A DNS–curve in a two state capital accumulation model: a numerical analysis†

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Abstract

In this paper we study a capital accumulation model in an optimal control theoretic framework, where the capital stock and the investment rate are modeled as state variables and the change in the investment rate as control. Adjustment costs are introduced for investment rate and its change. Moreover, we model network externalities by a convex segment in the revenue function, which implies the existence of two long-run optimal steady states, one with a low level and the other with a high level capital stock. It depends on the initial capital endowment and initial investment rate to which steady state it is optimal to converge. We numerically compute a curve in the state plane, for which it holds that, when starting from a point on this curve, the decision-maker is indifferent between going to either one of these steady states, and identify this curve as the DNS–curve. The negative slope of the DNS–curve indicates that there is a trade-off between the initial capital endowment and initial investment rate.

Key words: DNS–curve; capital accumulation; optimal control; numerical methods; multiple steady states.

JEL classification: C61, C62, D92

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†We wish to thank two anonymous referees for comments and suggestions. We thank Carolyn Oates and Yalçın Kaya for English language proof-reading. Remaining inadequacies are our own responsibility. An earlier version of this article was presented at the Seventh Viennese Workshop (Vienna, Austria, May 2000) and this version of the article was presented at the SCE conference (New Haven, Conn., USA, June 2001). This research was partly financed by the Austrian Science Foundation (FWF) under grant No. 10850-OEK (" Nonlinear Dynamics and Economic Growth ").

1 Introduction

Dechert and Nishimura (1983) analyze a discrete time Ramsey model in which the production function is convex-concave. They show that, provided that the interest rate has an intermediate value, the optimal path converges to a steady state only if the initial capital stock is above a critical value, otherwise it converges to zero. The first time this phenomenon appears in the literature is Skiba (1978), who studies a one-dimensional continuous time growth model with a non-convex technology. Dechert (1983) and Davidson and Harris (1981) analyze a firm capital accumulation model, where the revenue function contains a convex segment. They show that under a particular scenario a threshold value of the initial capital stock exists, above which it is optimal to converge to the larger saddle point (i.e. the saddle point with the larger capital stock level) and below which convergence to the smaller saddle point is preferable. Honoring these contributions we want to call such a threshold as the DNS–(Dechert-Nishimura-Skiba–)point. We define a DNS–point as a point at which there are two optimal paths, where each of them approaches a different steady state. The decision maker is indifferent concerning the choice of these two paths.

All these papers have in common the fact that the optimal paths are history dependent caused by (local) non-convexities. An unstable steady state is crucial for determining the threshold separating the starting points of the optimal trajectories leading to the different long run outcomes. In a recent paper by Wirl and Feichtinger

(1999) it is found, however, that (local) non-convexities are by no means necessary for the occurrence of such thresholds. They provide two mechanisms, viz, growth and control state interactions, which can lead to history dependence in a strictly concave framework.

All the above mentioned contributions consider a dynamic optimization problem with a one-dimensional structure. Intuition suggests that extending this feature to the class of optimal control models with two state variables will lead to the occurrence of a set of DNS–points, which we will call DNS–set. In the literature contributions that deal with this topic are scarce. Brock and Dechert (1983) prove the existence of a DNS–set, but the exact location of it is not determined. To best authors' knowledge no paper has appeared in which the shape of a DNS–set is worked out. This is maybe caused by the fact that finding a DNS–set analytically is not an easy task. In this paper we determine a DNS–set in a two-dimensional capital accumulation model by computing numerically the optimal trajectories.

The capital accumulation framework has been extensively analyzed in the literature assuming a constant or decreasing returns to scale technology and adjustment costs of investment (Eisner and Strotz, 1963; Lucas, 1967; Gould, 1968). One specific feature of our model is that, besides the prevalent adjustment costs associated with investment, additional costs are incurred for changes in the investment rate. Another specific feature in our model is that our formulation admits the existence of network externalities. By a network externality it is meant that the value of a good increases with the number of users (see Economides, 1996). An example would be that it is beneficial to the users of a certain software package, if this package is used by many other people. Similarly, for the user of some GSM network the value increases with the number of users because of the significantly reduced rates within the network. In the model this is reflected by an inverse demand function, which increases for intermediate quantities produced. This implies that the firm's revenue function is convex in a segment of intermediate capital stock values, whereas the firm's revenue function is concave for small and large capital stock values outside this segment. Similarly, in Davidson and Harris (1981) and Dechert (1983) a revenue function occurs, which contains one convex segment. In contrast to this, Barucci (1998) studies the classical framework with the exception that the revenue function is strictly convex throughout.

Our model leads to the existence of two long run optimal steady states. It depends on the firm's initial capital endowment and initial investment rate to which steady state it is optimal to converge. In the present paper we numerically compute a DNS–set and it turns out that it is a one-dimensional curve. All the numerical computation is done by Mathematica (Version 4.0.1.0, ©Wolfram Research, Inc.)

The organization of the paper is as follows. The next section specifies the model. In Section 3 we analyze the model and we interpret the results economically. Section 4 concludes the paper.

2 Model Formulation

Consider a firm that needs capital goods to produce commodities, which are sold on an output market. The more capital goods the firm owns the more commodities can be sold and thus the more revenue is obtained. Of course, in case that the firm has some market power the output price decreases with the number of goods that are sold, which implies that decreasing returns to scale will be present, especially if the capital stock is sufficiently large. On the other hand scale economies can cause increasing returns to scale. To analyze the effect of this on optimal firm behavior we assume that there exists an interval of capital stock values for which there are increasing returns to scale.

Denoting revenue by R and capital stock by K[1], it is imposed that $R(K)$ is a positive, twice continuously differentiable, and increasing function with one convex segment for intermediate values of the capital stock (see, e.g., Davidson and Harris, 1981, Figure 2b). This convex segment arises due to the fact that for these values of the capital stock the firm's production technology exhibits increasing returns to scale.

