4.49)\\
K_1(t_f) &= K_{1f} && (4.50)\\
K_2(t_f) &= K_{2f} && (4.51)
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= B_1R_{11}^{-1}B_1',\\
S_2 &= B_2R_{22}^{-1}B_2',\\
S_{01} &= B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1',\\
S_{02} &= B_2R_{22}^{-1}R_{12}R_{22}^{-1}B_2'.
\end{aligned}$$

Then the pair of strategies $(\gamma_1^*(x,t), \gamma_2^*(x,t)) := \left(-R_{11}^{-1}B_1'K_1(t)x, -R_{22}^{-1}B_2'K_2(t)x\right)$ is a feedback Nash equilibrium. The functions ψ_i of theorem 4.18 are given by $\psi_i(t) = K_i(t)x(t)$.

Proof :(*outline*) Write (ψ_1, ψ_2) in theorem 4.18 as $\psi_i = K_ix$. Then the Nash equilibrium is given by $\gamma_i^*(x,t) = -R_{ii}^{-1}B_i'\psi_i(t) = -R_{ii}^{-1}B_i'K_ix$.
Obviously $(\gamma_1^*, \gamma_2^*) \in \Gamma_1^{fb} \times \Gamma_2^{fb}$. Moreover, it is easily verified that K_1 and K_2 satisfy the coupled Riccati equations (4.48–4.51). □

Remark 4.21 When we allow for more general (e.g. nonlinear feedback) strategies, there may exist many more Nash equilibria for the MPS information structure.

In the previous section we have characterized the open-loop Nash equilibrium by means of a linear differential system, as in the papers Abou-Kandil and Bertrand (1986); Abou-Kandil et al. (1993); Engwerda and Weeren (1995b). In that way, it was easier to investigate the asymptotic properties of the open-loop Nash equilibrium and it also enabled us to calculate the solutions of (4.9–4.10). We will now see what happens if we try to rewrite the Riccati equations (4.48–4.51) for the feedback Nash equilibrium as a linear system, following a similar approach as in the previous section. For the feedback Nash equilibrium the functions (ψ_1, ψ_2), as described by theorem 4.18, satisfy the following differential equations:

$$\dot{\psi}_1 = -Q_1 x - (A' - K_2 S_2)\,\psi_1 - K_2 S_{02} \psi_2, \tag{4.52}$$

$$\dot{\psi}_2 = -Q_2 x - K_1 S_{01} \psi_1 - (A' - K_1 S_1)\,\psi_2. \tag{4.53}$$

This gives for the matrix M

$$M = M(K_1, K_2) = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A' - K_2 S_2 & K_2 S_{02} \\ Q_2 & K_1 S_{01} & A' - K_1 S_1 \end{pmatrix}. \tag{4.54}$$

In the (realistic) case $R_{12} = 0$, $R_{21} = 0$, (4.54) simplifies to

$$M = M(K_1, K_2) = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A' - K_2 S_2 & 0 \\ Q_2 & 0 & A' - K_1 S_1 \end{pmatrix}. \tag{4.55}$$

Note that, even in this special case, M depends on (K_1, K_2), so that the resulting equations are still nonlinear. In the rest of this section we will study in detail the quadratic system of Riccati differential equations for the feedback Nash equilibrium in the most simple case where all parameters are scalar and $R_{12} = 0$, $R_{21} = 0$. This analysis will show that the situation for the feedback Nash equilibrium is much more complicated than in the open-loop case.

4.3.2 The scalar case

Below we restrict our attention to the case in which all the system parameters are scalar[1]. Furthermore, we shall confine ourselves to the case where q_1, q_2, s_1 and s_2 are all strictly

[1] To emphasize the fact that all system parameters are scalar we put them in lower case, e.g. q_1 instead of Q_1.

positive. If we rewrite the terminal value problem for $\big(k_1(t), k_2(t)\big)$ as an initial value problem for $\big(k_1(\tau), k_2(\tau)\big)$, we get

$$\dot{k}_1 = 2ak_1 + q_1 - s_1k_1^2 - 2s_2k_1k_2, \tag{4.56}$$
$$\dot{k}_2 = 2ak_2 + q_2 - s_2k_2^2 - 2s_1k_1k_2, \tag{4.57}$$
$$k_1(0) = k_{1f}, \tag{4.58}$$
$$k_2(0) = k_{2f}. \tag{4.59}$$

Now define

$$\sigma_i := s_iq_i, \ i = 1, 2, \tag{4.60}$$
$$\kappa_i := s_ik_i, \ i = 1, 2. \tag{4.61}$$

Then we get the following system of quadratic differential equations:

$$\dot{\kappa}_1 = 2a\kappa_1 + \sigma_1 - \kappa_1^2 - 2\kappa_1\kappa_2, \tag{4.62}$$
$$\dot{\kappa}_2 = 2a\kappa_2 + \sigma_2 - \kappa_2^2 - 2\kappa_1\kappa_2. \tag{4.63}$$

The study of planar quadratic systems in general is a very complicated topic, as e.g. can be seen in the survey papers Coppel (1966); Reyn (1987). For example the famous 16th Hilbert problem, to determine the maximal number of limit cycles, H_d, for dth degree polynomial planar systems, is still unsolved even for quadratic systems ($d = 2$). Hence, in general we can expect complicated dependence on the parameters for the quadratic system (4.62–4.63); for instance in Reyn (1987), Reyn finds 101 topologically different global phase portraits for a 6-parameter family of quadratic systems. In the following subsections we address some of the characteristics of the quadratic system (4.62–4.63) that one typically is interested in.

4.3.3 Periodic solutions

The first question we address is the determination of the maximal number of limit cycles for the quadratic system (4.62–4.63). This leads to the question of existence of periodic solutions. We recall a famous criterion due to Bendixson (introduced in the paper Bendixson (1901), see also e.g. Perko (1991)):

Theorem 4.22 (Bendixson) *Let $f \in C^1(E \to \mathbb{R}^2)$, where E is a simply connected region in $\mathbb{R}^2$. If the divergence $\nabla \cdot f$ of the vector field f is not identically zero and does not change sign in E, then the planar system $\dot{x} = f(x)$ has no periodic solution lying entirely in E.*

Recall that the divergence of a vector field in $\mathbb{R}^2$ is given by the trace of the Jacobian matrix. The divergence of the quadratic system (4.62–4.63) is given by

$$\nabla \cdot f = 4a - 4\kappa_1 - 4\kappa_2. \tag{4.64}$$

Hence, the divergence equals zero on the line

$$a - \kappa_1 - \kappa_2 = 0. \tag{4.65}$$

It follows from theorem 4.22 that if there would exist a periodic solution of the quadratic system (4.62–4.63), this solution would have to cross the line (4.65) at least two times. However, on the line (4.65) we have

$$\dot{\kappa}_1 = \sigma_1 + \kappa_1^2 > 0,$$
$$\dot{\kappa}_2 = \sigma_2 + \kappa_2^2 > 0,$$

and hence any solution of (4.62–4.63) can cross the line (4.65) at most once. We conclude therefore that there does not exist any periodic solution to the quadratic system (4.62–4.63), and thus there are no limit cycles.

4.3.4 Critical points

The question of determining critical points of the quadratic system (4.62–4.63) is closely related to the question of the existence of stationary feedback Nash equilibria. The critical points of the differential equations (4.62–4.63) are the intersection points of the hyperbolas given by

$$2a\kappa_1 + \sigma_1 - \kappa_1^2 - 2\kappa_1\kappa_2 = 0, \tag{4.66}$$
$$2a\kappa_2 + \sigma_2 - \kappa_2^2 - 2\kappa_1\kappa_2 = 0. \tag{4.67}$$

Simple calculations show that hyperbola (4.66) has the asymptotes $\kappa_1 = 0$ and $\kappa_2 = a - \frac{1}{2}\kappa_1$ and hyperbola (4.67) has the asymptotes $\kappa_2 = 0$ and $\kappa_2 = 2a - 2\kappa_1$. Furthermore hyperbola (4.66) intersects the κ_1-axis in the points where $\kappa_1 = a \pm \sqrt{a^2 + \sigma_1}$, and hyperbola (4.67) intersects the κ_2-axis in the points where $\kappa_2 = a \pm \sqrt{a^2 + \sigma_2}$. We are now able to prove the following lemma:

Lemma 4.23 *The hyperbolas (4.66) and (4.67) can only intersect in the first or third quadrant of the* (κ_1, κ_2) *plane.*

Proof : Suppose (4.66) and (4.67) intersect in a point $S = \big(\bar{\kappa}_1, \bar{\kappa}_2\big)$ where $\bar{\kappa}_1 > 0$. Hyperbola (4.67) intersects the κ_2-axis in the points $\kappa_2 = a \pm \sqrt{a^2+\sigma_2}$. Because the κ_1-axis is an asymptote of (4.67), in S we have either $\bar{\kappa}_2 > 0$ or $\bar{\kappa}_2 < 0$ and then $\bar{\kappa}_2 < a - \sqrt{a^2+\sigma_2}$, i.e. S might be located either on the upper curve or on the lower curve of (4.67). Suppose, in the intersection point S, $\bar{\kappa}_2 < a-\sqrt{a^2+\sigma_2}$, i.e. S is located on the lower curve of (4.67). Elementary calculus shows that on (4.66) for $\kappa_1 > 0$, $\kappa_2 < a - \sqrt{a^2+\sigma_2}$ iff $\kappa_1 > \sqrt{a^2+\sigma_2} + \sqrt{a^2+\sigma_1+\sigma_2}$. Hence, necessarily $\bar{\kappa}_1 > \sqrt{a^2+\sigma_2} + \sqrt{a^2+\sigma_1+\sigma_2}$. Moreover, because the intersection point S lies on (4.66) to the right of the κ_2-axis, we know S has to be located above the asymptote $\kappa_2 = a - \frac{1}{2}\kappa_1$, hence $\bar{\kappa}_2 > a - \frac{1}{2}\bar{\kappa}_1$. Similarly, S has to be located on (4.67) below the asymptote $\kappa_2 = 2a - 2\kappa_1$. Hence $\bar{\kappa}_1 < \frac{2a}{3}$. But this contradicts the fact that $\bar{\kappa}_1 > \sqrt{a^2+\sigma_2} + \sqrt{a^2+\sigma_1+\sigma_2}$. Therefore $\bar{\kappa}_2 > 0$, and thus there exists no intersection point in the fourth quadrant. Along the same lines one can prove that there exists no intersection point in the second quadrant. □

We can identify two square regions in $\mathbb{R}^2$ in which critical points can be located. Define the following two regions in $\mathbb{R}^2$,

$$G_1 := \left(0, a+\sqrt{a^2+\sigma_1}\right) \times \left(0, a+\sqrt{a^2+\sigma_2}\right), \tag{4.68}$$

$$G_2 := \left(a-\sqrt{a^2+\sigma_1}, 0\right) \times \left(a-\sqrt{a^2+\sigma_2}, 0\right). \tag{4.69}$$

We have the following lemma:

Lemma 4.24 *The critical points of the quadratic system (4.62–4.63) are located in the regions G_1 and G_2. Moreover, each of the regions contains at least one critical point.*

Proof : The region G_1 lies entirely in the first quadrant. (4.66) intersects the κ_1–axis, in the point where $\kappa_1 = a + \sqrt{a^2+\sigma_1}$, and hence any critical point in the first quadrant has to be located to the left of $\kappa_1 = a + \sqrt{a^2+\sigma_1}$. Similarly, any critical point in the first quadrant has to be located below the line $\kappa_2 = a + \sqrt{a^2+\sigma_2}$. Hence, any critical point in the first quadrant has to be located in G_1, and similarly any critical point in the third quadrant has to be located in G_2. Furthermore, it is easily seen that (4.66) enters G_1 in the point $\big(0, a+\sqrt{a^2+\sigma_2}\big)$, and leaves G_1 through the line $\kappa_1 = a + \sqrt{a^2+\sigma_1}$. Hyperbola (4.67) enters G_1 through the line $\kappa_2 = a+\sqrt{a^2+\sigma_2}$, and leaves G_1 in the point $\big(a+\sqrt{a^2+\sigma_1}, 0\big)$. Necessarily, (4.66) and (4.67) have to intersect at least once[2] in G_1 (and similarly at least once in G_2). □

Lemma 4.25 *In every critical point $S = (\bar{\kappa}_1, \bar{\kappa}_2)$ located in G_1, we have $a - \bar{\kappa}_1 - \bar{\kappa}_2 < 0$, and in every critical point $T = (\tilde{\kappa}_1, \tilde{\kappa}_2)$ located in G_2, we have $a - \tilde{\kappa}_1 - \tilde{\kappa}_2 > 0$.*

[2]Note that, counting multiplicities, we even have that the number of intersection points in G_1 necessarily is odd.

Proof : Let $S = (\bar{\kappa}_1, \bar{\kappa}_2)$ be a critical point in G_1. Because S is located on the hyperbola (4.66), we know S is located above the asymptote $\kappa_2 = a - \frac{1}{2}\kappa_1$, and thus

$$\bar{\kappa}_2 > a - \frac{1}{2}\bar{\kappa}_1.$$

And hence

$$a - \bar{\kappa}_1 - \bar{\kappa}_2 < a - \bar{\kappa}_1 - a + \frac{1}{2}\bar{\kappa}_1 = -\frac{1}{2}\bar{\kappa}_1 < 0.$$

The proof that for every critical point $T = (\tilde{\kappa}_1, \tilde{\kappa}_2)$ located in G_2, we have $a - \tilde{\kappa}_1 - \tilde{\kappa}_2 > 0$, goes along the same lines. □

Remark 4.26 The property $a - \bar{\kappa}_1 - \bar{\kappa}_2 < 0$ means that the closed-loop system obtained by applying the linear stationary feedback strategies

$$\bar{\gamma}_i(x) = -\frac{b_i}{r_{ii}}\bar{k}_i x$$

is asymptotically stable.

We see that the system (4.62–4.63) has at least two critical points. Because the system is quadratic we also know that the system (4.62–4.63) has at most four critical points. Furthermore, the system (4.62–4.63) can only have critical points of multiplicity up to 3, because of the location of the critical points in the areas G_1 and G_2.

Lemma 4.27 *If the quadratic system (4.62–4.63) has a critical point of multiplicity* 2 *or* 3, *then the system parameters have to satisfy the equation*

$$\begin{aligned} a^8 + \left(6\sigma_1\sigma_2 - 6\sigma_1^2 - 6\sigma_2^2\right)a^4 + \left(12\sigma_1^2\sigma_2 + 12\sigma_1\sigma_2^2 - 8\sigma_1^3 - 8\sigma_2^3\right)a^2 & \\ -9\sigma_1^2\sigma_2^2 + 6\sigma_1^3\sigma_2 + 6\sigma_1\sigma_2^3 - 3\sigma_1^4 - 3\sigma_2^4 &= 0 \end{aligned} \tag{4.70}$$

Proof : Suppose S is a critical point of higher multiplicity of the system (4.62–4.63). Then the tangent of (4.66) and the tangent of (4.67) in S have to coincide. Note that, because of the fact that on the line $a - \kappa_1 - \kappa_2 = 0$, both $\dot{\kappa}_1 > 0$ and $\dot{\kappa}_2 > 0$, in any critical point $a - \kappa_1 - \kappa_2 \neq 0$. We find that in S necessarily

$$\begin{cases} 2a\kappa_1 + \sigma_1 - \kappa_1^2 - 2\kappa_1\kappa_2 &= 0 \\ 2a\kappa_2 + \sigma_2 - \kappa_2^2 - 2\kappa_1\kappa_2 &= 0 \\ (a - \kappa_1 - \kappa_2)^2 - \kappa_1\kappa_2 &= 0 \end{cases} \tag{4.71}$$

Using a Gröbner basis (calculated with Maple V) for the system of equations (4.71) κ_1 and κ_2 are eliminated from the equations (4.71), and we find that the system parameters have to satisfy the equation (4.70). □

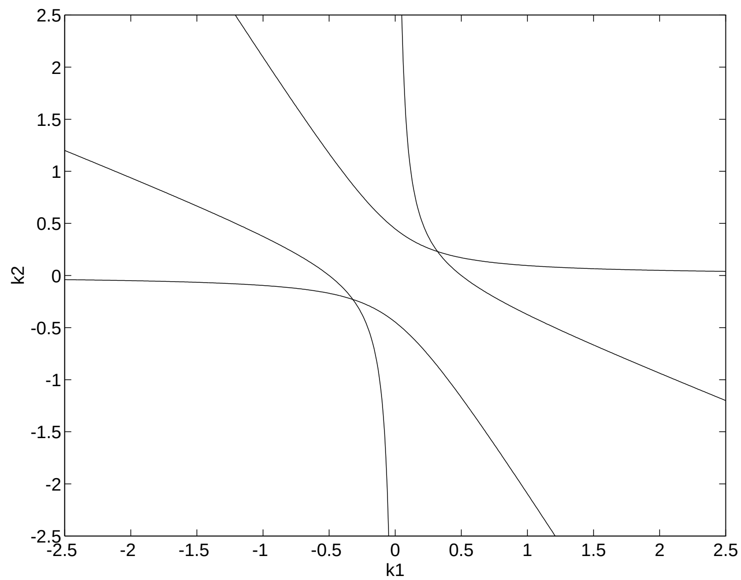


Figure 4.1: Case (i), two critical points

From the previous lemma, we see that bifurcations can occur where the system parameters satisfy equation (4.70). When equation (4.70) is not satisfied, all critical points will have multiplicity 1. In points where equation (4.70) is satisfied two (or three) critical points may coincide. In case all critical points have multiplicity 1, there will be either two or four critical points. Noting that the number of critical points in G_1 or G_2 necessarily has to be odd (counting multiplicities), we find three possibilities:

(i) The system (4.62–4.63) has exactly two critical points, one of them lies in G_1, the other in G_2. (see figure 4.1)

(ii) The system has four different critical points, one of them lies in G_1 and all the others in G_2. (see figure 4.2)

(iii) The system has four different critical points, one of them lies in G_2 and all the others in G_1. (see figure 4.3)

We will illustrate the above results in an example.

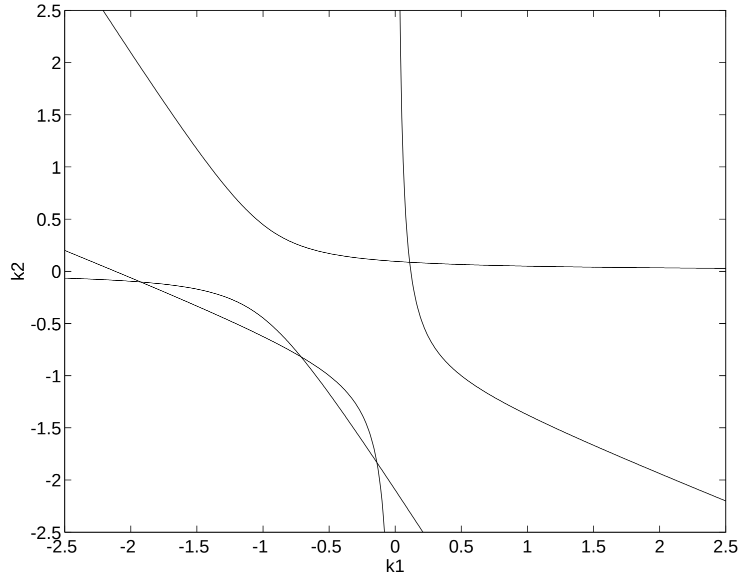


Figure 4.2: Case (ii), four critical points, one in G_1.

