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Vol. 78, No. 1, January 1998
Printed in Belgium

Protective and Prudent Behaviour in Games

Gloria Fiestras-Janeiro

Departamento de Estadística e Investigación Operativa, Universidad de Vigo, Vigo, Spain

and

Peter Borm and Freek van Megen

Department of Econometrics and CentER for Economic Research, Tilburg University, Tilburg, The Netherlands

Received January 6, 1996; revised June 16, 1997

This paper introduces the notions of protective and prudent equilibria in the context of finite games in strategic form. It turns out that for games both notions are in fact equivalent. Moreover, it is shown that for matrix games the set of protective equilibria equals the set of proper equilibria. *Journal of Economic Literature* Classification Number: C72. © 1998 Academic Press

1. INTRODUCTION

The notion of protective behaviour was first introduced in [1] in the context of implementation of social choice functions for social choice situations with a finite number of alternatives. Here, an agent behaves in a protective way if he reveals his preferences so as to protect himself from the worst eventuality as far as possible. This concept is closely related to the notion of prudent behaviour as formulated in [7]. The main difference is that the latter assumes that each agent considers all possible preference profiles of other agents equally likely, whereas the former does not.

Protective behaviour as a binary decision criterium on the set of all finite-dimensional vectors of real numbers is axiomatically characterized in [2]. Here, also an axiomatic comparison with the maximin decision criterium is offered. An application of protective behaviour towards matching models is presented in [3].

0022-0531/98 $25.00

In this paper we want to proceed on this line of research by considering protective and prudent behaviour in mixed extensions of finite games in strategic form. First we define both protective and prudent strategies in a strategic form game based on the ideas of these two concepts in social choice situations. It is shown that the sets of prudent and protective strategies for each player in a strategic form game coincide. A protective equilibrium is defined to be a strategy combination that consists of a protective strategy for each player. Existence of protective equilibria is shown and it is seen that each protective strategy is also a maximin strategy. So, in particular, for matrix games, protective equilibria offer a refinement of the Nash equilibrium concept. Moreover, it is proved that each protective strategy is a Dresher optimal strategy ([5]), and conversely. Hence, for matrix games, protective equilibria coincide with proper equilibria à la Myerson ([8], cf. [10]), and the nucleolus introduced by [9] for matrix games.

2. PROTECTIVE AND PRUDENT EQUILIBRIA

Let $N=\{1, ..., n\}$ denote the set of players. A finite game Γ in strategic form with player set N is represented by $\Gamma=\langle\{S_i\}_{i\in N}, \{K_i\}_{i\in N}\rangle$ where for each $i\in N$, the finite set S_i denotes the set of pure strategies for player i and $K_i\colon \prod_{j=1}^n S_j\to\mathbb{R}$ denotes the payoff function for player i.

Considering mixed strategies we let $\Delta_j=\Delta(S_j)$ represent the set of all probability measures on S_j for all $j\in N$. The payoff functions $\{K_i\}_{i\in N}$ are extended to the set $\prod_{j\in N}\Delta_j$ of all mixed strategies combinations in the obvious way.

For notational convenience we set, for $i\in N$,

$$S=\prod_{j\in N} S_j,\quad \Delta=\prod_{j\in N}\Delta_j,\quad S_{-i}=\prod_{j\in N\setminus\{i\}} S_j\quad \text{and}\quad \Delta_{-i}=\prod_{j\in N\setminus\{i\}}\Delta_j.$$

A pure strategy combination is denoted by $s\in S$, a mixed strategy combination by $\sigma\in\Delta$. Sometimes, given $i\in N$, we will write $s=(s_{-i}, s_i)$ and $\sigma=(\sigma_{-i}, \sigma_i)$.

A mixed strategy combination $\tilde{\sigma}$ is called a Nash equilibrium of Γ if for each player $i\in N$, $K_i(\tilde{\sigma})\geqslant K_i(\tilde{\sigma}_{-i}, \sigma_i)$ for all $\sigma_i\in\Delta_i$. It is well known that every finite game in strategic form has at least one combination of mixed strategies which is a Nash equilibrium.

A mixed strategy $\tilde{\sigma}_i\in\Delta_i$ is called a maximin strategy for player i if

$$\min_{\sigma_{-i}\in\Delta_{-i}} K_i(\sigma_{-i}, \tilde{\sigma}_i)=\max_{\sigma_i\in\Delta_i}\min_{\sigma_{-i}\in\Delta_{-i}} K_i(\sigma_{-i}, \sigma_i).$$

For each player $i \in N$, the set Δ_i is compact and closed, and the payoff-function K_i is continuous. These properties guarantee the existence of maximin strategy combinations for every finite person game in strategic form.

As maximin strategies assure some payoff to a player, we will see that the notion of protectiveness and prudentness we introduce below, offers a possibility to select interesting strategies out of this set.

DEFINITION 2.1. Let $\Gamma = \langle \{S_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle$. Let $i \in N$ and $\sigma_i \in \Delta_i$. Recursively, we define $a_i^r(\sigma_i) \in \mathbb{R}$ and $S_{-i}^r(\sigma_i) \subset S_{-i}$ by

(i) for $r = 1$,

$$a_i^1(\sigma_i) = \min\{K_i(s_{-i}, \sigma_i) \mid s_{-i} \in S_{-i}\}$$
$$S_{-i}^1(\sigma_i) = \{s_{-i} \in S_{-i} \mid K_i(s_{-i}, \sigma_i) = a_i^1(\sigma_i)\},$$

(ii) for $r > 1$,

$$a_i^r(\sigma_i) = \min\left\{K_i(s_{-i}, \sigma_i) \,\middle|\, s_{-i} \in S_{-i} \middle\backslash \bigcup_{k=1}^{r-1} S_{-i}^k(\sigma_i)\right\}$$
$$S_{-i}^r(\sigma_i) = \{s_{-i} \in S_{-i} \mid K_i(s_{-i}, \sigma_i) = a_i^r(\sigma_i)\}.$$

DEFINITION 2.2. Let Γ be a finite game in strategic form. Let $i \in N$ and $\sigma_i, \tilde{\sigma}_i \in \Delta_i$. We say that $\tilde{\sigma}_i$ protectively dominates σ_i, in notation $\tilde{\sigma}_i \succ_{pro} \sigma_i$, if there exists an $l \in \mathbb{N}$, such that

(i) $a_i^r(\sigma_i) = a_i^r(\tilde{\sigma}_i)$ and $S_{-i}^r(\sigma_i) = S_{-i}^r(\tilde{\sigma}_i)$ for all $r \in \mathbb{N}$, $r < l$, and

(ii) $a_i^l(\sigma_i) < a_i^l(\tilde{\sigma}_i)$ or both $a_i^l(\sigma_i) = a_i^l(\tilde{\sigma}_i)$ and $S_{-i}^l(\tilde{\sigma}_i) \subsetneq S_{-i}^l(\sigma_i)$.

A mixed strategy $\hat{\sigma}_i \in \Delta_i$ is called protective for player i in Γ if it is undominated w.r.t. the protective dominance relation, i.e., if there does not exist a mixed strategy $\tilde{\sigma}_i \in \Delta_i$ such that $\tilde{\sigma}_i \succ_{pro} \hat{\sigma}_i$.

A combination of mixed strategies σ is called a protective equilibrium of Γ if σ_i is a protective strategy for player i for all $i \in N$.

So, a strategy is protective if it consecutively maximizes the worst possible payoff, thereby taking into account the sets of pure strategy combinations of the opponents which yield that minimal amount. One readily verifies that each protective strategy is maximin.

Even though the protective dominance relation need not be complete, the next lemma reveals that a protective strategy is dominant, up to payoff equivalence, with respect to the $\succ_{pro}$ relation.

LEMMA 2.3. *Let $\Gamma = \langle \{S_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle$. Let $\tilde{\sigma}_i \in \Delta_i$ be a protective strategy of player $i \in N$ and let $\sigma_i \in \Delta_i$ be an arbitrary mixed strategy for* *player i. Then, either $\tilde{\sigma}_i$ and σ_i are payoff equivalent for player i or $\tilde{\sigma}_i \succ_{pro} \sigma_i$.*

Proof. Assume that $\tilde{\sigma}_i$ and σ_i are not payoff equivalent and suppose that $\tilde{\sigma}_i$ does not protectively dominate σ_i. Taking into account that $\tilde{\sigma}_i$ is a protective strategy of player i, σ_i does not protectively dominate $\tilde{\sigma}_i$. Then, according to Definition 2.2, there exists an $l \in \mathbb{N}$ such that

(i) $a_i^r(\sigma_i) = a_i^r(\tilde{\sigma}_i)$ and $S_{-i}^r(\sigma_i) = S_{-i}^r(\tilde{\sigma}_i)$ for all $r \in \mathbb{N}$, $r < l$, and

(ii) $a_i^l(\sigma_i) = a_i^l(\tilde{\sigma}_i)$, $S_{-i}^l(\tilde{\sigma}_i) \backslash S_{-i}^l(\sigma_i) \neq \varnothing$ and $S_{-i}^l(\sigma_i) \backslash S_{-i}^l(\tilde{\sigma}_i) \neq \varnothing$.