An alternative explanation for the convex segment in the revenue function is that the inverse demand function is locally increasing, which arises due to a network externality (see Economides, 1996). A network externality implies that the value of a good increases with the number of users of this good. An example would be that it is beneficial to the users of a certain word processing program, if this package is used

[1]Since the model is dynamic, all variables are functions of time, i.e. $K = K(t)$. For notational convenience, in what follows this dependence of variables on time is not explicitly denoted.

by many other people, because, e.g., it makes it easier to exchange files. Similarly, for the user of some cellular phone network the value increases with the number of users because of the significantly reduced rates within the network. In the model this is reflected by an inverse demand function which decreases for small and large quantities produced, but has an increasing segment for intermediate quantities produced. A relevant specification is

$$P(Q) = \frac{\gamma_1}{\sqrt{Q}} - \frac{\gamma_3}{1+\gamma_2 Q^4}, \tag{1}$$

where P denotes the price and Q is the quantity produced. Figure 1 depicts the graph of this inverse demand function.

====== Figure 1 about here ======

If we assume that the production function is linear, $Q(K) = \beta K$, say, then the firm's revenue function $R(K) = P(Q(K))Q(K)$ is the following:

$$R(K) = k_1\sqrt{K} - k_3 \frac{K}{1+k_2 K^4}, \tag{2}$$

with $k_1 = \gamma_1\sqrt{\beta}, k_2 = \gamma_2\beta^4$ and $k_3 = \gamma_3\beta$. Figure 2 depicts the graph of the firm's revenue function. In the segment between $K \approx 1.2$ and $K \approx 3.3$ the network externality leads to a convex segment of the revenue function.

====== Figure 2 about here ======

The firm can increase its capital stock by investing, where the investment rate is denoted by I. Besides the purchase costs, the cost of investments $c(I)$ also consists of adjustment costs which are assumed to be strictly convex, i.e. $c(0) = 0$ and $c'' > 0$. For the numerical analysis below we adopt the quadratic specification imposed by, e.g., Barucci (1998):

$$c(I) = c_1 I + \frac{c_2}{2} I^2. \tag{3}$$

As usual, capital stock increases with investments and decreases with depreciation. Assuming a constant depreciation rate $\delta > 0$, the following state equation for capital stock arises:

$$\dot{K} = I - \delta K. \tag{4}$$

The concept of investment adjustment costs is refined here by explicitly modelling that changes in the investment rate are costly (see also Jorgensen and Kort, 1993). Such costs can arise in cases where an organization is used to a certain rate of investment, so that it has to reorganize when changes in this investment rate occur. Representing the change of investment by v, these costs equal $g(v)$. Assuming a quadratic cost function is quite common in the literature (compare, e.g., Salop, 1973; Steindl et al., 1986). Even in the case of non-symmetric, piecewise linear costs with a kink at the level zero a quadratic approximation is reasonable (see Holt et al., 1960). Here we also adopt the quadratic form. Admittedly it simplifies the calculations, but it is not essential:

$$g(v) = \frac{\alpha}{2} v^2,$$

where

$$\dot{I} = v. \tag{5}$$

In order to include (5) in the optimization problem, the investment I will be modelled as a state variable.

The firm's objective is to maximize the discounted cash flow stream over an infinite planning horizon. Collecting the revenue function and both types of adjustment costs described above, and assuming a constant discount rate ρ, we arrive at the following expression for the criterion function:

$$\max_{v} \int_0^\infty e^{-\rho t} \left[R(K) - c(I) - \frac{\alpha}{2} v^2 \right] dt. \tag{6}$$

The optimal control model now consists of the expressions (4)-(6). It has two state variables, K and I, and one control variable, v. Summarizing, the following model is obtained:

$$\begin{aligned} \max_{v} \quad & \int_0^\infty e^{-\rho t} \left[R(K) - c(I) - \frac{\alpha}{2} v^2 \right] dt, \\ \text{s.t.} \quad & \dot{K} = I - \delta K, \\ & \dot{I} = v. \end{aligned}$$

If $\alpha = 0$, the control v is costless and adjustment of I can be done instantaneously. Thus, v can be deleted and the problem becomes an optimal control model with state variable K and control variable I. Then we are back in the classical capital accumulation models. For linear or concave $R(K)$ and convex $c(I)$ the basic framework arises, which is analyzed extensively (e.g. Lucas, 1967; Gould, 1968). In Dechert (1983) the existence of a DNS–point is proved for this model with a convex-concave $R(K)$, while in Davidson and Harris (1981) the implications of convex segments in $R(K)$ and concave segments in $c(I)$ are studied. Barucci (1998) considers convex quadratic functions for both $R(K)$ and $c(I)$.

3 Analysis of the Model

This section consists of five subsections. First, in Subsection 3.1 the optimality conditions are listed, after which in Subsection 3.2 the stability behavior around the steady states is studied. In Subsection 3.3 the stable manifolds are analyzed, while in Subsection 3.4 the DNS–curve is determined. Finally, economic intuitions of the mathematical results are provided in Subsection 3.5.

3.1 Optimality Conditions

To apply Pontryagin's Maximum Principle (see, e.g., Leonard and Long, 1988), we start out by stating the current value Hamiltonian:

$$H = R(K) - c(I) - \frac{\alpha}{2}v^2 + \lambda_1 (I - \delta K) + \lambda_2 v.$$

Maximization of the Hamiltonian with respect to the control variable v gives:

$$v = \frac{1}{\alpha}\lambda_2. \tag{7}$$

Further application of Pontryagin's Maximum Principle and taking into account (4) and (5), lead to the following dynamic system:

$$
\begin{aligned}
\dot{K} &= I - \delta K, \\
\dot{I} &= v = \tfrac{1}{\alpha}\lambda_2, \\
\dot{\lambda}_1 &= (\rho + \delta)\lambda_1 - R'(K), \\
\dot{\lambda}_2 &= \rho\lambda_2 - \lambda_1 + c'(I).
\end{aligned}
\tag{8}
$$

3.2 Stability Analysis

From $\dot{K} = \dot{I} = 0$ it is straightforward to see why $\lambda_2 = 0$ and $I = \delta K$ is required for a steady state. Additional to these conditions the equations $\dot{\lambda}_1 = \dot{\lambda}_2 = 0$ imply that steady state values also have to satisfy

$$
\lambda_1 = c'(\delta K) = \frac{1}{(\rho + \delta)} R'(K), \tag{9}
$$

which is illustrated in Figure 3.