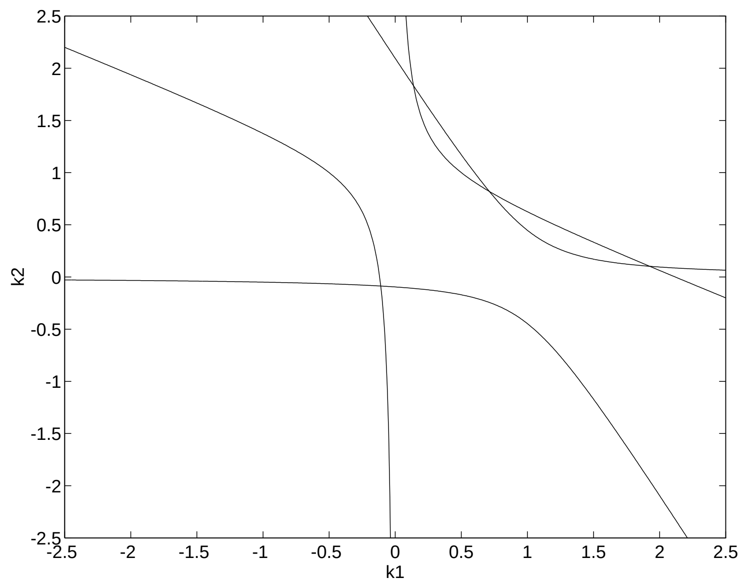


Figure 4.3: Case (iii): four critical points, three in G_1.

Example 4.28
In this example we take $\sigma_1 = 0.25$ and $\sigma_2 = 0.2$. Then (4.62–4.63) is given by

$$\dot{\kappa}_1 = 2a\kappa_1 + 0.25 - \kappa_1^2 - 2\kappa_1\kappa_2,$$
$$\dot{\kappa}_2 = 2a\kappa_2 + 0.2 - \kappa_2^2 - 2\kappa_1\kappa_2.$$

The equation (4.70) is given by

$$16000a^8 - 50400a^4 + 12960a^2 - 1323 = 0,$$

which has the (real) solutions $a \approx \pm 0.6383$. First we study the case $a = 1$. In this case the critical points are

$$P_1 \approx (-0.110, -0.087),$$
$$P_2 \approx (1.925, 0.102),$$
$$P_3 \approx (0.712, 0.820),$$
$$P_4 \approx (0.139, 1.832).$$

Now the case $a = 0$. Then the critical points are

$$P_1 \approx (-0.321, -0.230),$$
$$P_2 \approx (0.321, 0.230).$$

Finally, we study the case $a = -1$. The critical points are:

$$P_1 \approx (-0.139, -1.832),$$
$$P_2 \approx (-0.712, -0.819),$$
$$P_3 \approx (-1.925, -0.102),$$
$$P_4 \approx (0.110, 0.087).$$

In the bifurcation at $a \approx 0.6383$, the system changes from having four critical points (for $a > 0.6383$) towards a situation in which there are two critical points (for $a < 0.6383$). In the bifurcation at $a \approx -0.6383$ the system changes again from two critical points (for $a > -0.6383$) to four critical points (for $a < -0.6383$).

For the case $a = 1$ we have calculated some solutions of the differential equations (4.62–4.63) (see figure 4.4).

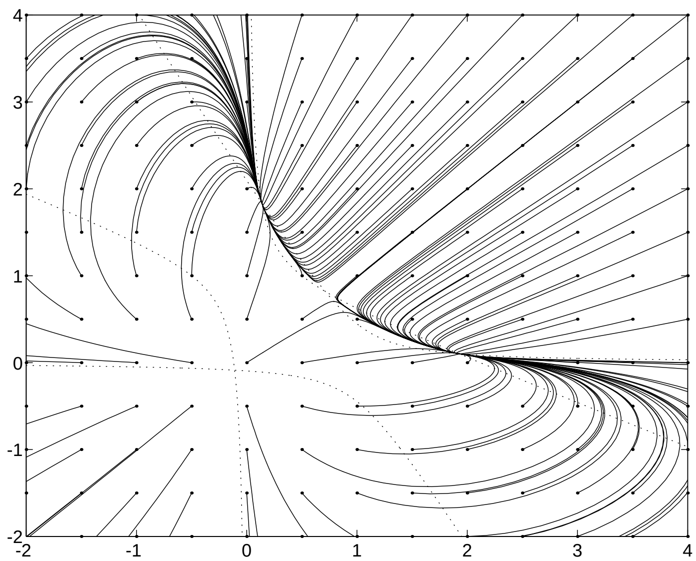


Figure 4.4: Solutions of the differential equations for $a = 1$, $\sigma_1 = 0.25$ and $\sigma_2 = 0.2$.

4.3.5 The behavior at infinity

Finally we analyze the critical points at infinity. We study the behavior of trajectories "at infinity" by studying the flow of the quadratic system (4.62–4.63) on the so-called Poincaré sphere. This approach was introduced by Poincaré in the paper Poincaré (1881). A description of this theory can be found in (Perko, 1991, pp. 248–269). When we consider a flow of a dynamical system on $\mathbb{R}^2$, given by

$$\left\{ \begin{array}{lcl} \dot{x} & = & P(x,y) \\ \dot{y} & = & Q(x,y) \end{array} \right. \tag{4.72}$$

where P and Q are polynomial functions of x and y, then the critical points at infinity for the polynomial system (4.72) occur at the points $(X, Y, 0)$ on the equator of the Poincaré sphere where $X^2 + Y^2 = 1$ and

$$XQ_m(X,Y) - YP_m(X,Y) = 0. \tag{4.73}$$

Here, m is the maximal degree of the terms in P and Q, P_m and Q_m denote the polynomials consisting of the terms of degree m. The solutions X, Y of (4.73), with $X^2 + Y^2 = 1$, can be found at the polar angles θ_j and $\theta_j + \pi$ satisfying

$$G_{m+1}(\theta) \equiv Q_m(\cos\theta, \sin\theta)\cos\theta - P_m(\cos\theta, \sin\theta)\sin\theta = 0. \tag{4.74}$$

For the system (4.62–4.63), $m = 2$ and the polynomials P and Q are given by

$$P(x,y) = 2ax + \sigma_1 - x^2 - 2xy, \tag{4.75}$$
$$Q(x,y) = 2ay + \sigma_2 - y^2 - 2xy. \tag{4.76}$$

Thus, P_2 and Q_2 become

$$P_2(x,y) = -x^2 - 2xy, \tag{4.77}$$
$$Q_2(x,y) = -y^2 - 2xy. \tag{4.78}$$

The critical points at infinity for the system (4.62–4.63) can now be found by solving the following equations:

$$XQ_2(X,Y) - YP_2(X,Y) = 0, \tag{4.79}$$
$$X^2 + Y^2 = 1, \tag{4.80}$$

which is equivalent to

$$XY^2 - YX^2 = 0, \tag{4.81}$$
$$X^2 + Y^2 = 1. \tag{4.82}$$

The critical points at infinity are listed in table 4.1.

	X	Y	θ	nature
P_1	1	0	0	saddle
P_2	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{4}\pi$	unstable node
P_3	0	1	$\frac{1}{2}\pi$	saddle
P_4	-1	0	π	saddle
P_5	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	$\frac{5}{4}\pi$	stable node
P_6	0	-1	$\frac{3}{2}\pi$	saddle

Table 4.1: Critical points at infinity

Note that the behavior at infinity of the system (4.62–4.63) is independent of the parameters a, σ_1, σ_2.

4.3.6 The nature of the critical points

In this subsection we study the nature of the finite critical points of the quadratic system (4.62–4.63). The Jacobian of the system (4.62–4.63) is given by

$$Df(\kappa_1, \kappa_2) = \begin{pmatrix} 2(a - \kappa_1 - \kappa_2) & -2\kappa_1 \\ -2\kappa_2 & 2(a - \kappa_1 - \kappa_2) \end{pmatrix} \tag{4.83}$$

The eigenvalues of $Df(\kappa_1, \kappa_2)$, for (κ_1, κ_2) in G_1 or G_2, are given by

$$\lambda_{1,2} = 2(a - \kappa_1 - \kappa_2) \pm 2\sqrt{\kappa_1 \kappa_2}. \tag{4.84}$$

In any critical point of multiplicity 1, $(a - \kappa_1 - \kappa_2)^2 \neq \kappa_1 \kappa_2$, hence any critical point of multiplicity 1 is hyperbolic. Moreover, since λ_1 and λ_2 are both real, all hyperbolic critical points are either nodes or saddles, there are no foci.

On the projective plane (which can be thought of as the projection of the upper hemisphere of the Poincaré sphere onto the unit disk), we know for the vector field, defined by (4.62–4.63), by the Poincaré Index theorem, that $n - s = 1$, where n is the number of nodes and s is the number of saddles. In the previous section we determined the nature of the critical points at infinity (2 saddles and 1 node), hence

$$n_f - s_f = 1 - n_\infty + s_\infty = 2, \tag{4.85}$$

where n_f, s_f are the number of "finite nodes" and "finite saddles", respectively, and n_∞, s_∞ are the number of nodes and saddles respectively at infinity.

Case (i): two finite critical points of multiplicity 1. In case there are exactly two finite critical points, we deduce from (4.85) that these points necessarily have to be nodes. Because of the fact that for the critical point in G_1, by lemma 4.25, $a - \kappa_1 - \kappa_2 < 0$, we know that in this point, in agreement with (4.84), the eigenvalues of the Jacobian λ_1 and λ_2 are both negative. Hence the critical point in G_1 is a stable node. Similarly, the critical point in G_2 is an unstable node.

Case (ii): four finite critical points, one of them in G_1. We have four finite critical points, hence $n_f + s_f = 4$. Moreover, by (4.85) we know $n_f - s_f = 2$. Hence, $n_f = 3$ and $s_f = 1$: there are three nodes and one saddle. Denote the critical point in G_1 by $(\bar{\kappa}_1, \bar{\kappa}_2)$. We will show that this point is a stable node. We can (locally) interpret hyperbola (4.66) as a function, given by $\kappa_2 = h_1(\kappa_1)$, and similarly we can (locally) interpret hyperbola

(4.67) as a function given by $\kappa_2 = h_2(\kappa_1)$. Then, the derivative of h_1 in a point $\big(\kappa_1, \kappa_2\big)$ on hyperbola (4.66), is given by

$$h_1'(\kappa_1, \kappa_2) = \frac{a - \kappa_1 - \kappa_2}{\kappa_1}, \tag{4.86}$$

and the derivative of h_2 in a point $\big(\kappa_1, \kappa_2\big)$ on hyperbola (4.67), is given by

$$h_2'(\kappa_1, \kappa_2) = \frac{\kappa_2}{a - \kappa_1 - \kappa_2}. \tag{4.87}$$

Furthermore, the determinant of the Jacobian (4.83) is given by

$$\det Df(\kappa_1, \kappa_2) = 4\left((a - \kappa_1 - \kappa_2)^2 - \kappa_1\kappa_2\right). \tag{4.88}$$

Since, in G_1, $\kappa_1 > 0$, $\kappa_2 > 0$ and $\big(a - \kappa_1 - \kappa_2\big) < 0$ we find

$$\det Df(\kappa_1, \kappa_2) > 0 \Leftrightarrow h'(\kappa_1, \kappa_2) < 0, \tag{4.89}$$

where

$$h(\kappa_1) = h_1(\kappa_1) - h_2(\kappa_1). \tag{4.90}$$

Now it is easily verified, that the critical point in G_1 is the point where $h(\bar{\kappa}_1) = 0$, and moreover that $h(\kappa_1)$ changes sign from positive to negative when κ_1 is increased. Hence,

$$h'(\bar{\kappa}_1, \bar{\kappa_2}) < 0, \tag{4.91}$$

and thus

$$\det Df(\bar{\kappa}_1, \bar{\kappa}_2) > 0, \tag{4.92}$$

meaning that the eigenvalues λ_1 and λ_2 are both negative. Thus, $\big(\bar{\kappa}_1, \bar{\kappa}_2\big)$ is a stable node.

Case (iii): four finite critical points, one of them in G_2. Similarly as in case (ii), we find $n_f = 3$ and $s_f = 1$. The only critical point in G_2 is an unstable node, the remaining three critical points in G_1 now consist of two stable nodes and one saddle.

For the three cases, with finite critical points of multiplicity 1, we find the global phase portraits, as sketched in figures 4.5, 4.6 and 4.7.

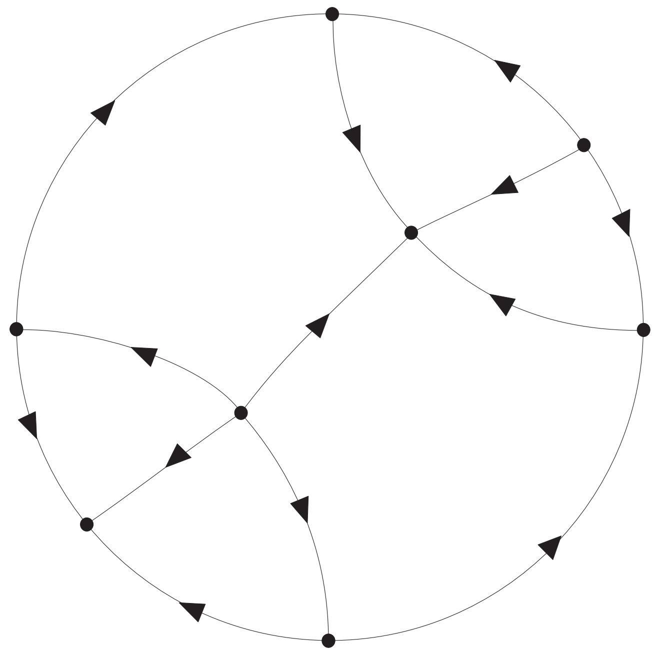


Figure 4.5: Case (i), two finite critical points.

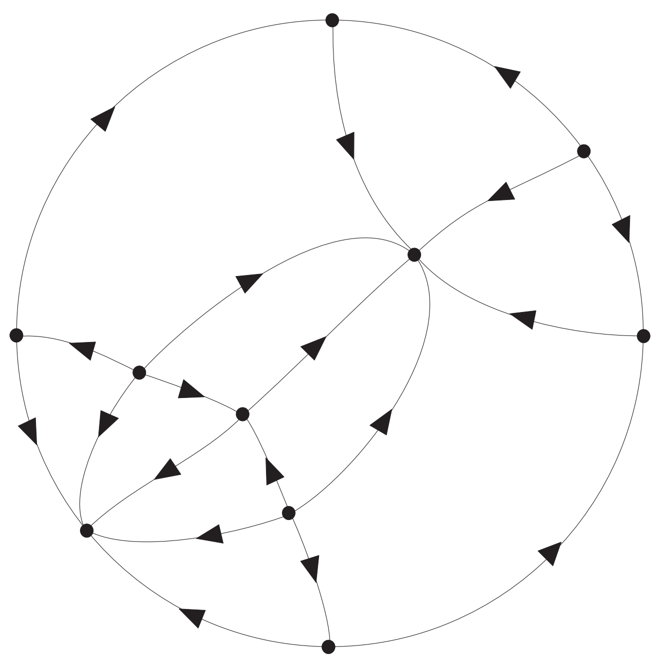


Figure 4.6: Case (ii), four finite critical points, one in G_1.

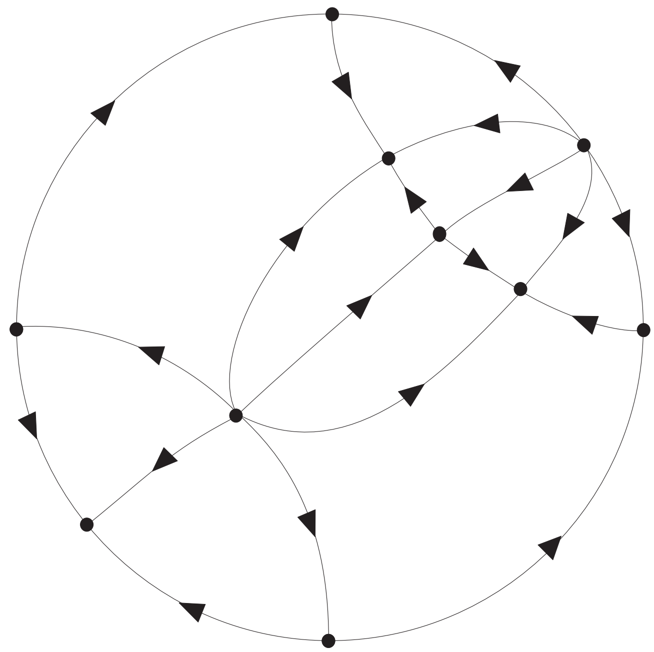


Figure 4.7: Case (iii), four finite critical points, three in G_1.

Remark 4.29 On the axis $\kappa_1 = 0$ we find that $\dot{\kappa}_1 > 0$ and similarly on $\kappa_2 = 0$ we find $\dot{\kappa}_2 > 0$. From this and the global phase portraits it follows now that for any pair of initial conditions $\big(k_{1f}, k_{2f}\big)$, with $k_{1f} \geqslant 0,\ k_{2f} \geqslant 0$, the solutions of (4.62–4.63) converge to (one of) the critical point(s) in the first quadrant. Moreover, all solutions of (4.62–4.63) starting in the first quadrant, will never leave the first quadrant.

Apart from the bifurcations, we see that there exist three topologically different phase portraits of the system (4.62–4.63). Even more possibilities can be expected when the dimension of the state space or the number of players is increased. The analysis in this section can not straightforwardly be generalized to more dimensions or to the general N-player case. In our opinion it is a rather complicated task to perform a more general multidimensional, N-player, analysis. However, since in the 1-player case (the LQ optimal control problem) we know that the behavior in the multivariable case is similar to the behavior in the scalar case, we believe that our analysis provides some clues to what can be expected in the more general case.

The most important observation we have made is that it is possible that there exist several different stable critical points for the system of coupled Riccati differential equations (4.48–4.51). In that case, even over a longer (finite) horizon, the solutions of the system of coupled Riccati differential equations depend heavily on the terminal conditions $\big(K_{1f}, K_{2f}\big)$.

4.3.7 An illustrative example

We end this section with an example, illustrating the results obtained in this section. The example is inspired by Tabellini (1986). Tabellini studies in Tabellini (1986) a differential game, played between fiscal and monetary authorities. In this model the law of motion of public debt is given by the government budget constraint

$$\dot{d}(t) = rd(t) + f(t) - m(t), \tag{4.93}$$

where all variables have been scaled to nominal income, and where d is the stock of outstanding public debt, f is the fiscal deficit net of interest payments, m is the creation of monetary base against liabilities of the Treasury, and where r can be shown to be the difference between the real rate of interest net of taxes and the rate of growth of real income. The cost functionals[3] L_i are given by

$$\begin{aligned} L_1 &= \int_0^{t_f} \left\{(m(t) - \bar{m})^2 + \tau d(t)^2\right\} dt + k_{1f} d(t_f)^2, & (4.94)\\ L_2 &= \int_0^{t_f} \left\{\left(f(t) - \bar{f}\right)^2 + \lambda d(t)^2\right\} dt + k_{2f} d(t_f)^2, & (4.95)\end{aligned}$$

in which $\tau, \lambda > 0$, $\bar{m}$ and $\bar{f}$ are given targets for $m(t)$ and $f(t)$. Now introduce

$$\begin{aligned} u_1(t) &:= m(t) - \bar{m}, & (4.96)\\ u_2(t) &:= f(t) - \bar{f}, & (4.97)\\ c &:= \bar{f} - \bar{m}, & (4.98)\\ x(t) &:= d(t). & (4.99)\end{aligned}$$

Then (4.93) can be rewritten as

$$\dot{x}(t) = rx(t) - u_1(t) + u_2(t) + c, \tag{4.100}$$

and the cost functionals L_i as

$$\begin{aligned} L_1 &= \int_0^{t_f} \left\{\tau x(t)^2 + u_1(t)^2\right\} dt + k_{1f} x(t_f)^2, & (4.101)\\ L_2 &= \int_0^{t_f} \left\{\lambda x(t)^2 + u_2(t)^2\right\} dt + k_{2f} x(t_f)^2. & (4.102)\end{aligned}$$

[3] In Tabellini (1986) discounted infinite-horizon criteria are used. In this section however we will study the asymptotic behavior of the finite-horizon undiscounted equivalents.

