GAMES AND ECONOMIC BEHAVIOR **10,** 355–367 (1995)

A Transfer Property of Equilibrium Payoffs in Economies with Land

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Received December 3, 1992

NTU-games and TU-games are often used as tools to analyze economic models. Legut *et al.* (1994) investigated the properties of TU-games associated with economies with "land." In this paper we especially give attention to the NTU-games connected with these kinds of economies. The main result is that equilibrium payoffs in the NTU-model are connected to equilibrium payoff in the TU-model by b-transfer, a concept introduced by Shapley (1969). © 1995 Academic Press, Inc.

1. INTRODUCTION

We consider in this paper an exchange economy of Debreu type with only one commodity, land. This commodity is modeled by a measure space $(L, \mathcal{B}, \nu)$, wherein $\mathcal{B}$ is a σ-algebra of subsets of L and $\nu: \mathcal{B} \to R_+$ is a non-trival finite measure on L: $0 < \nu(L) < \infty$ (cf. Berliant (1982) and Dunz (1984)).

* Financial support under KBN Grant PB 664/2/91 is gratefully acknowledged.

0899-8256/95 $12.00

Utilities are given as integrals of "utility density functions" f_i over ν-measurable parcels $C \in \mathfrak{B}$: $U_i(C) = \int_C f_i \, d\nu$.

An exchange economy with land $\mathfrak{E}$ consists of

- a set of agents N,
- A $\mathfrak{B}$-partition $\{A_i\}_{i \in N}$ of L, the initial endowments of the agents,
- a set $\{f_i\}_{i \in N}$ of nonnegative bounded ν-measurable functions f_i: $L \to R_+$, the utility densities of the agents.

A *$\mathfrak{B}$-partition of L* is a collection subsets $A_i \in \mathfrak{B}$ such that $\nu(A_i \cap A_j) = 0$ if $i \neq j$ and $\nu(\bigcup_{i \in N} A_i) = \nu(L)$. Utilities are given by $U_i(C) := \int_C f_i \, d\nu$ for every parcel $C \in \mathfrak{B}$:

$$\mathfrak{E} = \{N, \{A_i, f_i\}_{i \in N}\}.$$

To investigate these types of economies we introduce (cf. Legut *et al.* (1992)) a transferable utility game $(N, v_{\mathfrak{E}})$ defined by

$$v_{\mathfrak{E}}(S) := \sup \left\{ \sum_{i \in S} \int_{C_i} f_i \, d\nu \mid \{C_i\}_{i \in S} \text{ is a } \mathfrak{B}\text{-partition of } A(S) = \bigcup_{i \in S} A_i \right\}$$

and a non-transferable utility game $(N, V_{\mathfrak{E}})$ with

$$V_{\mathfrak{E}}(S) := \left\{ x \in R^S \mid \text{there is a } \mathfrak{B}\text{-partition } \{C_i\}_{i \in S} \text{ of } A(S) \text{ with } x_i \leq \int_{C_i} f_i \, d\nu \text{ for all } i \in S \right\}.$$

In this paper we will prove a connection between the equilibrium payoffs of the TU-game $(N, v_{\mathfrak{E}})$ and the NTU-game $(N, V_{\mathfrak{E}})$.

The interpretation of the functions f_i is different depending on the point of view we take. If we consider these functions from a TU-point of view, we are assuming that there is an exogeneously given medium, "money," by which utility can be transfered from one agent to another. Then the function values of f_i have the dimension $[\$/m^2]$, wherein \$ is a unit for the exogeneously given medium "money." The utility of a parcel C has the dimension [\$]. If we take an NTU-point of view, the values of f_i have the dimension $[\mu_i/m^2]$, where $[\mu_i]$ is the unit of subjective utility of player i. Then $U_i(C) = \int_C f_i \, d\nu$ has the dimension $[\mu_i]$. We must assume that the preferences of agent i over parcels C determine the utility density f_i

up to positive affine transformations (otherwise the integrals $\int_C f_i \, d\nu$ makes no sense). The utilities must be *cardinally determined.*

There are more differences between both interpretations of the utilities. In an NTU-setting, e.g., an economy in which *one* player owns all the land ($A_i = L$ and $A_j = \emptyset$ if $j \neq i$) is trivial, as no trade can take place. In a TU-interpretation this economy is not trivial, as players $j \neq i$ can buy parcels from player i by means of "money."

Further, the concept of *competitive equilibrium* is different in both contexts. In an NTU-setting a collection of $\mathfrak{B}$-subsets $X = \{X_i\}_{i \in N}$ is a competitive equilibrium if there is a (price) density g: $L \to R_+$ such that

(i) $\{X_i\}_{i \in N}$ is a partition (the market clears),

(ii) $\int_{X_i} g \, d\nu \leq \int_{A_i} g \, d\nu$ for all $i \in N$ (budget constraint),

(iii) if $\int_C f_i \, d\nu > \int_{X_i} f_i \, d\nu$, then $\int_C g \, d\nu > \int_{A_i} g \, d\nu$ (maximality condition).

The *NTU-equilibrium payoff associated with the competitive equilibrium* $\{X_i\}_{i \in N}$ is the N-vector x with coordinates $x_i := \int_{X_i} f_i \, d\nu$ for all agents $i \in N$.

In a TU-setting condition (i) stays and condition (ii) disappears: there is no budget constraint, as "money" can be used to balance any difference between selling and purchasing. Condition (iii) becomes

(iii)′ $\int_{X_i} (f_i - g) \, d\nu \geq \int_C (f_i - g) \, d\nu$ for every parcel $C \in \mathfrak{B}$ (maximality).

The *TU-equilibrium payoff associated with* $\{X_i\}_{i \in N}$ has the coordinates $x_i := \int_{X_i} (f_i - g) \, d\nu + \int_{A_i} g \, d\nu$.

Note that in condition (iii) only integrals of the same densities are compared with each other. In (iii)′ the functions f_i and g are subtracted and must therefore have the same dimension [\$/m^2]. By the way, the dimension of g in the NTU-setting can be [$\cdot/m^2$], where "$\cdot$" stands for a numeral or any other unit of measurement.

In this paper we are going to compare the TU- and NTU-equilibrium payoffs. We shall prove that these equilibrium payoff are connected by "b-transfer."

First we recall the definition of the b-transfer of an NTU-game. Let (N, V) be an NTU-game and $b \in R_+^N$. The *b-transfer of the game* (N, V) is the TU-game $t_b(V)$ with values

$$t_b(V)(S) = \sup\{\langle b, x\rangle \mid x \in V(S)\}.$$

The set $\{\langle b, x\rangle \mid x \in V(S)\}$ is not always bounded for every coalition, but

if it is, the b-transfer can be defined. In Shapley (1969) the b-transfer of a game is used to *define* the Shapley value for NTU-games. A point $x \in V(N)$ is a point of the Shapley NTU-value if there is a transfer vector $b \in R_+^N$ such that the b-transfer $t_b(V)$ exists and $(b_i x_i)_{i \in N}$ is the Shapley value of the TU-game $(N, t_b(V))$.

Introducing a transfer vector $b \in R_+^N$ is the same as introducing an exchange rate between the utilities of different players. If $(x_1, \ldots, x_n)$ is a point of $V(N)$, the vector $b \star x := (b_i x_i)_{i \in N}$ is meant as a TU-payoff vector of which the coordinates are compared and added. So the dimension of the coordinate b_i must be $[\cdot/\mu_i]$ or $[\$/\mu_i]$.

In this paper we prove that, under mild conditions on $\mathcal{E}$, a vector $x \in V_{\mathcal{E}}(N)$ is an NTU-equilibrium payoff if and only if there is a nonnegative transfer vector b such that $b \star x$ is a TU-equilibrium payoff of $t_b(V_{\mathcal{E}})$. Furthermore, $t_b(V_{\mathcal{E}})$ is the TU-game associated with the economy $b \star \mathcal{E}$, the economy with land, wherein each utility density f_i is multiplied with b_i.

2. Main Results

As before we start with a measurable space $(L, \mathcal{B}, \nu)$, wherein L is a nonempty set, $\mathcal{B}$ is a σ-algebra of subsets of L, and $\nu : \mathcal{B} \to R_+$ is a finite non-trivial measure.

An *economy with land* consists of a finite set of agents N, and for each agent $i \in N$, a ν-measurable initial endowment $A_i \subset L$ and a nonnegative (essentially) bounded ν-measurable utility density f_i on L. The integral $U_i(C) := \int_C f_i \, d\nu$ gives the appreciation of agent i for the measurable set C. We assume that the following conditions hold:

(a) The measure ν is *non-atomic*; i.e., for every set $E \in \mathcal{B}$ with $\nu(E) > 0$ there is a subset $F \subset E$ with $F \in \mathcal{B}$ and $0 < \nu(F) < \nu(E)$.

(b) The utilities f_i are almost everywhere positive: $\nu(\{x \in L \mid f_i(x) = 0\}) = 0$ for all agents $i \in N$.

(c) The economy $\mathcal{E}$ is *non-trivial*; i.e., at least two agents have an initial endowment of *positive* ν-measure. An agent i with $\nu(A_i) = 0$ is called a *dummy*.

Let $\{X_i\}_{i \in N}$ be a $\mathcal{B}$-partition of L. A coalition S *can improve upon* $\{X_i\}_{i \in N}$ if there is a $\mathcal{B}$-partition $\{Y_i\}_{i \in S}$ of $A(S) := \bigcup_{i \in S} A_i$ such that $\int_{Y_i} f_i \, d\nu \geq \int_{X_i} f_i \, d\nu$ for all agents $i \in S$ and there is at least one strict inequality. A coalition S *can strictly improve upon* $\{X_i\}_{i \in N}$ if there is a $\mathcal{B}$-partition $\{Y_i\}_{i \in S}$ of $A(S)$ such that $\int_{Y_i} f_i \, d\nu > \int_{X_i} f_i \, d\nu$ for all agents $i \in S$. A $\mathcal{B}$-partition $\{X_i\}_{i \in N}$ of L is *Pareto optimal* if the coalition N has no strict improvement upon $\{X_i\}_{i \in N}$. A $\mathcal{B}$-partition $\{X_i\}_{i \in N}$ of L is *strictly Pareto*

optimal if the coalition N has no improvement upon $\{X_i\}_{i\in N}$. A $\mathfrak{B}$-partition $\{X_i\}_{i\in N}$ of L is a *core allocation* if no coalition S has a strict improvement upon $\{X_i\}_{i\in N}$. It is a *strict core allocation* if no coalition S has an improvement.

The following lemma states that in economies satisfying conditions (a) and (b) Pareto optimality is the same as strict Pareto optimality and the core is the same as the strict core.

LEMMA 1. *If $\mathscr{E}$ is an economy with land satisfying the conditions (a) and (b), then every improvement upon $\{X_i\}_{i\in N}$ can be changed into a strict improvement.*

Proof. Let $\{Y_i\}_{i\in S}$ be an improvement of coalition S upon $\{X_i\}_{i\in N}$. Then $\int_{Y_i} f_i\,d\nu \geq \int_{X_i} f_i\,d\nu$ for all agents $i \in S$ and there is a strict inequality for $i_0 \in S$.

By non-atomicity of ν there is a subset $E \subset Y_{i_0}$ with positive ν-measure and $\int_E f_{i_0}\,d\nu < \int_{Y_{i_0}} f_{i_0}\,d\nu - \int_{X_{i_0}} f_{i_0}\,d\nu$. By non-atomicity we can divide E into $s-1$ parcels E_i, $i \in S\backslash\{i_0\}$ with positive ν-measure. By the positivity of f_i (condition (b)) we have $\int_{E_i} f_i\,d\nu > 0$. Then $Z_{i_0} := Y_{i_0}\backslash E$ together with $Z_i := Y_i \cup E_i$ for $i \in S$, $i \neq i_0$ is a strict improvement of S. ∎

In the proof we used the theorem of Lyapounov (1940) for the one-dimensional case, saying that if $(L, \mathfrak{B}, \nu)$ is a non-atomic finite measure space, $E \in \mathfrak{B}$ is a ν-measurable set, and f is an essentially bounded nonnegative ν-measurable function, then for every real number β with $0 \leq \beta \leq \alpha := \int_E f\,d\nu$ there is a ν-measurable set $F \subset E$ with $\int_F f\,d\nu = \beta$.

It is the goal of this paper to characterize equilibrium payoffs by b-transfer. In the following proposition we characterize, as a first step, the Pareto optimal allocations by b-transfer.

PROPOSITION 2. *A $\mathfrak{B}$-partition $\{X_i\}_{i\in N}$ of L is Pareto optimal if and only if there exists a vector $b \in R_+^N$ such that $b_i > 0$ if $\nu(X_i) > 0$ and $X_i \subset' \{x \in L \mid b_i f_i(x) = \max_{j\in N} b_j f_j(x)\}$. The notation $E \subset' F$ means $\nu(E\backslash F) = 0$.*

Proof. If b is as in the proposition and $\{Y_i\}_{i\in N}$ is a strict improvement upon $\{X_i\}_{i\in N}$, then

$$\sum_{i\in N} b_i \int_{X_i} f_i\,d\nu < \sum_{i\in N} b_i \int_{Y_i} f_i\,d\nu \leq \sum_{i\in N} \int_{Y_i} \max_{j\in N} b_j f_j\,d\nu = \int_L \max_{j\in N} b_j f_j\,d\nu.$$

Since for (almost) all points $x \in X_i$ $b_i f_i(x) = \max_{j\in N} b_j f_j(x)$, we find $\sum_{i\in N} \int_{X_i} b_i f_i\,d\nu = \int_L \bar{f}_b\,d\nu$, where $\bar{f}_b = \max_{j\in N} b_j f_j$. Therefore, there is no strict improvement upon $\{X_i\}_{i\in N}$ and $\{X_i\}_{i\in N}$ is Pareto optimal.

Conversely, let $\{X_i\}_{i\in N}$ be a Pareto optimal allocation. For agents j with $\nu(X_j) = 0$, we define $b_j = 0$. Let A be the set of agents i whose parcel X_i has positive ν-measure. For $i, j \in A$, $i \neq j$, we define a real number a_{ij} by

$$a_{ij} := \inf \operatorname{ess}_{X_i} f_i/f_j$$

$$:= \sup\{t \in R \mid \text{the set } \{x \in X_i \mid f_i/f_j(x) \leq t\} \text{ has } \nu\text{-measure zero}\}.$$

Further we define $a_{ii} = 1$ for all $i \in A$.

Claim 1. For every pair (i, j), $i, j \in A$, we have $a_{ij} < \infty$.

If $a_{ij} = \infty$ then the sequence of ν-measurable sets $\{x \in X_i \mid f_i/f_j(x) \leq 2^n\}$ has ν-measure zero. Then also the set $X_i \backslash f_j^{-1}(0)$ has ν-measure zero. By property (b) we find $\nu(X_i) = 0$.

Claim 2. If σ is a permutation of A, then $\prod_{i\in A} a_{i,\sigma(i)} \geq 1$.

It is sufficient to prove that $\prod_{i\in C} a_{i,\sigma(i)} \geq 1$ for every cycle C of the permutation σ of length ≥ 2. Without loss of generality we assume that $C = \{1, 2, \ldots, p\}$ and $\sigma(i) = i + 1$ for $i = 1, \ldots, p - 1$ and $\sigma(p) = 1$. Suppose that $\prod_{i\in C} a_{i,\sigma(i)} < 1$. Choose $\bar{a}_{i,\sigma(i)} > a_{i,\sigma(i)}$ such that $\prod_{i\in C} \bar{a}_{i,\sigma(i)} = 1$. From the definition of a_{ij} we infer that the sets $E_i := \{x \in X_i \mid f_i/f_{\sigma(i)}(x) < \bar{a}_{i,\sigma(i)}\}$ have a positive ν-measure for all $i \in C$. Let $\varepsilon > 0$ be a positive number such that $\prod_{j=1}^{i} \bar{a}_{j,\sigma(j)} \int_{E_i} f_i \, d\nu \geq \varepsilon$ for $i = 1, \ldots, p$. As ν is atomless and f_i is positive almost everywhere, there are subsets $F_i \subset E_i$ with $\int_{F_i} f_i \, d\nu = (\prod_{j<i} \bar{a}_{j,\sigma(j)})^{-1}\varepsilon$ for all $i \in C$. The integral $\int_{F_i} f_{\sigma(i)} \, d\nu > \bar{a}_{i,\sigma(i)}^{-1} \int_{F_i} f_i \, d\nu = \int_{F_{\sigma(i)}} f_{\sigma(i)} \, d\nu$ for all $i \in C$. If we take $Y_i = (X_i \backslash F_i) \cup F_{\sigma^{-1}(i)}$ for $i \in C$ and $Y_j = X_j$ for $j \notin C$ we have an improvement upon $\{X_i\}_{i\in N}$ and $\{X_i\}_{i\in N}$ is not (strictly) Pareto optimal.

Claim 3. There is a positive vector $b \in R^A_{++}$ such that $b_j/b_i \leq a_{ij}$ for all i and $j \in A$.

Take $x \in R^A_{++}$ arbitrarily and define the mapping $\varphi : R^A_{++} \to R^A$ by $\varphi(x)_i := \min_{j\in A} x_j a_{ji}$. As $a_{ij} > 0$ for all $i, j \in A$ we have $\varphi(x) > 0$ and as $a_{ii} = 1$ we have $\varphi(x) \leq x$. Repeating the mapping φ we find a weakly decreasing sequence

$$x \geq \varphi(x) \geq \varphi^2(x) \geq \cdots \geq \varphi^k(x) \geq \cdots.$$

On the other side, if $k > a := |A|$, we find for each $i \in A$

$$\varphi^k(x)_i = x_{t(k)} \cdot a_{t(k),t(k-1)} \cdots a_{t(2),t(1)} \cdot a_{t(1),t(0)}$$

for some sequence $t(0) = i, t(1), \ldots, t(k) \in A$. As $k > a$ the sequence $t(0), t(1), \ldots, t(k)$ contains repetitions and therefore by Claim 2 we have $\varphi^k(x)_i \geq \varphi^l(x)_i$ for some $l = l(i) < k$. Therefore the sequence $\varphi^k(x)$ is constant for $k \geq a$. We find a vector $b := \varphi^a(x) > 0$ with $b_i = \min_{j \in A} b_j a_{ji}$ for all $i \in A$.

Now we can finish the proof as follows. For almost all points of X_i, $i \in A$, we have $f_i/f_j \geq a_{ij} \geq b_j/b_i$. Then, for agents $i \in A$, $b_i f_i(x) = \max_{j \in A} b_j f_j(x)$ for almost every point of X_i. We may replace $\max_{j \in A}$ by $\max_{j \in N}$, as $b_j = 0$ when $j \notin A$. For $i \notin A$ the equality $b_i f_i = \max_{j \in N} b_j f_j$ almost everywhere on X_i is an empty condition. ∎

If $b \in R_+^N$ we introduce the notation

$$\bar{f}_b^S = \bigvee_{i \in S} b_i f_i, \qquad \bar{f}_b := \bar{f}_b^N, \qquad \underline{f}_b := \bigwedge_{i \in N} \bar{f}_b^{N \setminus i},$$

where $\bigwedge_{i \in S} b_i f_i$ and $\bigvee_{i \in S} b_i f_i$ denote the pointwise minimum and maximum of the functions $b_i f_i$, $(i \in S)$. The function $\underline{f}_b$ gives in every point the second highest value.

If $\mathcal{E} = \{N, \{A_i, f_i\}_{i \in N}\}$ is an economy with land, $b \star \mathcal{E}$ is the economy with the same set of agents and initial endowments as in $\mathcal{E}$ but with utility densities $\{b_i f_i\}_{i \in N}$ instead of $\{f_i\}_{i \in N}$. The economy $\overline{\mathcal{E}}$ has the functions $f_i \vee \underline{f}$ as utility densities.

It is typical for economies with land that the trades necessary to reach a reallocation $\{X_i\}_{i \in N}$ can be reconstructed from the reallocation $\{X_i\}_{i \in N}$. The parcels $A_i \cap X_j$ are sold by agent i to agent j and the payments (under a price regime g) are $\int_{A_i \cap X_j} g \, d\nu$. Therefore, it is not surprising that we need, in the proof of the main theorem, a theorem from the theory of flows. In fact we use the following theorem of Hoffman (1960).

PROPOSITION 3 (Hoffman (1960)). *If N is a finite set and L and C are $N \times N$-matrices with $L \leq C$, then the following statements are equivalent:*

(i) *There is an $N \times N$-matrix Y with $L \leq Y \leq C$ and $\sum_{j \neq i} Y_{ij} = \sum_{j \neq i} Y_{ji}$ for all $i \in N$.*

(ii) *For all subsets $S \subset N$ we have $\sum_{i \in S} \sum_{j \in N \setminus S} L_{ij} \leq \sum_{i \in S} \sum_{j \in N \setminus S} C_{ji}$.*

In the application of this proposition L_{ij} will be $\int_{A_i \cap X_j} \underline{f}_b \, d\nu$ and C_{ij} will be $\int_{A_i \cap X_j} \bar{f}_b \, d\nu$. The matrix coefficients Y_{ij} are the integrals $\int_{A_i \cap X_j} g \, d\nu$. If $\underline{f}_b \leq g \leq \bar{f}_b$ the first condition of (i) is satisfied. The second part of (i) says that $\int_{A_i \cap X_{N \setminus i}} g \, d\nu = \int_{A_{N \setminus i} \cap X_i} g \, d\nu$. If we add $\int_{A_i \cap X_i} g \, d\nu$ we find the budget equality $\int_{A_i} g \, d\nu = \int_{X_i} g \, d\nu$.

After these preparations we formulate the rather technical first version of the main theorem:

PROPOSITION 4. *A $\mathcal{B}$-partition $\{X_i\}_{i\in N}$ of L is an NTU-competitive equilibrium in the non-trivial economy $\mathcal{E}$ if and only if there is a vector $b \in R^N_+$ with $b \neq 0$ such that*

(i) $X_i \subset' \{x \in L \mid b_i f_i(x) = \bar{f}_b(x)\}$,

(ii) $\sum_{i\in S} \int_{X_i} \bar{f}_b \, d\nu \geq \sum_{i\in S} \int_{A_i} (\bar{f}^S_b \vee \underline{f}_b) \, d\nu$.

Moreover, $b_i > 0$ if $\nu(A_i) > 0$.

Proof. Let $\{X_i\}_{i\in N}$ be an NTU-competitive equilibrium with price density g. Let b_i be the infimum of the real numbers t such that the set $\{x \in X_i \mid g(x) \geq t f_i(x)\}$ has ν-measure zero: $\sup \operatorname{ess}_{X_i} g/f_i$. Note that $b_i = 0$ if $\nu(X_i) = 0$. Let $c_i := \inf \operatorname{ess}_{L\setminus X_i} g/f_i$, i.e., the supremum of the numbers t with the property that the set $\{x \in L\setminus X_i \mid g(x) \leq t f_i(x)\}$ has ν-measure zero.

Claim 1. $0 \leq b_i \leq c_i \leq \infty$ for all agents $i \in N$.

Suppose that $c_i < b_i$ for some agent $i \in N$. Take $\bar{b}_i$ and $\bar{c}_i$ such that $c_i < \bar{c}_i < \bar{b}_i < b_i$. The sets $Y := \{x \in X_i \mid g(x) \geq \bar{b}_i f_i(x)\}$ and $Z := \{x \in L\setminus X_i \mid g(x) \leq \bar{c}_i f_i(x)\}$ have positive ν-measure. There is a measurable set $E \subset Y$ with $0 < \int_E f_i \, d\nu < \int_Z f_i \, d\nu$ because of the non-atomicity of ν and $f_i > 0$ almost everywhere in X_i. From the same property we infer that there is a measurable set $F \subset Z$ with $\int_E f_i \, d\nu < \int_F f_i \, d\nu \leq \bar{b}_i/\bar{c}_i \int_E f_i \, d\nu$. If we replace X_i by $\bar{X}_i = (X_i\setminus E) \cup F$ we have $\int_{\bar{X}_i} f_i \, d\nu > \int_{X_i} f_i \, d\nu$ and

$$\int_E g \, d\nu \geq \int_E \bar{b}_i f_i \, d\nu > \bar{c}_i \int_F f_i \, d\nu \geq \int_F g \, d\nu.$$

Then g is not an equilibrium price for the competitive equilibrium $\{X_i\}_{i\in N}$. This finishes the proof of the claim.

Claim 2. If $b_i = 0$, then $\nu(A_i) = 0$; if $b_i = \infty$, then $\nu(A_j) = 0$ for every $j \neq i$.

If $b_i = 0$ for some player $i \in N$, then $g(x) = 0$ for almost every point of X_i. If $\nu(X_i) > 0$, then every player $j \neq i$ has $f_j > 0$ almost everywhere on X_i. This means that agent j can obtain X_i costless and his appreciation for X_i is positive. Then $(\{X_i\}, g)$ is not a price equilibrium. Therefore, $\nu(X_i) = 0$. If $\nu(A_i) > 0$ (agent i is not a dummy), we have $\int_{X_i} f_i \, d\nu = 0 < \int_{A_i} f_i \, d\nu$ and $(\{X_i\}_{i\in N}, g)$ is not a price equilibrium.

If $b_i = \infty$, we also have $c_i = \infty$. For every number $t > 0$ we have $t f_i(x) \leq g(x)$ for almost every point of $L\setminus X_j$. As g is essentially bounded, we find $f_i = 0$ almost everywhere on $L\setminus X_i$. This means that $\nu(L\setminus X_i) = 0$ or $\nu(X_j) = 0$ for every agent $j \neq i$. As before we conclude that $\nu(A_j) = 0$ for all agents $j \neq i$.

In Claims 1 and 2 we proved that $0 \leq b_i < \infty$ for all agents $i \in N$ if the economy $\mathscr{E}$ is *not trivial*. Further, $b_i > 0$ if agent i is not a *dummy*.

From the definition of b_i we infer that, if i is not a dummy, $b_i f_i \geq g$ almost everywhere on X_i and from the definition of c_i and Claim 1 we find $b_j f_j \leq c_j f_j \leq g$ almost everywhere on X_i for agents $j \neq i$. This means that $b_i f_i = \bar{f}_b$ almost everywhere on X_i and $\underline{f}_b \leq g \leq \bar{f}_b$ almost everywhere on L.

As $\int_{X_i} g\, d\nu = \int_{A_i} g\, d\nu$ for all agents $i \in N$ (budget constraint) the matrix $Y_{ij} = \int_{A_i \cap X_j} g\, d\nu$ is a circulation flow with lower capacity $L_{ij} = \int_{A_i \cap X_j} \underline{f}_b\, d\nu$ and upper capacity $C_{ij} = \int_{A_i \cap X_j} \bar{f}_b\, d\nu$. By the theorem of Hoffman we find $\int_{A(S) \cap X(N \setminus S)} \underline{f}_b\, d\nu \leq \int_{A(N \setminus S) \cap X(S)} \bar{f}_b\, d\nu$ for all $S \subset N$. Adding to both sides $\int_{A(S) \cap X(N \setminus S)} \bar{f}_b\, d\nu$, we find $\int_{X(S)} \bar{f}_b\, d\nu \geq \int_{A(S)} (\bar{f}_b^S \vee \underline{f}_b)\, d\nu$ for all $S \subset N$, as almost everywhere on $X(N \setminus S)$ the function $\bar{f}_b^S \leq \underline{f}_b$ and on $X(S)$ the function $\bar{f}_b^S = \bar{f}_b \geq \underline{f}_b$.

Conversely, if there is a vector b satisfying the conditions of the theorem, we prove that there is a bounded ν-measurable function g such that

$$\underline{f}_b \leq g \leq \bar{f}_b \qquad \text{and} \qquad \int_{X_i} g\, d\nu = \int_{A_i} g\, d\nu.$$

If we define L_{ij} and C_{ij} as before, condition (ii) of Proposition 4 implies that the condition in the theorem of Hoffman is satisfied and there is a circulation flow $Y = (Y_{ij})$ with $L_{ij} \leq Y_{ij} \leq C_{ij}$ for all $i \neq j$. If $Y_{ij} = \lambda_{ij} L_{ij} + (1 - \lambda_{ij}) C_{ij}$ with $\lambda_{ij} \in [0, 1]$, we define $g_{|A_i \cap X_j}$ by $(\lambda_{ij} \underline{f}_b + (1 - \lambda_{ij}) \bar{f}_b)_{|A_i \cap X_j}$. Then $\underline{f}_b \leq g \leq \bar{f}_b$ and $\int_{A_i \cap X_j} g\, d\nu = Y_{ij}$. On $A_i \cap X_i$ the function g can be chosen arbitrarily between $\underline{f}_b$ and $\bar{f}_b$.

Finally, we prove that the price density g satisfies the maximality condition (iii). Let i be any agent and let $C \in \mathfrak{B}$ be a parcel of land with $\int_C f_i\, d\nu > \int_{X_i} f_i\, d\nu$. Then $C \setminus X_i$ has a positive measure and $b_i f_i \leq \underline{f}_b$ almost everywhere on $C \setminus X_i$. Further, $b_i f_i = \bar{f}_b$ almost everywhere on X_i.

Then we have, if $b_i > 0$,

$$\begin{aligned} \int_{C \setminus X_i} g\, d\nu - \int_{X_i \setminus C} g\, d\nu &\geq \int_{C \setminus X_i} \underline{f}_b\, d\nu - \int_{X_i \setminus C} \bar{f}_b\, d\nu \\ &\geq \int_{C \setminus X_i} b_i f_i\, d\nu - \int_{X_i \setminus C} b_i f_i\, d\nu > 0. \end{aligned}$$

If $b_i = 0$, then $\nu(X_i) = 0$ and $\int_{C \setminus X_i} g\, d\nu \geq \int_{C \setminus X_i} \underline{f}_b\, d\nu > 0$ if $\underline{f}_b > 0$ almost everywhere on C. This is clearly true if at least two players have a positive coordinate in b. If the economy is not trivial, the price of C is larger than

the price of X_i (and of A_i). The function g is a price density to the $\mathcal{B}$-partition $\{X_i\}_{i\in N}$. ■

Before we come to the final version of the main theorem we have to recall some results of Legut *et al.* (1994).

(α) In Legut *et al.* (1994) it is proved that a $\mathcal{B}$-partition $\{X_i\}_{i\in N}$ is a TU-competitive equilibrium if and only if, for all agents $i \in N$, $X_i \subset' \{x \in L \mid f_i(x) = \bar{f}(x)\}$. As $\{X_i\}_{i\in N}$ is a $\mathcal{B}$-partition, the set $Y_i := \{x \in L \mid f_i(x) = \bar{f}(x) > \underline{f}(x)\} \subset' X_i$. The equilibrium price can be any function between $\underline{f}$ and $\bar{f}$. Condition (i) of Proposition 4 states that $\{X_i\}_{i\in N}$ is a TU-competitive equilibrium in the economy $b \star \mathcal{E}$.

(β) For an economy with land $\mathcal{E}$ the TU-game $v_{\mathcal{E}}$ as defined in the Introduction has the following values:

$$v_{\mathcal{E}}(S) = \int_{A(S)} \bar{f}^S \, d\nu.$$

Rewriting condition (ii) we find for the right hand side

$$\sum_{i\in S} \int_{A_i} (\bar{f}_b^S \vee \underline{f}_b) \, d\nu = \int_{A(S)} (\bar{f}_b^S \vee \underline{f}_b) \, d\nu = v_{\overline{b\star\mathcal{E}}}(S),$$

where $\overline{b \star \mathcal{E}}$ is the economy with the same set of agents and initial endowments as $\mathcal{E}$ but with utilities $b_i f_i \vee \underline{f}_b$, $i \in N$. Condition (ii) of Proposition 4 says that $\{\int_{X_i} \bar{f}_b \, d\nu\}_{i\in N}$ is a core element of the TU-game $v_{\overline{b\star\mathcal{E}}}$.

(γ) Finally, for a TU-competitive equilibrium $\{X_i\}_{i\in N}$, the price density g (arbitrarily chosen between $\underline{f}$ and $\bar{f}$) determines the TU-equilibrium payoff

$$x_i := \int_{X_i} (f_i - g) \, d\nu + \int_{A_i} g \, d\nu \qquad \text{for all } i \in N$$

completely. It is proved that $\{x_i\}_{i\in N}$ is a TU-equilibrium payoff if and only if $x \in \text{Core}(v_{\overline{\mathcal{E}}})$. Hence, as $\bar{f}_b = b_i f_i$ almost everywhere on X_i, condition (ii) of Proposition 4 is equivalent with

$$\left(b_i \int_{X_i} f_i \, d\nu\right)_{i\in N} \text{ is a TU-equilibrium payoff with respect to}$$

$$\{X_i\}_{i\in N} \text{ in the economy } b \star \mathcal{E}.$$

Summarizing we have

THEOREM 5. *A $\mathcal{B}$-partition $\{X_i\}_{i\in N}$ is an NTU-competitive equilibrium of a non-trivial economy with land $\mathcal{E} = \{N, \{A_i, f_i\}_{i\in N}\}$ if and only if there is a transfer vector $b \in R_+^N$ (with $b_i > 0$ if $\nu(A_i) > 0$) such that*

(i) *$\{X_i\}_{i\in N}$ is a TU-competitive equilibrium of the economy $b \star \mathcal{E}$,*

(ii) *$b \star (\int_{X_i} f_i \, d\nu)_{i\in N}$ is a TU-equilibrium payoff with respect to $\{X_i\}_{i\in N}$ in the economy $b \star \mathcal{E}$.*

For equilibrium payoffs we have

COROLLARY. *If $\mathcal{E}$ is a non-trivial economy with land and x is an element of $V_{\mathcal{E}}(N)$, then x is an NTU-equilibrium payoff iff there is a transfer vector $b \in R_+^N$ (with $b_i > 0$ if $\nu(A_i) > 0$) such that $b \star x$ is a TU-equilibrium payoff in the economy $b \star \mathcal{E}$.*

Proof. Let $x \in V_{\mathcal{E}}(N)$ and let $b \in R_+^N$ be a vector such that $b_i > 0$ if $\nu(A_i) > 0$ and $b \star x$ is a TU-equilibrium payoff of $b \star \mathcal{E}$.

As $x \in V_{\mathcal{E}}(N)$, there is $\mathcal{B}$-partition $\{X_i\}_{i\in N}$ such that $x_i \leq \int_{X_i} f_i \, d\nu$ for all agents $i \in N$. We prove that $\{X_i\}_{i\in N}$ satisfies the conditions of Theorem 5.

If we multiply these inequalities with b_i and sum over $i \in N$, we find

$$\langle b, x\rangle \leq \sum_{i\in N} \int_{X_i} b_i f_i \, d\nu.$$

Since $b \star x$ is a TU-equilibrium payoff in $b \star \mathcal{E}$, we have in particular that $\langle b, x\rangle = v_{b\star\mathcal{E}}(N) = \int_L \bar{f}_b \, d\nu$ and therefore all inequalities in the preceding formula are equalities: $b_i x_i = \int_{X_i} b_i f_i \, d\nu = \int_{X_i} \bar{f}_b \, d\nu$ for all agents $i \in N$. This means that $X_i \subset' \{x \in L \mid b_i f_i(x) = \bar{f}_b(x)\}$ and $\{X_i\}_{i\in N}$ is a TU-competitive equilibrium in $b \star \mathcal{E}$ by (α) (condition (i)). Moreover, $b \star x = b \star (\int_{X_i} f_i \, d\nu)_{i\in N}$ is a TU-equilibrium payoff in the economy $b \star \mathcal{E}$ (condition (ii)). ∎

3. CONCLUDING REMARKS

In this final section we will answer three fundamental questions about the model we used in this paper.[1]

First the meaning of a TU-model and particularly of TU-equilibria can be questioned.

[1] We are grateful to one of the referees who raised these questions.

(1) *What is the meaning of TU-models and especially of TU-equilibria? Are price equilibria without budget constraint not strange to all thinking about economies?*

Indeed, a TU-model needs many preassumptions to make sense. It assumes, e.g., that there is an *externality* like money, equally appreciated by all the agents, and that in the economy to be considered "money is not the problem" or, to say it differently, every player must have enough "outside assets" to balance the difference between his buyings and his sellings. Under these circumstances it makes sense to fix a price that "clears the market" and gives after all the agents a payoff, partially in goods (land) and partially in money, that is as high as possible. This is exactly the TU-equilibrium concept we propose. In fact, there is some tradition in modeling economic situations by TU-games (see, e.g., the house market model of Shapley and Shubik (1972) and Kaneko (1976), (1982)). Finally this paper shows that TU-equilibria are tightly connected with economically not suspect NTU-equilibria.

The second question raises the issue of how far a commodity like "land" differs from traditional commodities.

(2) *What is special to the commodity "land" compared with "normal commodities"?*

The main difference is the uniqueness of parcels of land. A parcel can be bought from the owner and from nobody else. This makes it possible to trace back who bought what parcel from whom after the trades are made. In an economy with normal commodities (like crude oil or milk) this cannot be done uniquely. In the paper we use this idea by comparing the trading behavior with a flow of goods and money. Loosely speaking, land can be seen as an overwhelming number of indivisible goods (the points of L) that can be traded only in some combinations (the measurable sets).

The last question is about the kind of utility functions we considered.

(3) *What do the utility functions have the form they have?*

Indeed, a difficult commodity like land requires a more sophisticated utility function. In our first paper on this subject (Legut *et al.* (1992)) we referred to the Ph.D. dissertations of Berliant (1982) and Dunz (1984), wherein attempts are made to consider more general utility functions. But even the most general utility concept we know of (utility functions only continuous with respect to an atomless measure (Dunz (1984)) cannot deal with all the intricacies of a commodity like land. Here global geometrical or topological concepts like shape of parcels, connectedness, or 1-connectedness (there no enclaves owned by somebody else) play an important role. Therefore we used the simpler utility functions, as the more sophisticated ones are not doing the job better.

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