====== Figure 3 about here ======

Here we see that in the case of a convex cost function $c(I)$ at most one steady state exists, if the revenue function $R(K)$ is concave, as it is the case in the basic capital accumulation models. However, due to the convex segment for intermediate values of the capital stock, the revenue function that we consider has a concave-convex-concave shape. Given the functional forms in (2) and (3), it is easily seen from (9) that depending upon the choice of the parameters for the functions $R(K)$ and $c(I)$ there are at most three steady states (see also Figure 3).

We continue with the investigation of the situation of three different steady states $\mathcal{K}_i = (K_i, I_i, v_i, \lambda_{1i}, \lambda_{2i}), i = 1, 2, 3,$ where $\mathcal{K}_1$ corresponds to the steady state with the smallest capital stock, and the steady state with the largest capital stock value is denoted by $\mathcal{K}_3$.

Proposition 1 *In the case of the existence of three steady states, the stability analysis gives the following results:*

- *The stable invariant manifold of the steady state $\mathcal{K}_2$ is one dimensional.*
- *The stable invariant manifolds of the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$ are two dimensional.*
- *The steady state $\mathcal{K}_i$, $i = 1, 3$, is a saddle point with transient oscillations, if*

$$[\alpha \delta (\rho + \delta) - c_2]^2 + 4R''(K_i)\alpha < 0,$$

otherwise it is a saddle point with real eigenvalues.

Proof: To analyze the stability behavior of the dynamic system around the steady states, we follow the lines of Dockner (1985), p. 96. To do this, we first compute the Jacobian (for reasons of readability we do not yet substitute $R(K)$ given in (2)).

$$\det J(K) = \begin{vmatrix} -\delta & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \\ -R''(K) & 0 & \rho+\delta & 0 \\ 0 & c_2 & -1 & \rho \end{vmatrix} = \frac{1}{\alpha} [\delta (\rho + \delta) c_2 - R''(K)] .$$

It is easily seen that the Jacobian is positive in the first and the third steady state, $\mathcal{K}_1$ and $\mathcal{K}_3$, while it is negative in the second steady state, $\mathcal{K}_2$. This implies that $\mathcal{K}_2$ is unstable, except for a one-dimensional manifold (see, e.g., Feichtinger et al., 1994, Figure 1).

To determine the signs of the eigenvalues, besides the Jacobian, we also have to compute a specific quantity κ as defined by Dockner (1985), p. 96:

$$\kappa = \begin{vmatrix} -\delta & 0 \\ -R''(K) & \rho+\delta \end{vmatrix} + \begin{vmatrix} 0 & \frac{1}{\alpha} \\ c_2 & \rho \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = -\delta\,(\rho+\delta) - \frac{c_2}{\alpha} < 0.$$

Given the fact that the Jacobian is positive and κ is negative in $\mathcal{K}_1$ and $\mathcal{K}_3$, we know that both steady states are saddle points. Now, if

$$\frac{\kappa^2}{4} - \det J(K) = \frac{1}{4\alpha^2}\left[(\delta\,(\rho+\delta)\,\alpha - c_2)^2 + 4R''(K)\alpha\right] \geq 0,$$

then the steady state is a saddle point with real eigenvalues. Otherwise a saddle point convergence occurs with transient oscillations (damped cycles). ○

Definition 1 *The region of optimal convergence of a steady state is the set of all initial states, from which it is optimal to choose a trajectory converging to the steady state.*

The stable invariant manifold of $\mathcal{K}_2$ is one-dimensional (it would be practically impossible to have initial states in this sparse set). This result leads us to the conclusion that only $\mathcal{K}_1$ and $\mathcal{K}_3$ are the candidates for steady states with a region of optimal convergence, which has a positive measure.

Definition 2 *A DNS(Dechert-Nishimura-Skiba)-point is a point in the state space which belongs to at least two different regions of optimal convergence. The DNS-set is the set of all DNS-points.*

The aim of this paper is to numerically determine the regions of optimal convergence of the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$, respectively, and the DNS–set, which separates these regions. To do so the parameter values are fixed as follows (Note that it is not really necessary to choose eight different parameter values; see Appendix (a)):

$$\rho = \frac{1}{25};\ k_1 = 2;\ k_2 = \frac{3}{256};\ k_3 = 1;\ c_1 = \frac{3}{4};\ c_2 = \frac{5}{2};\ \alpha = 12;\ \delta = \frac{1}{4}. \qquad (10)$$

The numerical values of the steady states are listed in Table 1. For the following considerations it will suffice to retain the K-I-v – values of the first and of the third steady state, $\mathcal{K}_1 = \{0.58, 0.15, 0.0\}$ and $\mathcal{K}_3 = \{4.1, 1.0, 0.0\}$. These steady states are saddle points with transient oscillations.

====== Table 1 about here ======

3.3 Stable Invariant Manifolds

Figure 4 and 5 visualize the stable invariant manifolds of the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$. More precisely, these figures show projections of higher dimensional surfaces into the three dimensional state, namely the state-control space (K, I, v). In the following discussion of the figures we omit the phrase "...projection of...into the (K, I, v) space".

====== Figure 4 about here ======

====== Figure 5 about here ======

We determine these surfaces by computing the eigenvectors of the linearization of the canonical system in the steady states. Proposition 1 states that steady states $\mathcal{K}_1$ and $\mathcal{K}_3$ have exactly one conjugate complex pair of eigenvalues with negative real parts, respectively. We take the real and imaginary part of one of the eigenvectors, which correspond to eigenvalues with negative real parts. Which eigenvector is used, is not significant, because the eigenvectors are conjugate complex. These two vectors (real and imaginary part of one of the eigenvectors) define a linear approximation of the stable invariant manifold locally at the steady state (see, e.g., Guckenheimer and Holmes, 1983). With the aid of these two vectors we define an ellipse, which has the steady state as center and which is situated in an ε–ball around the steady state; we get this ellipse by normalizing these two vectors and by multiplying one by $\varepsilon\, cos\,\phi$ and the other by $\varepsilon\, sin\,\phi$, $\phi \in [0, 2\pi]$.

It is evident that this ellipse is situated in the linear approximation of the stable invariant manifold. Choosing starting points on this ellipse and going back in time (by multiplying the r.h.s. of the state equations by -1) we numerically compute an approximation of the stable manifold. The quality of this approximation depends on the value of ε and on the smoothness of the manifold in the ε–ball around the steady state.

In Figure 4 the shape of the stable manifold of the steady state $\mathcal{K}_1$ is plotted. The depicted trajectories converge to the steady state $\mathcal{K}_1$ located in the upper-right corner of the figure. The lines between the trajectories are drawn to form polygons so as to display the shape of the surface. We outline the position of the steady state $\mathcal{K}_3$ by a bold face bar (instead of a dot or circle), which crosses orthogonally the state plane at the steady state values of $\mathcal{K}_3$. We decided to choose this bold face bar to visualize the position of the steady state $\mathcal{K}_3$ relative to the stable manifold of the steady state $\mathcal{K}_1$. Now, it is easily seen that the stable manifold does not include steady state $\mathcal{K}_3$, since the stable manifold folds back far away before crossing the bold face bar.

In Figure 5 the shape of the stable manifold of the steady state $\mathcal{K}_3$ is plotted. The depicted trajectories converge to this steady state, which is located in the lower-right corner of the figure. We outline the position of the other steady state, now $\mathcal{K}_1$, by a bold face bar, which crosses orthogonally the state plane at the steady state values of $\mathcal{K}_1$. Here again, the stable manifold does not include the other steady state.

3.4 The DNS–curve

Emanating from the initial states we need to find a trajectory that converges to a steady state. This is equivalent to complementing the given initial state values with values for the co-state and/or control variables in a way that the resulting initial point lies on the stable manifold of that steady state, where the trajectory converges to. Then the value of the objective functional corresponding to this trajectory can be determined by dividing the Hamiltonian by the discount factor (see Michel, 1982).

So it is important to compute the stable invariant manifolds first. We have seen in Figures 4 and 5 that both stable invariant manifolds corresponding to $\mathcal{K}_1$ and $\mathcal{K}_3$ do not include the other long run equilibrium points $\mathcal{K}_3$ and $\mathcal{K}_1$, respectively. Hence, in this particular example it can never be optimal to start from one saddle point and converge to the other. Under the assumption that optimal trajectories will converge to one of the two steady states, it is obvious that there must be a division of the state space into two different regions of optimal convergence of the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$.

Next, we compute the set in the state space that separates these regions of optimal convergence. Right on this DNS–set the firm is indifferent between converging to either one of the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$, since the values of the objective functional of the two resulting trajectories are exactly the same. Here we solely rely on numerical evidence that this set is actually a curve, a DNS–curve. Here, it is left to be proven that this DNS–set is in fact a one-dimensional manifold.

Apparently this "indifference" curve is situated in an area of the state space, where it is possible (for a given set of initial state values) to choose a trajectory starting with these initial states and converging to the steady state $\mathcal{K}_1$ and to choose another trajectory starting with these initial state and converging to the steady state $\mathcal{K}_3$.

================ Figure 6 about here ==================

As we can see from Figure 6 the state space (K, I) is split into three regions:

Region I: for initial values in this region there exist no trajectories converging to the steady state $\mathcal{K}_3$, and thus $\mathcal{K}_1$ is the only candidate for the long run optimal equilibrium.

Region IIa + IIb: for initial values in this region there exist trajectories converging to the steady state $\mathcal{K}_3$, and trajectories converging to the steady state $\mathcal{K}_1$. Here both trajectories are candidates for representing the optimal solution.

Region III: for initial values in this region there exist no trajectories converging to the steady state $\mathcal{K}_1$, and thus $\mathcal{K}_3$ is the only candidate for the long run optimal equilibrium.

In order to specify the regions mentioned, for each unit square of the state space used in the scaling of Figure 6 we put a lattice of an array of 80 times 80 rectangular cells. Numerically, for each cell of the lattice we try to find trajectories starting inside the cell and converging to the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$. This is done indirectly by considering many starting points on the boundary of the above mentioned ellipses in ε-neighborhoods of $\mathcal{K}_1$ and $\mathcal{K}_3$ and going back in time. We mark the cells of the lattice that are only reached by those trajectories starting from $\mathcal{K}_1$ as belonging to Region I, the cells that are only reached by those trajectories starting from $\mathcal{K}_3$ as belonging to Region III, and the cells which are reached by those trajectories starting from $\mathcal{K}_1$, as well as those starting from $\mathcal{K}_3$ as belonging to Region II.

For each cell of the lattice in the Region II, we compare the values of the objective functionals for the trajectories converging to the steady state $\mathcal{K}_1$ and for the trajectories converging to the steady state $\mathcal{K}_3$. It turns out that for the cells in the Region IIa (see

Figure 6), it is better (in the sense of maximizing the objective functional) to converge to the steady state $\mathcal{K}_1$. On the other hand, for the cells in the Region IIb, it is better to converge to the steady state $\mathcal{K}_3$. The region of optimal convergence of the steady state $\mathcal{K}_1$ consists of the Regions I and IIa, the region of optimal convergence of the steady state $\mathcal{K}_3$ consists of the Regions IIb and III, whereas the boundary between the Regions IIa and IIb is the DNS–curve. It so happens that the unstable steady state $\mathcal{K}_2$ is situated in the interior of the Region IIa. For some other features of the solution the interested reader can see Appendix (b).

3.5 Economic Analysis

For the parameter values concerned (see (10)) the optimal solution consists of two steady states to which it can be optimal to converge in the long run. In the steady state with the larger capital stock the revenue is larger, but on the other hand more replacement investment has to be undertaken to remain in this steady state, which implies that the adjustment costs are larger too.

Both steady states have their own region of optimal convergence, whereas a DNS-curve forms the boundary between these regions. This DNS–curve consists of all points in the state space for which converging to each of the steady states leads to exactly the same value of the objective. Converging to the larger steady state requires an increase of the investment rate, while investments have to decrease in case the firm starts to approach the smaller steady state. This implies that exactly on this curve the firm's policy function is discontinuous: the control $v(K, I)$ is positive on the trajectory that approaches the larger steady state, while $v(K, I)$ is negative on the trajectory that will converge to the smaller steady state. Because the stable manifolds of the equilibria $\mathcal{K}_1$ and thus $\mathcal{K}_3$ are smooth, it implicitly follows that the feedback function $v(K, I)$ is continuous and smooth everywhere with the exception of the DNS–curve.

It thus depends on the initial levels of the capital stock and the investment rate to which steady state it is optimal to converge to. From Figure 6 it can be concluded that the DNS–curve is decreasing in the (K, I)–plane. From an economic point of view this can be explained as follows. In Figure 3 we see that for capital stock values between, say, 0.6 and 2.0, marginal revenue is low compared to marginal investment costs. Therefore, if the firm starts out with a low capital stock value and sufficiently low investment rate it is not profitable to enter a growth phase that passes this interval of capital stock values. This implies that convergence to the lower steady state is optimal for low values of investment and capital stock. Convergence to the larger steady state is optimal for sufficiently large initial values of capital stock and investment. This is especially caused by the fact that (1) changing the investment rate is costly (also in a negative direction), and (2) marginal revenue is large compared to marginal investment costs for capital stock values around 3 (see Figure 3).

The situation in Figure 6 occurs for a relatively large value of α, which is the parameter in the adjustment cost function for the rate of change of investment. For lower values of α the costs of $\dot{I}$ are less, so that vertical motions in the state space are less costly. Then it will turn out that only one of the stable steady states will remain optimal so that the DNS–curve disappears. For this model with the parameters chosen, numerical experiments show that reducing α leads to an upward movement of the DNS–curve, so that for a larger domain of initial values of capital stock and investment it becomes optimal to converge to the lower steady state. If α is sufficiently low, it will never be optimal to end up in the larger steady state. This can be explained by the fact that (for the parameters chosen) in the steady state $\mathcal{K}_1$ the per capita revenue is relatively high compared to the other equilibrium, which can be see in Figure 2.

4 Concluding Remarks

This paper considers two main features. First, and most important, in a two state variable optimal control model with two long run equilibria the location of a DNS–curve is numerically determined and the economic intuition is provided. The DNS–curve connects all points in the state space on which the decision maker is indifferent concerning to which of the two long run equilibria to converge to. While in Brock and Dechert (1983) the existence of a DNS–curve is proved, to our knowledge this contribution is the first one in which a DNS–curve is computed.

Second, our paper contributes to the literature of capital accumulation models. Especially the effects of increasing returns to scale for an intermediate interval of capital stock values are investigated. As such the paper extends the analysis of Davidson and

Harris (1981) to a two dimensional framework. Furthermore the concept of adjustment costs is refined by making changes in the investment rate costly.

Finally, we list some suggestions for future research. First, consider the work of Ladron-de-Guevara et al. (1999). Building upon some basic results on dynamic programming, Ladron-de-Guevara et al. (1999) present a new procedure for the characterization of optimal trajectories. To make their proof more transparent, the authors first basically restrict themselves to a single state variable. In general, however, their method of analysis which is based upon the construction of a "candidate" value function, applies to two-state models. In particular Ladron-de-Guevara et al. (1999) analyze a two-state endogenous growth model with physical and human capital in which leisure enters the utility function. By assuming that human capital does not affect the quality of leisure, while it influences production and investment, the analysis leads to multiple balanced growth paths. Using their method the authors show that unstable balanced growth paths with complex roots are non-optimal. Moreover, they are able to study continuity and discontinuity of the optimal policy function. The method developed by the authors should be of interest in related non-concave optimization framework. It would be a useful task to apply this method also to the analysis of the model described in the present paper.

Second, using the approach followed in this paper it must be possible to extend the analysis of two dimensional problems with non-concavities. In this way, the economic knowledge concerning the effects of increasing returns to scale can be increased considerably.

Appendix

(a) In fact the parameter space of our problem is only 5-dimensional. This can be seen by considering the following dimensionless transformation:

$$\tau = \delta t,\ k_2' = \sqrt[4]{k_2},\ \bar{\rho} = \frac{\rho}{\delta},\ a_1 = \frac{k_3}{k_1\sqrt{k_2'}},\ a_2 = \frac{c_1\delta}{k_1\sqrt{k_2'}},\ a_3 = \frac{c_2\delta^2}{k_1 k_2'\sqrt{k_2'}},\ a_4 = \frac{\alpha\delta^4}{k_1 k_2'\sqrt{k_2'}}.$$

With the transformed model variables

$$x(\tau) = k_2' K\left(\frac{\tau}{\delta}\right),\ y(\tau) = \frac{k_2'}{\delta} I\left(\frac{\tau}{\delta}\right),\ u(\tau) = \frac{k_2'}{\delta^2} v\left(\frac{\tau}{\delta}\right),$$

our problem can be written as:

$$\max_u \quad \int_0^\infty e^{-\bar{\rho}\tau}\left[\sqrt{x} - a_1\frac{x}{1+x^4} - a_2 y - \frac{a_3}{2}y^2 - \frac{a_4}{2}u^2\right] d\tau,$$

$$\text{s.t.} \quad \dot{x} = y - x,$$

$$\dot{y} = u.$$

With the specification (10) the transformed parameters are:

$$k_2' = 0.329\ \ \bar{\rho} = 0.160,\ a_1 = 0.872,\ a_2 = 0.163,\ a_3 = 0.414,\ a_4 = 0.124.$$

We would like to thank an anonymous referee for pointing this out to us.

(b) For this particular numerical example both stable invariant manifolds do not include the other steady state. For this model this property is necessary for the existence of a DNS–curve, because the maximized Hamiltonian is strictly convex in the co-state variables λ_1 and λ_2, and for any fixed states K, I it attains its minimum at the $\dot{K} = \dot{I} = 0$ surface. Since the steady states essentially lie on this surface, it follows that, if a stable invariant manifold included the other steady state, it would never be optimal to remain in that steady state (motivation: the objective functional can be determined by dividing the Hamiltonian by the discount factor).

Regardless of the fact that the stable invariant manifold folds in this example, optimal homoclinic cycles are not possible. Since a homoclinic cycle has to pass the fold of the stable invariant manifold, its projection onto the state space traverses Region II and has to cross the DNS–curve before it converges. However, we have computed the stable invariant manifolds up to the folds, and we see that the DNS-curve splits Region II into two different regions of optimal convergence (Regions IIa and IIb). Then the existence of a homoclinic cycle would contradict the Principle of Optimality.

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List of Tables

	K	I	v	λ_1	λ_2
$\mathcal{K}_1$	0.6	0.1	0.0	1.1	0.0
$\mathcal{K}_2$	2.3	0.6	0.0	2.2	0.0
$\mathcal{K}_3$	4.1	1.0	0.0	3.3	0.0

Table 1: Numerical values of the steady states

Figure Captions

Figure 1: The inverse demand function $P(Q)$ for $\gamma_1 = 2, \gamma_2 = \frac{3}{256}$, and $\gamma_3 = 1$. In the segment between $Q \approx 2.2$ and $Q \approx 4.9$ network externalities lead to an increasing shape.

Figure 2: The firm's revenue function with the inverse demand function $P(Q)$ from Figure 1 and $\beta = 1$ (i.e. $k_1 = 2, k_2 = \frac{3}{256}$, and $k_3 = 1$).

Figure 3: Three steady states satisfy $R'(K)/(\rho+\delta) = c'(\delta K)$. The points of intersection of $R'(K)/(\rho+\delta)$ (solid) and $c'(\delta K)$ (dotted) are the steady states.

Figure 4: Stable invariant manifold of the steady state $\mathcal{K}_1$.

Figure 5: Stable invariant manifold of the steady state $\mathcal{K}_3$.

Figure 6: Regions of (optimal) convergence and DNS–curve. The bold line separating Regions IIa and IIb is the DNS–curve. The typical motions are illustrated by dashed

(stable invariant manifold of the steady state $\mathcal{K}_1$) and by dotted curves (stable invariant manifold of the steady state $\mathcal{K}_3$).

A DNS-curve in a two state capital accumulation model
by Haunschmied, Kort, Hartl, Feichtinger
JEDC #4000

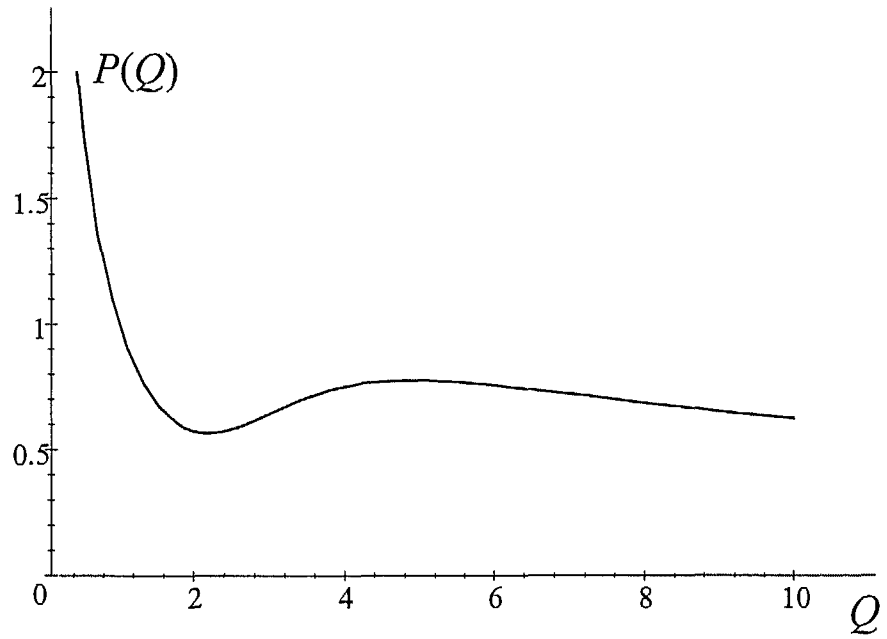

Figure 1

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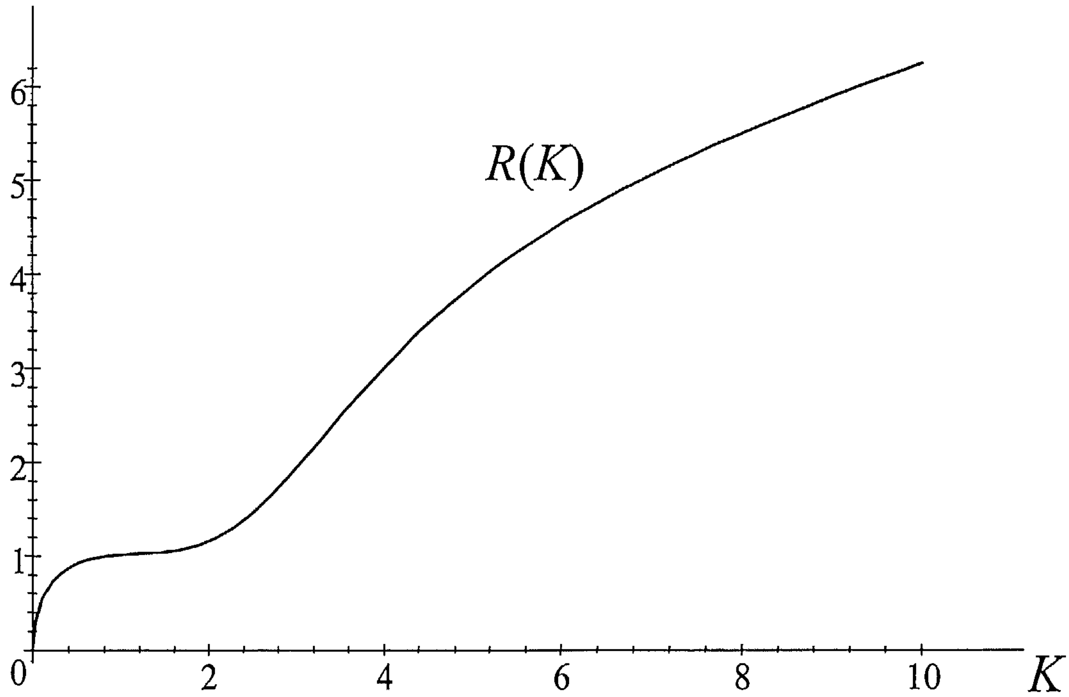

Figure 2

A DNS-curve in a two state capital accumulation model
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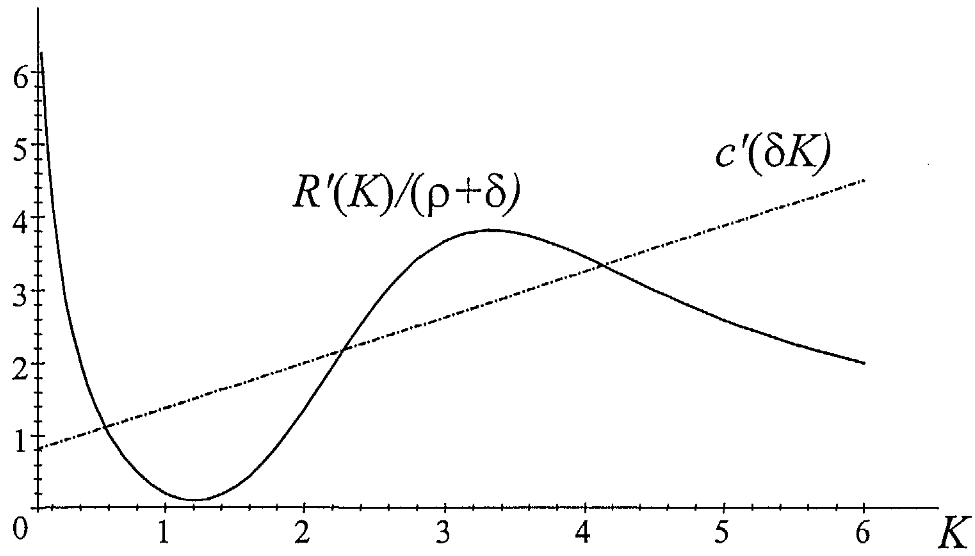

Figure 3

A DNS - curve in a two state capital accumulation model : a numerical analysis

by Haunschmied, Kort, Hartl, and Feichtinger

JEDC #4000

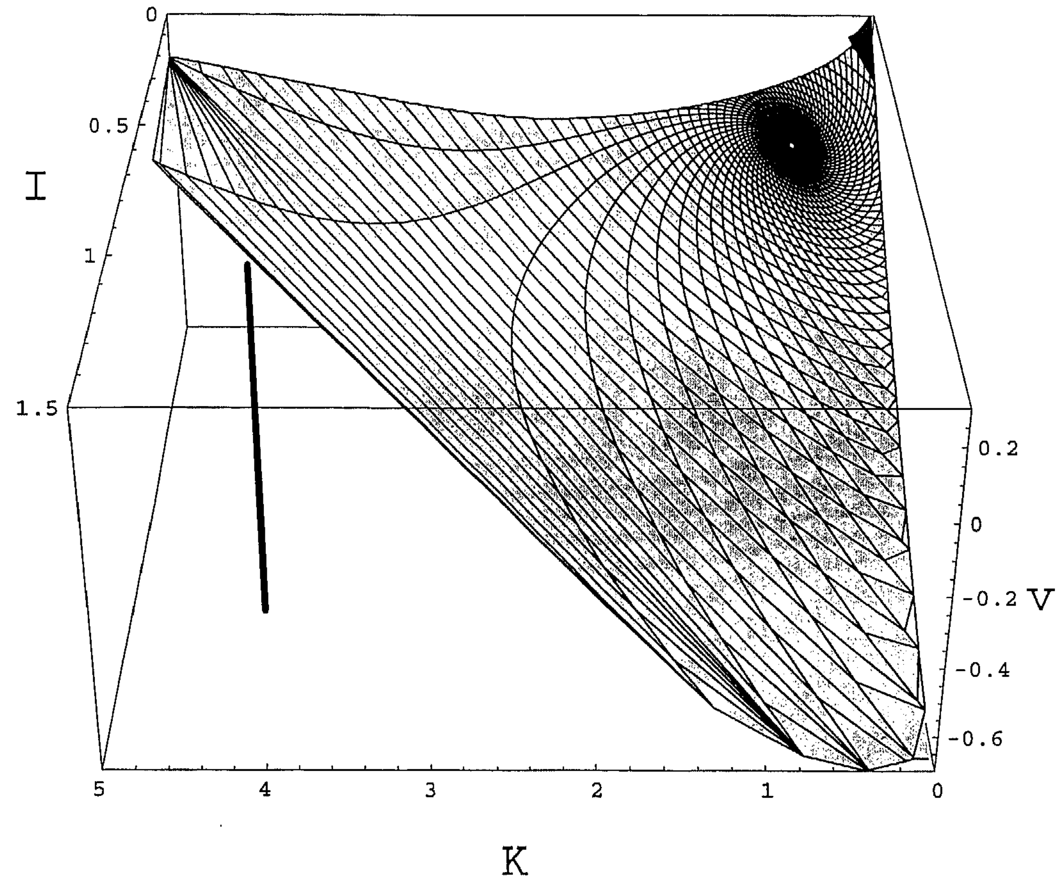

Figure 4

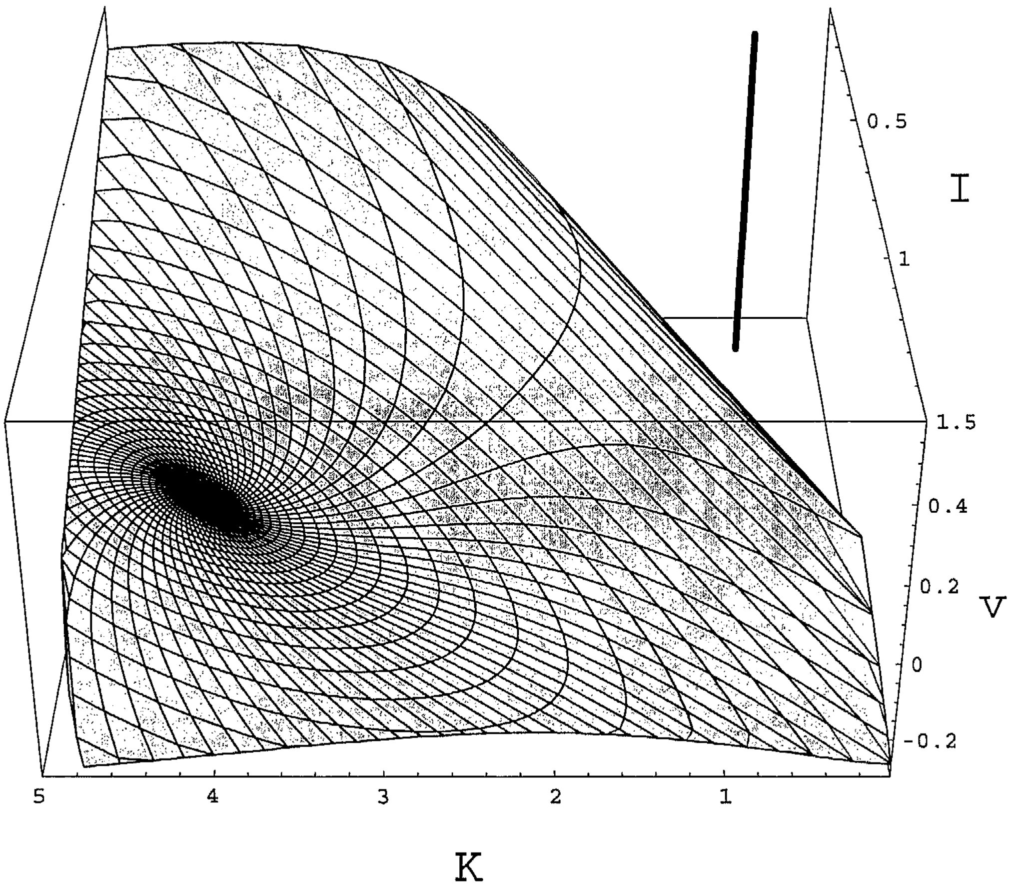

Figure 5

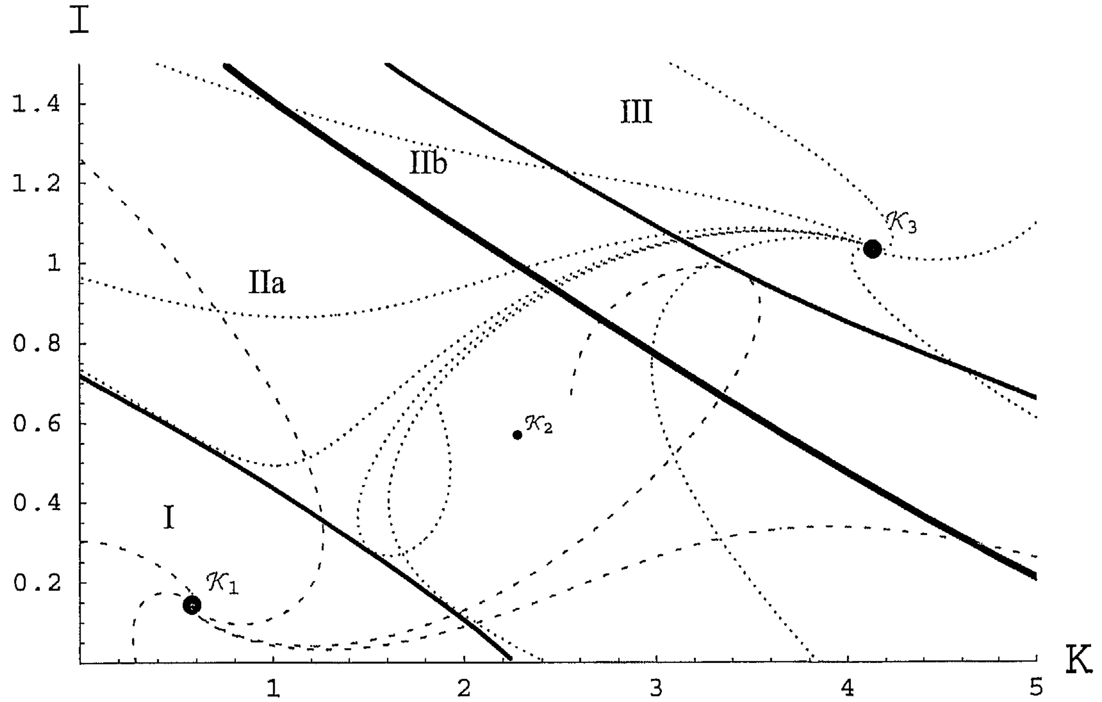
I
1.4
1.2
1
0.8
0.6
0.4
0.2
III
IIb
IIa
I
$\mathcal{K}_3$
$\mathcal{K}_2$
$\mathcal{K}_1$
1
2
3
4
5
K

Figure 6