Let $0 < \alpha < 1$ and let $\hat{\sigma}_i = \alpha\tilde{\sigma}_i + (1-\alpha)\sigma_i \in \Delta_i$ with the obvious interpretation. We will prove that $\hat{\sigma}_i \succ_{pro} \tilde{\sigma}_i$.

Clearly, from (i), $a_i^r(\sigma_i) = a_i^r(\tilde{\sigma}_i) = a_i^r(\hat{\sigma}_i)$ and $S_{-i}^r(\tilde{\sigma}_i) = S_{-i}^r(\sigma_i) \subset S_{-i}^r(\hat{\sigma}_i)$, for all $r < l$. One can even show that $S_{-i}^r(\tilde{\sigma}_i) = S_{-i}^r(\sigma_i) = S_{-i}^r(\hat{\sigma}_i)$ for all $r < l$.

Next, we will show that either $a_i^l(\hat{\sigma}_i) > a_i^l(\tilde{\sigma})$ or both $S_{-i}^l(\hat{\sigma}_i) \subsetneqq S_{-i}^l(\tilde{\sigma}_i)$ and $a_i^l(\hat{\sigma}_i) = a_i^l(\tilde{\sigma}_i)$. Consider the following two cases:

(a) Let $S_{-i}^l(\tilde{\sigma}_i) \cap S_{-i}^l(\sigma_i) = \varnothing$. We will show that $a_i^l(\hat{\sigma}_i) > a_i^l(\tilde{\sigma}_i)$. Take $\tilde{s}_{-i} \in S_{-i}^l(\tilde{\sigma}_i)$. Then, there is an $r > l$ such that $\tilde{s}_{-i} \in S_{-i}^r(\sigma_i)$ and so,

$$K_i(\tilde{s}_{-i}, \sigma_i) > K_i(\tilde{s}_{-i}, \tilde{\sigma}_i) = a_i^l(\sigma_i) = a_i^l(\tilde{\sigma}_i).$$

Hence,

$$K_i(\tilde{s}_{-i}, \hat{\sigma}_i) > K_i(\tilde{s}_{-i}, \tilde{\sigma}_i) = a_i^l(\tilde{\sigma}_i).$$

For all $\bar{s}_{-i} \in S_{-i}^t(\tilde{\sigma}_i)$ for some $t > l$, it holds that

$$K_i(\bar{s}_{-i}, \tilde{\sigma}_i) > a_i^l(\tilde{\sigma}_i) \qquad \text{and} \qquad K_i(\bar{s}_{-i}, \sigma_i) \geqslant a_i^l(\sigma_i) = a_i^l(\tilde{\sigma}_i)$$

and, hence, $K_i(\bar{s}_{-i}, \hat{\sigma}_i) > a_i^l(\tilde{\sigma}_i)$. We may conclude that $a_i^l(\hat{\sigma}_i) > a_i^l(\tilde{\sigma}_i)$.

(b) Let $S_{-i}^l(\tilde{\sigma}_i) \cap S_{-i}^l(\sigma_i) \neq \varnothing$.
Clearly, for every $\bar{s}_{-i} \in S_{-i}^l(\tilde{\sigma}_i) \cap S_{-i}^l(\sigma_i)$

$$a_i^l(\tilde{\sigma}_i) = K_i(\bar{s}_{-i}, \tilde{\sigma}) = K_i(\bar{s}_{-i}, \hat{\sigma}_i) = K_i(\bar{s}_{-i}, \sigma_i) = a_i^l(\sigma_i),$$

which implies that $a_i^l(\hat{\sigma}_i) = a_i^l(\tilde{\sigma}_i)$.