Then, by (Başar and Olsder, 1995, corollary 6.5), the feedback Nash equilibrium for the differential game is given by

$$\gamma_1(x,t) = k_1(t_f - t)x + \zeta_1(t_f - t), \tag{4.103}$$
$$\gamma_2(x,t) = -k_2(t_f - t) - \zeta_2(t_f - t), \tag{4.104}$$

in which k_1 and k_2 satisfy the Riccati differential equations

$$\dot{k}_1 = 2rk_1 + \tau - k_1^2 - 2k_1k_2, \; k_1(0) = k_{1f}, \tag{4.105}$$
$$\dot{k}_2 = 2rk_2 + \lambda - k_2^2 - 2k_1k_2, \; k_2(0) = k_{2f}, \tag{4.106}$$

and ζ_1, ζ_2 are given by

$$\dot{\zeta}_1 = ck_1 + (r - k_1 - k_2)\zeta_1 - k_1\zeta_2, \; \zeta_1(0) = 0, \tag{4.107}$$
$$\dot{\zeta}_2 = ck_2 - k_2\zeta_1 + (r - k_1 - k_2)\zeta_2, \; \zeta_2(0) = 0. \tag{4.108}$$

Now it is easily verified that the critical points of the system (4.105–4.108) can be identified with the critical points of the Riccati differential system (4.105–4.106), which is exactly of the form (4.62–4.63). For any critical point $(\bar{k}_1, \bar{k}_2)$ of (4.105–4.106) the critical point of (4.105–4.108) is given by $(\bar{k}_1, \bar{k}_2, \bar{\zeta}_1, \bar{\zeta}_2)$, in which $(\bar{\zeta}_1, \bar{\zeta}_2)$ are given by

$$\bar{\zeta}_1 = \frac{c\bar{k}_1(\bar{k}_1 - r)}{\lambda + \tau + r^2 - 3\bar{k}_1\bar{k}_2}, \tag{4.109}$$
$$\bar{\zeta}_2 = \frac{c\bar{k}_2(\bar{k}_2 - r)}{\lambda + \tau + r^2 - 3\bar{k}_1\bar{k}_2}. \tag{4.110}$$

We find the following lemma:

Lemma 4.30 *Let $S := (\bar{k}_1, \bar{k}_2)$ be a critical point of (4.105–4.106) and $\mathcal{S} := (\bar{k}_1, \bar{k}_2, \bar{\zeta}_1, \bar{\zeta}_2)$ the corresponding critical point of (4.105–4.108). Then $\mathcal{S}$ is a(n) (un)stable node of (4.105–4.108) if and only if S is a(n) (un)stable node of (4.105–4.106).*

Proof : Let $S := (\bar{k}_1, \bar{k}_2)$ be a critical point of (4.105–4.106) and $\mathcal{S} := (\bar{k}_1, \bar{k}_2, \bar{\zeta}_1, \bar{\zeta}_2)$ the corresponding critical point of (4.105–4.108). Then the Jacobian D_f of (4.105–4.108) in $\mathcal{S}$ is given by

$$D_f(\mathcal{S}) = \begin{pmatrix} 2(r - \bar{k}_1 - \bar{k}_2) & -2\bar{k}_1 & 0 & 0 \\ -2\bar{k}_2 & 2(r - \bar{k}_1 - \bar{k}_2) & 0 & 0 \\ c - \bar{\zeta}_1 - \bar{\zeta}_2 & -\bar{\zeta}_1 & r - \bar{k}_1 - \bar{k}_2 & -\bar{k}_1 \\ -\bar{\zeta}_2 & c - \bar{\zeta}_1 - \bar{\zeta}_2 & -\bar{k}_2 & r - \bar{k}_1 - \bar{k}_2 \end{pmatrix}.$$

The eigenvalues of D_f are given by

$$\sigma(D_f(\mathcal{S})) = \left\{ r - \bar{k}_1 - \bar{k}_2 \pm \sqrt{\bar{k}_1 \bar{k}_2}, 2\left(r - \bar{k}_1 - \bar{k}_2 \pm \sqrt{\bar{k}_1 \bar{k}_2}\right) \right\}.$$

Note that the eigenvalues of the Jacobian of (4.105–4.106) in S are

$$\lambda_{1,2} = 2\left(r - \bar{k}_1 - \bar{k}_2 \pm \sqrt{\bar{k}_1 \bar{k}_2}\right),$$

which proves our claim. □

By lemma 4.25 we note that in every critical point of (4.105–4.108), for which $\bar{k}_1 > 0$ and $\bar{k}_2 > 0$, the associated closed loop system

$$\dot{x} = (r - \bar{k}_1 - \bar{k}_2)x + c - \bar{\zeta}_1 - \bar{\zeta}_2, \tag{4.111}$$

is asymptotically stable with respect to the steady state

$$\bar{x} = -\frac{c - \bar{\zeta}_1 - \bar{\zeta}_2}{r - \bar{k}_1 - \bar{k}_2}. \tag{4.112}$$

From the previous results in this section we note that exist values of the parameters r, λ and τ, such that there are different critical points of the system (4.105–4.108) with $\bar{k}_1, \bar{k}_2 > 0$. In that case, especially when the game is played over a longer horizon, the behavior of the feedback Nash equilibrium critically depends on the specified terminal conditions k_{1f}, k_{2f}. This suggests that in the infinite-horizon case there can exist multiple linear stationary feedback Nash equilibria (see also next section) depending on the parameters r, τ and λ.

Remark 4.31 Tabellini does not find multiple feedback Nash-equilibria in the infinite-horizon discounted case, in apparent contrast with the results that we find here. With the results of the next section it can be shown that for all values of r, and for all $\tau > 0$ and $\lambda > 0$ there exists a linear stationary feedback Nash equilibrium for the undiscounted infinite-horizon game[4]. We will show that every critical point of the system (4.105–4.108) with $\bar{k}_1, \bar{k}_2 > 0$ corresponds to a linear stationary feedback Nash equilibrium for the infinite-horizon undiscounted game, showing nonuniqueness. Moreover (see remark 4.26), in every linear stationary feedback Nash equilibrium the resulting closed-loop system is asymptotically stable, showing that also the steady states are likely to be nonunique.

[4]From this we might conclude that for the purpose of obtaining stationary feedback Nash equilibria, there is no need to introduce discounting into the cost functionals.

4.4 The infinite-horizon feedback Nash equilibrium

In this section we consider the feedback Nash equilibrium in linear stationary strategies for the differential game over an infinite time horizon. We study the following cost functionals:

$$\mathcal{L}_1(u_1, u_2) = \int_0^\infty \{x(t)'Q_1x(t) + u_1(t)'R_{11}u_1(t)\}\, dt, \tag{4.113}$$

$$\mathcal{L}_2(u_1, u_2) = \int_0^\infty \{x(t)'Q_2x(t) + u_2(t)'R_{22}u_2(t)\}\, dt, \tag{4.114}$$

with $Q_i \geqslant 0$ and $R_{ii} > 0$.

Before we can state the main result of this section we first need some results from linear-quadratic optimal control theory.

Definition 4.32 Consider the system

$$\Sigma : \begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

Σ is called *output stabilizable*, if there exists a state feedback control law $u(t) = Fx(t)$, such that the corresponding output

$$y_F(t) = (C + DF)x(t)$$

converges to zero as t tends to infinity for every x_0.

Then the following theorem can be proved (see Geerts (1989); Geerts and Hautus (1990)):

Theorem 4.33 *Consider the system $\dot{x} = Ax + Bu$ together with the linear quadratic cost functional*

$$\mathcal{J}(x_0, u) = \int_0^\infty \{x(t)'Qx(t) + u(t)'Ru(t)\}\, dt,$$

with $Q \geqslant 0$ and $R > 0$. Factorize

$$\begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} C' \\ D' \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix}.$$

Then the following statements are equivalent:

(i) *For every $x_0 \in \mathbb{R}^n$ there exists a u such that $\mathcal{J}(x_0, u) < \infty$,*

(ii) *The algebraic Riccati equation*

$$A'P + PA + Q - PBR^{-1}B'P = 0, \tag{4.115}$$

has a real symmetric positive semidefinite solution P*,*

(iii) *The system* $\big(A, B, C, D\big)$ *is output stabilizable.*

Assume that these conditions hold. Then there exists a smallest real symmetric positive semidefinite solution of the algebraic Riccati equation (4.115), i.e. there exists a real symmetric solution $P^- \geqslant 0$ *such that for every real symmetric solution* $P \geqslant 0$ *we have* $P^- \leqslant P$*. For every* x_0 *we have*

$$\mathcal{J}^*(x_0) := \inf\big\{\mathcal{J}(x_0, u)\big\} = x_0'P^-x_0.$$

Furthermore, for every x_0 *there is exactly one optimal input function, i.e. a function* u^* *such that* $\mathcal{J}(x_0, u^*) = \mathcal{J}^*(x_0)$*. This optimal input is generated by the time-invariant feedback law*

$$u^*(t) = -R^{-1}B'P^-x(t).$$

We find the following proposition:

Proposition 4.34 *Suppose* $\big(C_i, D_i\big)$ *are such that* $C_i'C_i = Q_i$*,* $D_i'C_i = 0$ *and* $D_i'D_i = R_{ii}$*. Suppose that there exist* $\big(K_1, K_2\big)$ *satisfying the coupled algebraic Riccati equations*

$$A'K_1 + K_1A + Q_1 - K_1S_1K_1 - K_1S_2K_2 - K_2S_2K_1 = 0, \tag{4.116}$$
$$A'K_2 + K_2A + Q_2 - K_2S_2K_2 - K_2S_1K_1 - K_1S_1K_2 = 0, \tag{4.117}$$

such that K_1 *is the smallest real positive semidefinite solution of (4.116) for given* K_2 *and* K_2 *is the smallest real positive semidefinite solution of (4.117) for given* K_1*, and moreover* K_1 *and* K_2 *are such that the systems*

$$\big(A - B_2R_{22}^{-1}B_2'K_2, B_1, C_1, D_1\big) \text{ and } \big(A - B_1R_{11}^{-1}B_1'K_1, B_2, C_2, D_2\big)$$

are both output stabilizable. Then the strategies γ_i*, given by*

$$u_i = \gamma_i(x) = -R_{ii}^{-1}B_i'K_ix,$$

constitute a feedback Nash equilibrium in linear stationary strategies.

Proof : Suppose the second player plays some linear stationary feedback strategy $\gamma_2(x) = F_2x$, where F_2 is such that the system $\big(A + B_2F_2, B_1, C_1, D_1\big)$ is output stabilizable. To obtain the best reply for player 1, player 1 has to solve the linear-quadratic optimal control problem

$$\min_{u_1} \int_0^\infty \{x(t)'Q_1x(t) + u_1(t)R_{11}u_1(t)\}\,dt,$$

subject to

$$\dot{x} = (A + B_2F_2)\,x + B_1u_1, x(0) = x_0.$$

Because $\big(A+B_2F_2, B_1, C_1, D_1\big)$ is output stabilizable, we know from theorem 4.33 that the optimal u_1 is given by the linear stationary feedback strategy $u_1 = \gamma_1(x) = -R_{11}^{-1}B_1'Px$, where P is the smallest real positive semidefinite solution of the algebraic Riccati equation, given by

$$\big(A + B_2F_2\big)'P + P\big(A + B_2F_2\big) - PS_1P + Q_1 = 0.$$

Now suppose player 2 plays the strategy $\gamma_2(x) = -R_{22}^{-1}B_2'K_2x$, for some K_2, such that the system $\big(A - B_2R_{22}^{-1}B_2'K_2, B_1, C_1, D_1\big)$ is output stabilizable. Then, the best reply against this strategy for player 1 is to play the strategy $\gamma_1(x) = -R_{11}^{-1}B_1'K_1x$, where K_1 is the smallest real positive semidefinite solution of (4.116) for given K_2. Similarly, for given K_1 such that the system $\big(A - B_1R_{11}^{-1}B_1'K_1, B_2, C_2, D_2\big)$ is output stabilizable, the best reply of player 2 against $\gamma_1(x) = -R_{11}^{-1}B_1'K_1x$ is to play $\gamma_2(x) = -R_{22}^{-1}B_2'K_2x$, where K_2 is the smallest real postive semidefinite solution of (4.117) for given K_1. Hence, $\big(\gamma_1(x), \gamma_2(x)\big) = \big(-R_{11}^{-1}B_1'K_1x, -R_{22}^{-1}B_2'K_2x\big)$ is a Nash equilibrium in linear stationary feedback strategies. □

Remark 4.35 Note that, although we require the smallest real symmetric solutions of the coupled Riccati equations, the lemma in no way implies uniqueness of equilibria.

Remark 4.36 Taking a closer look at the proof of this lemma, we see that the best reply against any linear stationary feedback strategy is again a linear stationary feedback strategy, i.e. the class of linear stationary feedback strategies is closed under best replies (see definition 3.4).

In the scalar case analyzed in the previous section, all (dynamic) equilibria in the first quadrant are also linear stationary feedback Nash equilibria. This illustrates the possible nonuniqueness of linear stationary feedback Nash equilibria. We also see that the criterion of dynamic stability does distinguish between these equilibria, but only partly: the nonuniqueness is reduced, but not eliminated completely. Moreover (see remark 4.26) in the scalar case all the linear stationary feedback Nash equilibria stabilize the closed-loop system.

4.5 Conclusions

In this chapter we have studied the asymptotic properties of some different Nash equilibria in two-player, nonzero-sum, linear-quadratic differential games. In section 4.2 we have presented a detailed analysis of the open-loop Nash equilibrium. We analyzed the open-loop Nash equilibrium starting from its basics: the Hamiltonian equations. Necessary and sufficient conditions for the existence of a unique open-loop Nash equilibrium were derived. Moreover, a sufficient condition is given under which the open-loop Nash equilibrium can be obtained via the solutions of the Riccati differential equations (4.9–4.10). In theorem 4.13 we showed that, under some well-posedness assumptions, the open-loop Nash equilibrium converges to a unique solution when the horizon t_f tends to infinity.

In section 4.3 we have seen that the situation for the feedback Nash equilibrium is more complicated. We found out that even in the scalar case it is possible that there exist different (stable) critical points of the system of Riccati differential equations. This implies in particular that the asymptotic behavior of feedback Nash equilibria depends critically on the specified terminal conditions, i.e. on the weights put on the terminal values of the state $x(t_f)$.

Finally, in section 4.4, we have studied linear stationary feedback Nash equilibria for games over an infinite time horizon. We saw that, although in the finite-horizon case there is a generically unique feedback Nash equilibrium, in the infinite-horizon case nonuniqueness can be expected, even within the class of linear stationary feedback strategies. This possible nonuniqueness seems to be in contradiction with the generic uniqueness of finite-horizon feedback Nash equilibrium. The explanation of this phenomenon can be found in the critical dependence on the weights put on terminal values of the state. Furthermore, we have seen that the criterion of dynamic stability of the critical points is not sufficient to eliminate this nonuniqueness. Nonuniqueness can be reduced using the criterion of dynamic stability, but can not be eliminated completely.

Chapter 5

Continuously repeated games

5.1 Introduction

In this chapter, we provide a general model for studying the role of a coordinator in reaching a prespecified global control objective in a continuously repeated game. The model, which is the continuous-time counterpart of the model as introduced in section 3.4, allows the individual players to react in a strategic fashion to the behavior of the coordinator. This chapter is mainly based on Weeren et al. (1995). In chapter two, we recapitulated models in hierarchical control (see also Jamshidi (1983); Singh (1980); Weeren (1993)) in which a coordinator is introduced as a mechanism for finding a Pareto efficient equilibrium for a dynamic hierarchical control system. As already pointed out in section 2.4 and in Weeren (1993), it is necessary for all players to commit themselves to cooperate with the coordinator in order that the coordination can be successful. Therefore, the models as described in chapter two can be viewed as cooperative hierarchical control models. The model as proposed in section 3.4 and in this chapter is a first step towards the incorporation of strategic behavior into the hierarchical control framework.

The model in this chapter is based upon a two-player static game, which is played repeatedly in continuous time. We introduce the notion of coordination and arrive at a differential game with nonlinear dynamics. In chapter three we have concluded that it is desirable to formulate the model in continuous time over an infinite horizon. Unfortunately, it is impossible to handle the model analytically, but we will show how the model can be handled numerically. We consider this the second main contribution of this chapter. We will show how stationary feedback Nash equilibria of a general nonlinear differential game over an infinite time horizon, with a scalar state, can be obtained numerically. As is well known (see Başar and Olsder (1995); Feichtinger and Wirl (1993); Klompstra (1992); Tsutsui and Mino (1990)) these feedback Nash equilibria can be described by the so-called Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations. In this chapter we propose to solve these equations directly using recently developed methods for solving differential-algebraic equations (see Brenan et al. (1989); Griepentrog and März (1986); Hairer et al. (1989); Hairer and Wanner (1991)).

The outline of this chapter is as follows. In section 5.2 we will develop the general model describing the strategic interactions between players and coordinator. In section 5.3 we will discuss solution methods for differential-algebraic equations in general and for HJBI equations in particular. Then in section 5.4 and 5.5 we present two mechanisms fitting the general model of section 5.2. In section 5.4 we present a redistribution mechanism in which the coordinator is given direct control over the distribution of the payoffs between the individual players. We will discuss how one can numerically obtain all stationary feedback Nash equilibria of the resulting differential game using the methods developed in section 5.3, and illustrate this by a worked example. In section 5.5 we present another possible mechanism fitting the general model, which we refer to as the Pareto mechanism. In this case the coordinator influences the choice of Pareto efficient strategy, in such a way that the resulting differential game describes a movement along the Pareto frontier of the underlying static game. Finally, in section 5.6 we present some conclusions.

5.2 General model formulation

As in section 3.4, we consider the following situation. Two players repeatedly play a nonzero-sum game G. It is again assumed that the game G depends in some way (through the payoffs that the players receive, or through the strategy spaces that are available to them) on a parameter $\alpha \in [0, 1]$ that varies in time, where the value of α is determined by a "coordinator" through some decision rule that takes the actions of the players into account. In this way, the decisions of the players can influence their future payoffs, which now gives rise to a differential game which we shall again refer to as the "controlled game". As suggested in remarks 3.48 and 3.51 we study the controlled game over an infinite horizon. Then we can compare the asymptotic values of the equilibria of the controlled game to the possible modes of play in the original game G, which allows us to conclude whether the decision rule chosen by the coordinator is effective in establishing a global control objective or not.

Similar as in section 3.4 we formalize this idea as follows. Consider a two-player static game G in strategic form, with strategy spaces Γ_i and payoff functions π_i, in which the objective for player i is the maximization of his payoff π_i. From this game G we construct a new game, $G(\alpha)$, for every α in $[0,1]$, where α is the variable that is manipulated by the coordinator. Denote by $\Gamma_i(\alpha)$ the strategy spaces of $G(\alpha)$ and by $\nu_i(\alpha, \gamma_1, \gamma_2)$ the payoffs. Now assume that the coordinator can observe the strategies $\gamma_i(\alpha)$ chosen by the individual players, and uses a decision rule

$$\dot{\alpha} = f(\alpha, \gamma_1(\alpha), \gamma_2(\alpha))$$

to determine the future values of α. Finally, by choosing as a criterion either

$$\mathcal{L}_i = \int_0^{t_f} \nu_i(\alpha(t), \gamma_1(\alpha(t)), \gamma_2(\alpha(t)))dt$$

or

$$\mathcal{L}_i = \int_0^{\infty} e^{-rt}\nu_i(\alpha(t), \gamma_1(\alpha(t)), \gamma_2(\alpha(t)))dt$$

for some $r > 0$, a differential game is specified, which we refer to as the controlled game. So the construction of a controlled game from a static game G is done in the following steps:

Step 1: construction of a coordination mechanism $G \rightarrowtail G(\alpha)$,

Step 2: specification of a decision rule

$$\dot{\alpha} = f(\alpha, \gamma_1(\alpha), \gamma_2(\alpha)),$$

for the coordinator,

Step 3: choice between a finite-horizon criterion and an infinite-horizon discounted criterion, and in the latter case specification of $r > 0$.

Regarding step 3, for reasons as specified in remarks 3.48 and 3.51, we choose an infinite-horizon discounted criterion.

5.2.1 Construction of a controlled game

In this subsection, we will construct a class of controlled games we will use in the remainder of this chapter. First we make some assumptions on the underlying static game G.

Assumption 5.1
The strategy spaces $\Gamma_i \subseteq \mathbb{R}^k$ are convex.
The payoff functions $\pi_i : \Gamma_1 \times \Gamma_2 \to \mathbb{R}$, $i = 1, 2$, are twice differentiable and strictly concave, i.e. $\begin{pmatrix} \frac{\partial^2 \pi_i}{\partial \gamma_1^2} & \frac{\partial^2 \pi_i}{\partial \gamma_1 \partial \gamma_2} \\ \frac{\partial^2 \pi_i}{\partial \gamma_1 \partial \gamma_2} & \frac{\partial^2 \pi_i}{\partial \gamma_2^2} \end{pmatrix} < 0.$

By $\bar{\gamma} = (\bar{\gamma}_1, \bar{\gamma}_2) \in \Gamma_1 \times \Gamma_2$, we denote a Nash equilibrium of the game G.

Denote by $\gamma_i^*(\alpha)$ the cooperative strategy for player i, to be played when the coordinator selects α. Furthermore, denote by $\gamma_i^a(\alpha)$ an alternative strategy, that player i would play when playing noncooperatively. We have to make a choice for the alternative strategy γ^a. The issue on how to choose such an alternative strategy is closely related to the issue of choosing threatpoints or disagreement strategies in bargaining theory (see e.g. Houba (1994); Osborne and Rubinstein (1991)). A possible choice of alternative strategy is a Nash equilibrium for the underlying game G. Especially in the case that G has a unique Nash equilibrium this seems a good choice, for the Nash equilibrium is the standard equilibrium concept in noncooperative situations (see section 3.2).

As before, we introduce $c_i(t)$, which is a parameter reflecting the willingness of player i to play cooperatively at time instant t. If $c_i(t) = 0$ then player i chooses to play the alternative strategy $\gamma_i^a(\alpha(t))$ and if $c_i(t) = 1$ then player i chooses to play the strategy $\gamma_i^*(\alpha(t))$. We allow the players to hesitate between cooperative and noncooperative play by allowing the parameter $c_i(t)$ to take values between 0 and 1. For given c_i, the strategy played by player i is given by $u_i(c_i) := c_i\gamma_i^*(\alpha) + (1 - c_i)\,\gamma_i^a(\alpha)$.

Now we assume that the coordinator, by observing the actions of both players at time-instant t, can exactly determine the values of $c_i(t)$. Using this information the coordinator adjusts the value of $\alpha(t)$. The process of coordination is described by a decision rule

$$\dot{\alpha}(t) = f(\alpha(t), c_1(t), c_2(t)). \tag{5.1}$$

This decision rule has to satisfy some properties:

1. f is sufficiently smooth, i.e. f is at least twice differentiable w.r.t. c_i, and at least differentiable w.r.t. α,
2. $\forall_{c_1,c_2} f(0, c_1, c_2) \geqslant 0, f(1, c_1, c_2) \leqslant 0$,
3. $\frac{\partial^2 f}{\partial c_i \partial c_j} = 0$, $\frac{\partial f}{\partial c_i} \not\equiv 0$.

The smoothness condition is imposed in order to prevent some technical difficulties in the sequel of this chapter. Clearly this condition might be weakened at the expense of some technical difficulties. The second condition is crucial, in the sense that it guarantees that $\alpha(t)$ remains in $[0, 1]$ for all t. Note that, due to this property, every nontrivial choice for f will be nonlinear. Finally, the third condition is sufficient to guarantee that the optimization problems we will encounter are strictly concave, and that the mechanism is not trivial. Obviously also this condition might be weakened, and in this case a more delicate analysis would be required. An example of a coordination rule satisfying properties 1 to 3 is

$$f(\alpha, c_1, c_2) = \beta\alpha\big(1-\alpha\big)\big(c_2 - c_1\big),$$

where $\beta \in \big(0, \infty\big)$ is an arbitrary constant. This decision rule reflects the intuition that whenever one of the players shows less willingness to cooperate, the coordinator might try to convince this player to play more cooperatively in the future by choosing a new α, which is more favorable for that particular player. When β is chosen in $\big(-\infty, 0\big)$, the decision rule is such that the coordinator punishes any player who is not playing cooperatively.

A further assumption we make is that both players exactly know the mechanism f the coordinator is using. This creates a possibility for strategic behavior by both players. By choosing c_1 and c_2 the players can influence the behavior of the coordinator. A nonlinear differential game emerges, where α is the state variable, c_1 and c_2 are the controls, and with the payoff functionals

$$L_i = \int_0^\infty e^{-rt}\nu_i(\alpha(t), u_1(c_1(t)), u_2(c_2(t)))dt, \tag{5.2}$$

in which $u_i(c_i(t)) = c_i(t)\gamma_i^*(\alpha(t)) + \big(1 - c_i(t)\big)\gamma_i^a(\alpha(t))$. We refer to this newly defined differential game as the controlled game. The fact that both players at time instant t know exactly which $\alpha(t)$ is selected by the coordinator, justifies the assumption of memoryless perfect state information (see section 3.2).

Note that by introducing $u_i(c_i) = c_i\gamma_i^*(\alpha) + (1-c_i)\,\gamma_i^a(\alpha)$, the payoff for player i at time instant t is given by $\nu_i(\alpha(t), u_1(c_1(t)), u_2(c_2(t)))$, which we will sometimes write with some abuse of notation as $\nu_i(\alpha(t), c_1(t), c_2(t))$. In the sequel of this chapter we will assume that ν_1 and ν_2 are strictly concave in (c_1, c_2).

5.2.2 Equilibria of the controlled game

A natural solution concept to consider for the controlled game is the feedback Nash equilibrium (see section 3.2). As argued in section 3.4, we will consider the controlled game over an infinite time horizon, with discounted payoffs. This produces the payoff functionals

$$L_i = \int_0^\infty e^{-rt}\nu_i(\alpha(t), u_1(c_1(t)), u_2(c_2(t)))dt, \tag{5.3}$$

where $u_i(c_i(t)) = c_i(t)\gamma_i^*(\alpha(t)) + \big(1 - c_i(t)\big)\gamma_i^a(\alpha(t))$. Moreover we shall restrict attention to stationary feedback Nash equilibria[1] corresponding to continuously differentiable value

[1] In Feichtinger and Wirl (1993); Maskin and Tirole (1994); Tsutsui and Mino (1990) these are called Markov perfect Nash equilibria.

functions.

The Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations describing stationary feedback Nash equilibria (see appendix A and e.g. Feichtinger and Wirl (1993); Tsutsui and Mino (1990)) are given by

$$\begin{aligned} rV_1(\alpha) &= \max_{c_1\in[0,1]} \left\{V_1'(\alpha)f(\alpha,c_1,c_2)+\nu_1(\alpha,u_1(c_1),u_2(c_2))\right\}, &(5.4)\\ rV_2(\alpha) &= \max_{c_2\in[0,1]} \left\{V_2'(\alpha)f(\alpha,c_1,c_2)+\nu_2(\alpha,u_1(c_1),u_2(c_2))\right\}. &(5.5)\end{aligned}$$

Remark 5.2 The characterization of stationary feedback Nash equilibria by the HJBI equations (5.4–5.5) must be understood in the following sense. It can be shown (see appendix A) that if $\left(\bar{V}_1,\bar{V}_2,\bar{c}_1,\bar{c}_2\right)$ are continuously differentiable solutions of (5.4–5.5) such that $\bar{V}_1$ and $\bar{V}_2$ are bounded, then the pair of strategies $(\bar{c}_1,\bar{c}_2)$ is a stationary feedback Nash equilibrium (see also Tsutsui and Mino (1990)).

Remark 5.3 Note that by requiring stationary feedback Nash equilibria, Folk-theorem-like results do not immediately hold, for trigger strategies are not admissible (see Maskin and Tirole (1994)). Nevertheless, stationary feedback Nash equilibria are in general not unique (see Feichtinger and Wirl (1993); Tsutsui and Mino (1990), and section 4.4).

Equivalent (by the concavity assumptions) to (5.4–5.5) is the system

$$\begin{aligned} \frac{\partial f}{\partial c_1}(\alpha,c_1,c_2)V_1'(\alpha)+\frac{\partial \nu_1}{\partial c_1}(\alpha,c_1,c_2) &= \eta_1, &(5.6)\\ \frac{\partial f}{\partial c_2}(\alpha,c_1,c_2)V_2'(\alpha)+\frac{\partial \nu_2}{\partial c_2}(\alpha,c_1,c_2) &= \eta_2, &(5.7)\\ f(\alpha,c_1,c_2)V_1'(\alpha)+\nu_1(\alpha,c_1,c_2)-rV_1(\alpha) &= 0, &(5.8)\\ f(\alpha,c_1,c_2)V_2'(\alpha)+\nu_2(\alpha,c_1,c_2)-rV_2(\alpha) &= 0, &(5.9)\\ 0\leqslant c_1\leqslant 1,\ \big(1-c_1\big)\eta_1\leqslant 0,\ c_1\eta_1\geqslant 0,\\ 0\leqslant c_2\leqslant 1,\ \big(1-c_2\big)\eta_2\leqslant 0,\ c_2\eta_2\geqslant 0.\end{aligned}$$

In case no constraint on c_1 and c_2 is active, this results in the system of differential-algebraic equations

$$\begin{aligned} \frac{\partial f}{\partial c_1}(\alpha,c_1,c_2)V_1'(\alpha)+\frac{\partial \nu_1}{\partial c_1}(\alpha,c_1,c_2) &= 0, &(5.10)\\ \frac{\partial f}{\partial c_2}(\alpha,c_1,c_2)V_2'(\alpha)+\frac{\partial \nu_2}{\partial c_2}(\alpha,c_1,c_2) &= 0, &(5.11)\\ f(\alpha,c_1,c_2)V_1'(\alpha)+\nu_1(\alpha,c_1,c_2)-rV_1(\alpha) &= 0, &(5.12)\\ f(\alpha,c_1,c_2)V_2'(\alpha)+\nu_2(\alpha,c_1,c_2)-rV_2(\alpha) &= 0. &(5.13)\end{aligned}$$

Similar equations may be written down when one or both of the constraints are active.

5.3 Treatment of HJBI-DAEs

5.3.1 General DAEs

In the current section we will discuss the (numerical) treatment of DAEs in general and the HJBI-DAEs in particular. For a more extensive treatment of general DAEs the interested reader is referred to Brenan et al. (1989); Griepentrog and März (1986); Hairer et al. (1989); Hairer and Wanner (1991).

By a differential-algebraic equation (DAE) is meant an equation of the form

$$F(t, y, y') = 0, \tag{5.14}$$

in which y is a function of t, and y' is the first derivative of y with respect to t. Regarding this DAE we can consider the system of equations

$$\begin{aligned} F(t, y, y') &= 0 \\ \frac{d}{dt} F(t, y, y') &= 0 \\ &\vdots \\ \frac{d^{j-1}}{dt^{j-1}} F(t, y, y') &= 0 \end{aligned} \tag{5.15}$$

which can be written as

$$\mathbf{F}_j(t, y, \mathbf{y}_j) = 0, \tag{5.16}$$

where

$$\mathbf{y}_j = \begin{pmatrix} y' \\ \vdots \\ y^{(j)} \end{pmatrix}. \tag{5.17}$$

Then the (differential) index of (5.14) is defined in the following way (see Brenan et al. (1989); Gear (1988)).

Definition 5.4 The index of (5.14) is the smallest ν such that $\mathbf{F}_{\nu+1}(t, y, \mathbf{y}_j) = 0$ uniquely determines the variable y' as a continuous function of y, t.

The index is of crucial importance in selecting a numerical solution method for a given DAE. Backward differentiation formulas (BDF) have emerged as the most popular and best understood class of linear multistep methods for DAEs (see Brenan et al. (1989); Hairer and Wanner (1991)). In general, multistep methods and Runge-Kutta methods are not stable for higher-index DAE systems. In the case of a system of DAEs of index 0 or 1 it is always possible to use these methods. A well known implementation of the BDF technique is provided in the Fortran package DASSL (as described in Brenan et al. (1989)). For the treatment of higher-index systems the reader is referred to Brasey and Hairer (1993); Brenan et al. (1989); Gear (1988); Hairer et al. (1989); Hairer and Wanner (1991).

5.3.2 The index of HJBI-DAEs

Returning to the system of coupled DAEs (5.6–5.9), we note that whenever these systems have index 0 or 1, they can directly be solved with the use of DASSL. First we take a look at the system of HJBI-DAEs (5.10–5.13), i.e. the system of HJBI equations in case the constraints $c_1 \geqslant 0$, $c_1 \leqslant 1$, $c_2 \geqslant 0$ and $c_2 \leqslant 1$ are not active. For ease of notation, we will ignore the arguments of f,V_i and ν_i in the rest of this section. Define

$$y := \begin{pmatrix} V_1 \\ V_2 \\ c_1 \\ c_2 \end{pmatrix}. \tag{5.18}$$

To determine the index we first compute the Jacobian $F_{y'}$,

$$F_{y'} = \begin{pmatrix} \frac{\partial f}{\partial c_1} & 0 & 0 & 0 \\ 0 & \frac{\partial f}{\partial c_2} & 0 & 0 \\ f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \end{pmatrix}. \tag{5.19}$$

Clearly $F_{y'}$ is not invertible, hence the index of the system of HJBI-DAEs is at least 1. Differentiating the system of HJBI-DAEs once, using $\frac{\partial^2 f}{\partial c_i \partial c_j} = 0$ and (5.10–5.11), we obtain the additional equations

$$\frac{\partial^2 f}{\partial \alpha \partial c_1} V_1' + \frac{\partial f}{\partial c_1} V_1'' + \frac{\partial^2 \nu_1}{\partial \alpha \partial c_1} + \frac{\partial^2 \nu_1}{\partial c_1^2} c_1' + \frac{\partial^2 \nu_1}{\partial c_1 \partial c_2} c_2' = 0, \tag{5.20}$$

$$\frac{\partial^2 f}{\partial \alpha \partial c_2} V_2' + \frac{\partial f}{\partial c_2} V_2'' + \frac{\partial^2 \nu_2}{\partial \alpha \partial c_2} + \frac{\partial^2 \nu_2}{\partial c_1 \partial c_2} c_1' + \frac{\partial^2 \nu_2}{\partial c_2^2} c_2' = 0, \tag{5.21}$$

$$f V_1'' - \left(\frac{\partial f}{\partial \alpha} - r \right) \left(\frac{\partial \nu_1}{\partial c_1} / \frac{\partial f}{\partial c_1} \right) + \frac{\partial \nu_1}{\partial \alpha} + \left(\frac{\partial \nu_1}{\partial c_2} - \frac{\partial f}{\partial c_2} \cdot \frac{\partial \nu_1}{\partial c_1} / \frac{\partial f}{\partial c_1} \right) c_2' = 0, \tag{5.22}$$

$$f V_2'' - \left(\frac{\partial f}{\partial \alpha} - r \right) \left(\frac{\partial \nu_2}{\partial c_2} / \frac{\partial f}{\partial c_2} \right) + \frac{\partial \nu_2}{\partial \alpha} + \left(\frac{\partial \nu_2}{\partial c_1} - \frac{\partial f}{\partial c_1} \cdot \frac{\partial \nu_2}{\partial c_2} / \frac{\partial f}{\partial c_2} \right) c_1' = 0. \tag{5.23}$$

In this way we find the following lemma:

Lemma 5.5 *The system of HJBI-DAEs (5.10–5.13) has index 1 if and only if the equations (5.10–5.13) together with the equations (5.20–5.23) determine y' uniquely as a continuous function of y and α.*

Note that we can eliminate V_1'' and V_2'' from (5.22–5.23) using equations (5.20–5.21). Using this elimination we can straightforwardly derive that (5.22–5.23) constitute an implicit ODE for c_1' and c_2' if and only if the matrix

$$\mathcal{J} := \begin{pmatrix} -f \cdot \frac{\partial^2 \nu_1}{\partial c_1^2} \Big/ \frac{\partial f}{\partial c_1} & \frac{\partial \nu_1}{\partial c_2} - \frac{\partial}{\partial c_2}\left(f \cdot \frac{\partial \nu_1}{\partial c_1}\right) \Big/ \frac{\partial f}{\partial c_1} \\ \frac{\partial \nu_2}{\partial c_1} - \frac{\partial}{\partial c_1}\left(f \cdot \frac{\partial \nu_2}{\partial c_2}\right) \Big/ \frac{\partial f}{\partial c_2} & -f \cdot \frac{\partial^2 \nu_2}{\partial c_2^2} \Big/ \frac{\partial f}{\partial c_2} \end{pmatrix} \tag{5.24}$$

is nonsingular. So we now have the following result:

Proposition 5.6 *The system of HJBI-DAEs (5.10–5.13) has index 1 if and only if the matrix $\mathcal{J}$ given by (5.24) is nonsingular.*

The systems of DAEs which emerge when one of the constraints $c_1 \geqslant 0$, $c_1 \leqslant 1$, $c_2 \geqslant 0$ or $c_2 \leqslant 1$ becomes active, can be shown to have index at least one in a similar fashion. Moreover, conditions as described in proposition 5.6 can be derived.

In case two constraints are active, the equations are either index 0 (i.e. implicit ODEs) or algebraic, depending on whether $f(\alpha, c_1, c_2) = 0$ or not.

Remark 5.7 Note that the method of studying HJBI equations via the so-called auxiliary equations as introduced in Tsutsui and Mino (1990) is closely related to the setup described in this section. The model considered in Tsutsui and Mino (1990) is of a more special form than the one considered here, which makes it possible to obtain explicit expressions for the equilibrium feedback strategies and to substitute these in the HJBI equations. Then, by differentiating the HJBI equations (implicit) ODEs are obtained. These ODEs have the property that they do not depend on V_1 and V_2. After deriving the ODEs, symmetry conditions are used to reduce the system of ODEs to a single first order ODE in $y = V_1' = V_2'$. This ODE is then solved analytically. In Feichtinger and Wirl (1993) a similar setup is used.

5.4 A redistribution mechanism

As already discussed in sections 5.2 and 3.4, there are several ways in which the coordination parameter α may affect the underlying static game G. In this section we consider the case in which the payoffs depend on α and the strategy spaces do not.

5.4.1 A symmetric redistribution game

We make the following assumptions about the underlying static game G.

Assumption 5.8
(i) *The game G is symmetric, i.e. $\Gamma_1 = \Gamma_2$ and $\pi_1(\gamma_1, \gamma_2) = \pi_2(\gamma_2, \gamma_1)$,*

(ii) *G has a unique Nash equilibrium $\big(\bar{\gamma}_1, \bar{\gamma}_2\big)$, with equilibrium payoffs $\big(\bar{\pi}_1, \bar{\pi}_2\big)$,*

(iii) *the unique Nash equilibrium of G is not Pareto efficient.*

The symmetry suggests restricting our attention to Pareto efficient strategies $\hat{\gamma}(\frac{1}{2})$ corresponding to $\alpha_i = \frac{1}{2}$ (see theorem 3.41). So for the cooperative strategy we choose $\gamma_i^*(\alpha) = \hat{\gamma}_i(\frac{1}{2})$. The second assumption, that G has a unique Nash equilibrium, justifies the choice of this Nash equilibrium as the alternative strategy, i.e. $\gamma_i^a(\alpha) = \bar{\gamma}_i$. Note that both the cooperative strategies γ^* and the alternative strategies γ^a do not depend on α in this case. The extra payoffs from playing $u_i(c_i) = c_i\hat{\gamma}_i(\frac{1}{2}) + \big(1-c_i\big)\bar{\gamma}_i$ are given by

$$\pi^*(c_1, c_2) := \pi_1(u_1(c_1), u_2(c_2)) + \pi_2(u_1(c_1), u_2(c_2)) - \pi_1(\bar{\gamma}_1, \bar{\gamma}_2) - \pi_2(\bar{\gamma}_1, \bar{\gamma}_2). \tag{5.25}$$

Now suppose that these extra payoffs are redistributed over the players by the coordinator, according to the rule

$$\nu_1(\alpha, c_1, c_2) \;:=\; \alpha\pi^*(c_1, c_2), \tag{5.26}$$
$$\nu_2(\alpha, c_1, c_2) \;:=\; \big(1-\alpha\big)\pi^*(c_1, c_2). \tag{5.27}$$

Then the HJBI equations describing the stationary feedback Nash equilibria of the controlled game are given by

$$rV_1(\alpha) \;=\; \max_{c_1 \in [0,1]} \left\{V_1'(\alpha) f(\alpha, c_1, c_2) + \alpha\pi^*(c_1, c_2)\right\}, \tag{5.28}$$
$$rV_2(\alpha) \;=\; \max_{c_2 \in [0,1]} \left\{V_2'(\alpha) f(\alpha, c_1, c_2) + \big(1-\alpha\big)\pi^*(c_1, c_2)\right\}, \tag{5.29}$$

or, as long as the constraints $c_1 \geqslant 0$, $c_1 \leqslant 1$, $c_2 \geqslant 0$, $c_2 \leqslant 1$ are not active, in the form (5.10–5.13):

$$
\begin{aligned}
\frac{\partial f}{\partial c_1} V_1' + \alpha \frac{\partial \pi^*}{\partial c_1} &= 0, && (5.30)\\
\frac{\partial f}{\partial c_2} V_2' + \big(1-\alpha\big) \frac{\partial \pi^*}{\partial c_2} &= 0, && (5.31)\\
f V_1' + \alpha \pi^* - r V_1 &= 0, && (5.32)\\
f V_2' + \big(1-\alpha\big)\pi^* - r V_2 &= 0. && (5.33)
\end{aligned}
$$

As the coordinator's decision rule, we take

$$f(\alpha, c_1, c_2) = \beta\alpha\big(1-\alpha\big)\big(c_2 - c_1\big),$$

with $\beta \neq 0$.

Now, if we solve (5.30–5.31) for $\big(V_1', V_2'\big)$, and then substitute the result in (5.32–5.33), we obtain

$$
\begin{aligned}
-\beta\alpha\big(1-\alpha\big)V_1' + \alpha \frac{\partial \pi^*}{\partial c_1} &= 0, && (5.34)\\
\beta\alpha\big(1-\alpha\big)V_2' + \big(1-\alpha\big) \frac{\partial \pi^*}{\partial c_2} &= 0, && (5.35)\\
\alpha\big(c_2 - c_1\big)\frac{\partial \pi^*}{\partial c_1} + \alpha\pi^* - rV_1 &= 0, && (5.36)\\
-\big(1-\alpha\big)\big(c_2 - c_1\big)\frac{\partial \pi^*}{\partial c_2} + \big(1-\alpha\big)\pi^* - rV_2 &= 0. && (5.37)
\end{aligned}
$$

The matrix $\mathcal{J}$ (see (5.24)) is given by

$$\mathcal{J} := \begin{pmatrix} \alpha\big(c_2 - c_1\big)\frac{\partial^2 \pi^*}{\partial c_1^2} & \alpha\left(\frac{\partial \pi^*}{c_1} + \frac{\partial \pi^*}{\partial c_2} + \big(c_2 - c_1\big)\frac{\partial^2 \pi^*}{\partial c_1 \partial c_2}\right) \\ \big(1-\alpha\big)\left(\frac{\partial \pi^*}{c_1} + \frac{\partial \pi^*}{\partial c_2} - \big(c_2 - c_1\big)\frac{\partial^2 \pi^*}{\partial c_1 \partial c_2}\right) & -\big(1-\alpha\big)\big(c_2 - c_1\big)\frac{\partial^2 \pi^*}{\partial c_2^2} \end{pmatrix}. \tag{5.38}$$

Note that $\mathcal{J}$ is nonsingular for all $\alpha \in \big(0, 1\big)$ if and only if the matrix $\tilde{\mathcal{J}}$ given by

$$\tilde{\mathcal{J}} := \begin{pmatrix} 0 & \frac{\partial \pi^*}{\partial c_1} + \frac{\partial \pi^*}{\partial c_2} \\ -\left(\frac{\partial \pi^*}{\partial c_1} + \frac{\partial \pi^*}{\partial c_2}\right) & 0 \end{pmatrix} + \big(c_2 - c_1\big)\begin{pmatrix} \frac{\partial^2 \pi^*}{\partial c_1^2} & \frac{\partial^2 \pi^*}{\partial c_1 \partial c_2} \\ \frac{\partial^2 \pi^*}{\partial c_1 \partial c_2} & \frac{\partial^2 \pi^*}{\partial c_2^2} \end{pmatrix} \tag{5.39}$$

is nonsingular.

Before we show that the system of HJBI-DAEs (5.34–5.37) is an index 1 system, we first need the following lemma:

Lemma 5.9 *The function π^* defined in (5.25) is strictly concave, i.e. the matrix*

$$\begin{pmatrix} \frac{\partial^2\pi^*}{\partial c_1^2} & \frac{\partial^2\pi^*}{\partial c_1 \partial c_2} \\ \frac{\partial^2\pi^*}{\partial c_1 \partial c_2} & \frac{\partial^2\pi^*}{\partial c_2^2} \end{pmatrix}$$

is negative definite.

Proof : The symmetry of the game G implies $\bar{\gamma}_1 = \bar{\gamma}_2 =: \bar{\gamma}$ and $\hat{\gamma}_1(\frac{1}{2}) = \hat{\gamma}_2(\frac{1}{2}) =: \hat{\gamma}$. Then elementary calculus shows that

$$\begin{pmatrix} \frac{\partial^2\pi^*}{\partial c_1^2} & \frac{\partial^2\pi^*}{\partial c_1 \partial c_2} \\ \frac{\partial^2\pi^*}{\partial c_1 \partial c_2} & \frac{\partial^2\pi^*}{\partial c_2^2} \end{pmatrix} = (\hat{\gamma} - \bar{\gamma})^2 \left(\begin{pmatrix} \frac{\partial^2\pi_1}{\partial \gamma_1^2} & \frac{\partial^2\pi_1}{\partial \gamma_1 \partial \gamma_2} \\ \frac{\partial^2\pi_1}{\partial \gamma_1 \partial \gamma_2} & \frac{\partial^2\pi_1}{\partial \gamma_2^2} \end{pmatrix} + \begin{pmatrix} \frac{\partial^2\pi_2}{\partial \gamma_1^2} & \frac{\partial^2\pi_2}{\partial \gamma_1 \partial \gamma_2} \\ \frac{\partial^2\pi_2}{\partial \gamma_1 \partial \gamma_2} & \frac{\partial^2\pi_2}{\partial \gamma_2^2} \end{pmatrix} \right).$$

From the strict concavity of π_1 and π_2 it follows that

$$\begin{pmatrix} \frac{\partial^2\pi_i}{\partial \gamma_1^2} & \frac{\partial^2\pi_i}{\partial \gamma_1 \partial \gamma_2} \\ \frac{\partial^2\pi_i}{\partial \gamma_1 \partial \gamma_2} & \frac{\partial^2\pi_i}{\partial \gamma_2^2} \end{pmatrix} < 0,$$

for $i = 1, 2$, and hence

$$\begin{pmatrix} \frac{\partial^2\pi^*}{\partial c_1^2} & \frac{\partial^2\pi^*}{\partial c_1 \partial c_2} \\ \frac{\partial^2\pi^*}{\partial c_1 \partial c_2} & \frac{\partial^2\pi^*}{\partial c_2^2} \end{pmatrix} < 0$$

since $\hat{\gamma} \neq \bar{\gamma}$ by assumption 5.8 (iii). □

Proposition 5.10 *The system of HJBI-DAEs (5.34–5.37) has index* 1 *on its domain of validity* $\big(0, 1\big) \times \big(0, 1\big)$.

Proof : We will show that the matrix $\tilde{\mathcal{J}}$ appearing in (5.39) is nonsingular for all $\alpha \in \big(0, 1\big)$. We will consider two cases, first the case $c_1 \neq c_2$, and secondly the case $c_1 = c_2$. In the case $c_1 \neq c_2$ we note that $\tilde{\mathcal{J}}$ is the sum of a skew-symmetric matrix and a matrix that is, depending on the sign of $c_2 - c_1$, either positive or negative definite. Hence, for $c_1 \neq c_2$ $\tilde{\mathcal{J}}$ is nonsingular.

In the case $c_1 = c_2$, we see that

$$\tilde{\mathcal{J}} = \begin{pmatrix} 0 & \frac{\partial\pi^*}{\partial c_1} + \frac{\partial\pi^*}{\partial c_2} \\ -\left(\frac{\partial\pi^*}{\partial c_1} + \frac{\partial\pi^*}{\partial c_2}\right) & 0 \end{pmatrix}.$$

Note that $\pi^*(c_1, c_2) = \pi^*(c_2, c_1)$, and hence $\tilde{\mathcal{J}}$ is singular if and only if $\frac{\partial\pi^*}{\partial c_1} = \frac{\partial\pi^*}{\partial c_2} = 0$. Elementary calculus shows that $\frac{\partial\pi^*}{\partial c_1} = \frac{\partial\pi^*}{\partial c_2} = 0$ if and only if

$$\begin{aligned} \frac{\partial\pi_1}{\partial\gamma_1}(u_1(c_1), u_2(c_2)) + \frac{\partial\pi_2}{\partial\gamma_1}(u_1(c_1), u_2(c_2)) &= 0, \\ \frac{\partial\pi_1}{\partial\gamma_2}(u_1(c_1), u_2(c_2)) + \frac{\partial\pi_2}{\partial\gamma_2}(u_1(c_1), u_2(c_2)) &= 0. \end{aligned}$$

Note that these last equations are (see theorem 3.41) precisely the first order conditions characterizing $\hat{\gamma}(\frac{1}{2})$, and hence satisfied if and only if $c_1 = c_2 = 1$. However, $c_1 = c_2 = 1$ lies outside the domain of validity $\big(0,1\big) \times \big(0,1\big)$. □

Because the system of HJBI-DAEs (5.34–5.37) has index 1, we can derive ODEs for c_1 and c_2 by differentiating (5.34–5.37) once. The resulting ODEs are given by

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \mathcal{J}^{-1} \begin{pmatrix} \Big(-\big(c_2 - c_1\big) - \frac{r}{\beta\big(1-\alpha\big)}\Big) \frac{\partial \pi^*}{\partial c_1} + \pi^* \\ \Big(\big(c_2 - c_1\big) - \frac{r}{\beta\alpha}\Big) \frac{\partial \pi^*}{\partial c_2} + \pi^* \end{pmatrix} \tag{5.40}$$

We shall be interested in particular in *symmetric* solutions, i.e. those for which $c_1(\alpha) = c_2(1-\alpha)$ and $V_1(\alpha) = V_2(1-\alpha)$. These solutions can be characterized as follows.

Lemma 5.11 *A solution $\big(V_1, V_2, c_1, c_2\big)$ of the HJBI-DAEs (5.34–5.37) is symmetric if and only if it satisfies $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$.*

Proof : By writing the HJBI-DAEs (5.34–5.37) and its first derivatives, evaluated in $\alpha = \frac{1}{2}$, it is easily verified that there are only 2 degrees of freedom in specifying consistent initial[2] conditions, i.e. when 2 variables out of

$$\Big\{V_1(\frac{1}{2}), V_2(\frac{1}{2}), c_1(\frac{1}{2}), c_2(\frac{1}{2}), V_1'(\frac{1}{2}), V_2'(\frac{1}{2}), c_1'(\frac{1}{2}), c_2'(\frac{1}{2})\Big\}$$

are chosen, the other variables are fixed by the system of HJBI-DAEs (5.34–5.37) and its first derivatives, evaluated in $\alpha = \frac{1}{2}$.

Now let $\big(V_1(\alpha), V_2(\alpha), c_1(\alpha), c_2(\alpha)\big)$ be a solution of the HJBI-DAEs (5.34–5.37) corresponding to the initial conditions $\big(c_1(\frac{1}{2}), c_2(\frac{1}{2})\big)$. Then it can straightforwardly be shown that $\big(V_2(1-\alpha), V_1(1-\alpha), c_2(1-\alpha), c_1(1-\alpha)\big)$ is a solution of the HJBI-DAEs (5.34–5.37) corresponding to the initial conditions $\big(c_2(\frac{1}{2}), c_1(\frac{1}{2})\big)$. From (5.40) we see, because

$$g(\alpha, c_1, c_2) := \mathcal{J}^{-1} \begin{pmatrix} \Big(-\big(c_2 - c_1\big) - \frac{r}{\beta\big(1-\alpha\big)}\Big) \frac{\partial \pi^*}{\partial c_1} + \pi^* \\ \Big(\big(c_2 - c_1\big) - \frac{r}{\beta\alpha}\Big) \frac{\partial \pi^*}{\partial c_2} + \pi^* \end{pmatrix}$$

is a C^1 function and hence satisfies a Lipschitz condition, that whenever $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$, necessarily $c_1(\alpha) = c_2(1-\alpha)$ for all $\alpha \in \big(0,1\big)$. □

[2] Note that in this case we consider $\alpha = \frac{1}{2}$ as the 'starting point', i.e. initial conditions are specified in $\alpha = \frac{1}{2}$.

5.4.2 A worked example

As an example of a redistribution controlled game we choose for G a Cournot duopoly, in which the prices are determined by

$$p(y) = \begin{cases} 120 - y & \text{if} \quad y \leqslant 120 \\ 0 & \text{if} \quad y > 120 \end{cases} \tag{5.41}$$

and production costs are given by

$$c_i(y_i) = y_i^2. \tag{5.42}$$

Then the payoffs of G are given by (see e.g. Gibbons (1992); Takayama (1985)):

$$\pi_i(y_1, y_2) = y_i\,(120 - y_1 - y_2) - y_i^2. \tag{5.43}$$

The Nash equilibrium $\bar{\gamma}$ of G, with payoff $\bar{\pi}$, and the Pareto efficient strategy $\hat{\gamma}(\frac{1}{2})$, with payoff $\hat{\pi}$ are given by

$$\bar{\gamma} = 24, \tag{5.44}$$

$$\bar{\pi} = 1152, \tag{5.45}$$

$$\hat{\gamma}(\frac{1}{2}) = 20, \tag{5.46}$$

$$\hat{\pi} = 1200. \tag{5.47}$$

Hence, the additional payoffs after redistribution (5.25) are

$$\pi^*(c_1, c_2) = 96\,(c_1 + c_2) - 32\left(c_1^2 + c_1 c_2 + c_2^2\right). \tag{5.48}$$

Remark 5.12 The controlled game constructed in this way, can be interpreted as follows. Consider two firms who produce an identical product. Instead of selling the products themselves, the goods are sold on an instantaneously clearing market by a separate institution (the coordinator), who distributes the payoffs between the two firms using the decision rule f.

From (5.34–5.37) we find the HJBI-DAEs

$$V_1' = \frac{96 - 64c_1 - 32c_2}{\beta\big(1 - \alpha\big)}, \tag{5.49}$$

$$V_2' = \frac{-96 + 32c_1 + 64c_2}{\beta\alpha}, \tag{5.50}$$

$$V_1 = \frac{\alpha}{r}\left(\big(c_2 - c_1\big)\big(96 - 64c_1 - 32c_2\big) + \pi^*(c_1, c_2)\right), \tag{5.51}$$

$$V_2 = \frac{1 - \alpha}{r}\left(\big(c_1 - c_2\big)\big(96 - 32c_1 - 64c_2\big) + \pi^*(c_1, c_2)\right). \tag{5.52}$$

Similar HJBI-DAEs can be derived for the cases where one or more constraints on c_i become active.

By starting the integration at $\alpha = \frac{1}{2}$ and using symmetry, the systems of HJBI-DAEs are solved using DASSL (see Brenan et al. (1989)). In fact, we only calculate V_1, V_2, c_1 and c_2 for α from $\alpha = \frac{1}{2}$ to $\alpha = 0.999$, (thus avoiding the singularity at $\alpha = 1$), and then by using symmetry (i.e. $c_1(\alpha) = c_2(1-\alpha)$) we obtain the results for $\alpha = 0.001$ to $\alpha = 0.999$. The DASSL-output is then fed into Matlab, where we use spline interpolation and the built-in Runge-Kutta ODE solver to simulate the resulting closed-loop dynamics of the controlled game.

We have already seen in the proof of lemma 5.11 that in specifying consistent initial conditions the degree of freedom is 2, i.e. when one of the pairs of variables $\big(V_1(\frac{1}{2}), V_2(\frac{1}{2})\big)$ or equivalently $\big(c_1(\frac{1}{2}), c_2(\frac{1}{2})\big)$ is chosen, the others are fixed by the system of HJBI-DAEs. By requiring the extra symmetry condition $c_1(\alpha) = c_2(1-\alpha)$, $V_1(\alpha) = V_2(1-\alpha)$, i.e. $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$, the degree of freedom is reduced to 1. In the experiments we have started by fixing the initial value of $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$, which then fully determines the consistent initial conditions.

In the experiments we have fixed the parameters $\beta = \frac{1}{3}$ and $r = 1$. We have varied the initial conditions $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$ (see table 5.1). Consistent values $V_1(\frac{1}{2}) = V_2(\frac{1}{2})$ are then calculated using the HJBI-DAEs (5.49-5.52) evaluated in $\alpha = \frac{1}{2}$.

$c_1(\frac{1}{2}) = c_2(\frac{1}{2})$	$V_1(\frac{1}{2}) = V_2(\frac{1}{2})$
0.75	45
0.79	45.8832
0.796	46.002432
0.8	46.08
0.9	47.52
0.99	47.9952

Table 5.1: Some consistent initial conditions for $\beta = \frac{1}{3}$, $r = 1$

The results of the experiments are shown in figures 5.1–5.6. In figure 5.1 and figure 5.2 we see that for the initial conditions corresponding to $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.75$ and $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.79$, the solutions V_1 and V_2 become unbounded, and hence do not correspond to a stationary feedback Nash equilibrium of the controlled game[3]. In the other cases (see figures 5.3,5.4,5.5,5.6), V_1 and V_2 are continuously differentiable and bounded,

[3]Note that π^* is bounded and hence also $\alpha\pi^*$ and $(1-\alpha)\pi^*$. Using the fact that the discount factor r is positive, it immediately follows that any value function is necessarily bounded.

and hence by theorem A.3 correspond to stationary feedback Nash equilibria for the controlled game. Note that in particular we can conclude from this that all initial conditions $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) \in [0.795, 1)$ specify a valid stationary feedback Nash equilibrium, implying that this differential game allows for uncountably many stationary feedback Nash equilibria. In the figures 5.3,5.4,5.5,5.6 we have plotted $V_1(\alpha)$, $c_1(\alpha)$, the closed-loop mechanism $f(\alpha, c_1(\alpha), c_2(\alpha))$ and the simulated closed-loop dynamics $\dot{\alpha}(t) = f(\alpha(t), c_1(\alpha(t)), c_2(\alpha(t)))$ for $\alpha(0) = 0.35, 0.4, 0.45, 0.5, 0.55, 0.6$ and 0.65.

In figure 5.3 and 5.4 we see that the corresponding stationary feedback Nash equilibria support five different steady states; three of these are unstable ($\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$), and two are stable ($\alpha \approx 0.4$ and $\alpha \approx 0.6$). Finally, in figure 5.5 and figure 5.6, the stationary feedback Nash equilibria support only three steady states; two of these are unstable ($\alpha = 0$ and $\alpha = 1$) and one is stable ($\alpha = 0.5$). This suggests that somewhere between $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.9$ and $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.8$ a bifurcation takes place. To confirm this, we calculated the steady states, belonging to equilibria corresponding to initial conditions $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$ ranging from 0.8 to 0.9. The outcomes are plotted in figure 5.7. In all the stable steady states we have also determined the corresponding values of the cooperation parameters $c_1 = c_2$. These values are plotted in figure 5.8. In figure 5.9 we have plotted the payoffs for player 1 and 2 in the stable steady states which are greater than or equal to $\frac{1}{2}$. The dotted line in this figure is the payoff in the steady state $\frac{1}{2}$.

In all stationary feedback Nash equilibria that we find, convergence takes place to a quite cooperative situation; a threshold value of approximately 0.84 is found for the cooperation coefficients. However, there are two ways in which this cooperation is achieved. If the players are already cooperative above the threshold value in a situation of equal distribution of profits ($\alpha = \frac{1}{2}$), then a symmetric solution is obtained. This symmetry breaks down however if the players are less cooperative at $\alpha = \frac{1}{2}$; the slightest deviation of the initial value $\alpha(0) = \frac{1}{2}$ will cause a process in which convergence takes place to a situation in which both players are equally cooperative but take unequal shares in the revenues of cooperation.

In a second experiment we have fixed the parameters at $r = 1$ and $\beta = -\frac{1}{3}$. In this case it is easily verified that the strategies $c_1 = c_2 \equiv 1$ give a stationary feedback Nash equilibrium, with corresponding value functions $V_1(\alpha) = 96\alpha$ and $V_2(\alpha) = 96\big(1 - \alpha\big)$. Moreover, we calculated solutions of the HJBI equations corresponding to several initial conditions $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) \in \big[0, 1\big)$. In all these calculated solutions V_1 and V_2 turn out to be unbounded. This suggests that the only symmetric stationary feedback Nash equilibrium is given by $c_1 = c_2 \equiv 1$. Apparently the mechanism in which the coordinator punishes any deviation from a joint cooperative strategy (i.e. the mechanism with $\beta < 0$) is more effective, in the sense that it supports full cooperation (i.e. $c_1 = c_2 \equiv 1$) as the only

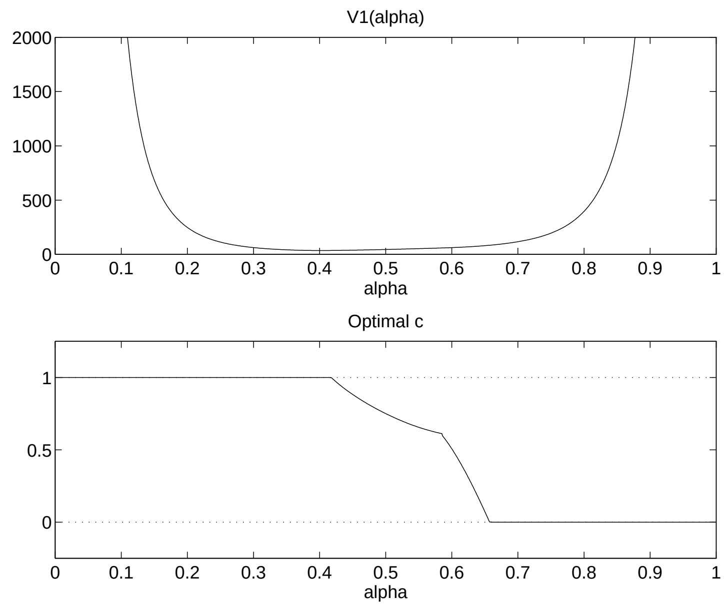


Figure 5.1: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.75$

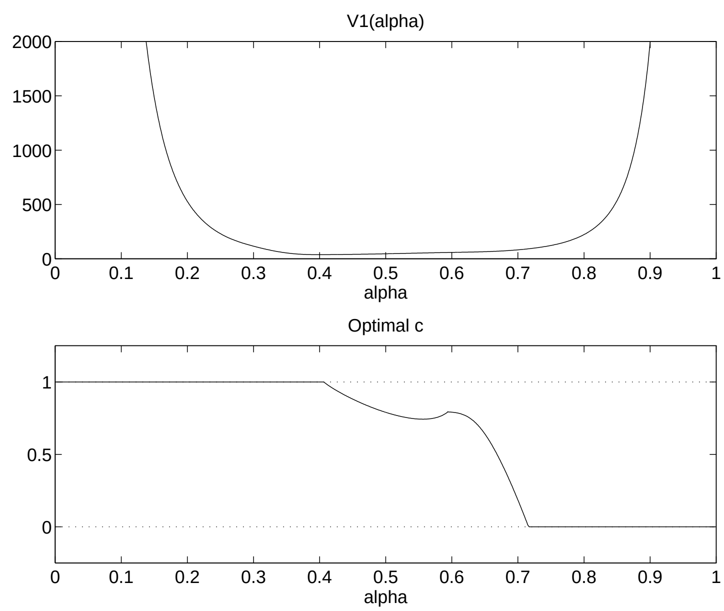


Figure 5.2: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.79$

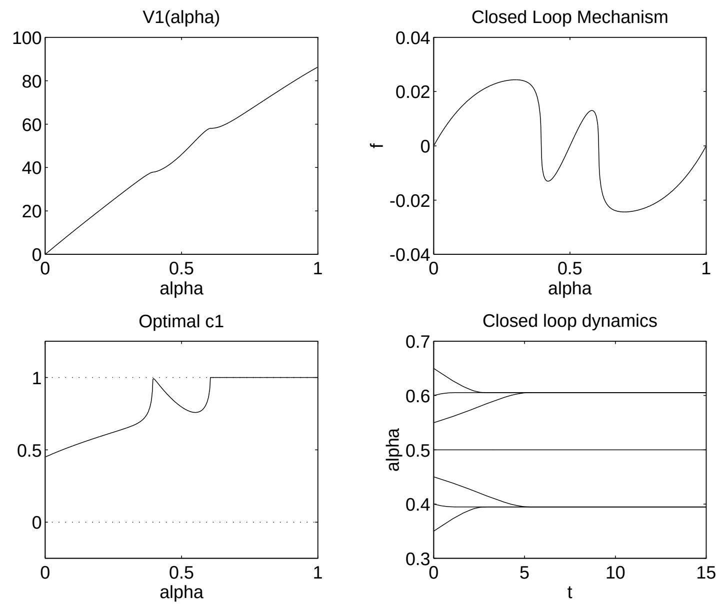


Figure 5.3: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.796$

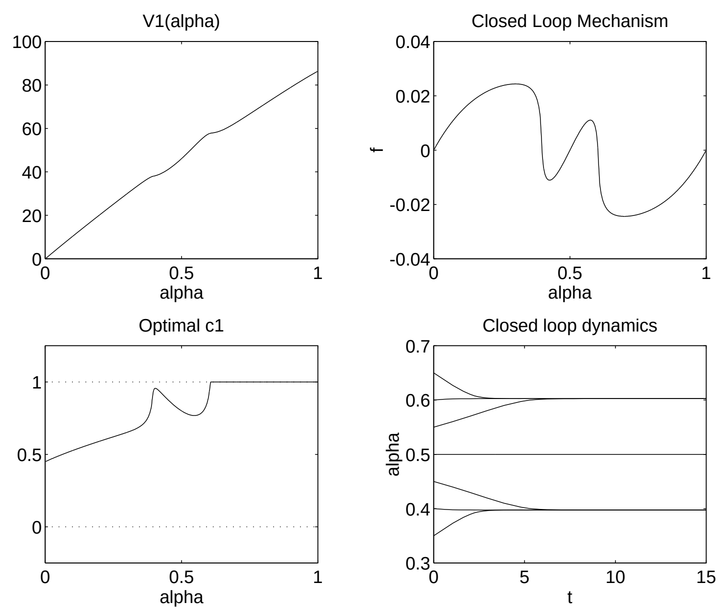


Figure 5.4: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.8$

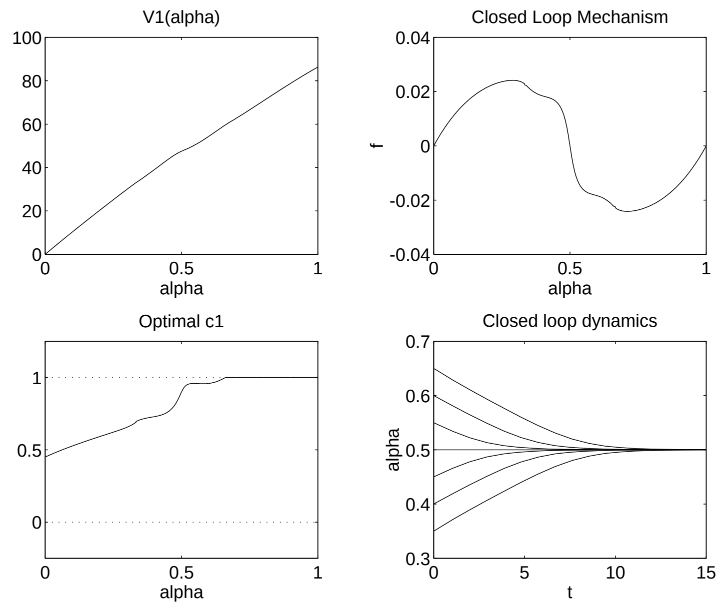


Figure 5.5: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.9$

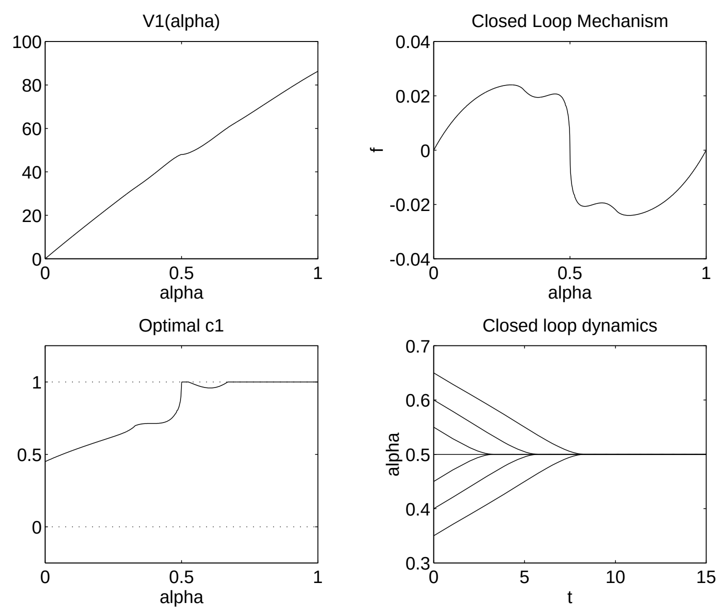


Figure 5.6: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.99$

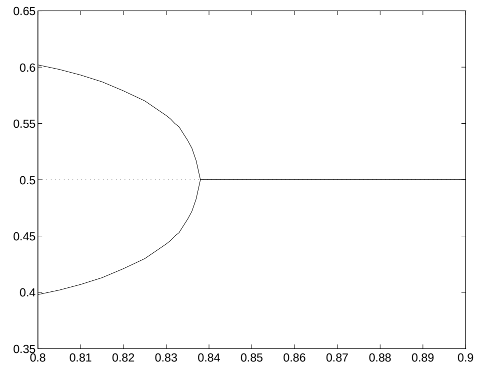


Figure 5.7: Bifurcation of steady states

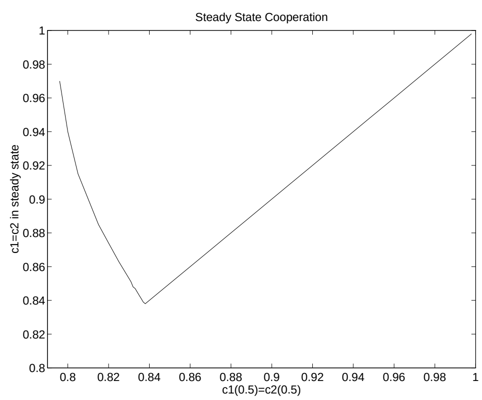


Figure 5.8: Steady state cooperation

symmetric stationary feedback Nash equilibrium of the controlled game.

5.5 A Pareto mechanism

In this section we will consider a situation in which the coordination parameter α affects the underlying static game G not through the payoffs but rather through the strategy spaces of both players. We motivate the choice of coordination mechanism by the following result (see theorem 3.41, see also e.g. Takayama (1985))

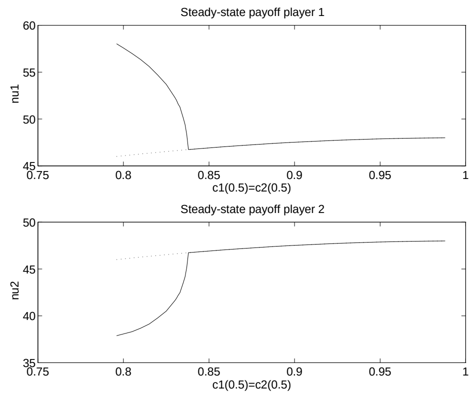


Figure 5.9: Steady-state payoffs

Theorem 5.13 *For all* $\mu \in \big(0,1\big)$ *holds that if* $\big(\hat{\gamma}_1, \hat{\gamma}_2\big) \in \Gamma_1 \times \Gamma_2$ *satisfies*

$$\big(\hat{\gamma}_1, \hat{\gamma}_2\big) \in \arg \max_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} \big\{\mu\pi_1(\gamma_1,\gamma_2) + \big(1-\mu\big)\pi_2(\gamma_1,\gamma_2)\big\} ,$$

then $\big(\hat{\gamma}_1, \hat{\gamma}_2\big)$ *is Pareto efficient.*
Moreover, if Γ_1, Γ_2 *are convex, and* π_1, π_2 *are concave, then for all Pareto efficient* $\big(\hat{\gamma}_1, \hat{\gamma}_2\big)$ *there exists a* $\mu \in \big[0,1\big]$*, such that*

$$\big(\hat{\gamma}_1, \hat{\gamma}_2\big) \in \arg \max_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} \big\{\mu\pi_1(\gamma_1,\gamma_2) + \big(1-\mu\big)\pi_2(\gamma_1,\gamma_2)\big\} .$$

We will no longer assume that G is symmetric. Now the task of the coordinator is to choose the Pareto efficient strategy to be considered by the individual players, i.e. the coordinator determines the choice of μ according to theorem 5.13 at time instant t. The cooperative strategies to be considered are $\gamma_i^*(\alpha) = \hat{\gamma}_i(\alpha)$. In this section we will exclude the possibility of sidepayments or redistribution, by taking

$$\nu_i(\alpha, \gamma_1, \gamma_2) = \pi_i(\gamma_1, \gamma_2). \tag{5.53}$$

The HJBI equations describing the stationary feedback Nash equilibria of the controlled game, are given by

$$rV_1(\alpha) = \max_{c_1\in[0,1]} \left\{V_1'(\alpha)f(\alpha, c_1, c_2) + \pi_1(u_1(c_1), u_2(c_2))\right\} , \tag{5.54}$$

$$rV_2(\alpha) = \max_{c_2\in[0,1]} \left\{V_2'(\alpha)f(\alpha, c_1, c_2) + \pi_2(u_1(c_1), u_2(c_2))\right\} . \tag{5.55}$$

We find the following proposition (compare with proposition 3.46):

Proposition 5.14 *Suppose G has a unique Nash equilibrium $(\bar{\gamma}_1, \bar{\gamma}_2)$. Furthermore, suppose the alternative strategy γ^a is such that for all $\alpha \in (0,1)$ the system of equations*

$$\begin{aligned} c_1(\alpha)\gamma_1^*(\alpha) + (1 - c_1(\alpha))\,\gamma_1^a(\alpha) &= \bar{\gamma}_1, \\ c_2(\alpha)\gamma_2^*(\alpha) + (1 - c_2(\alpha))\,\gamma_2^a(\alpha) &= \bar{\gamma}_2, \end{aligned}$$

has a unique solution $(\bar{c}_1(\alpha), \bar{c}_2(\alpha))$, with $0 \leqslant \bar{c}_i(\alpha) \leqslant 1$. Then a stationary feedback Nash equilibrium of the controlled game is given by

$$(c_1(\alpha), c_2(\alpha)) = \big(\bar{c}_1(\alpha), \bar{c}_2(\alpha)\big).$$

The actions $(u_1(\bar{c}_1(\alpha(t))), u_2(\bar{c}_2(\alpha(t))))$ played at every time instant t are equal to the Nash equilibrium $(\bar{\gamma}_1, \bar{\gamma}_2)$ of G.

Proof : For all $\alpha \in (0,1)$ the Nash equilibrium of G is recovered for

$$(c_1, c_2) = (\bar{c}_1(\alpha), \bar{c}_2(\alpha))\,,$$

more precisely

$$\begin{aligned} \bar{c}_1(\alpha) &\in \arg\max_{c_1} \{\pi_i(u_1(c_1), u_2(\bar{c}_2(\alpha)))\}\,, \\ \bar{c}_2(\alpha) &\in \arg\max_{c_2} \{\pi_i(u_1(\bar{c}_1(\alpha)), u_2(c_2))\}\,. \end{aligned}$$

Note that, for all $\alpha \in (0,1)$, $\pi_i(u_1(\bar{c}_1(\alpha)), u_2(\bar{c}_2(\alpha))) = \pi_i(\bar{\gamma}_1, \bar{\gamma}_2)$, which does not depend on α. The HJBI equations are given by

$$\begin{aligned} rV_i(\alpha) &= \pi_i(\bar{\gamma}_1, \bar{\gamma}_2), \\ V_i'(\alpha) &= 0. \end{aligned}$$

$\square$

Remark 5.15 Note that although the actions at every time instant t equal the actions corresponding to the unique Nash equilibrium of G, they emerge from a different strategy. Moreover, these equilibrium strategies can give rise to a nontrivial dynamic behavior of α. So in this sense a coordination process does take place, but is never successful, because both players attain the same payoff as in the case of noncooperative play.

Remark 5.16 In general the conditions of proposition 5.14 will not be satisfied for all $\alpha \in \big(0,1\big)$. In that case a stationary feedback Nash equilibrium of the controlled game will allow for different actions to be played. Also in the case of multiple Nash equilibria for the game G different actions can be expected. Furthermore it is important to note that, even in the case that all the conditions of proposition 5.14 are fulfilled, this stationary feedback Nash equilibrium is not necessarily unique.

In the Cournot duopoly example that we introduced in the previous section, it can straightforwardly be shown that the stationary feedback Nash equilibrium described by proposition 5.14, in case the alternative strategies γ_i^a are chosen to be equal to the Nash strategies $\bar{\gamma}_i$, is in fact unique. This is a consequence of the fact that $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0$ provides the only consistent initial conditions for the HJBI-DAEs. In this case the determinant of the matrix $\mathcal{J}$ (see (5.24)) is given by

$$\det \mathcal{J} = -331776 \frac{16\alpha^4 - 32\alpha^3 + 3\alpha^2 + 13\alpha - 4}{\left(1 - 16\alpha + 16\alpha^2\right)^2}, \tag{5.56}$$

so that $\det \mathcal{J} = 0$ for $\alpha = -\frac{1}{8} + \frac{1}{8}\sqrt{17} \approx 0.390$ or for $\alpha = \frac{9}{8} - \frac{1}{8}\sqrt{17} \approx 0.610$, and moreover the denominator of $\det \mathcal{J}$ equals 0 for $\alpha = \frac{1}{2} \pm \frac{1}{4}\sqrt{3}$. Hence the system of HJBI-DAEs is locally of higher index. As a consequence the DASSL code can not be used to obtain the solution. However, using the code RADAU5 (see Hairer et al. (1989); Hairer and Wanner (1991)), which is also suited for semi-explicit index 2 and index 3 systems, it is possible to find the solution efficiently[4].

5.6 Conclusions

In this chapter we have reintroduced the model as specified in section 3.4 for the process of coordination, now in continuous time and over an infinite horizon. The most important aspect in this setup is that we have allowed for strategic behavior by the individual players, influencing the outcome of the coordination process. We have obtained a nonlinear differential game, with state variable α, which we called the controlled game. We have taken a closer look at two special cases of such a controlled game, namely a redistribution controlled game and a Pareto controlled game. Using recently developed methods for differential-algebraic equations (DAEs), we have described in what way for such differential games all stationary feedback Nash equilibria can be calculated. In a worked example of a repeated symmetric Cournot duopoly, we have illustrated the numerical method for the redistribution controlled game. We saw that for this example there are several qualitatively different stationary feedback Nash equilibria. In this example we saw that if the players are sufficiently willing to cooperate at the point where all extra payoffs are divided equally between the players, this point is supported by the stationary feedback Nash equilibria as the only stable steady state. However, in case the players are not sufficiently willing to cooperate at this point, the stability of the steady state is lost. Moreover, we have seen that if the coordinator's decision rule is changed in such a way that deviations from

[4] Of course in this case we did not need to use any numerical method to find the solution. However, we can use this case as a testcase for the different numerical methods for DAEs.

a cooperative strategy are punished by the coordinator, a symmetric stationary feedback Nash equilibrium exists that supports full cooperation of both players. In the case of the Pareto controlled game, we found for the same example that the unique stationary feedback Nash equilibrium does not support any cooperation at all; in this case the coordination mechanism is too weak to stimulate cooperation. We can conclude that the choice of coordination mechanism and the choice of decision rule for the coordinator can be viewed as a control problem; by choosing the appropriate mechanism and decision rule a global control objective can be pursued.

Chapter 6

Conclusions

In this last chapter we take the opportunity to briefly recapitulate the main conclusions from the different chapters of this dissertation and based on these conclusions we propose some directions for future research. In particular we will state some open problems and briefly comment on how they might be tackled.

6.1 Summary

In this dissertation we have dealt with hierarchical systems. We started this study by recapitulating models for hierarchical control which are known in the literature on large-scale systems (see e.g. Mesarovic et al. (1970); Jamshidi (1983); Singh (1980); Weeren (1993)). We noted that there are some omissions in these models, namely the lack of a realistic information flow, i.e. one that takes place in real time, and the fact that the given model formulation presupposes full cooperation of the individual decision units. This second issue is taken as the starting point of our research. We noted that the presupposition of full cooperation causes the models of Mesarovic et al. (1970); Jamshidi (1983); Singh (1980) to be not very suitable for hierarchical situations where the individual decision units have possibly conflicting interests. This led to the main question to be answered: *how can we incorporate strategic behavior by the individual decision units into the hierarchical control framework?*

To answer this question the key instrument is dynamic noncooperative game theory. Inspired by strategic bargaining models (see section 3.3), in which a specific bargaining procedure is formulated as a dynamic game whose equilibria are studied, in section 3.4

a model is developed describing strategic behavior. Based on a repeatedly played static game, a stylized model is obtained, which enables us to concentrate on the strategic aspects of coordination and neglects other aspects like e.g. incomplete information or dynamics of the underlying system. This model involves the specification of a difference game, the so-called controlled game. Determination of the feedback Nash equilibria of the controlled game in section 3.4 showed (see remarks 3.48 and 3.51) that it is desirable to rephrase the model over an infinite horizon. Furthermore, in section 3.5 we concluded that for computational reasons it is desirable to rephrase the model in continuous time.

Before reformulating and analyzing the controlled game as an infinite-horizon differential game, we have studied in chapter four the asymptotic properties of Nash equilibria in linear-quadratic differential games in order to get a better perspective on the relation between finite-horizon and infinite-horizon results. In section 4.2 we derived necessary and sufficient conditions for the existence of an open-loop Nash equilibrium. In theorem 4.13 we showed that, under some well-posedness assumptions, the open-loop Nash equilibrium converges to a unique solution when the horizon t_f tends to infinity. In contrast to this result we have seen in section 4.3 that the asymptotic behavior of linear feedback Nash equilibria depends critically on the weights put on the terminal values of the state $x(t_f)$. In section 4.4 we have studied linear stationary feedback Nash equilibria for games over an infinite horizon. The most important conclusion is that in the infinite-horizon case nonuniqueness can be expected.

Chapter five was completely devoted to the analysis of the continuous-time, infinite-horizon controlled game. In this chapter we have proposed a new method to obtain stationary feedback Nash equilibria for infinite-horizon differential games with a one-dimensional state space. This method is based on the direct calculation of solutions of the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations using techniques for numerically solving differential-algebraic equations. We showed how, by using this method, the controlled game can be analyzed, and we illustrated this by a worked example. We can conclude that we have succeeded in constructing a stylized model that describes strategic behavior in a hierarchical control framework and that is amenable to analysis.

6.2 Future research

In the course of this dissertation we have signalled a number of open problems. In this section we will discuss some of these open problems and the perspectives on tackling these problems. In subsection 6.2.1 we will discuss the informational problems we have mentioned in section 2.4, i.e. the problems encountered when the coordination process as described in section 2.2 is reformulated such that coordination takes place while the hierarchical system is controlled instead of before it is controlled. In subsection 6.2.2 we discuss possible extensions to the model as introduced in chapter five. We conclude this section on future research by considering the control problem of choosing an appropriate coordination mechanism and a suitable decision rule in the specification of a controlled game.

6.2.1 Information in hierarchical control

In section 2.4 we noted that it would be more realistic to formulate the hierarchical control model as stated in section 2.2, in such a way that coordination takes place while the system is controlled, instead of before. We already noted there that this would result in the information structures

$$\eta_i(t) = \{x_i(t), \lambda(t-1)\}\,, \tag{6.1}$$

for the individual decision units, and

$$\eta^c(t) = \{x_i(t), u_i(t), z_i(t+1) \mid i = 1, \ldots, N\}\,, \tag{6.2}$$

for the coordinator. The first-level problem consists of determining $u_i(t)$ and $z_i(t+1)$ based on $\eta_i(t)$, such that the modified cost functionals L_i (see (2.7)) are minimized by the individual policymakers. On the second level $\lambda(t)$ is manipulated by the coordinator, in order to minimize[1] the interaction error $\|e\|$ (see (2.6)), based on the information $\eta^c(t)$. To give an impression of the problems involved with the given information structures consider the following situation. If policymaker i at time instant t would have the information $\tilde{\eta}_i(t) = \{x(t), \lambda(t)\}$, and in the first-level problem he would have to choose $u_i(t)$ and $z_i(t)$, the first-level problems could be solved using dynamic programming, i.e. by solving

$$\begin{aligned} V_i(x_i, t) = \min_{u_i, z_i} \Big\{ & x_i' Q_i x_i + u_i' R_i u_i + \lambda_i' z_i - \sum_{j \neq i} \lambda_j' A_{ji} x_i \\ & + V_i(A x_i + B_i u_i + \sum_{j \neq i} A_{ij} x_j, t+1) \Big\} \end{aligned} \tag{6.3}$$

$$V_i(x_i, t_f) = x_i' Q_i x_i. \tag{6.4}$$

Assuming the second-level problem can be solved, we conclude that the first-level problems where $\eta_i(t)$ is replaced with $\tilde{\eta}_i(t)$ are solvable. However, the individual policymakers do

[1] Note that in general we can not expect to achieve interaction error zero; therefore we have replaced the statement that the interaction error should be zero in the second-level problem by the more reasonable demand that it should be minimized in some sense.

not have the information $\tilde{\eta}_i(t)$ but they only have partial state information (they know $x_i(t)$ instead of $x(t)$ as a whole), and there is a delay in the information they get from the coordinator (they know $\lambda(t-1)$ instead of $\lambda(t)$). Moreover, they have to select $z_i(t+1)$ rather than $z_i(t)$. Therefore the solution of the originally stated first-level problem, i.e. selection of $u_i(t)$ and $z_i(t+1)$ based on $\eta_i(t)$, also involves making predictions $\tilde{x}(t)$ for $x(t)$ and $\tilde{\lambda}(t)$ for $\lambda(t)$ based on the information $\eta_i(t)$. One might be tempted to think that the only problem to be solved is the (separate) problem of prediction of the needed information $\tilde{\eta}_i(t)$ based on the available information $\eta_i(t)$, and plug in the estimates in the (solved) optimization problem. Problems of this kind are for instance adressed in Witsenhausen (1971)[2], where it is shown that in general the problem of estimating the necessary information ($\tilde{\eta}_i$ in our case) and the solution of the optimization problem can not be separated, except in some very special cases. Due to this lack of a separation principle, which is also the reason that in general differential or difference games with incomplete information cannot be handled yet, the solution of the first-level problems is a very difficult, if not unsolvable, problem. However, James et al. (1994) propose in the context of nonlinear H_∞ problems a different approach towards incomplete information in difference games using the so-called "information state". Also Maskin and Tirole (1994) show that there might exist a relation between the issue of incomplete information in difference games and realization problems. A possible way to get around informational problems as discussed above, is to look for suboptimal solutions, as is for instance done in Sethi and Zhang (1994)[3].

6.2.2 Extensions to the controlled game

Finally, in this subsection we discuss possible extensions to the model as introduced in chapter five. A number of possible extensions come to mind:

(i) allowing for more than two players,

(ii) replacement of the repeated game by a diffential game with a nontrivial state space,

(iii) inclusion of stochastic elements,

(iv) incomplete information.

We will briefly discuss the possible extensions below.

[2]In Witsenhausen (1971) a stochastic setup is used, but the conclusion remains valid also in a deterministic setup.

[3]Sethi and Zhang (1994) use the concept of asymptotic optimality in a stochastic framework, with one decision unit on the lower level.

The N-player case

As mentioned already in chapter one, handling the N-player case in its full generality, involves considering coalitions. Already in the three-player case the complexity introduced by coalitions is enormous, as can for instance be concluded from the discussion on three-player strategic bargaining problems in Houba (1994). In our setup however, there is a way out. We can make the assumption that the individual players can only communicate with the coordinator, and hence only have to decide whether or not to cooperate with that coordinator. The setup in chapter five can then be extended straightforwardly to N players. This would lead to a controlled game, in which the coordination variable α is a vector on the unit simplex S^N. Hence the controlled game is a nonlinear infinite-horizon differential game with an $(N-1)$-dimensional state space. Using similar techniques as in appendix A, we can show that also in this case the stationary feedback Nash equilibria are given by HJBI equations, which are now partial differential equations with algebraic constraints:

$$rV_1(\alpha) = \max_{0 \leqslant c_1 \leqslant 1} \left\{ V_{1,\alpha}(\alpha) f(\alpha, c_1, \ldots, c_N) + \nu_1(\alpha, c_1, \ldots, c_N) \right\}, \tag{6.5}$$

$$rV_2(\alpha) = \max_{0 \leqslant c_2 \leqslant 1} \left\{ V_{2,\alpha}(\alpha) f(\alpha, c_1, \ldots, c_N) + \nu_2(\alpha, c_1, \ldots, c_N) \right\}, \tag{6.6}$$

$$\vdots$$

$$rV_N(\alpha) = \max_{0 \leqslant c_N \leqslant 1} \left\{ V_{N,\alpha}(\alpha) f(\alpha, c_1, \ldots, c_N) + \nu_N(\alpha, c_1, \ldots, c_N) \right\}, \tag{6.7}$$

in which

$$V_{i,\alpha}(\alpha) = \begin{pmatrix} \frac{\partial V_i}{\partial \alpha_1}(\alpha) & \cdots & \frac{\partial V_i}{\partial \alpha_N}(\alpha) \end{pmatrix}. \tag{6.8}$$

Although it is possible to characterize the stationary feedback Nash equilibria by the HJBI equations, these HJBI equations are no longer equivalent to differential-algebraic equations. Therefore it is not possible to use the same numerical techniques as discussed in section 5.3 to obtain the equilibria of the controlled game. Hence with the assumption that the individual players only communicate via the coordinator the solution of the general N-player case calls for work on partial differential equations of the form (6.5–6.7).

Differential games instead of repeated games

Ultimately, it would be nice if we could make a synthesis of the models as discussed in section 2.2 and the models as introduced in chapter five. To accomplish this we should be able to build the model as described in chapter five on a differential game with a nontrivial state space instead of a continuously repeated game. It is easily verified that the construction of a controlled game based on a differential game will lead to a nonlinear differential game with a state space of dimension greater than one. Hence, analysis of controlled games based on general differential games calls for work on partial differential equations of the following type (for the two-player case):

$$rV_1(\alpha,x) = \max_{0\leqslant c_1\leqslant 1}\left\{\frac{\partial V_1}{\partial\alpha}(\alpha,x)f(\alpha,c_1,c_2)+V_{1,x}(\alpha,x)F(x,u_1(c_1,x),u_2(c_2,x)) \right. \\ \left. +\nu_1(\alpha,x,u_1(c_1,x),u_2(c_2,x))\right\}, \tag{6.9}$$

$$rV_2(\alpha,x) = \max_{0\leqslant c_2\leqslant 1}\left\{\frac{\partial V_2}{\partial\alpha}(\alpha,x)f(\alpha,c_1,c_2)+V_{2,x}(\alpha,x)F(x,u_1(c_1,x),u_2(c_2,x)) \right. \\ \left. +\nu_2(\alpha,x,u_1(c_1,x),u_2(c_2,x))\right\}, \tag{6.10}$$

in which

$$V_{i,x}(\alpha,x) = \begin{pmatrix} \frac{\partial V_i}{\partial x_1}(\alpha,x) & \cdots & \frac{\partial V_i}{\partial x_n}(\alpha,x) \end{pmatrix}. \tag{6.11}$$

Stochastic models

Throughout this dissertation we have worked in a completely deterministic framework. We recall the following fact. Consider the stochastic differential game (see (Başar and Olsder, 1995, section 6.7)) given by the stochastic differential equation

$$dx_t = F(t,x_t,u_1(t),\ldots,u_N(t))dt + \sigma(t,x_t)dw_t,\ x_t|_{t=0} = x_0, \tag{6.12}$$

and the cost functionals

$$L_i(u_1,\ldots,u_N) = \int_0^{t_f} g_i(t,x_t,u_1(t),\ldots,u_N(t))dt + q_i(x(t_f)). \tag{6.13}$$

Now by considering memoryless perfect state information[4], i.e. $\eta_i(t) = (x_0, x_t)$, the expected cost functionals for the game in strategic form are given by

$$J_i(\gamma_1,\ldots,\gamma_N) = \mathbb{E}\left[L_i(u_1,\ldots,u_N) \mid u_j(\cdot) = \gamma_j(\cdot,\eta_j), j=1,\ldots,N\right]. \tag{6.14}$$

Here $\mathbb{E}[\cdot]$ denotes the expectation operation taken with respect to the statistics of the standard Wiener process w_t. Then, it can be shown that the Nash equilibria of this

[4]Note that we do not allow for noisy measurements here. Stochastic differential games with noisy measurements cannot be solved yet, due to inavailability of a separation principle (see Başar and Olsder (1995); Witsenhausen (1971)). However, as noted before in section 6.2.1, a promising approach towards difference games with incomplete or imperfect information, making use of the concept of "information state" can be found in James et al. (1994).

differential game coincide with the feedback Nash equilibria of the equivalent deterministic differential game (see (Başar and Olsder, 1995, theorem 6.24)). From this fact we can conclude that such a stochastic reformulation of a differential game does not change the analysis substantially. In particular, the set of interesting Nash equilibria does not change. Hence, a stochastic reformulation in this way does not add anything to our model, while a stochastic reformulation involving noisy measurements calls for research on dynamic games with imperfect information.

Incomplete information

Throughout the model formulation in section 3.4 and in chapter five we have assumed that the coordinator can perfectly observe the actions of the individual players. This assumption is crucial because the actions of the players are inputs for the coordinator's decision rule. Evidently, when the coordinator is not able to perfectly observe their actions, individual players can exploit this fact to their own benefit. It is however not difficult to incorporate this kind of imperfect information into the model of chapter five. Assume that the coordinator observes the actions of the individual players through observation functions $h_i : \Gamma_i(\alpha) \to \Upsilon_i(\alpha)$. Then the coordinator's decision rule f can be specified as a function of α and the observations, i.e. f is now a function mapping $(0,1) \times \Upsilon_1(\alpha) \times \Upsilon_2(\alpha)$ to $(0,1)$. Obviously, introduction of such a decision rule does not change the analysis in chapter five substantially. However, the nature of the model does change essentially. As noted in section 5.6 the choice of decision rule of the coordinator can be viewed as a control problem. When making the modification as mentioned above, the choice of decision rule must also take into account the imperfect observations of the coordinator.

The considerations on extensions to the model as introduced in chapter five given above can briefly be summarized as follows. Allowing for more than two players or replacement of the underlying continuously repeated game by a differential game leads to a formulation in which the analysis of partial differential equations of the form (6.5–6.7) or of the form (6.9–6.10) plays an important role. Inclusion of stochastic elements into the controlled game either leads to a situation that does not add anything to the analysis or to a situation that requires the analysis of a differential game with incomplete or imperfect information. Finally, inclusion of imperfect information at the coordinator's end can straightforwardly be achieved, although it has some consequences for the control problem of choosing a suitable decision rule for the coordinator.

6.2.3 The control problem

We have seen in section 3.4 and in chapter five that the construction of a controlled game involved the specification of a coordination mechanism and of a decision rule for the coordinator. Given some prespecified global control objective and a coordination mechanism transforming G into $G(\alpha)$, the challenge is to design a decision rule for the coordinator such that the global control objective is reached. In the Cournot duopoly example as discussed in chapter five, we saw that for the redistribution mechanism, the choice of a decision rule which punishes deviations from a cooperative strategy, establishes a controlled game which succeeds in reaching the global control objective of full cooperation. From a consumers point of view one can also consider the following global control objective: is it possible to force the firms to produce such amounts that the marketprice equals the marginal production costs? Obviously, in order to reach this global control objective a different decision rule must be designed.

As argued before, the problem of choosing an appropriate decision rule can be viewed as a control problem. In general, the design of a decision rule involves the specification of a (nonlinear) compensator which takes the strategies γ_1 and γ_2 as its inputs and has α as its output. Such a compensator can be described in the following way:

$$\dot{z} = f(z, \gamma_1, \gamma_2), \tag{6.15}$$

$$\alpha = h(z). \tag{6.16}$$

Note that the decision rules as introduced in chapter five are a special case of this more general form, in which $h(z) = z$. Of course, the analysis of a controlled game, using a more general compensator of the form (6.15–6.16) as its decision rule, can involve some problems. In particular, one might want to assume that the players know at every time-instant t the state $z(t)$, thus assuring memoryless perfect state information. If the players would only be able to observe α, we again face problems involving incomplete information.

Appendix A

Derivation of HJBI equations

In this appendix we derive the HJBI equations associated to stationary feedback Nash equilibria of general nonlinear differential games, where the state space is an open and bounded subinterval of $\mathbb{R}$. The results of this appendix can straightforwardly be generalized to more general state spaces. Although the results of this appendix may be considered essentially well known (see Feichtinger and Wirl (1993); Tsutsui and Mino (1990)), we have not been able to find suitable references in the existing literature. Therefore we have decided to provide the proofs in this appendix. Similar results for optimal control problems can be found in Fleming and Soner (1993).

First we consider the optimal control problem

$$\max_u \int_0^\infty e^{-r\tau}\pi(x(\tau), u(\tau))d\tau,$$

under the conditions

$$\begin{aligned} \dot{x} &= f(x, u), \\ x(0) &= x_0, \end{aligned}$$

in which $r > 0$.

Assume that for all x_0,

$$\max_u \int_0^\infty e^{-r\tau}\pi(x(\tau), u(\tau))d\tau < \infty.$$

Now define the value function

$$V(x_0) := \max_u \int_0^\infty e^{-r\tau}\pi(x(\tau), u(\tau))d\tau.$$

Then we find the following lemma:

Lemma A.1 *Assume that the value function V is continuously differentiable. Then V satisfies the Hamilton-Jacobi-Bellman equation*

$$rV(x_0) = \max_{u_0} \{\pi(x_0, u_0) + V'(x_0) f(x_0, u_0)\}.$$

Proof : Let $\Delta t > 0$ and let $x(\cdot)$ be any admissible trajectory satisfying $x(0) = x_0$. Then, because V is continuously differentiable

$$V(x(t)) = V(x_0) + V'(x_0) f(x_0, u(0))\Delta t + o(\Delta t),$$

for $\Delta t \to 0$.

Now we find

$$\begin{aligned}
V(x_0) &= \max_u \int_0^\infty e^{-r\tau}\pi(x(\tau), u(\tau))d\tau \\
&= \max_u \left\{\int_0^{\Delta t} e^{-r\tau}\pi(x(\tau), u(\tau))d\tau + \int_{\Delta t}^\infty e^{-r\tau}\pi(x(\tau), u(\tau))d\tau\right\} \\
&= \max_u \left\{\int_0^{\Delta t} e^{-r\tau}\pi(x(\tau), u(\tau))d\tau + e^{-r\Delta t}\int_0^\infty e^{-r\tau}\pi(x(\tau+\Delta t), u(\tau+\Delta t))d\tau\right\} \\
&= \max_u \left\{\int_0^{\Delta t} e^{-r\tau}\pi(x(\tau), u(\tau))d\tau + e^{-r\Delta t}V(x(\Delta t))\right\} \\
&= \max_{u_0} \left\{\pi(x_0, u_0)\Delta t + e^{-r\Delta t}V(x_0) + e^{-r\Delta t}V'(x_0)f(x_0, u_0)\Delta t + o(\Delta t)\right\} \\
&= \max_{u_0} \{\pi(x_0, u_0)\Delta t + (1 - r\Delta t)V(x_0) + V'(x_0)f(x_0, u_0)\Delta t + o(\Delta t)\} \\
&= (1 - r\Delta t)V(x_0) + \max_{u_0} \{(\pi(x_0, u_0) + V'(x_0)f(x_0, u_0))\,\Delta t + o(\Delta t)\},
\end{aligned}$$

for $\Delta t \to 0$.

This immediately implies

$$\forall_{\Delta t > 0}\; rV(x_0) = \max_{u_0} \{\pi(x_0, u_0) + V'(x_0)f(x_0, u_0) + o(\Delta t)/\Delta t\},$$

and hence necessarily

$$rV(x_0) = \max_{u_0} \{\pi(x_0, u_0) + V'(x_0) f(x_0, u_0)\}.$$

□

Now consider the differential game

$$\dot{x} = f(x, u_1, u_2),$$

with payoff functionals

$$\begin{aligned} L_1(u_1,u_2) &:= \int_0^\infty e^{-r\tau}\pi_1(x(\tau),u_1(\tau),u_2(\tau))d\tau, \\ L_2(u_1,u_2) &:= \int_0^\infty e^{-r\tau}\pi_2(x(\tau),u_1(\tau),u_2(\tau))d\tau. \end{aligned}$$

Because we are interested in stationary feedback Nash equilibria, we limit the admissible strategies to the class of stationary feedback strategies. As in (Başar and Olsder, 1995, section 5.3), we demand that the strategies satisfy a Lipschitz continuity condition to ensure well-posedness.

Now define the value functions V_1, V_2 by

$$\begin{aligned} V_1(x_0,\gamma_2) &:= \max_{u_1}\int_0^\infty e^{-r\tau}\pi_1(x(\tau),u_1(\tau),\gamma_2(x(\tau)))d\tau, \\ V_2(x_0,\gamma_1) &:= \max_{u_2}\int_0^\infty e^{-r\tau}\pi_2(x(\tau),\gamma_1(x(\tau)),u_2(\tau))d\tau. \end{aligned}$$

We assume that for all $\gamma_1 \in \Gamma_1$ and for all $\gamma_2 \in \Gamma_2$:

$$\begin{aligned} \max_{u_1}\int_0^\infty e^{-r\tau}\pi_1(x(\tau),u_1(\tau),\gamma_2(x(\tau)))d\tau &< \infty, \\ \max_{u_2}\int_0^\infty e^{-r\tau}\pi_2(x(\tau),\gamma_1(x(\tau)),u_2(\tau))d\tau &< \infty. \end{aligned}$$

Then we immediately find, using lemma A.1, provided V_1 and V_2 are continuously differentiable, that V_1 and V_2 satisfy the stationary Hamilton-Jacobi-Bellman-Isaacs equations

$$rV_1(x_0,\gamma_2) = \max_{u_0}\left\{\pi_1(x_0,u_0,\gamma_2(x_0)) + \frac{\partial V_1}{\partial x}(x_0,\gamma_2)f(x_0,u_0,\gamma_2(x_0))\right\},$$
$$rV_2(x_0,\gamma_1) = \max_{u_0}\left\{\pi_2(x_0,\gamma_1(x_0),u_0) + \frac{\partial V_2}{\partial x}(x_0,\gamma_1)f(x_0,\gamma_1(x_0),u_0)\right\}.$$

This leads to the following lemma:

Lemma A.2 *If $(\bar\gamma_1,\bar\gamma_2)$ is a stationary feedback Nash equilibrium, with continuously differentiable value functions V_1 and V_2, then V_1 and V_2 satisfy the HJBI equations*

$$\begin{aligned} rV_1(x_0,\bar\gamma_2) &= \max_{u_0}\left\{\pi_1(x_0,u_0,\bar\gamma_2(x_0)) + \frac{\partial V_1}{\partial x}(x_0,\bar\gamma_2)f(x_0,u_0,\bar\gamma_2(x_0))\right\}, \\ rV_2(x_0,\bar\gamma_1) &= \max_{u_0}\left\{\pi_2(x_0,\bar\gamma_1(x_0),u_0) + \frac{\partial V_2}{\partial x}(x_0,\bar\gamma_1)f(x_0,\bar\gamma_1(x_0),u_0)\right\}. \end{aligned}$$

Moreover

$$\begin{aligned}\bar{\gamma}_1(x_0) &\in \arg\max_{u_0}\left\{\pi_1(x_0,u_0,\bar{\gamma}_2(x_0)) + \frac{\partial V_1}{\partial x}(x_0,\bar{\gamma}_2)f(x_0,u_0,\bar{\gamma}_2(x_0))\right\},\\ \bar{\gamma}_2(x_0) &\in \arg\max_{u_0}\left\{\pi_2(x_0,\bar{\gamma}_1(x_0),u_0) + \frac{\partial V_2}{\partial x}(x_0,\bar{\gamma}_1)f(x_0,\bar{\gamma}_1(x_0),u_0)\right\}.\end{aligned}$$

With a slight abuse of notation we write $V_i(x_0)$ instead of $V_i(x_0,\bar{\gamma}_j)$.

We now find the following theorem:

Theorem A.3 (Verification theorem) *Suppose* $(\bar{V}_1,\bar{V}_2,\bar{\gamma}_1,\bar{\gamma}_2)$ *are solutions of the HJBI equations*

$$\begin{aligned}rV_1(x_0) &= \max_{u_0}\left\{\pi_1(x_0,u_0,\bar{\gamma}_2(x_0)) + V_1'(x_0)f(x_0,u_0,\bar{\gamma}_2(x_0))\right\},\\ rV_2(x_0) &= \max_{u_0}\left\{\pi_2(x_0,\bar{\gamma}_1(x_0),u_0) + V_2'(x_0)f(x_0,\bar{\gamma}_1(x_0),u_0)\right\},\end{aligned}$$

with

$$\begin{aligned}\bar{\gamma}_1(x_0) &\in \arg\max_{u_0}\left\{\pi_1(x_0,u_0,\bar{\gamma}_2(x_0)) + V_1'(x_0)f(x_0,u_0,\bar{\gamma}_2(x_0))\right\},\\ \bar{\gamma}_2(x_0) &\in \arg\max_{u_0}\left\{\pi_2(x_0,\bar{\gamma}_1(x_0),u_0) + V_2'(x_0)f(x_0,\bar{\gamma}_1(x_0),u_0)\right\},\end{aligned}$$

and $\bar{V}_1,\bar{V}_2$ *continuously differentiable and bounded.*

Then, $\bar{V}_1$ *and* $\bar{V}_2$ *are value functions, i.e.*

$$\begin{aligned}\bar{V}_1(x_0) &= \max_{\gamma_1}\int_0^\infty e^{-rt}\pi_1(x(t),\gamma_1(x(t)),\bar{\gamma}_2(x(t)))dt\\ \bar{V}_2(x_0) &= \max_{\gamma_2}\int_0^\infty e^{-rt}\pi_2(x(t),\bar{\gamma}_1(x(t)),\gamma_2(x(t)))dt\end{aligned}$$

Proof : We find

$$\begin{aligned}&\int_0^\infty e^{-rt}\pi_1(x(t),\bar{\gamma}_1(x(t)),\bar{\gamma}_2(x(t)))dt\\ =& \int_0^\infty e^{-rt}\left[r\bar{V}_1(x(t)) - f(x(t),\bar{\gamma}_1(x(t)),\bar{\gamma}_2(x(t)))\bar{V}_1'(x(t))\right]dt\\ =& \int_0^\infty \left[-\frac{d}{dt}\left[e^{-rt}\right]\bar{V}_1(x(t)) - e^{-rt}\dot{x}(t)\bar{V}_1'(x(t))\right]dt\\ =& \int_0^\infty -\frac{d}{dt}\left[e^{-rt}\bar{V}_1(x(t))\right]dt\\ =& \bar{V}_1(x_0).\end{aligned}$$

Similarly, we can derive $\bar{V}_2(x_0) = \int_0^\infty e^{-rt}\pi_2(x(t), \bar{\gamma}_1(x(t)), \bar{\gamma}_2(x(t)))dt$.

Now suppose γ_1 is any admissible stationary feedback strategy. Then we find for all x:

$$\pi_1(x, \gamma_1(x), \bar{\gamma}_2(x)) + f(x, \gamma_1(x), \bar{\gamma}_2(x))\bar{V}_1'(x) \leqslant r\bar{V}_1(x),$$

and hence:

$$\begin{aligned}
& \int_0^\infty e^{-rt}\pi_1(x(t), \gamma_1(x(t)), \bar{\gamma}_2(x(t)))dt \\
\leqslant\ & \int_0^\infty e^{-rt}\left[r\bar{V}_1(x(t)) - f(x(t), \gamma_1(x(t)), \bar{\gamma}_2(x(t)))\bar{V}_1'(x(t))\right]dt \\
=\ & \int_0^\infty \left[-\frac{d}{dt}\left[e^{-rt}\right]\bar{V}_1(x(t)) - e^{-rt}\dot{x}(t)\bar{V}_1'(x(t))\right]dt \\
=\ & \int_0^\infty -\frac{d}{dt}\left[e^{-rt}\bar{V}_1(x(t))\right]dt \\
=\ & \bar{V}_1(x_0).
\end{aligned}$$

This implies

$$\bar{V}_1(x_0) = \max_{\gamma_1} \int_0^\infty e^{-rt}\pi_1(x(t), \gamma_1(x(t)), \bar{\gamma}_2(x(t)))dt,$$

and similarly we find

$$\bar{V}_2(x_0) = \max_{\gamma_2} \int_0^\infty e^{-rt}\pi_2(x(t), \bar{\gamma}_1(x(t)), \gamma_2(x(t)))dt.$$

□

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Samenvatting

Veel processen, zowel economische als technische, bestaan uit elkaar beïnvloedende deelprocessen, elk bestuurd door een afzonderlijke besturingseenheid. Vaak bestaat er ook nog een afzonderlijke besturingseenheid die het gehele proces beïnvloedt. Een dergelijk proces noemen we een hiërarchisch systeem. Enkele voorbeelden van hiërarchische systemen zijn computernetwerken, sommige produktieprocessen, het ecologisch systeem en de Europese Unie. In al deze voorbeelden bestaan er verschillende, afzonderlijk bestuurde deelprocessen, terwijl een centrale eenheid (in de bovenstaande voorbeelden het centrale systeem, het produktie management, de nationale overheid, de Europese raad) het geheel probeert te sturen. Formeel spreken we over coördinatie; de centrale eenheid (coördinator) bestuurt het gehele systeem door de acties van de afzonderlijke besturingseenheden te coördineren.

In dit proefschrift bespreken we modellen voor coördinatie binnen een hiërarchische besturingsopzet. Bestaande modellen voor coördinatie gaan er van uit dat de afzonderlijke besturingseenheden volledig samenwerken met de coördinator. In de praktijk echter is het meestal niet erg realistisch om volledige samenwerking te veronderstellen. Een individuele besturingseenheid kan zich verbeteren door op een strategische manier te reageren op de acties van de coördinator en van de andere besturingseenheden. We gebruiken de term “strategisch” om aan te duiden dat de afzonderlijke besturingseenheden hun acties niet alleen gebruiken om het systeem te besturen, maar eveneens om het gedrag van de coördinator en de andere besturingseenheden te beïnvloeden. In dit proefschrift introduceren we een nieuwe opzet voor hiërarchische besturingsmodellen, waarin we strategisch gedrag van de afzonderlijke besturingseenheden toestaan. Hierbij maken we gebruik van dynamische niet-coöperatieve speltheorie (zie bijv. Başar and Olsder (1995)). Deze nieuwe opzet wordt eerst geïntroduceerd binnen de context van twee-speler herhaalde spelen en wordt later uitgebreid geanalyseerd in de context van continu herhaalde spelen. In dit laatste model moet een oneindige-horizon differentiaalspel geanalyseerd worden. Daarom gaan we in dit proefschrift uitgebreid in op de analyse van stationaire feedback Nash-evenwichten in dergelijke spelen.

Het proefschrift heeft de volgende indeling:

Hoofdstuk 1: Inleiding,

Hoofdstuk 2: Modellen voor coördinatie,

Hoofdstuk 3: Herhaalde spelen,

Hoofdstuk 4: Nash-evenwichten van differentiaalspelen,

Hoofdstuk 5: Continu herhaalde spelen,

Hoofdstuk 6: Conclusies.

In hoofdstuk twee beginnen we met het vastleggen van de hiërarchische besturingsopzet. Deze opzet behelst de besturing van een grootschalig systeem, dat opgesplitst kan worden in N elkaar beïnvloedende deelsystemen. Voor elk deelsysteem moet een aparte beleidmaker beslissen hoe zijn deelsysteem bestuurd moet worden. Om een vooraf gespecificeerde globale regeldoelstelling te bereiken wordt een coördinator geïntroduceerd. Deze coördinator wisselt informatie uit met de afzonderlijke beleidsmakers om uiteindelijk het globale doel te bereiken. In sectie 2.2 recapituleren we een model voor zo'n coördinatie-proces, gebaseerd op Mesarovic et al. (1970); Jamshidi (1983); Singh (1980). We concluderen (zie ook Weeren (1993)) dat een dergelijk coördinatie-proces alleen kan werken als alle beleidsmakers volledig samenwerken met de coördinator. Dit heeft tot gevolg dat in de opzet zoals gedefinieerd in sectie 2.2 strategisch gedrag van de individuele beleidsmakers niet mogelijk is. Immers, strategisch gedrag zou betekenen dat de individuele beleidsmakers af kunnen wijken van de coöperatieve strategie, om zo de coördinator en de andere beleidsmakers te beïnvloeden. Dit is dan ook het doel van dit proefschrift: we bestuderen de mogelijkheid om strategisch gedrag in een hiërarchische besturingsopzet toe te staan.

In hoofdstuk drie beginnen we met de constructie van een eenvoudig model dat strategisch gedrag in een hiërarchische besturingsopzet mogelijk maakt. We richten ons op herhaalde spelen in discrete tijd, omdat we ons zo kunnen concentreren op strategische aspecten van coördinatie, zonder dat we ons druk hoeven te maken over andere zaken zoals bijvoorbeeld informationele aspecten. Het belangrijkste gereedschap waar we gebruik van maken om het model te specificeren en te analyseren is dynamische niet-coöperatieve speltheorie (zie Başar and Olsder (1995)). Daarom beginnen we hoofdstuk drie met een korte introductie van dynamische niet-coöperatieve speltheorie. Na deze introductie in de speltheorie geven we ook nog een korte bespreking van strategische onderhandelingstheorie (zie bijv. Houba (1994); Rubinstein (1982)). Deze theorie behelst de specificatie van onderhandelingsprocedures als een dynamisch spel en de analyse van de evenwichten van dat spel. Geïnspireerd door dit idee ontwikkelen we in sectie 3.4 een model voor strategisch gedrag in een hiërarchische besturingsopzet. Dit model bestaat uit een differentiespel, dat we het "bestuurde spel" noemen, gebouwd op een statisch twee-speler spel dat herhaald gespeeld wordt. Analyse van dit bestuurde spel toont aan dat het wenselijk is om het model opnieuw te specificeren als een differentiaalspel dat gespeeld wordt over een oneindige horizon. De specificatie en analyse van het bestuurde spel in continue tijd wordt uitgesteld tot hoofdstuk vijf.

In hoofdstuk vier beschouwen we een speciale klasse van niet-nulsom differentiaalspelen, namelijk niet-nulsom lineair-kwadratische differentiaalspelen. In het geval van open-loop informatie, dat wil zeggen dat iedere speler op tijdstip t slechts de begintoestand x_0 kent, leiden we noodzakelijke en voldoende voorwaarde af voor de existentie van een uniek open-loop Nash-evenwicht. Bovendien geven we een voldoende voorwaarde zodat het open-loop Nash-evenwicht op de gebruikelijke manier gevonden kan worden, namelijk via de oplossingen van een gekoppeld systeem van asymmetrische Riccati differentiaalvergelijkingen. Bovendien laten we zien dat, onder de voorwaarde dat het probleem goed gedefinieerd is, het open-loop Nash-evenwicht convergeert naar een unieke oplossing als de tijdshorizon naar oneindig gaat.

We laten eveneens zien dat het asymptotisch gedrag van het zogenaamde feedback Nash-evenwicht ingewikkelder is dan in het open-loop geval. We geven een gedetailleerde analyse voor het meest eenvoudige geval, namelijk het geval waarin de dynamica scalair is. We laten zien dat voor het feedback Nash-evenwicht het stelsel van gekoppelde Riccati differentiaalvergelijkingen mogelijk meerdere stabiele kritieke punten heeft. Dit betekent in het bijzonder dat het asymptotische gedrag van feedback Nash-evenwichten gevoelig af kan hangen van de weging op de eindwaarden van de toestand $x(t_f)$.

Als laatste onderwerp in hoofdstuk vier beschouwen we lineaire stationaire feedback Nash-evenwichten van lineair-kwadratische differentiaalspelen die gespeeld worden over een oneindige horizon. In tegenstelling tot de generieke uniciteit van lineaire feedback Nash-evenwichten van lineair-kwadratische spelen over een vaste eindige horizon, vinden we in het geval van een oneindige horizon dat zelfs binnen de klasse van lineaire feedback strategieën niet-uniciteit verwacht kan worden. De verklaring van deze schijnbare tegenstelling ligt in de gevoelige afhankelijkheid van de wegingen op de eindwaarden van de toestand in het geval van een eindige horizon. We laten verder nog zien dat het criterium van dynamische stabiliteit van kritieke punten niet voldoende is om de niet-uniciteit te elimineren.

Na het uitstapje in hoofdstuk vier naar de Nash-evenwichten van LQ-spelen, herformuleren we in hoofdstuk vijf het model van hoofdstuk drie in continue tijd. Uitgaande van een twee-speler statisch spel dat herhaald gespeeld wordt, introduceren we een coördinatie-mechanisme en een beslissingsregel voor de coördinator die de uitbetalingen en strategieën van het onderliggende statische spel beïnvloeden. Dit mondt uit in een niet-lineair differentiaalspel, waarvan de toestandsruimte dimensie een heeft, waaraan we refereren als het bestuurde spel. Helaas is het in het algemeen onmogelijk om een dergelijk niet-lineair differentiaalspel analytisch te bestuderen. We laten echter zien hoe stationaire feedback Nash-evenwichten van twee-speler niet-lineaire differentiaalspelen met een een-dimensionale toestandsruimte numeriek verkregen kunnen worden. Zoals bekend worden feedback Nash-evenwichten van differentiaalspelen beschreven met behulp van Hamilton-Jacobi-Bellman-Isaacs (HJBI) vergelijkingen. In hoofdstuk vijf laten we zien hoe oplossingen van deze HJBI-vergelijkingen numeriek kunnen worden verkregen in het geval van stationaire feedback Nash-evenwichten voor scalaire differentiaalspelen. In dat geval vormen de HJBI vergelijkingen een stelsel van differentiaal-algebraïsche vergelijkingen, en kunnen recentelijk ontwikkelde numerieke methoden hiervoor gebruikt worden. In een uitgewerkt voorbeeld, waarin het onderliggende statische spel een Cournot duopolie is, illustreren we de numerieke methode en bespreken we de gevonden stationaire feedback Nash-evenwichten. In het bijzonder merken we op dat in dit voorbeeld er sprake is van overaftelbaar veel verschillende stationaire feedback Nash-evenwichten. We besluiten hoofdstuk vijf met de observatie dat de keuze van coördinatie-mechanisme en beslissingsregel van de coördinator gezien kan worden als een regelprobleem; door het kiezen van het juiste mechanisme en de juiste beslissingsregel kan de coördinator sturen in de richting van een globale besturingsdoelstelling.

Tenslotte laten we in hoofdstuk zes de gevonden resultaten nogmaals de revue passeren. We trekken een aantal conclusies en geven aan in welke richting toekomstig onderzoek kan worden verricht. In het bijzonder bespreken we mogelijke uitbreidingen van het model zoals geïntroduceerd in hoofdstuk vijf.