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Effectivity Functions and Associated Claim Game Correspondences*

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In this paper upper cycle free effectivity functions are implemented by strategic claim game correspondences. Conditions are provided such that an effectivity function coincides with the α- and β-effectivity functions corresponding to its associated claim game correspondence. Furthermore, the notion of the core of an effectivity function is extended to subsets of alternatives, and it is shown that for upper cycle free effectivity functions this so-called setcore is always nonempty. Moreover, given a preference profile, relations between the setcore of an effectivity function and strong Nash equilibria of the associated claim game correspondence are established. *Journal of Economic Literature* Classification Numbers: 025, 026. © 1995 Academic Press, Inc.

1. INTRODUCTION AND NOTATION

The first to indicate connections between different types of games were von Neumann and Morgenstern (1944). They showed that it was possible to construct a cooperative game with side payments (or: a game in coalitional form) from a game in strategic form, and conversely. Later, others also established relationships between cooperative and non-cooperative games (see, for example, Nash, 1950; Aumann, 1961, 1967).

Borm and Tijs (1992) introduced a "claim" game in strategic form corresponding to an NTU-game. In the claim game strategies of players can be interpreted as claims on coalitions and payoffs. Among others

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Borm and Tijs showed that if the NTU-game is superadditive, strong core elements of the NTU-game correspond to strong Nash equilibria of the associated claim game.

In the domain of social choice theory there are several papers which provide relations between cooperative and non-cooperative situations. An important example is the implementation of effectivity functions by game forms in Moulin and Peleg (1982). Effectivity functions describe coalitional power in a society. The interpretation is that if a coalition is effective for a subset of alternatives, it can force the outcome within that set.

Game forms describe non-cooperative situations and are introduced by Gibbard (1973). Peleg (1984b) introduced game correspondences as an extension of game forms. In a game form the outcome function assigns to every strategy vector one alternative, while in a game correspondence the outcome function assigns to each strategy vector a subset of alternatives. Similar to social choice correspondences the use of game correspondences is particularly useful in situations where there are many ties and it therefore is impossible to select a single alternative without violating certain elementary equity requirements such as anonymity and neutrality.

The aim of this paper is to implement effectivity functions by game correspondences using techniques similar to those of Borm and Tijs (1992). Therefore we extend the definition of the core (at a given preference profile), introduced by Moulin and Peleg (1982), to be a collection of subsets of alternatives rather than only one subset. This is the subject of Section 3. Moreover, Section 3 shows that this extended core, which we call the setcore of an effectivity function, is never empty if the effectivity function is upper cycle free.

First, in Section 2 we recall the definition of an effectivity function and establish some preliminary results.

Section 4 discusses game correspondences and, given a preference profile, formally defines the notion of strong Nash equilibrium. Moreover, we recall the definition of the α-effectivity function E_α^G, and that of the β-effectivity function E_β^G, associated with a game correspondence G, as introduced in Peleg (1984b).

Section 5 is the central part of the paper. We construct the claim game correspondence $G(E)$ associated with an upper cycle free effectivity function E and we show that the α- and β-effectivity functions associated with $G(E)$ are identical, i.e., that the claim game correspondence is tight. We also show that the α-effectivity function associated with $G(E)$ equals E if and only if E is weakly A-monotonic and superadditive. Furthermore, Section 5 establishes relations between cooperative and non-cooperative solution concepts. It is shown that if E is superadditive, the setcore elements of E at a given preference profile exactly correspond to the outcomes of strong Nash equilibria of $G(E)$ at that profile.

In Section 6 we discuss a process for deriving a game form from a claim game correspondence in such a way that the α-effectivity function does not change. As a result of this process we obtain an alternative proof of the fact that, given an effectivity function E, it is possible to construct a game form G such that $E = E_\alpha^G$ if and only if E is superadditive and A-monotonic (cf. Moulin, 1983).

Notation. Let D be a set. By $\mathcal{P}(D)$ we denote the power set of D, i.e., $\mathcal{P}(D) = \{C \mid C \subset D\}$, and by 2^D we denote the set of all non-empty subsets of D. So $2^D = \mathcal{P}(D)\backslash\{\varnothing\}$.

A *preference relation* R on D is a subset of $D \times D$ which satisfies

(i) completeness: for all $a, b \in D$, $(a, b) \in R$ or $(b, a) \in R$; and
(ii) transitivity: for all $a, b, c \in D$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

We use the notation $a\ R\ b$ if $(a, b) \in R$, and $a\ P\ b$ if $a\ R\ b$ and not $b\ R\ a$.

If A is a set of alternatives, with $a, b \in A$, N a finite set of individuals, and R^i a preference relation on A for all $i \in N$, then $a\ R^i\ b$ is to be interpreted as "alternative a is at least as good as alternative b according to R^i." Furthermore, $R^S := (R^i)_{i \in S}$, $P^S := (P^i)_{i \in S}$, and R^N is called a *(preference) profile* on A.

2. Effectivity Functions

Let A be a finite set of alternatives and let N be the set $\{1, \ldots, n\}$. N is called a *society*, members of N are called *players* or *voters*, and non-empty subsets of N are called *coalitions*.

Definition 2.1. An *effectivity function* is a map E: $2^N \to \mathcal{P}(2^A)$ such that

(i) $E(N) = 2^A$,
(ii) $A \in E(S)$ for all $S \in 2^N$.

The interpretation of E is as follows: if $B \in E(S)$, then S can force the final decision within the subset B of alternatives. By definition the society N can force every outcome.

Effectivity functions were introduced by Moulin and Peleg (1982), but the idea of effectiveness of coalitions was proposed earlier by Rosenthal (1972). In the following definition we collect several properties of effectivity functions which we use later on. With the exception of (ii) these properties can all be found in Abdou and Keiding (1991).

DEFINITION 2.2. Let E: $2^N \to \mathcal{P}(2^A)$ be an effectivity function.

(i) E is *A-monotonic* if for all $S \in 2^N$ and all B, $B' \in 2^A$ with $B \subset B'$ and $B \in E(S)$, we have $B' \in E(S)$.

(ii) E is *weakly A-monotonic* if for all $S \in 2^N$ and all B, $B' \in 2^A$, with $B \subset B'$: if $B \in E(S)$, then there exists a partition $\{S_1, \ldots, S_k\}$ of S and there are $B'_r \in E(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $B' = \bigcap_{r=1}^{k} B'_r$.

(iii) E is *N-monotonic* if for all S, $S' \in 2^N$ with $S \subset S'$, and all $B \in 2^A$, with $B \in E(S)$, we have $B \in E(S')$.

(iv) E is *superadditive* if for all S_1, $S_2 \in 2^N$, with $S_1 \cap S_2 = \varnothing$, and all $B_1 \in E(S_1)$, $B_2 \in E(S_2)$, we have $B_1 \cap B_2 \in E(S_1 \cup S_2)$.

(v) E is *upper cycle free* if for all $S_1, \ldots, S_k \in 2^N$ with $S_r \cup S_t = \varnothing$ for all r, $t \in \{1, \ldots, k\}$, $r \neq t$ and all $B_1, \ldots, B_k \in 2^A$ with $B_r \in E(S_r)$ for all $r \in \{1, \ldots, k\}$, we have $\bigcap_{r=1}^{k} B_r \neq \varnothing$.

Clearly, if E is superadditive, then E is upper cycle free and N-monotonic, and if E is A-monotonic, then E is weakly A-monotonic.

Now we define the superadditive cover $\overline{E}$ of an effectivity function E (Peleg, 1984a).

DEFINITION 2.3. Let E: $2^N \to \mathcal{P}(2^A)$ be an effectivity function. The *superadditive cover* $\overline{E}$ of E is a map that assigns to each $S \in 2^N$ a subset of $\mathcal{P}(A)$ such that it satisfies the following property: $B \in \mathcal{P}(A)$ is an element of $\overline{E}(S)$ if and only if there exist a partition $\{S_1, \ldots, S_k\}$ of S and sets $B_r \in E(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $B = \bigcap_{r=1}^{k} B_r$.

So, $\overline{E}$ is a function from 2^N to $\mathcal{P}(\mathcal{P}(A))$. It is easy to see that $E \subset \overline{E}$ for all effectivity functions E, i.e., $E(S) \subset \overline{E}(S)$ for all $S \in 2^N$. It is clear that $\overline{E}$ is an effectivity function ($\varnothing \notin \overline{E}(S)$ for all $S \in 2^N$) if and only if E is upper cycle free. The name "superadditive cover" is explained in the next two lemmas.

LEMMA 2.4. *Let E: $2^N \to \mathcal{P}(2^A)$ be an effectivity function. The following three properties are equivalent.*

(i) E is upper cycle free.

(ii) $\overline{E}$ is a superadditive effectivity function.

(iii) There exists a superadditive effectivity function E' such that $E \subset E'$.

Proof. (i) ⇒ (ii) Let E be upper cycle free. Then by definition $\varnothing \notin \overline{E}(S)$ for all $S \in 2^N$. Hence $\overline{E}$ is an effectivity function. Let S, $T \in 2^N$ with $S \cap T = \varnothing$ and let $B \in \overline{E}(S)$ and $D \in \overline{E}(T)$. Then there are partitions $\{S_1, \ldots, S_k\}$ of S and $\{T_1, \ldots, T_l\}$ of T and there are $B_1, \ldots, B_k$ and $D_1, \ldots, D_l$ with $B_r \in E(S_r)$ for all $r \in \{1, \ldots, k\}$ and $D_s \in E(T_s)$ for all $s \in \{1, \ldots, l\}$ such that $B = \bigcap_{r=1}^{k} B_r$ and $D = \bigcap_{s=1}^{l} D_s$. Then

$\{S_1, \ldots S_k, T_1, \ldots, T_l\}$ is a partition of $S \cup T$ and therefore by definition of $\overline{E}$ we obtain $B \cap D \in \overline{E}(S \cup T)$. So $\overline{E}$ is superadditive.

(ii) $\Rightarrow$ (iii) Trivial because $E \subset \overline{E}$.

(iii) $\Rightarrow$ (i) Let E' be a superadditive effectivity function such that $E \subset E'$. Since E' is superadditive, E' is upper cycle free. But then E, too, is upper cycle free because $E \subset E'$. ∎

The following lemma shows that for an upper cycle free effectivity function E, $\overline{E}$ is the smallest superadditive effectivity function containing E.

LEMMA 2.5. *Let* E: $2^N \to \mathcal{P}(2^A)$ *be an effectivity function. Then for each superadditive effectivity function* E' *with* $E \subset E'$ *we have* $E \subset \overline{E} \subset E'$.

Proof. Let E' be a superadditive effectivity function such that $E \subset E'$. Since $E \subset E'$ and E' is superadditive, it follows from Lemma 2.4 that E is upper cycle free and that $\overline{E}$ is superadditive. Let $S \in 2^N$ and let $B \in \overline{E}(S)$. Then there is a partition $\{S_1, \ldots, S_k\}$ of S and there are $B_1, \ldots, B_k$ with $B_r \in E(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $B = \bigcap_{r=1}^{k} B_r$. Then by superadditivity of E' and $E \subset E'$ it follows that $B \in E'(S)$. ∎

Lemma 2.6, which we will use in Section 5, describes monotonicity relations between an upper cycle free effectivity function and its superadditive cover.

LEMMA 2.6. *Let* E: $2^N \to \mathcal{P}(2^A)$ *be an upper cycle free effectivity function. Then* E *is weakly* A*-monotonic if and only if* $\overline{E}$ *is* A*-monotonic.*

Proof. ($\Rightarrow$) Let E be weakly A-monotonic. Let $B, B' \in 2^A$ with $B \subset B'$ and suppose $B \in \overline{E}(S)$ for some $S \in 2^N$. We prove that $B' \in \overline{E}(S)$. Since $B \in \overline{E}(S)$, there is a partition $\{S_1, \ldots, S_k\}$ of S and there are $B_r \in E(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $B = \bigcap_{r=1}^{k} B_r$. Define $B'_r := B_r \cup B'$ for all $r \in \{1, \ldots, k\}$. Then $B_r \subset B'_r$ for all $r \in \{1, \ldots, k\}$ and $\bigcap_{r=1}^{k} B'_r = B'$. E is weakly A-monotonic, so for each $r \in \{1, \ldots, k\}$ there exist a partition $\{S_{r1}, \ldots, S_{rk_r}\}$ of S_r and $B'_{rs} \in E(S_{rs})$ for all $s \in \{1, \ldots, k_r\}$ such that $B'_r = \bigcap_{r=1}^{k_r} B'_{rs}$. But then $\{S_{rs} \mid s \in \{1, \ldots, k_r\}, r \in \{1, \ldots, k\}\}$ is a partition of S, and since $B'_{rs} \in E(S_{rs})$ for all $s \in \{1, \ldots, k_r\}$ and $r \in \{1, \ldots, k\}$, we obtain by definition of $\overline{E}$ that $B' = \bigcap_{r=1}^{k} \bigcap_{s=1}^{k_r} B'_{rs} \in \overline{E}(S)$. So $\overline{E}$ is A-monotonic.

($\Leftarrow$) Let $\overline{E}$ be A-monotonic. Let $B, B' \in 2^A$ with $B \subset B'$ and suppose $B \in E(S)$ for some $S \in 2^N$. Since $\overline{E}$ is A-monotonic and $E \subset \overline{E}$, we have $B' \in \overline{E}(S)$. Hence, there is a partition $\{S_1, \ldots, S_k\}$ of S and there are $B'_r \in E(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $B' = \bigcap_{r=1}^{k} B'_r$. So E is weakly A-monotonic. ∎

3. The Setcore of an Effectivity Function

Let E: $2^N \to \mathcal{P}(2^A)$ be an effectivity function and R^N a profile on A. The problem is, given all the preferences of the players, to find an alternative, or a set of alternatives, which every player can agree upon. One important solution concept is the core, defined (cf. Moulin and Peleg, 1982) as the subset of A which consists of all "undominated" elements with respect to the profile R^N. Other solution concepts which assign to an effectivity function and a profile a subset of alternatives are the nucleus (Holzman, 1987) and the supernucleus (Fristrup and Keiding, 1988).

Definition 3.1. (cf. Moulin and Peleg, 1982). Let E: $2^N \to \mathcal{P}(2^A)$ be an effectivity function, and let R^N be a profile on A. The *core of E at R^N*, Core(E, R^N), consists of all alternatives $a \in A$ for which there do not exist $S \in 2^N$ and $B \in E(S)$ with $B \not\ni a$ and $b\ P^S\ a$ for all $b \in B$.

We will now modify the notion of the core in the sense that it will assign to an effectivity function and a preference profile a collection of subsets of A rather than one subset. This modification of the core is called the setcore.

For this aim we first must extend preference profiles on A to preference profiles on 2^A.

Definition 3.2. Let R be a preference relation on A. We define the extension $\tilde{R}$ of R to 2^A by: for all B, $B' \in 2^A$ we have $B'\ \tilde{R}\ B$ if and only if

(i) for all $b' \in B' \backslash B$ and all $b \in B$ we have $b'\ R\ b$,
(ii) for all $b' \in B'$ and all $b \in B \backslash B'$ we have $b'\ R\ b$.

Furthermore, we define $B'\ \tilde{P}\ B$ if and only if $B \backslash B' \neq \emptyset$ and

(iii) for all $b' \in B' \backslash B$ and all $b \in B$ we have $b'\ P\ b$,
(iv) for all $b' \in B'$ and all $b \in B \backslash B'$ we have $b'\ P\ b$.

We write $\tilde{R}^S$ instead of $(\tilde{R}^i)_{i \in S}$ and $\tilde{P}^S$ instead of $(\tilde{P}^i)_{i \in S}$.

It readily follows that, if we restrict $\tilde{R}$ to the singletons of 2^A, then this restriction can be identified with R by identifying a singleton with its unique element. Note that $\tilde{R}$ is also reflexive and transitive but it need not be complete. According to this definition $B' \in 2^A$ is preferred to $B \in 2^A$ if, in going from B to B', the elements added to B (i.e., $B' \backslash B$) are better than the ones already present, and the elements of B that are dropped (i.e., $B \backslash B'$) are worse than the elements of B which are kept (i.e., $B \cap B'$).

Note that $B'\ \tilde{P}\ B$ implies that B' is not a set that strictly contains B.

Now we are able to give the definition of the setcore.

DEFINITION 3.3. Let $E: 2^N \to \mathcal{P}(2^A)$ be an effectivity function and let R^N be a profile on A. Let $B \in 2^A$ be a subset of alternatives. B is an element of the *setcore of E at* R^N, Setcore(E, R^N), if there do not exist $S \in 2^N$, and $B' \in E(S)$, such that $B' \tilde{P}^S B$.

Remark 3.4. The setcore of E at R^N definitionally extends the core to subsets of alternatives. One easily checks that $a \in$ Core(E, R^N) if and only if $\{a\} \in$ Setcore(E, R^N). Moreover, it holds that, if $B \subset$ Core(E, R^N), $B \neq \emptyset$, then $B \in$ Setcore(E, R^N).

For, suppose $B \notin$ Setcore(E, R^N). Then there exist a coalition $S \in 2^N$ and a $B' \in E(S)$ such that $B' \tilde{P}^S B$. Hence, $B \backslash B' \neq \emptyset$. Take $a \in B \backslash B'$. Since $B' \tilde{P}^S B$, we obtain by definition of $\tilde{P}^S$ that $b' P^S a$ for all $b' \in B'$. Hence $a \notin$ Core(E, R^N), which leads to a contradiction.

From Remark 3.4 it follows that the setcore of an effectivity function (at a profile) is non-empty, whenever the core of the effectivity function (at this profile) is non-empty. However, Example 3.5 shows that if the core of an effectivity function at a profile is empty, then the setcore is not necessarily empty.

EXAMPLE 3.5. Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$.
Define $E: 2^N \to \mathcal{P}(2^A)$ as follows. For $S \in 2^N$

$$E(S) := \begin{cases} \{A\} & \text{if } |S| = 1 \\ 2^A & \text{if } |S| > 1. \end{cases}$$

(E represents a situation in which the majority decides.) Clearly, E is an upper cycle free effectivity function.

Define the preference profile $R^N = (R^1, R^2, R^3)$ on A by

$$R^1 = a\ b\ c,$$
$$R^2 = b\ c\ a,$$
$$R^3 = c\ a\ b.$$

Here there are no indifferences and the preferences of the players are denoted in decreasing order, i.e., player 1 likes a the most, then b, and then c, etc.

One easily checks that Core$(E, R^N) = \emptyset$. However, the setcore of E at R^N is non-empty: Setcore$(E, R^N) = \{A\}$, which is a rather natural solution.

Now we formulate the main theorem of this section, which yields an existence theorem of the setcore on the class of upper cycle free effectivity functions.

THEOREM 3.6. *Let E: $2^N \to \mathcal{P}(2^A)$ be an upper cycle free effectivity function, and let R^N be a profile on A. Then* Setcore$(E, R^N) \neq \emptyset$.

Proof. The proof is by induction on $|A|$. Clearly, if $|A| = 1$, then Setcore$(E, R^N) \neq \emptyset$. Let $|A| > 1$, and assume that Setcore$(E', R'^N) \neq \emptyset$ for all upper cycle free effectivity functions E': $2^N \to \mathcal{P}(2^{A'})$ with $|A'| < |A|$ and all profiles R'^N on A'. Suppose Setcore$(E, R^N) = \emptyset$. Then, in particular, there exist a coalition $T \in 2^N$ and an $A' \in E(T)$ such that $A' \tilde{P}^T A$. By definition of $\tilde{P}^T$ it follows that $|A'| < |A|$ and that $A' \tilde{P}^T (A \backslash A')$.

Define E': $2^N \to \mathcal{P}(2^{A'})$ by $E'(N) := 2^{A'}$ and for $S \in 2^N \backslash \{N\}$ and $B \in 2^{A'}$

$$B \in E'(S) \Leftrightarrow B = A' \text{ or there exist } C \subset A \backslash A' \text{ and } D \subset A', D \neq \emptyset \text{ such that}$$

$$B \cup C \in E(S) \qquad \text{and} \qquad (B \cup C) \tilde{P}^S D.$$

Clearly, E' is an effectivity function. Furthermore, let P'^N be the restriction of P^N to A'. We will show that

(I) Setcore$(E', R'^N) = \emptyset$;
(II) E' is upper cycle free.

(I) Let $B' \in 2^{A'}$. Since Setcore$(E, R^N) = \emptyset$, there are $S \in 2^N$ and $B \in E(S)$ such that $B \tilde{P}^S B'$. Let $X := B \cap A'$.

CLAIM. *$X \in E'(S)$ and $X \tilde{P}'^S B'$.*

From the claim it immediately follows that $B' \notin$ Setcore(E', R'^N). Hence, Setcore$(E', R'^N) = \emptyset$.

Proof of the claim. First we show that $X \neq \emptyset$. Suppose $X = \emptyset$. Then $B \subset A \backslash A'$ and therefore, $B' \cap B = \emptyset$. Since $A' \tilde{P}^T (A \backslash A')$, it follows that $S \cap T = \emptyset$. Since $A' \cap B = \emptyset$, this leads to a contradiction with upper cycle freeness of E. Hence, $X \neq \emptyset$.

Since $X \cup (B \backslash A') = B \in E(S)$, and $B \tilde{P}^S B'$ it follows that $X \in E'(S)$. Using the fact that $B \tilde{P}^S B'$ and $B' \cap B \subset X \subset B$, it follows from the definition of $\tilde{P}'^S$ that $X \tilde{P}'^S B'$.

(II) Let $S_1, \ldots, S_k \in 2^N$ with $S_r \cap S_t = \emptyset$ for all $r, t \in \{1, \ldots, k\}$, $r \neq t$, and $B_1, \ldots, B_k \in 2^{A'}$ be such that $B_r \in E'(S_r)$ for all $r \in \{1, \ldots, k\}$. It suffices to prove that $\bigcap_{r=1}^k B_r \neq \emptyset$.

Suppose that $\bigcap_{r=1}^k B_r = \emptyset$. We assume, w.l.o.g., that $B_r \neq A'$ for all r. From the definition of E' it follows that there are $C_1, \ldots, C_k \subset A \backslash A'$

and $D_1, \ldots, D_k \subset A'$ such that $B_r \cup C_r \in E(S_r)$ and $(B_r \cup C_r)\, \tilde{P}^{S_r}\, D_r$ for all r. Since E is upper cycle free, it follows that $\bigcap_{r=1}^{k} (B_r \cup C_r) = \bigcap_{r=1}^{k} C_r \neq \emptyset$. Let $b \in \bigcap_{r=1}^{k} C_r$. Since $b \notin A'$ and $(B_r \cup C_r)\tilde{P}^{S_r}\, D_r$ for all r, it follows that $\{b\}\tilde{P}^{S_r}\, D_r$ for all r. As $A'\ \tilde{P}^T\ (A \backslash A')$, we have $S_r = \emptyset$ for all r.

Consider $T, S_1, \ldots, S_k$ and $A' \in E(T)$, $B_1 \cup C_1 \in E(S_1), \ldots, B_k \cup C_k \in E(S_k)$. Since $\bigcap_{r=1}^{k} (B_r \cup C_r) \cap A' = \emptyset$, it follows that E is not upper cycle free. This leads to a contradiction and hence E' is upper cycle free.

Statements (I) and (II) are in contradiction with the induction hypothesis. So, Setcore$(E, R^N) \neq \emptyset$. ■

Example 3.7 shows that upper cycle freeness is not a necessary property for guaranteeing non-emptyness of the setcore at every profile.

EXAMPLE 3.7. Let $N = \{1, 2, 3\}$ and $A = \{a_0, \ldots, a_5\}$. Let $B_1 = \{a_0, a_1, a_3\}$, $B_2 = \{a_1, a_2, a_4\}$, $B_3 = \{a_3, a_4, a_5\}$. Define the effectivity function E by

$$E(\{i\}) = \{B_i, A\}, \qquad i = 1, 2, 3$$

$$E(S) = \{A\} \qquad \text{if } |S| = 2$$

$$E(N) = 2^A.$$

Clearly, E is not upper cycle free.

We leave it to the reader to verify that Setcore$(E, R^N) \neq \emptyset$ for all profiles R^N.

4. GAME CORRESPONDENCES AND ASSOCIATED EFFECTIVITY FUNCTIONS

In this section we formally consider game correspondences and their associated α- and β-effectivity functions. Let $S \in 2^N$ be a coalition. For all $i \in S$, let X_i be a non-empty set. We denote the Cartesian product $\Pi_{i \in S} X_i$ by X_S. If $\sigma_i \in X_i$ for all $i \in S$, then we write σ_S instead of $(\sigma_i)_{i \in S}$.

DEFINITION 4.1. (cf. Peleg, 1984b). A *game correspondence* is an $(n + 2)$-tuple $G = (X_1, \ldots, X_n, A, \pi)$, where X_i is a non-empty set of strategies for each $i \in N$, A is a finite set of alternatives, and π: $X_N \rightarrow 2^A$ is non-imposed, i.e., for each $a \in A$ there is a strategy $\sigma_N := (\sigma_1, \ldots, \sigma_n) \in X_N$ such that $\pi(\sigma_N) = \{a\}$.

The interpretation of G is as follows: given the choice $\sigma_i \in X_i$ of each player $i \in N$, the outcome multifunction π determines a non-empty subset $\pi(\sigma_N)$ of alternatives.

This definition is almost similar to the definition of a game form introduced by Gibbard (1973), where the map π is a surjective function from X_N to A.

Moulin and Peleg (1982) defined the α- and β-effectivity functions associated with a game form. Peleg (1984b) extended this definition to game correspondences.

DEFINITION 4.2. Let $G = (X_1, \ldots, X_n, A, \pi)$ be a game correspondence. The *α-effectivity function E_α^G and the β-effectivity function E_β^G associated with* G are defined as follows. Let $S \in 2^N$. Then

$$E_\alpha^G(S) := \{B \in 2^A \mid \exists \sigma_S \in X_S \, \forall \tau_{N \setminus S} \in X_{N \setminus S} : \pi(\sigma_S, \tau_{N \setminus S}) \subset B\}$$

$$E_\beta^G(S) := \{B \in 2^A \mid \forall \tau_{N \setminus S} \in X_{N \setminus S} \, \exists \sigma_S \in X_S : \pi(\sigma_S, \tau_{N \setminus S}) \subset B\}.$$

Note that when one defines the α- and β-effectivity functions associated with a game form $\pi(\sigma_S, \tau_{N \setminus S}) \subset B$ changes into $\pi(\sigma_S, \tau_{N \setminus S}) \in B$.

The reader can easily verify that E_α^G and E_β^G are indeed effectivity functions (since π is non-imposed) and that $E_\alpha^G(S) \subset E_\beta^G(S)$ for all $S \in 2^N$ and all game correspondences G. A game correspondence G is called *tight* if $E_\alpha^G = E_\beta^G$. Furthermore, it is easy to see that for every game correspondence G, E_α^G and E_β^G are both N-monotonic and A-monotonic and that E_α^G is superadditive.

Some solution concepts for game correspondences at a given profile are recalled in

DEFINITION 4.3. Let $G = (X_1, \ldots, X_n, A, \pi)$ be a game correspondence and R^N a profile. A strategy vector $\sigma_N \in X_N$ is a *Nash equilibrium of* G *at* R^N if there do not exist $i \in N$ and $\tau_i \in X_i$ such that $\pi(\sigma_{N \setminus \{i\}}, \tau_i) \tilde{P}^i \pi(\sigma_N)$. Further, σ_N is called a *strong Nash equilibrium of* G *at* R^N if there do not exist $S \in 2^N$ and $\tau_S \in X_S$ such that $\pi(\sigma_{N \setminus S}, \tau_S)$ $\tilde{P}^S \pi(\sigma_N)$.

5. IMPLEMENTATION RESULTS

This section shows how for an upper cycle free effectivity function E, one can construct a game correspondence $G(E)$ such that $G(E)$ is tight and $E \subset E_\alpha^{G(E)}$. The game correspondence $G(E)$ is called the claim game correspondence associated with E. We provide necessary and sufficient conditions on E such that $E = E_\alpha^{G(E)}$. Finally, this section establishes relationships between the setcore of E at a preference profile R^N and the set of strong Nash equilibria of the claim game correspondence $G(E)$ at R^N.

DEFINITION 5.1. Let $E: 2^N \rightarrow \mathcal{P}(2^A)$ be an upper cycle free effectivity function. The *claim game correspondence* $G(E)$ *associated with* E is given by $G(E) = (X_1, \ldots, X_n, A, \pi)$, where for each $i \in N$

$$X_i := \{(S, B) \in 2^N \times 2^A \mid i \in S, B \in E(S)\},$$

and for $\sigma_N = (S_i, B_i)_{i \in N} \in X_N$, $\pi(\sigma_N)$ is defined by

$$\pi(\sigma_N) := \begin{cases} \cap\{B \in 2^A \mid B \in F(\sigma_N)\} & \text{if } F(\sigma_N) \neq \emptyset \\ A & \text{if } F(\sigma_N) = \emptyset. \end{cases}$$

Here $F: X_N \rightarrow 2^A$ is defined as follows. For $\sigma_N \in X_N$

$$F(\sigma_N) := \{B \in 2^A \mid \exists S \in 2^N[B \in E(S)], \forall i \in S[\sigma_i = (S, B)]\}.$$

Note that $\pi(\sigma_N) \neq \emptyset$ for all $\sigma_N \in X_N$ because E is upper cycle free.

In the claim game correspondence $G(E)$ the strategy (S_i, B_i) of player $i \in N$ can be interpreted as a claim in the following way. Player i wants to form coalition $S_i \ni i$ and he wants the final outcome to be in a subset B_i of alternatives for which S_i is effective. According to the outcome function π, the final outcome will certainly be in B_i if all the players in S_i have exactly the same claims.

EXAMPLE 5.2. Let $N = \{1, 2, 3, 4\}$ and $A = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$. Define $E: 2^N \rightarrow \mathcal{P}(2^A)$ by

$$E(\{1\}) = \{\{a_0, a_4\}, \{a_1, a_2, a_4\}, A\}, E(\{1, 2\}) = \{\{a_1, a_2, a_5, a_6\}, A\},$$
$$E(\{2, 3\}) = \{\{a_0, a_1, a_2, a_5\}, A\}, E(\{3, 4\}) = \{\{a_1, a_3, a_4, a_6\}, A\},$$
$$E(N) = 2^A, \qquad \text{and} \qquad E(S) = \{A\} \text{ else.}$$

Then E is upper cycle free, so $G(E) = (X_1, \ldots, X_n, A, \pi)$ is well-defined. Note that E is not superadditive. Define the strategy $\sigma_N \in X_N$ by $\sigma_1 = (\{1\}, \{a_0, a_4\})$, $\sigma_2 = \sigma_3 = (\{2, 3\}, \{a_0, a_1, a_2, a_5\})$, $\sigma_4 = (\{3, 4\}, \{a_1, a_3, a_4, a_6\})$. Then $F(\sigma_N) = \{\{a_0, a_4\}, \{a_0, a_1, a_2, a_5\}\}$ and therefore, $\pi(\sigma_N) = \{a_0, a_4\} \cap \{a_0, a_1, a_2, a_5\} = \{a_0\}$.

Tightness of claim game correspondences associated with upper cycle free effectivity functions follows from

PROPOSITION 5.3. *Let the effectivity function* $E: 2^N \to \mathcal{P}(2^A)$ *be upper cycle free. Then*

$$E \subset E_\alpha^{G(E)} = E_\beta^{G(E)}.$$

Proof. Let $S \in 2^N$ and $B \in E(S)$. Define $\sigma_S \in X_S$ by $\sigma_i := (S, B)$ for all $i \in S$. Then for all $\tau_{N\setminus S} \in X_{N\setminus S}$ we have $B \in F(\sigma_S, \tau_{N\setminus S})$. Hence, $\pi(\sigma_S, \tau_{N\setminus S}) \subset B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. Hence, $B \in E_\alpha^{G(E)}(S)$.

To prove tightness of $G(E)$ it suffices to show that $E_\beta^{G(E)}(S) \subset E_\alpha^{G(E)}(S)$ for all $S \in 2^N$. Let $S \in 2^N$ and $B \in E_\beta^{G(E)}(S)$. Define $\hat{\tau}_{N\setminus S} \in X_{N\setminus S}$ by $\hat{\tau}_i := (\{i\}, A)$ for all $i \in N\setminus S$. Since $B \in E_\beta^{G(E)}(S)$, there is a $\sigma_S \in X_S$ such that $\pi(\sigma_S, \hat{\tau}_{N\setminus S}) \subset B$. Hence, $\cap\{D \in 2^A \mid D \in F(\sigma_S, \hat{\tau}_{N\setminus S})\} \subset B$. Because $(F(\sigma_S, \hat{\tau}_{N\setminus S})\setminus\{A\}) \subset F(\sigma_S, \tau_{N\setminus S})$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$, we obtain $\cap\{D \in 2^A \mid D \in F(\sigma_S, \tau_{N\setminus S})\} \subset B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. Hence, $B \in E_\alpha^{G(E)}(S)$. ∎

The next proposition shows the importance of the condition of upper cycle freeness.

PROPOSITION 5.4. *Let* $E: 2^N \to \mathcal{P}(2^A)$ *be an effectivity function. There exists a game correspondence G such that* $E \subset E_\alpha^G$ *if and only if E is upper cycle free.*

Proof. (⇒) Since E_α^G is superadditive this follows immediately from Lemma 2.4.

(⇐) This follows immediately from Proposition 5.3. ∎

Theorem 5.6 below characterizes the properties an effectivity function E must satisfy for coincidence of E and $E_\alpha^{G(E)}$. For this we need the monotonicity result of

LEMMA 5.5. *Let E and E' be upper cycle free effectivity functions and let* $G(E) = (X_1, \ldots, X_n, A, \pi)$ *and* $G(E') = (X'_1, \ldots, X'_n, A, \pi')$ *be the associated claim game correspondences. If* $E \subset E'$ *then* $E_\alpha^{G(E)} \subset E_\alpha^{G(E')}$.

Proof. Let $B \in E_\alpha^{G(E)}(S)$ for some $S \in 2^N$. If $B = A$, then $B \in E_\alpha^{G(E')}(S)$. Suppose $B \neq A$. There is a strategy $\sigma_S = (S_i, B_i)_{i \in S} \in X_S$ such that $\pi(\sigma_S, \tau_{N\setminus S}) \subset B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. Take $\tau_{N\setminus S} \in X_{N\setminus S}$ such that $\tau_i = (\{i\}, A)$ for all $i \in X_{N\setminus S}$. Let $S_0 := \{i \in S \mid \exists j \in S_i[S_j \neq S_i \text{ or } B_j \neq B_i]\}$. Since $\pi(\sigma_S, \tau_{N\setminus S}) \subset B \neq A$, we have $S_0 \neq S$ and there is a partition $\{S_1^*, \ldots, S_k^*\}$ of $S\setminus S_0$ and $B_r^* \in E(S_r^*)$ for all $r \in \{1, \ldots, k\}$ such that $(S_i, B_i) = (S_r^*, B_r^*)$ for all $i \in S_r^*$ and all $r \in \{1, \ldots, k\}$ and $\bigcap_{r=1}^k B_r^* \subset B$. Since $\sigma_S \in X'_S$, and $B_r^* \in F(\sigma_S, \tau'_{N\setminus S})$ for $r \in \{1, \ldots, k\}$ and all $\tau'_{N\setminus S} \in X'_{N\setminus S}$, we obtain $\pi'(\sigma_S, \tau'_{N\setminus S}) \subset B$ for all $\tau'_{N\setminus S} \in X'_{N\setminus S}$. So $B \in E_\alpha^{G(E')}(S)$. ∎

THEOREM 5.6. *Let E: $2^N \to \mathcal{P}(2^A)$ be an upper cycle free effectivity function. Then*

(i) $E_\beta^{G(E)} = E_\alpha^{G(E)} = E_\alpha^{G(\overline{E})} = E_\beta^{G(\overline{E})}$;
(ii) $\overline{E} = E_\alpha^{G(\overline{E})}$ if and only if $\overline{E}$ is A-monotonic;
(iii) $E = E_\alpha^{G(E)}$ if and only if E is A-monotonic and superadditive.

Proof. (i) Because of Proposition 5.3 it suffices to prove that $E_\alpha^{G(E)} = E_\alpha^{G(\overline{E})}$.

($\subset$) This follows by Lemma 5.5.

($\supset$) Let $G(E) := (X_1, \ldots, X_n, A, \pi)$ and $G(\overline{E}) := (\overline{X}_1, \ldots, \overline{X}_n, A, \overline{\pi})$. Let $S \in 2^N$ and $B \in E_\alpha^{G(\overline{E})}(S)$. If $B = A$, then $B \in E_\alpha^{G(E)}(S)$. Suppose $B \neq A$. Then there is a $\overline{\sigma}_S \in \overline{X}_S$ such that $\overline{\pi}(\overline{\sigma}_S, \overline{\tau}_{N\setminus S}) \subset B$ for all $\overline{\tau}_{N\setminus S} \in \overline{X}_{N\setminus S}$. Take $\overline{\tau}_{N\setminus S} \in \overline{X}_{N\setminus S}$ such that $\overline{\tau}_i = (\{i\}, A)$ for all $i \in N\setminus S$. As in the proof of Lemma 5.5, since $\overline{\pi}(\overline{\sigma}_S, \overline{\tau}_{N\setminus S}) \subset B$ one can find disjoint subcoalitions $S_1, \ldots, S_k$ of S and $B_r \in \overline{E}(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $\overline{\sigma}_i = (S_r, B_r)$ for all $i \in S_r$ and $r \in \{1, \ldots, k\}$, and $\bigcap_{r=1}^k B_r \subset B$. For all $r \in \{1, \ldots, k\}$ by definition of $\overline{E}$, there are partitions $\{S_{r1}, \ldots, S_{rk_r}\}$ of S_r and there are $B_{rs} \in E(S_{rs})$ for all $s \in \{1, \ldots, k_r\}$ such that $B_r = \bigcap_{s=1}^{k_r} B_{rs}$.

Define $\sigma_S \in X_S$ by $\sigma_i := (S_{rs}, B_{rs})$ for all $i \in S_{rs}$, $s \in \{1, \ldots, k_r\}$ and $r \in \{1, \ldots, k\}$. Then $B_{rs} \in F(\sigma_S, \tau_{N\setminus S})$ for all $s \in \{1, \ldots, k_r\}$, $r \in \{1, \ldots, k\}$ and $\tau_{N\setminus S} \in X_{N\setminus S}$. Since $B \supset \bigcap_{r=1}^k B_r = \bigcap_{r=1}^k \bigcap_{s=1}^{k_r} B_{rs}$ we obtain $\pi(\sigma_S, \tau_{N\setminus S}) \subset B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. Hence, $B \in E_\alpha^{G(E)}(S)$.

(ii) ($\Leftarrow$) Suppose $\overline{E}$ is A-monotonic. By Proposition 5.3 it is sufficient to prove that $\overline{E} \supset E_\alpha^{G(\overline{E})}$. Therefore, let $S \in 2^N$ and $B \in E_\alpha^{G(\overline{E})}(S)$. If $B = A$, then $B \in \overline{E}(S)$. