Linear Transformation of Products: Games and Economies

JUDITH TIMMER* PETER BORM* JEROEN SUIJS*

Abstract

In this paper we introduce situations involving the linear transformation of products, in short: LTP situations. LTP situations are production situations where each producer has a single linear transformation technique. We show that the corresponding LTP games are totally balanced. Next, we relate LTP situations to exchange economies in two different ways and we prove the existence of an equilibrium in these economies. Finally, we extend the LTP situation to one where a producer may have more than one linear transformation technique. We show that each totally balanced game with nonnegative values is an extended LTP game.

Journal of Economic Literature Classification Number: C71.

1991 Mathematics Subject Classification Number: 90A15, 90D12.

Keywords: linear transformation, cooperative games, economies, price equilibria.

1 Introduction

In OWEN (1975) linear production situations were introduced. These are production situations where each producer can use a pre-determined set of linear production techniques to produce goods. Each producer is endowed with a bundle of resources. A restriction of this model is that each production process can have only one output good, so, this model excludes linear production processes with by-products. In this paper we introduce situations involving the linear transformation of products (LTP situations) to deal with production techniques with at least one output good. We define the set of products to be the set of all goods including resources. In an LTP situation we have a set of producers and, for the moment, we assume that each producer controls only one transformation technique. Later, we will extend this to situations where each producer can control more than one linear transformation technique. Each producer owns a bundle of resource goods which he can use in his transformation

*CentER and Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands; e-mail: J.B.Timmer@kub.nl

(production) process or which he can sell directly on the market. The outcome of the transformation process, the produced goods, will also be sold on the market. The goal of each producer is to maximize his profit given his transformation technique, resource bundle and market prices.

The organization of this paper is as follows. First we give a formal description of LTP situations. Next we introduce LTP games and we show that these games are totally balanced. Then we relate LTP situations to exchange economies in two different ways. In the first model, producers may exchange their resources at endogenous exchange prices before production starts. Each producer wants to maximize his profit given the exogenous market prices. In the second model, producers may exchange their production at endogenous exchange prices. After this exchange, they can, once more, apply their transformation technique and sell the resulting bundle of goods at the exogenous market prices. Again each producer wants to maximize his profit. We show that under certain conditions both models allow for a market equilibrium. A new characterization of totally balanced games with nonnegative values is given next by means of extended LTP situations in which each producer can control more than one linear transformation technique. Existing characterizations of totally balanced games in the literature are the class of flow games by KALAI and ZEMEL (1982) and the class of market games, by SHAPLEY and SHUBIK (1969). A characterization of totally balanced games with nonnegative values is the class of linear production games (cf. OWEN (1975) and TIJS (1995)). Finally, an appendix contains the proofs that were omitted in the text.

2 LTP situations

We start this section with an example. In the chemical industry, a refinery process is used to manufacture from crude oil other, more useful, products like gasoline, kerosene and petroleum solvents. For example, suppose that 500 barrels of gasoline, 300 barrels of kerosene and 100 barrels of petroleum solvents can be manufactured from 1000 barrels of crude oil requiring 100 hours of labour. Assuming that the production process is linear, this production technique is represented by the following vector.

$$a = \begin{bmatrix} -100 \\ -1000 \\ 500 \\ 300 \\ 100 \end{bmatrix} \begin{array}{l} \text{labour hours} \\ \text{crude oil} \\ \text{gasoline} \\ \text{kerosene} \\ \text{petroleum solvents} \end{array}$$

So, labour and crude oil are the input goods in this production process while gasoline, kerosene and petroleum solvents are the output goods. Since the production technique is linear, any nonnegative multiple of a is a possible production technique. The value of this nonnegative multiplier is called the activity level. For instance, if a firm operates at activity level 3, she can manufacture 1500 barrels of gasoline, 900 barrels of kerosene and 300 barrels of petroleum solvents from 300 hours of labour and 3000 barrels of crude oil. Obviously, the activity level of a firm is restrained by the number of input goods at her disposal.

LTP situations describe production situations in which each producer controls a transformation technique, as described in the example above, and a bundle of (resource) goods. The transformation technique is modelled by a vector that describes which goods and how many the producer needs to produce other goods. Transformation techniques are linear, i.e. the output is a linear function of the input. A producer has to choose at which activity level his production process will operate. The choice of the activity level will depend on the resources owned by the producer. Given an activity level, the transformation technique describes how much input is needed. Then the producer can carry out his production process at a certain activity level only if his resources contain the required input. After production, the producer sells all the remaining goods, i.e. produced goods and resources not used in the transformation process, on the market. We assume that the market is insatiable, so that all goods can be sold. Furthermore, all producers are pricetakers. Their output does not influence the market prices. The goal of each producer is to maximize his profit from the sale of the remaining goods.

Next, we introduce some notation. Denote by M the finite set of goods and by N the finite set of producers. Each producer $i \in N$ is endowed with a bundle of goods $\omega(i) \in \mathbb{R}^M_+$. The vector $a^i \in \mathbb{R}^M$ describes the transformation technique of producer i in the following way. Producer i needs $-a^i_j$ units of each good j with $a^i_j \leq 0$ to produce a^i_k units of the goods k with $a^i_k \geq 0$. We assume that each vector a^i contains at least one positive and one negative element. Let y_i be the activity level of producer $i \in N$. Then for the production of the bundle $\{a^i_j y_i |\, a^i_j \geq 0\}$ he needs the resources $\{-a^i_j y_i |\, a^i_j \leq 0\}$. Since we assume the transformation technique to be irreversible, we have that activity levels are nonnegative, that is, $y_i \geq 0$.

We have seen that producer i uses good j as an input in his transformation process if $a^i_j \leq 0$ and that good j is an output if $a^i_j \geq 0$. The resources needed for the transformation process are thus described by the vector g^i with $g^i_j := \max\{0, -a^i_j\}$ for all $j \in M$, $i \in N$. So, at activity level y_i producer i uses the bundle $g^i y_i$ to produce $(a^i + g^i)y_i$. After production, producer i possesses the bundle $\omega(i) + (a^i + g^i)y_i - g^i y_i = \omega(i) + a^i y_i$ which he can sell at exogenously given market prices $p \in \mathbb{R}^M_+ \backslash \{0\}$. Since a producer cannot use more goods than he has available, it must hold that $g^i y_i \leq \omega(i)$. The profit maximization problem of producer $i \in N$ thus becomes:

$$\begin{array}{ll} \max & p^T(\omega(i) + a^i y_i) \\ \text{s.t.} & g^i y_i \leq \omega(i) \\ & y_i \geq 0 \end{array}$$

The transformation techniques and resources of all producers can be summarized by defining the transformation matrix $A \in \mathbb{R}^{M \times N}$, where the i^{th} column of A corresponds to the transformation vector a^i and the vector $\omega := (\omega(i))_{i \in N}$. In short, an LTP situation is described by a 4-tuple $\langle N, A, \omega, p \rangle$.

3 LTP games

By cooperating, producers can pool their transformation techniques and their resources. Each producer then gets a part of the total resources to use in his transformation process. We assume that when producers cooperate, they cannot use the output of other producers as resources for their production. Furthermore, the activity levels of the producers in this coalition should be such that the total resources cover the total input needed. After transformation, the coalition sells the remaining goods, i.e. produced goods and resources not used in any of the transformation processes, on the market at exogenous market prices. The goal of a coalition is to maximize its profit.

If a coalition $S \subset N$, $S \neq \emptyset$ cooperates, it collectively owns the resource bundle $\omega(S) := \sum_{i \in S} \omega(i)$ and moreover, this coalition can use each transformation technique a^i, $i \in S$. To produce $\sum_{i \in S}(a^i + g^i)y_i$ it needs the input $\sum_{i \in S} g^i y_i$. After transformation, coalition S can sell $\sum_{i \in S}(\omega(i) + a^i y_i) = \omega(S) + Ay$ where y is the vector of all activity levels with $y_i = 0$ if $i \notin S$. Since the coalition cannot use more goods than it has available, it should hold that $\sum_{i \in S} g^i y_i \leq \omega(S)$ or, equivalently, $Gy \leq \omega(S)$ with $y_i = 0$ if $i \notin S$. The profit maximization problem of the coalition thus equals

$$\begin{array}{rl} \max & p^T(\omega(S) + Ay) \\ \text{s.t.} & Gy \leq \omega(S) \\ & y \geq 0 \\ & y_i = 0 \text{ if } i \notin S \end{array} \tag{1}$$

So, an LTP situation gives rise to a cooperative game as the following definition shows. Let $\langle N, A, \omega, p \rangle$ be an LTP situation. Then the corresponding *LTP game* (N, v) is such that the characteristic function v assigns to each coalition $S \subset N$ the maximal profit it can obtain as given in (1) and $v(\emptyset) = 0$. The following example illustrates this definition.

Example 1 Consider the following LTP situation: $N = \{1, 2, 3\}, p = (1, 1, 1)^T$,

$$A = \begin{bmatrix} 1 & -4 & -1 \\ -1 & 1 & 3 \\ 0 & 2 & -1 \end{bmatrix}, \ \omega(1) = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \ \omega(2) = \begin{bmatrix} 12 \\ 2 \\ 0 \end{bmatrix} \text{ and } \omega(3) = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$$

The corresponding LTP game equals

$v(S)$:	optimal activity level y :
$v(\{1\}) = 9$	$y = (0, 0, 0)^T$
$v(\{2\}) = 14$	$y = (0, 0, 0)^T$
$v(\{3\}) = 17$	$y = (0, 0, 4)^T$
$v(\{1, 2\}) = 23$	$y = (y_1, 0, 0)^T, \ 0 \leq y_1 \leq 2$
$v(\{1, 3\}) = 30$	$y = (y_1, 0, 8)^T, \ 0 \leq y_1 \leq 4$
$v(\{2, 3\}) = 31$	$y = (0, 0, 4)^T$
$v(\{1, 2, 3\}) = 46$	$y = (y_1, 0, 10)^T, \ 0 \leq y_1 \leq 4$

The main issue of cooperative game theory is how to divide the benefits from cooperation. For LTP games, this means how cooperating producers divide their joint profit among each other. One way to share the joint profit from cooperation is to do this according to a core-allocation. The *core* of a game (N, v) is the set

$$C(v) = \left\{ x \in \mathbb{R}^N \middle| \sum_{i \in N} x_i = v(N) \text{ and} \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \backslash \{\emptyset\} \right\}$$

where $2^N \backslash \{\emptyset\}$ denotes the set of all nonempty subsets of N. When a core-element $x \in C(v)$ is proposed as a distribution of the total profit $v(N)$ where producer i gets payoff x_i, then coalition S will get at least as much as it can obtain on its own since $\sum_{i \in S} x_i \geq v(S)$. So, no coalition has an incentive to leave the grand coalition N. A game is *balanced* if it has a nonempty core and it is called *totally balanced* if each subgame $(S, v_{|S})$ is balanced, where $v_{|S}(T) := v(T)$ if $T \subset S$. The following theorem shows that an LTP game has a nonempty core.

Theorem 1 *Let $\langle N, A, \omega, p \rangle$ be an LTP situation. Then the corresponding LTP game (N, v) has a nonempty core.*

Proof. Based on the LTP game (N, v) we can define a new game (N, a) where $a(S) = p^T \omega(S)$. In this game each coalition gets the value of its endowment $\omega(S)$. Next, we define another game (N, u) where $u(S) = v(S) - a(S)$ or, equivalently,

$$\begin{aligned} u(S) = \max \quad & p^T A y \\ \text{s.t.} \quad & G y \leq \omega(S) \\ & y \geq 0 \\ & y_i = 0 \text{ if } i \notin S \end{aligned}$$

In this game, the value $u(S)$ is the net profit coalition S obtains over the value $p^T \omega(S)$ of its initial endowment by optimally using the transformation technique. In the case $S = N$ the above maximization problem reduces to

$$\begin{aligned} u(N) = \max \quad & p^T A y \\ \text{s.t.} \quad & G y \leq \omega(N) \\ & y \geq 0 \end{aligned}$$

To this problem corresponds the following dual minimization problem:

$$\begin{aligned} \min \quad & z^T \omega(N) \\ \text{s.t.} \quad & G^T z \geq A^T p \\ & z \geq 0 \end{aligned} \tag{2}$$

Let the minimum[1] of (2) be obtained in $\underline{z}$. First, we show that $x = (x_1, \ldots, x_n)^T$, where $x_i = \underline{z}^T \omega(i)$, is a core-element of (N, u). According to duality theory $\sum_{i \in N} x_i = \sum_{i \in N} \underline{z}^T \omega(i) =$

[1] Since the set of feasible solutions is closed, convex, non-empty and bounded from below by the zero-vector, the problem can be solved and a minimum exists.

$\underline{z}^T\omega(N) = u(N)$. So, x represents a distribution of $u(N)$ among the members of N. Notice that $\underline{z}$ is also a feasible solution of the problem $\min\{z^T\omega(S)|\, G^Tz \geq A^Tp;\, z \geq 0\}$ for all $S \subset N$. Thus,

$$\begin{aligned}
\underline{z}^T\omega(S) \geq &\ \min\{z^T\omega(S)|\, G^Tz \geq A^Tp;\, z \geq 0\} \\
= &\ \max\{p^TAy|\, Gy \leq \omega(S);\, y \geq 0\} \\
\geq &\ \max\{p^TAy|\, Gy \leq \omega(S);\, y \geq 0;\, y_i = 0 \text{ if } i \notin S\} \\
= &\ u(S)
\end{aligned}$$

This implies that $\sum_{i\in S} x_i = \sum_{i\in S} \underline{z}^T\omega(i) = \underline{z}^T\omega(S) \geq u(S)$ and thus $x \in C(u)$.

Next, we show that $x' \in C(v)$ with $x'_i = (\underline{z}+p)^T\omega(i)$. Since $v(S) = u(S) + a(S)$ it follows that $\sum_{i\in S} x'_i = \underline{z}^T\omega(S) + p^T\omega(S) \geq u(S) + a(S) = v(S)$ for all $S \subset N$. Furthermore, $\sum_{i\in N} x'_i = \underline{z}^T\omega(N) + p^T\omega(N) = u(N) + a(N) = v(N)$. Hence $x' \in C(v)$. $\square$

Theorem 1 implies that an LTP game is balanced. Since each subgame $(S, v_{|S})$ of an LTP game is another LTP game, an LTP game is totally balanced. Note that the proof of theorem 1 also indicates how to find a core-element of an LTP game.

When the minimum of (2) is attained in $\underline{z}$ then $\underline{z}+p$ is the vector containing the shadow prices of the resources. The vector $\underline{z}$ contains the prices that coalition N would want to pay for its resources in excess of p.

The goal of each coalition is to maximize its profit. From all the bundles of goods that the coalition can produce, it will choose the bundle that gives her maximal profit. We can use a multi-commodity game as studied in VAN DEN NOUWELAND, AARTS and BORM (1989) to describe for each coalition the set of bundles of goods it can sell. Let $\langle N, A, \omega, p\rangle$ be an LTP situation. Then the corresponding *multi-commodity game* (N, F) is a game where N is the player set and for all $S \subset N$

$$F(S) = \{x \in \mathbb{R}^M_+ \,|\, x \leq \omega(S) + Ay;\, Gy \leq \omega(S);\, y \geq 0;\, y_i = 0 \text{ if } i \notin S\}$$

and $F(\emptyset) = \{0\}$.

Note that if an LTP game (N, v) and a multi-commodity game (N, F) are both based on the same LTP situation $\langle N, A, \omega, p\rangle$, then we can write

$$\begin{aligned}
v(S) = \max &\quad p^Tx \\
\text{s.t.} &\quad x \in F(S)
\end{aligned}$$

We show that a multi-commodity game corresponding to an LTP situation is totally balanced. First, we define when multi-commodity games are (totally) balanced (cf. VAN DEN NOUWELAND et al. (1989)). For $S \subset N$ let $e^S \in \mathbb{R}^N$ denote the characteristic vector of S, so $e^S_i = 1$ if $i \in S$ and $e^S_i = 0$ otherwise. A map $\lambda : 2^N\backslash\{\emptyset\} \to \mathbb{R}_+$ is a balanced map if $\sum_{S\in 2^N\backslash\{\emptyset\}} \lambda(S)e^S = e^N$. A multi-commodity game (N, F) is *balanced* if for all balanced maps λ it holds that $\sum_{S\in 2^N\setminus\{\emptyset\}}\lambda(S)F(S)\subset F(N)$.[2] This game is *totally balanced* if for all $S\in 2^N\setminus\{\emptyset\}$ the restricted game $(S,F_{|S})$ is balanced, where $F_{|S}(U)=F(U)$ for all $U\subset S$.

Theorem 2 *Let $\langle N,A,\omega,p\rangle$ be an LTP situation. Then the corresponding multi-commodity game (N,F) is totally balanced.*

Proof. Let (N,F) be the multi-commodity game corresponding to the LTP situation $\langle N,A,\omega,p\rangle$ and let $S\in 2^N\setminus\{\emptyset\}$. We show that $\sum_{U\in 2^S\setminus\{\emptyset\}}\lambda(U)F(U)\subset F(S)$ for all balanced maps λ for coalition S.

Let $x^U\in F(U)$ for all $U\subset S$. Then there is a $y^U\in\mathbb{R}^N_+$ such that $x^U\le\omega(U)+Ay^U$, $Gy^U\le\omega(U)$ and $y^U_i=0$ if $i\notin U$. Define $z:=\sum_{U\in 2^S\setminus\{\emptyset\}}\lambda(U)x^U$ and $y:=\sum_{U\in 2^S\setminus\{\emptyset\}}\lambda(U)y^U$. Then

$$\begin{aligned} z= & \sum_{U\in 2^S\setminus\{\emptyset\}}\lambda(U)x^U\le\sum_{U\in 2^S\setminus\{\emptyset\}}\lambda(U)[\omega(U)+Ay^U]= \\ & \sum_{U\in 2^S\setminus\{\emptyset\}}\lambda(U)\omega(U)+A\sum_{U\in 2^S\setminus\{\emptyset\}}\lambda(U)y^U=\omega(S)+Ay \end{aligned}$$

where the last equality follows from

$$\begin{aligned} \sum_{U\in 2^S\setminus\{\emptyset\}}\lambda(U)\omega(U) & =\sum_{U\in 2^S\setminus\{\emptyset\}}\sum_{i\in U}\lambda(U)\omega(i)=\sum_{i\in S}\sum_{U:i\in U}\lambda(U)\omega(i) \\ & =\sum_{i\in S}\omega(i)=\omega(S). \end{aligned}$$

Furthermore, it holds that $y\ge 0$, $y_i=0$ if $i\notin S$ and

$$Gy=G\sum_{U\in 2^S\setminus\{\emptyset\}}\lambda(U)y^U=\sum_{U\in 2^S\setminus\{\emptyset\}}\lambda(U)Gy^U\le\sum_{U\in 2^S\setminus\{\emptyset\}}\lambda(U)\omega(U)=\omega(S).$$

Since $z\le\omega(S)+Ay$, $Gy\le\omega(S)$, $y\ge 0$ and $y_i=0$ if $i\notin S$ it follows that $z\in F(S)$.□

If a multi-commodity game is balanced then there exists a so-called stable outcome (cf. VAN DEN NOUWELAND et al.(1989)). The set $SO(F)$ of *stable outcomes* in a multi-commodity game (N,F) is defined by

$$SO(F)=\left\{x\in(\mathbb{R}^M_+)^N\;\middle|\;\begin{array}{l}\sum_{i\in N}x^i\in\mathrm{Par}(F(N)),\\ \sum_{i\in S}x^i\notin F(S)\setminus\mathrm{Par}(F(S))\text{ for all }S\in 2^N\setminus\{\emptyset\}\end{array}\right\}$$

where $\mathrm{Par}(X)=\{x\in X|\;\nexists z\in X:z\ge x\text{ and }z\ne x\}$ is the set of Pareto-optimal allocations in X. A stable outcome x is such that no coalition can obtain more units of each good than it gets according to x.

The next theorem shows the relationship between core-elements of an LTP game (N,v) and stable outcomes of a multi-commodity game (N,F).

Theorem 3 *Let $\langle N,A,\omega,p\rangle$ be an LTP situation with corresponding LTP game (N,v) and corresponding multi-commodity game (N,F). Let $\beta\in\mathbb{R}^N_+$ be a core-element of (N,v) and let $\alpha\in F(N)$ be such that $p^T\alpha=v(N)$. If $p>0$ and $v(N)>0$, then $x\in(\mathbb{R}^M_+)^N$ with $x^i=[\beta_i(v(N))^{-1}]\alpha$ for all $i\in N$, is a stable outcome of (N,F).*

[2] For $A,B\subset\mathbb{R}^M_+$, $\lambda\in\mathbb{R}$, $A+B:=\{x+y|\;x\in A,\;y\in B\}$ and $\lambda A:=\{\lambda x|\;x\in A\}$.

Proof. From the definition of x it follows that $\sum_{i\in N} x^i = \sum_{i\in N} \left[\beta_i (v(N))^{-1}\right] \alpha = \alpha \in F(N)$. If $S \in 2^N \backslash \{\emptyset\}$ and $z \in \mathbb{R}^M_+$ such that $z \geq \sum_{i\in S} x^i$, $z \neq \sum_{i\in S} x^i$ then $p^T z > p^T \sum_{i\in S} x^i = \sum_{i\in S} p^T x^i = \sum_{i\in S} \beta_i \geq v(S)$ so $z \notin F(S)$. We conclude that $x \in SO(F)$. $\square$

4 Exchange economies

In this section we relate LTP situations to exchange economies in two different ways. In the first model, called model 1, producers can exchange their resources before transformation starts. This exchange takes place in a separate market so that the endogenous price vector q in this exchange market may differ from the prices in the market where the producers sell their goods after production. After the exchange, each producer will use his new bundle of goods in his transformation process. After transformation, the remaining goods will be sold at exogenous prices p. The goal of each producer is to maximize his profit.

Let $\langle N, A, \omega, p\rangle$ be an LTP situation. If q denotes the price vector in the exchange market, then producer $i \in N$ exchanges his resource bundle $\omega(i)$ for a bundle $x(i)$ at price vector q. A producer cannot spend more money on the bundle $x(i)$ than the value of his resources $\omega(i)$: $q^T x(i) \leq q^T \omega(i)$. After the exchange producer i will use the bundle $x(i)$ as resources for his transformation process. When producer $i \in N$ operates his transformation process at an activity level y_i then he needs the resources $g^i y_i$. Since his resources now equal $x(i)$, we get the restriction $g^i y_i \leq x(i)$. Finally, producer i will sell the remaining goods $x(i) + a^i y_i$ on the market at given prices p. In short, the profit maximization problem of producer i in model 1 is given by

$$\begin{array}{rl} \max & p^T (x(i) + a^i y_i) \\ \text{s.t.} & g^i y_i \leq x(i) \\ & y_i \geq 0 \\ & q^T x(i) \leq q^T \omega(i) \\ & x(i) \geq 0 \end{array} \tag{3}$$

An equilibrium in this model consists of a bundle of goods $x^*(i)$, an activity level y_i^* for all $i \in N$ and a price vector q^* such that producer $i \in N$ maximizes his profit in $x^*(i)$ and y_i^* given q^* and such that total demand equals total supply: $\sum_{i\in N} x^*(i) = \sum_{i\in N} \omega(i)$.

Note that the prices p are exogenous while the prices q^* are determined by the equilibrium conditions. If q^* is an equilibrium price vector and λ is a positive real number then $q^{*T} x(i) \leq q^{*T} \omega(i)$ if and only if $(\lambda q^*)^T x(i) \leq (\lambda q^*)^T \omega(i)$ and thus is λq^* another equilibrium price vector. This implies that in our search for equilibrium price vectors, we can restrict our attention to prices in $\Delta^M = \{q \in \mathbb{R}^M_+ | \sum_{j\in M} q_j = 1\}$. Also note that if there is an equilibrium price vector q^* then we can always find a $\lambda > 0$ such that $\lambda q^* \geq p$. This new equilibrium price vector λq^* ensures that producers trade their resources instead of selling them on the market at exogenous prices p. We will now show that this model allows for an equilibrium.

Theorem 4 *Let $\langle N, A, \omega, p\rangle$ be an LTP situation. If $p \in \mathbf{R}_+^M$ and $\omega(i) \in \mathbf{R}_{++}^M$ for all $i \in N$ then there exists an equilibrium in model 1.*

Proof. The profit maximization problem of producer $i \in N$ is given by (3). The producer can solve this problem in two steps. When he knows that he will own $x(i)$ after the exchange then his maximization problem reduces to

$$\begin{array}{rl} \max & p^T(x(i) + a^i y_i) \\ \text{s.t.} & g^i y_i \leq x(i) \\ & y_i \geq 0 \end{array}$$

Since the objective function is continuous and the set $\{y_i | \, g^i y_i \leq x(i), \; y_i \geq 0\}$ is compact and non-empty, this reduced problem can be solved for all $x(i)$. Define

$$R^i(x(i)) = \max\{p^T(x(i) + a^i y_i) | \, g^i y_i \leq x(i); \; y_i \geq 0\} \tag{4}$$

Then we can rewrite (3) as

$$\begin{array}{rl} \max & R^i(x(i)) \\ \text{s.t.} & q^T x(i) \leq q^T \omega(i) \\ & x(i) \geq 0 \end{array}$$

In the appendix we show that R^i is a continuous, monotone and quasi-concave function. If we think of R^i as the utility function of producer i then this maximization problem equals the utility maximization problem of agent i in an exchange economy. DEBREU (1959) proves the existence of an equilibrium in such an exchange economy and this also proves the existence of an equilibrium in model 1. □

In the second model, called model 2, a producer can start by transforming his resource bundle, after which the producers can mutually exchange their products in a separate market. After the exchange, each producer will use his new bundle of goods in his transformation process and sell the remaining goods at exogenous prices p. Notice that in this model production takes place at two points in time. Again the goal of each producer is to maximize his profit.

For a formal description of model 2, let $\langle N, A, \omega, p\rangle$ be an LTP situation. Then producer $i \in N$ starts by transforming his resource bundle $\omega(i)$ into $\omega(i) + a^i \widehat{y}_i$ with $\widehat{y}_i$ such that $g^i \widehat{y}_i \leq \omega(i)$ and $\widehat{y}_i \geq 0$. Next, this producer exchanges his products $\omega(i) + a^i \widehat{y}_i$ for the bundle $x(i)$ at endogenous prices q. A producer cannot spend more money on the bundle $x(i)$ than the value of his products: $q^T x(i) \leq q^T(\omega(i) + a^i \widehat{y}_i)$. After the exchange has taken place producer i will use the bundle $x(i)$ as resources for his transformation process. He will sell the remaining goods $x(i) + a^i y_i$ on the market at endogenous prices p where y_i is such that $g^i y_i \leq x(i)$ and $y_i \geq 0$. In short, the profit maximization problem of producer i in model

2 is

$$\begin{aligned} \max \quad & p^T(x(i) + a^i y_i) \\ \text{s.t.} \quad & g^i y_i \le x(i) \\ & y_i \ge 0 \\ & q^T x(i) \le q^T(\omega(i) + a^i \widehat{y}_i) \\ & x(i) \ge 0 \\ & g^i \widehat{y}_i \le \omega(i) \\ & \widehat{y}_i \ge 0 \end{aligned} \tag{5}$$

An equilibrium in this model consists of a price vector q^* and for all $i \in N$ of a bundle of goods $x^*(i)$, a production level y_i^* and a production level $\widehat{y}_i^*$ such that producer $i \in N$ maximizes his profit in $x^*(i)$, y_i^* and $\widehat{y}_i^*$ given q^* and total demand equals total supply in the intermediate exchange market: $\sum_{i \in N} x^*(i) = \sum_{i \in N}(\omega(i) + a^i \widehat{y}_i^*)$.

As in model 1 it holds that if q^* is an equilibrium price vector and λ is a positive real number then λq^* is another equilibrium price vector. So in our search for equilibrium price vectors, we can restrict our attention to prices in $\Delta^M = \{q \in \mathbb{R}^M_+ \mid \sum_{j \in M} q_j = 1\}$.

Next, we define irreversibility of the transformation process. The transformation set T_i for the producer i is the set of all transformations possible for him:

$$T_i = \{x \in \mathbb{R}^M \mid x \le a^i y_i,\ g^i y_i \le \omega(i),\ y_i \ge 0\} \tag{6}$$

The set $T = \sum_{i \in N} T_i$ is called the total transformation set. We say that the transformation process is *irreversible* if $T \cap (-T) = \{0\}$, where $-T = \{x \mid -x \in T\}$. This means that for each transformation $x \in T$, $x \neq 0$, the transformation $-x$ is not possible; the transformation process cannot be reversed.

Theorem 5 *Let $\langle N, A, \omega, p\rangle$ be an LTP situation. If $p \in \mathbf{R}^M_+$, $\omega(i) \in \mathbf{R}^M_{++}$ for all $i \in N$ and the transformation process is irreversible then there exists an equilibrium in model 2.*

Proof. The profit maximization problem of producer $i \in N$ is given by (5). In lemma 3 in the appendix we show that this problem is equivalent to

$$\begin{aligned} \max \quad & p^T(x(i) + a^i y_i) & \text{where } v^q(i) = \max \quad & q^T(\omega(i) + a^i \widehat{y}_i) \\ \text{s.t.} \quad & g^i y_i \le x(i) & \text{s.t.} \quad & g^i \widehat{y}_i \le \omega(i) \\ & y_i \ge 0 & & \widehat{y}_i \ge 0 \\ & q^T x(i) \le v^q(i) & & \\ & x(i) \ge 0 & & \end{aligned}$$

We can write this as

$$\begin{aligned} \max \quad & R^i(x(i)) & \text{where } v^q(i) = q^T\omega(i) + \max \quad & q^T a^i \widehat{y}_i \\ \text{s.t.} \quad & q^T x(i) \le v^q(i) & \text{s.t.} \quad & g^i \widehat{y}_i \le \omega(i) \\ & x(i) \ge 0 & & \widehat{y}_i \ge 0 \end{aligned} \tag{7}$$

where, as in the proof of theorem 4, $R^i(x(i)) = \max\{p^T(x(i)+a^i y_i) |\, g^i y_i \leq x(i);\ y_i \geq 0\}$ is a continuous, monotone and quasi-concave function. Let $u^q(i)$ be the net profit producer i obtains over the value $q^T\omega(i)$ of his initial endowment: $u^q(i) = v^q(i) - q^T\omega(i)$, or, equivalently,

$$u^q(i) = \max \quad q^T a^i \widehat{y}_i \quad \text{s.t.} \quad g^i\widehat{y}_i \leq \omega(i), \ \widehat{y}_i \geq 0 \qquad = \max \quad q^T x \quad \text{s.t.} \quad x \in T_i$$

where T_i is the transformation set of producer i as defined in (6). If we substitute this in (7) we get

$$\begin{array}{rl} \max & R^i(x(i)) \\ \text{s.t.} & q^T x(i) \leq q^T\omega(i) + u^q(i) \\ & x(i) \geq 0 \end{array} \qquad \text{where } u^q(i) = \max \quad q^T x \quad \text{s.t.} \quad x \in T_i \tag{8}$$

If we think of R^i as the utility function of 'consumer' i then the left hand side of (8) is the utility maximization problem of consumer i in a private ownership economy as described in DEBREU (1959). This consumer cannot spend more money on the bundle $x(i)$ than the sum of the value of his endowment and the net profit of producer i. The right hand side of (8) is the net profit maximization problem of a producer in a private ownership economy. In this model, consumer i and producer i are the same person and the net profit of producer i goes to consumer i. Debreu proves the existence of an equilibrium in such an economy and this also proves the existence of an equilibrium in model 2. □

We illustrate both models with the following example.

Example 2 Consider an LTP situation with two producers, $N = \{1, 2\}$. The transformation matrix A equals

$$A = \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$$

Each producer owns one unit of each good: $\omega(1) = \omega(2) = (1,1)^T$. One unit of each good can be sold for 1 dollar: $p = (1,1)^T$. The value of each coalition in the LTP game is

$$\begin{array}{ll} v(\{1\}) = 3, & y_1^* = 1 \\ v(\{2\}) = 4, & y_2^* = 1 \\ v(\{1,2\}) = 8, & y_1^* = 0,\ y_2^* = 2 \end{array}$$

The core of this game is the set $C(v) = \{(x, 8-x) |\, 3 \leq x \leq 4\}$.

In model 1, the profit maximization problem of producer 1 is

$$\begin{array}{rl} \max & x(1)_1 + x(1)_2 + y_1 \\ \text{s.t.} & 0 \leq y_1 \leq x(1)_1 \\ & q^T x(1) \leq q^T\omega(1) \\ & x(1) \geq 0 \end{array}$$

Producer 1 will choose y_1 as high as possible, so $y_1^* = x(1)_1$. The maximization problem reduces to

$$\begin{aligned} \max \quad & 2x(1)_1 + x(1)_2 \\ \text{s.t.} \quad & q^T x(1) \leq q^T \omega(1) \\ & x(1) \geq 0 \end{aligned}$$

We restrict ourselves to prices q in $\Delta^2 = \{q \in \mathbf{R}^2 | \, q_1 + q_2 = 1\}$. For all $q \in \Delta^2$ it holds that $q^T \omega(1) = q_1 + q_2 = 1$. If we substitute this in the maximization problem we get

$$\begin{aligned} \max \quad & 2x(1)_1 + x(1)_2 \\ \text{s.t.} \quad & q^T x(1) \leq 1 \\ & x(1) \geq 0 \end{aligned}$$

Similarly we can reduce the profit maximization problem of producer 2 to

$$\begin{aligned} \max \quad & 3x(2)_1 + x(2)_2 \\ \text{s.t.} \quad & q^T x(2) \leq 1 \\ & x(2) \geq 0 \end{aligned}$$

where $y_2^* = x(2)_1$. In an equilibrium, demand should equal supply: $x^*(1) + x^*(2) = \omega(1) + \omega(2) = (2, 2)^T$. The unique equilibrium price in Δ^2 is $q^* = (\frac{2}{3}, \frac{1}{3})^T$. To ensure that producers exchange their endowments, we can take, e.g., price vector $\overline{q} = 3q^* = (2, 1)^T \geq p = (1, 1)^T$ in the exchange market. The equilibrium bundles are $x^*(1) = (\frac{1}{2}, 2)^T$, $x^*(2) = (1\frac{1}{2}, 0)^T$ and the equilibrium activity levels are $y_1^* = \frac{1}{2}$, $y_2^* = 1\frac{1}{2}$. Note that producer 2 would like to have as much units of good 1 as possible since he is the more efficient producer and can earn a lot of money by transforming them into units of good 2 and selling these on the market. To receive all the units of good 1 owned by producer 1, producer 2 has to offer in exchange the goods that producer 1 could have produced from his units of good 1. Thus, producer 2 will exchange two units of good 2 for one unit of good 1. But he owns just one unit of good 2 so he will exchange that unit for half a unit of good 1. Producer 2 now owns one and a half units of good 1 which he transforms into four and a half units of good 2. He sells these on the market and his profit equals $4\frac{1}{2}$. Producer 1 transforms half a unit of good 1 into one unit of good 2 and sells this together with his other two units of good 2 on the market. The profit of producer 1 equals $3 = v(\{1\})$, so he is indifferent between participating in the exchange and acting on his own. Producer 2 gains from the exchange, $4\frac{1}{2} > v(\{2\})$, thus both producers participating in the exchange is better than both producers acting on their own. However, $(3, 4\frac{1}{2}) \notin C(v)$ since $3 + 4\frac{1}{2} < v(N) = 8$. Working together results in a higher profit. In this example, both producers need good 1 in their transformation process. When producers need different goods in their transformation process then it may hold that the equilibrium payoffs in model 1 generate a core-element of the corresponding LTP game.

In model 2, the profit maximization problem of producer 1 is

$$\begin{array}{rl} \max & 2x(1)_1 + x(1)_2 \\ \text{s.t.} & q^T x(1) \le v^q(1) \\ & x(1) \ge 0 \end{array} \qquad v^q(1) = \begin{cases} 3q_2, & q_1 < \frac{2}{3} \quad (\widehat{y}_1^* = 1) \\ 1, & q_1 = \frac{2}{3} \quad (\widehat{y}_1^* \in [0,1]) \\ 1, & q_1 > \frac{2}{3} \quad (\widehat{y}_1^* = 0) \end{cases}$$

where $q \in \Delta^2$ and we substituted $y_1^* = x(1)_1$. For producer 2 it equals

$$\begin{array}{rl} \max & 3x(2)_1 + x(2)_2 \\ \text{s.t.} & q^T x(2) \le v^q(2) \\ & x(2) \ge 0 \end{array} \qquad v^q(2) = \begin{cases} 4q_2, & q_1 < \frac{3}{4} \quad (\widehat{y}_2^* = 1) \\ 1, & q_1 = \frac{3}{4} \quad (\widehat{y}_2^* \in [0,1]) \\ 1, & q_1 > \frac{3}{4} \quad (\widehat{y}_2^* = 0) \end{cases}$$

where $y_2^* = x(2)_1$. In an equilibrium it should hold that $x^*(1) + x^*(2) = \omega(1) + a^1\widehat{y}_1^* + \omega(2) + a^2\widehat{y}_2^*$. The unique equilibrium price in Δ^2 is $q^* = (\frac{3}{4}, \frac{1}{4})^T$. To ensure that producers exchange their endowments, we can take, e.g. $\overline{q} = 4q^* = (3,1)^T \ge p = (1,1)^T$. The equilibrium bundles are $x^*(1) = (0,4)^T$, $x^*(2) = (2 - \widehat{y}_2^*, 3\widehat{y}_2^* - 2)^T$ and the equilibrium activity levels are $y_1^* = 0$, $y_2^* = 2 - \widehat{y}_2^*$, $\widehat{y}_1^* = 0$ and $\frac{2}{3} \le \widehat{y}_2^* \le 1$. As in model 1, producer 2 would like to have as much units of good 1 as possible, therefore he starts by transforming good 1 into good 2. Producer 1 knows this and he starts by doing nothing. Producer 1 owns one unit of the scarce good and he can ask three units of good 2 in exchange. This is exactly what player 2 can produce from one unit of good 2. So, producer 1 exchanges one unit of good 1 for three units of good 2. Since producer 1 now has no units of good 1 he cannot produce so he sells his four units of good 2 on the market. His profit equals 4. Producer 2 owns $(2 - \widehat{y}_2^*, 3\widehat{y}_2^* - 2)^T$ after the exchange. He transforms $2 - \widehat{y}_2^*$ units of good 1 into $3(2 - \widehat{y}_2^*) = 6 - 3\widehat{y}_2^*$ units of good 2. This leaves him with $3\widehat{y}_2^* - 2 + 6 - 3\widehat{y}_2^* = 4$ units of good 2 to sell on the market. His profit equals 4. Note that $(4,4) \in C(v)$. However, there are LTP situations where the payoffs in model 2 do not generate a core-element of the corresponding LTP game.

5 A characterization of totally balanced games

One restriction of LTP situations is that each producer has only one transformation technique. This is not very realistic. We can think, for example, of a firm producing two goods by using two different transformation techniques. In this section we will extend LTP situations so that each producer may have more than one technique. We will call these situations extended LTP situations.

We assume now that a producer controls some resources and at least one transformation technique. He chooses an activity level for each of his techniques. These choices depend on his resources. Given an activity level, a transformation technique describes how much input is needed. The producer can carry out his production processes at the desired activity levels only if his resources contain the required inputs. After production, the producer sells the produced goods and unused resources in the insatiable market at exogenous prices.

We will now introduce some additional notation. A transformation technique is a vector in $\mathbb{R}^M$. Producer $i \in N$ can use a transformation technique a^k if and only if $k \in D_i$ where D_i denotes the set of all techniques controlled by producer i. The resources needed for this technique are described by the vector $g^k \in \mathbb{R}^M_+$ with $g^k_j = \max\{0, -a^k_j\}$. The transformation matrix A, with its k^{th} column corresponding to a^k, is an element of $\mathbb{R}^{M \times D}$ where $D := (D_i)_{i \in N}$ and the related matrix G, with its k^{th} column corresponding to g^k, is an element of $\mathbb{R}^{M \times D}_+$. The vector of activity levels $y \in \mathbb{R}^D_+$ describes for each transformation technique at which level it is operated. If we denote by $D(S) := \cup_{i \in S} D_i$ the set of all transformation techniques available to coalition S then the profit maximization problem of this coalition is

$$\begin{aligned} \max \quad & p^T(\omega(S) + Ay) \\ \text{s.t.} \quad & Gy \leq \omega(S) \\ & y \geq 0 \\ & y_i = 0 \text{ if } i \notin D(S) \end{aligned} \tag{9}$$

An extended LTP situation is described by a 5-tuple $\langle N, A, D, \omega, p \rangle$. Given such a situation we define the corresponding *extended LTP game* (N, v) by the player set N and a function v that assigns to each coalition $S \subset N$ the maximal profit it can obtain as in (9) where $v(\emptyset) = 0$.

These extended LTP games have some nice properties. First, they are balanced. The proof is similar to that of theorem 1. Since each subgame $(S, v_{|S})$ is another extended LTP game, these games are totally balanced. Moreover, we can write each totally balanced game (N, u) with nonnegative values, i.e. $u(S) \geq 0$ for all $S \subset N$, as an extended LTP game.

Theorem 6 *Each totally balanced game with nonnegative values is an extended LTP game.*

Proof. Let (N, u) be a totally balanced game with nonnegative values. We construct an extended LTP situation such that for the corresponding LTP game it holds that $v(S) = u(S)$ for all $S \subset N$.

The set of producers equals N. Assume that $N = \{1, \ldots, n\}$. Define $D_i = \{S \subset N | \, i \in S, \, j < i \Rightarrow j \notin S\}$ then each transformation technique of producer i is related to a coalition of which the producer is the 'first' member. So, each coalition is related to one producer. Producer i controls 2^{n-i} techniques and all the producers together control $2^n - 1$ techniques.

Define $n + 2^n - 1$ goods in M as follows. Each of the first n goods is related to a producer in N and each of the $2^n - 1$ goods is related to a nonempty coalition in N.

The transformation technique related to coalition S is denoted by a^S. Technique $a^S \in \mathbb{R}^{n+2^n-1}$ contains $-e^S$ on the first n rows and the remaining $2^n - 1$ rows are related to the nonempty coalitions such that $a^S_U = 1$ if $U = S$ and 0 otherwise. So, the transformation technique a^S uses one unit of each "good" j for all $j \in S$ to produce one unit of "good" S.

The transformation matrix A is an $(n + 2^n - 1) \times (2^n - 1)$-matrix. The related matrix G contains columns g^S with e^S on its first n rows and zeros in the remaining rows.

Producer i owns one unit of good i, so $\omega(i)$ is the resource bundle with $e^{\{i\}}$ on the first n rows and zeros in the other rows. As before, when players cooperate they pool their resources: $\omega(S) = \sum_{i \in S} \omega(i)$. The price vector $p \in \mathbf{R}^{n+2^n-1}$ is defined as follows. The first n goods, the inputs, have price zero, $p_j = 0$ if $1 \leq j \leq n$, and good S has value $u(S), p_S = u(S)$. For ease of notation, define the shortened price vector $p(u) \in \mathbf{R}^{2^n-1}$ by $p(u)_S := p_S$. The vector of activity levels $y \in \mathbf{R}_+^{2^n-1}$ describes the activity level of each transformation technique a^S, $y = (y_S)_{S \in 2^N \backslash \{\emptyset\}}$.

Take an $S \in 2^N \backslash \{\emptyset\}$. The value $v(S)$ of this coalition is defined by (9). From our construction it follows that $p^T A = p(u)^T$, $p^T \omega(S) = 0$,

$$\begin{cases} Gy \leq \omega(S) \\ y_i = 0 \text{ if } i \notin D(S) \end{cases} \Leftrightarrow \begin{cases} \sum_T e^T y_T \leq e^S \\ y_T = 0 \text{ if } T \cap S = \emptyset \end{cases} \Leftrightarrow \begin{cases} \sum_{T: T \cap S \neq \emptyset} e^T y_T \leq e^S \\ y_T = 0 \text{ if } T \cap S = \emptyset \end{cases}$$

So we get that

$$\begin{array}{rl} v(S) = \max & p(u)^T y \\ \text{s.t.} & \sum_{T: T \cap S \neq \emptyset} e^T y_T \leq e^S \\ & y_T \geq 0 \text{ for all } T \\ & y_T = 0 \text{ if } T \cap S = \emptyset \end{array}$$

According to the last constraint, we can rewrite this as

$$\begin{array}{rl} v(S) = \max & \sum_{T: T \cap S \neq \emptyset} u(T) y_T \\ \text{s.t.} & \sum_{T: T \cap S \neq \emptyset} e^T y_T \leq e^S \\ & y_T \geq 0 \text{ for all } T \\ & y_T = 0 \text{ if } T \cap S = \emptyset \end{array}$$

Since (N, u) is a totally balanced game with nonnegative values, it follows that $v(S) = u(S)$ and $y_T = 1$ if $T = S$ and $y_T = 0$ otherwise. $\square$

This theorem implies in particular that each linear production game, as introduced by OWEN (1975) and studied in CURIEL, DERKS and TIJS (1989), can be written as an extended LTP game. Since each totally balanced game with nonnegative values can also be written as a linear production game, the other way around also holds.

A Appendix

In this section we present the proofs that were omitted in section 4.

Lemma 1 *The function R^i as defined in (4) is continuous for all $i \in N$.*

Proof. This proof consists of six steps. Let $i \in N$.

(i) Define the multifunction F^i from $\mathbb{R}^M_+$ to $\mathbb{R}_+$ by $F^i(x) = \{y_i | \, g^i y_i \leq x; \; y_i \geq 0\}$ then $F^i(x) = [0, \overline{y}_i(x)]$ where $\overline{y}_i(x) = \max\{y_i | \, g^i y_i \leq x; \; y_i \geq 0\}$. Since g^i and x both contain a finite number of elements that are finite and nonnegative and since we assumed that g^i contains at least one positive element, the number $\overline{y}_i(x)$ is finite. So, F^i is a compact-valued multifunction.

(ii) We show that $\overline{y}_i(x)$ is a continuous function. Define the carrier set of g^i by $C(g^i) = \{j \in M | \, g^i_j > 0\}$. This set is nonempty. Next, consider the following observations:

- $j \notin C(g^i) \quad \Rightarrow \quad g^i_j \overline{y}_i(x) = 0 \leq x_j$
- $j \in C(g^i), g^i_j \overline{y}_i(x) = x_j \quad \Rightarrow \quad \overline{y}_i(x) = x_j / g^i_j$
- $j \in C(g^i), g^i_j \overline{y}_i(x) < x_j \quad \Rightarrow \quad \overline{y}_i(x) < x_j / g^i_j$

These observations imply that $\overline{y}_i(x) = \min\{x_j / g^i_j | \, j \in C(g^i)\}$. Since g^i is a fixed vector, $C(g^i)$ is a fixed set containing a finite number of elements, so $\overline{y}_i(x)$ is the minimum of a finite number of continuous functions. We conclude that $\overline{y}_i(x)$ is a continuous function.

(iii) We show that F^i is an upper semicontinuous (usc) multifunction. Let $x^i \in \mathbb{R}^M_+$ and let O be an open set in $\mathbb{R}_+$ such that $F^i(x^i) \subset O$. Then $\overline{y}_i(x^i) \in O$. Since $\overline{y}_i$ is a continuous function, $\overline{y}_i^{-1}(O)$ is an open set in $\mathbb{R}^M_+$. By definition of the inverse, for all $\widehat{x}^i \in \overline{y}_i^{-1}(O)$ it holds that $\overline{y}_i(\widehat{x}^i) \in O$ and thus $F^i(\widehat{x}^i) \subset O$. So, F^i is usc.

(iv) We show that F^i is a lower semicontinuous (lsc) multifunction. Let $x^i \in \mathbb{R}^M_+$ and let O be an open set in $\mathbb{R}_+$ such that $F^i(x^i) \cap O \neq \emptyset$. If $\overline{y}_i(x^i) = 0$ then $F^i(x^i) = \{0\}$ and $0 \in O$. Take an open set O_x in $\mathbb{R}^M_+$ such that $x^i \in O_x$. Then for all $\widehat{x}^i \in O_x$ it holds that

$$F^i(\widehat{x}^i) \cap O \supset \{0\} \cap O = \{0\} \neq \emptyset$$

If $\overline{y}_i(x^i) > 0$ then there is a $t \in F^i(x^i) \cap O$ such that $0 < t < \overline{y}_i(x^i)$. Define $\widetilde{x}^i = g^i t$ then $\overline{y}_i(\widetilde{x}^i) = t$. Since $\overline{y}_i(x^i) > t$ there is an $r > 0$ such that for all $\widehat{x}^i \in B(x^i, r)$, the sphere in $\mathbb{R}^M_+$ around x^i with radius r, it holds that $\overline{y}_i(\widehat{x}^i) \geq \overline{y}_i(\widetilde{x}^i) = t$. This implies that for all $\widehat{x}^i \in B(x^i, r)$

$$F^i(\widehat{x}^i) \cap O = [0, \overline{y}_i(\widehat{x}^i)] \cap O \supset [t, \overline{y}_i(\widehat{x}^i)] \cap O \supset \{t\} \neq \emptyset$$

So, F^i is lsc.

(v) Define $f^i(x^i, y_i) = p^T(x^i + a^i y_i)$. This function is the sum of two continuous functions, so f^i is continuous.

(vi) Since $F^i : \mathbb{R}^M_+ \to \mathbb{R}_+$ is a compact-valued usc and lsc multifunction and $f^i : \mathbb{R}^M_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, the Maximum theorem of BERGE (1963) says that $R^i(x^i) = \max\{f^i(x^i, y_i) | \, y_i \in F^i(x^i)\}$ is a continuous function for all $i \in N$. $\square$

Lemma 2 *The function R^i as defined in (4) is monotone and quasi-concave for all $i \in N$.*

Proof. Let $i \in N$. First, we show that R^i is monotone. Let $z, x \in \mathbb{R}_+^M$ such that $z \geq x$, $z \neq x$. If $g^i y_i \leq x$ then also $g^i y_i \leq z$ so $\{y_i |\ g^i y_i \leq x;\ y_i \geq 0\} \subset \{y_i |\ g^i y_i \leq z;\ y_i \geq 0\}$. We assumed in theorem 4 that $p \in \mathbb{R}_+^M$ so $p^T x \leq p^T z$. Now it holds that

$$\begin{aligned} R^i(x) = &\ \max\{p^T(x + a^i y_i) |\ g^i y_i \leq x;\ y_i \geq 0\} \\ \leq &\ \max\{p^T(z + a^i y_i) |\ g^i y_i \leq z;\ y_i \geq 0\} = R^i(z) \end{aligned}$$

so R^i is a monotone function for all $i \in N$.

Next, we show that R^i is quasi-concave, i.e. we show that for all $b, c \in \mathbb{R}_+^M$, $b \neq c$ and for all $\alpha \in (0, 1)$ it holds that $R^i(\alpha b + (1-\alpha)c) \geq \min\{R^i(b), R^i(c)\}$. Let $b, c \in \mathbb{R}_+^M$, $b \neq c$ and let $\alpha \in (0, 1)$.

If $p^T a^i \leq 0$ then

$$\begin{aligned} R^i(\alpha b + (1-\alpha)c) = &\ p^T(\alpha b + (1-\alpha)c) = \alpha p^T b + (1-\alpha) p^T c \\ \geq &\ \min\{p^T b, p^T c\} = \min\{R^i(b), R^i(c)\} \end{aligned}$$

If $p^T a^i > 0$ then $R^i(\alpha b + (1-\alpha)c) = p^T(\alpha b + (1-\alpha)c + a^i \overline{y}_i(\alpha b + (1-\alpha)c))$. By definition of $\overline{y}_i$ it holds that

$$\begin{aligned} \overline{y}_i(\alpha b + (1-\alpha)c) = &\ \min\{(\alpha b + (1-\alpha)c)_j / g^i_j |\ j \in C(g^i)\} \\ \geq &\ \min\{\alpha b_j / g^i_j |\ j \in C(g^i)\} + \min\{(1-\alpha)c_j / g^i_j |\ j \in C(g^i)\} \\ = &\ \alpha \min\{b_j / g^i_j |\ j \in C(g^i)\} + (1-\alpha) \min\{c_j / g^i_j |\ j \in C(g^i)\} \\ = &\ \alpha \overline{y}_i(b) + (1-\alpha) \overline{y}_i(c) \end{aligned}$$

This implies that

$$\begin{aligned} R^i(\alpha b + (1-\alpha)c) = &\ p^T(\alpha b + (1-\alpha)c + a^i \overline{y}_i(\alpha b + (1-\alpha)c)) \\ = &\ \alpha p^T b + (1-\alpha) p^T c + p^T a^i \overline{y}_i(\alpha b + (1-\alpha)c) \\ \geq &\ \alpha p^T b + (1-\alpha) p^T c + p^T a^i [\alpha \overline{y}_i(b) + (1-\alpha) \overline{y}_i(c)] \\ = &\ \alpha p^T(b + a^i \overline{y}_i(b)) + (1-\alpha) p^T(c + a^i \overline{y}_i(c)) \\ = &\ \alpha R^i(b) + (1-\alpha) R^i(c) \\ \geq &\ \min\{R^i(b), R^i(c)\} \end{aligned}$$

So, we conclude that R^i is a quasi-concave function for all $i \in N$. $\square$

The next lemma shows that there is another way to find the equilibrium solution $(x^*(i), y_i^*, \widehat{y}_i^*)$ of model 2. This result is used in the proof of theorem 5.

Lemma 3 *Let $\langle N, A, \omega, p \rangle$ be an LTP situation. Then the following two statements are equivalent.*

a) $(x^*(i), y_i^*, \widehat{y}_i^*)$ *is an optimal solution of*

$$\begin{array}{rl} \max & p^T(x(i) + a^i y_i) \\ \text{s.t.} & g^i y_i \le x(i) \\ & y_i \ge 0 \\ & q^T x(i) \le q^T(\omega(i) + a^i \widehat{y}_i) \\ & g^i \widehat{y}_i \le \omega(i) \\ & \widehat{y}_i \ge 0 \\ & x(i) \ge 0 \end{array}$$

b) $(x^*(i), y_i^*, \widehat{y}_i^*)$ *is an optimal solution of*

$$\begin{array}{rlrl} \max & p^T(x(i) + a^i y_i) & \text{where } v^q(i) = \max & q^T(\omega(i) + a^i \widehat{y}_i) \\ \text{s.t.} & g^i y_i \le x(i) & \text{s.t.} & g^i \widehat{y}_i \le \omega(i) \\ & y_i \ge 0 & & \widehat{y}_i \ge 0 \\ & q^T x(i) \le v^q(i) & & \\ & x(i) \ge 0 & & \end{array}$$

Proof. First, let $(x^*(i), y_i^*, \widehat{y}_i^*)$ be an optimal solution of a). It remains to show that $\widehat{y}_i^*$ is an optimal solution of $v^q(i)$, i.e.

$$\begin{array}{rl} q^T(\omega(i) + a^i \widehat{y}_i^*) = \max & q^T(\omega(i) + a^i \widehat{y}_i) \\ \text{s.t.} & g^i \widehat{y}_i \le \omega(i) \\ & \widehat{y}_i \ge 0 \end{array}$$

We show this by contradiction. Suppose that it is not true. Then there exists a $\widetilde{y}_i$ such that

$$\begin{array}{rl} q^T(\omega(i) + a^i \widetilde{y}_i) = \max & q^T(\omega(i) + a^i \widehat{y}_i) > q^T(\omega(i) + a^i \widehat{y}_i^*) \\ \text{s.t.} & g^i \widehat{y}_i \le \omega(i) \\ & \widehat{y}_i \ge 0 \end{array}$$

So $v^q(i) = q^T(\omega(i) + a^i \widetilde{y}_i)$. Producer i can spend more money and in particular he can buy a bundle of goods $\widetilde{x}(i)$ such that he has more of all goods: $\widetilde{x}(i)_j > x^*(i)_j$ for all $j \in M$. But since he now has more goods, he can produce more goods, reaching a higher activity level and receiving a higher profit: $p^T(\widetilde{x}(i) + a^i y_i) > p^T(x^*(i) + a^i y_i)$. This contradicts the assumption that $(x^*(i), y_i^*, \widehat{y}_i^*)$ maximizes the profit of producer i.

Next, let $(x^*(i), y_i^*, \widehat{y}_i^*)$ be an optimal solution of b). Then $(x^*(i), y_i^*, \widehat{y}_i^*)$ satisfies all the constraints in a). Since the maximization problem of a) contains more constraints than the problem of b), $(x^*(i), y_i^*, \widehat{y}_i^*)$ is also an optimal solution of a). □

References

[1] BERGE, C.: *Topological Spaces*, New York: Macmillan, 1963.

[2] CURIEL, I.J., J.J.M. DERKS and S.H. TIJS: ”On Balanced Games and Flow Games with Committee Control,” *OR Spektrum*, **11** (1989), p. 83–88.

[3] DEBREU, G.: *Theory of Value*, New Haven: Yale University Press, 1959.

[4] KALAI, E. and E. ZEMEL: ”Totally Balanced Games and Games of Flow,” *Mathematics of Operations Research*, **7** (1982), p. 476–478.

[5] NOUWELAND, A. VAN DEN, H. AARTS and P. BORM: ”Multi-commodity games,” *Methods of Operations Research*, **63** (1989), p. 329–338.

[6] OWEN, G.: ”On the Core of Linear Production Games,” *Mathematical Programming*, **9** (1975), p. 358–370.

[7] SHAPLEY, L.S. and M. SHUBIK: ”On Market Games,” *Journal of Economic Theory*, **1** (1969), p. 9–25.

[8] TIJS, S.H.: *Game Theory*, syllabus, Tilburg University, 1995.