Furthermore, for every $s_{-i} \in \Delta_{-i} \backslash (\bigcup_{r=1}^{l-1} S_{-i}^r(\sigma_i) \cup (S_{-i}^l(\tilde{\sigma}_i) \cap S_{-i}^l(\sigma_i)))$, either $K_i(s_{-i}, \tilde{\sigma}_i) > a_i^l(\tilde{\sigma}_i)$ or $K_i(s_{-i}, \sigma_i) > a_i^l(\sigma_i)$ and, consequently, $K_i(s_{-i}, \hat{\sigma}_i) > a_i^l(\tilde{\sigma}_i) = a_i^l(\hat{\sigma}_i)$ and $S_{-i}^l(\hat{\sigma}_i) = S_{-i}^l(\tilde{\sigma}_i) \cap S_{-i}^l(\sigma_i) \subsetneqq S_{-i}^l(\tilde{\sigma}_i)$.

In both cases $\hat{\sigma}_i \succ_{pro} \tilde{\sigma}_i$. A contradiction results and the assertation of the theorem holds. ▮

Next we define the prudent domination criterium and prudent strategies. The main difference between prudent and protective domination can be described as follows. Even though both criteria compare payoff levels, the former only compares, for each player, the cardinality of the sets of pure strategy combinations of the opponents where those payoff levels are achieved instead of the inclusion relation used in the latter. Formally, we have

DEFINITION 2.4. Let $\Gamma = \langle \{S_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle$. Let $i \in N$ and $\sigma_i, \tilde{\sigma}_i \in \Delta_i$. We say that $\tilde{\sigma}_i$ prudently dominates σ_i in notation $\tilde{\sigma}_i \succ_{pru} \sigma_i$, if there exists $l \in \mathbb{N}$ such that

(i) $a_i^r(\sigma_i) = a_i^r(\tilde{\sigma}_i)$ and $|S_{-i}^r(\sigma_i)| = |S_{-i}^r(\tilde{\sigma}_i)|$ for all $r \in \mathbb{N}$, $r < l$, and

(ii) $a_i^l(\sigma_i) < a_i^l(\tilde{\sigma}_i)$ or both $a_i^l(\sigma_i) = a_i^l(\tilde{\sigma}_i)$ and $|S_{-i}^l(\tilde{\sigma}_i)| < |S_{-i}^l(\sigma_i)|$.

A mixed strategy $\hat{\sigma}_i \in \Delta_i$ is called prudent for player i in Γ if it is undominated w.r.t. the prudent dominance relation, i.e., if there does not exist any mixed strategy $\tilde{\sigma}_i \in \Delta_i$ such that $\tilde{\sigma}_i \succ_{pru} \hat{\sigma}_i$.

Clearly, if a strategy of player i is prudent, then it is also protective because from the Definitions 2.2 and 2.4 it follows that $\hat{\sigma}_i \succ_{pro} \sigma_i$ implies $\hat{\sigma}_i \succ_{pru} \sigma_i$.

As a consequence of the Lemma 2.3, also the converse holds, as is stated in

THEOREM 2.5. *In a finite strategic form game a mixed strategy is protective if and only if it is prudent.*

Proof. Let $\tilde{\sigma}_i$ be a protective strategy of player i and suppose that $\tilde{\sigma}$ is not prudent. Then one can find a strategy σ_i such that $\sigma_i \succ_{pru} \tilde{\sigma}_i$. However, this implies that σ_i and $\tilde{\sigma}_i$ are not payoff equivalent and so, using Lemma 2.3, we would find that $\tilde{\sigma}_i \succ_{pro} \sigma_i$ and consequently also $\tilde{\sigma}_i \succ_{pru} \sigma_i$, which establishes a contradiction. ▌

Existence of prudent and protective equilibria is guaranteed:

THEOREM 2.6. *Every finite game in strategic form has at least one prudent (and hence protective) equilibrium.*

Proof. We prove this result in a constructive way finding the set of prudent strategies of a finite game in strategic form of an arbitrary player. In this manner, using Theorem 2.5, we also obtain the set of all protective strategies. Let $\Gamma = \langle \{S_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle$. We define for each $i \in N$,

$$M_i^1 := \{\hat{\sigma}_i \in \Delta_i \mid a_i^1(\hat{\sigma}_i) = \max\{a_i^1(\sigma_i) \mid \sigma_i \in \Delta_i\}\}$$

$$P_i^1 := \{\hat{\sigma}_i \in M_i^1 \mid |S_{-i}^1(\hat{\sigma}_i)| \leqslant |S_{-i}^1(\sigma_i)| \text{ for all } \sigma_i \in M_i^1\}$$

and for $r > 1$, we define

$$M_i^r := \{\hat{\sigma}_i \in P_i^{r-1} \mid a_i^r(\hat{\sigma}_i) = \max\{a_i^r(\sigma_i) \mid \sigma_i \in M_i^{r-1}\}\}$$

$$P_i^r := \{\hat{\sigma}_i \in M_i^r \mid |S_{-i}^r(\hat{\sigma}_i)| \leqslant |S_{-i}^r(\sigma_i)| \text{ for all } \sigma_i \in M_i^r\}.$$

Note that M_i^1 is the set of maximin strategies of player i and that $a_i^r(\sigma_i) = \infty$ and $|S_{-i}^r(\sigma_i)| = 0$ if $\bigcup_{k=1}^{r-1} S_{-i}^k(\sigma_i) = S_{-i}$.

Clearly, $M_i^r \neq \varnothing$ and $P_i^r \neq \varnothing$ for all $r \in \mathbb{N}$. Since $|S_{-i}^r(\bar{\sigma}_i)| = |S_{-i}^r(\tilde{\sigma}_i)|$ for all $\bar{\sigma}_i, \hat{\sigma}_i \in P_i^r$ and $r \geqslant 1$, and S_{-i} is a finite set, we can take the smallest $t \in \mathbb{N}$ such that $P_i^t = P_i^r$ for all $r \geqslant t$. By definition, P_i^t precisely contains all prudent strategies of player i in the game Γ. ∎

3. PROTECTIVE BEHAVIOUR IN MATRIX GAMES

In this section we consider finite zero-sum games $A = \langle S_1, S_2, K, -K\rangle$ in strategic form with the payoff function K for player 1 determined by an $m \times n$ matrix A in the following way: $K(p, q) := pAq$ for all $p \in \Delta_1$ and $q \in \Delta_2$.

Dresher [5] proposed a criterion to select Nash equilibria of a matrix game based on the assumption that each player follows a conservative plan of action and tries to maximize the minimum gain resulting from the opponent's deviations. For describing the Dresher procedure, we recall some basic facts of matrix games.

Let $A = \langle S_1, S_2, K, -K\rangle$ be an $m \times n$ matrix game where $S_1 = \{e_1, ..., e_m\}$ and $S_2 = \{f_1, ..., f_n\}$. Its value, $v(A)$, is given by

$$v(A) := \max_{p \in \Delta_1} \min_{1 \leqslant j \leqslant n} pAf_j = \min_{q \in \Delta_2} \max_{1 \leqslant i \leqslant m} e_iAq.$$

The sets of optimal strategies for player 1 and 2 are given by the polytopes

$$O_1(A) := \{p \in \Delta_1 \mid pAf_j \geqslant v(A) \text{ for all } j \in \{1, ..., n\}\}$$

and

$$O_2(A) := \{q \in \Delta_2 \mid e_iAq \leqslant v(A) \text{ for all } i \in \{1, ..., m\}\}.$$

Furthermore we define the carrier of a strategy $p \in \Delta_1$ by

$$C_1(p) := \{e_i \in \{1, ..., m\} \mid p(e_i) > 0\},$$

the carrier of the set of optimal strategies by

$$C_1(A) := \bigcup_{p \in O_1(A)} C_1(p)$$

and the equalizer set by

$$E_1(A) := \{e_i \in \{1, ..., m\} \mid e_i A q = v(A) \text{ for all } q \in O_2(A)\}.$$

The sets $C_2(q)$, $C_2(A)$ and $E_2(A)$ are defined in an analogous way. It is well known that $C_1(A) = E_1(A)$ and $C_2(A) = E_2(A)$ (cf. [4] and [6]).

Dresher [5] constructs a sequence of matrix games A^k in the following way. Let $A^1 = A$. In the game A^2 player 1 has as pure strategy set $S_1(2)$ the extreme points of $O_1(A)$ and the set of pure strategies for player 2 is given by $S_2(2) = S_2 \backslash C_2(A)$. The payoff functions in A^2 are just the restrictions of the original ones.

If A^{k-1} is defined and $C_2(A^{k-1}) \neq S_2(k-1)$, then the set $S_1(k)$ of pure strategies of player 1 in A^k is constituted by the extreme points of $O_1(A^{k-1})$ and the set $S_2(k)$ of pure strategies of player 2 is $S_2(k-1)\backslash C_2(A^{k-1})$. Clearly, after a finite number of steps t, A^t has been defined and $C_2(A^t) = S_2(t)$. Then $O_1(A^t)$ is called the set of D-optimal strategies of player 1 denoted by $D_1(A)$.

In a similar way, one defines the set $D_2(A)$ of D-optimal strategies of player 2.

The set of proper equilibria of a matrix game was characterized in [10] as the set of combinations of D-optimal strategies of the game. In [9] has been proved that the nucleolus of a matrix game equals its set of proper equilibria. We offer another characterization using the concept of protective equilibria.

THEOREM 3.1. *For every finite matrix game the set of protective equilibria coincides with the set of proper equilibria.*

Proof. Let $A = \langle S_1, S_2, K, -K \rangle$ be an $m \times n$ matrix game with $S_1 = \{e_1, ..., e_m\}$ and $S_2 = \{f_1, ..., f_n\}$.

Let $\bar{p}$ be a protective strategy of player 1 in A. We will prove that $\bar{p}$ is a D-optimal strategy. Clearly, $\bar{p} \in O_1(A)$. Assume that $\bar{p} \in O_1(A^k)$ for all $k \in \{1, ..., t-1\}$ for some $t \geqslant 2$ where A^k is as described in the Dresher procedure. It suffices to prove that $\bar{p} \in O_1(A^t)$ if $S_2(t) = S_2(t-1)\backslash C_2(A^{t-1}) \neq \varnothing$.

Let $\tilde{p} \in O_1(A^t)$. If $\bar{p}$ and $\tilde{p}$ are payoff equivalent, the proof is finished. Otherwise, according to Lemma 2.2 we have that $\bar{p} \succ_{pro} \tilde{p}$. We need to show that

$$\min_{f_j \in S_2(t)} \bar{p} A f_j = \min_{f_j \in S_2(t)} \tilde{p} A f_j.$$

We know that

$$\min_{f_j \in S_2(k)} \bar{p}Af_j = \min_{f_j \in S_2(k)} \tilde{p}Af_j$$

for all $k \in \{1, ..., t-1\}$ and

$$\min_{f_j \in S_2(t)} \bar{p}Af_j \leqslant \min_{f_j \in S_2(t)} \tilde{p}Af_j$$

Suppose we have a strict inequality. Then, since $\bar{p} \succ_{pro} \tilde{p}$ there is a game A^r with $r \in \{1, ..., t-1\}$ and an integer l such that $v(A^r) = a_r^l(\bar{p}) = a_1^l(\tilde{p})$, $S^l_{-1}(\bar{p}) \subsetneq S^l_{-1}(\tilde{p})$ and for all $k < l$, $a_1^k(\bar{p}) = a_1^k(\tilde{p})$ and $S^k_{-1}(\bar{p}) = S^k_{-1}(\tilde{p})$.

Choose $f_s \in S^l_{-1}(\tilde{p}) \backslash S^l_{-1}(\bar{p})$. By definition, $\bar{p}Af_s > \tilde{p}Af_s$ and, hence, $f_s \in S_2(t)$. For, suppose $f_s \notin S_2(t)$. Then $f_s \in C_2(A^k) = E_2(A^k)$ for some $k \in \{1, ..., t-1\}$ and, consequently, since $\bar{p} \in O_1(A^k)$ and $\tilde{p} \in O_1(A^k)$, it would hold that $\bar{p}Af_s = \tilde{p}Af_s$, which is a contradiction. We may conclude that

$$\min_{f_j \in S_2(t)} \tilde{p}Af_j = \tilde{p}Af_s = a_1^l(\tilde{p}) = a_1^l(\bar{p}) \leqslant \min_{f_j \in S_2(t)} \bar{p}Af_j$$

and, there is equality. Hence, every protective strategy of player 1 is D-optimal. We can proceed in an analogous way to prove that every protective strategy of player 2 is also a D-optimal strategy.

Since each combination of D-optimal strategies is a proper equilibrium, it follows that each protective equilibrium is also proper. Moreover, since A has at least one protective equilibrium, this game has a protective D-optimal combination of strategies. Taking into account that all D-optimal strategies are payoff equivalent [10], it follows that all proper equilibria of this game are protective. ∎

ACKNOWLEDGMENTS

Gloria Fiestras-Janeiro acknowledges the hospitality of the Department of Econometrics and the CentER for Economic Research at the Tilburg University as well as financial support from the Universidad de Vigo and from Xunta de Galicia Project XUGA20702B93.

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium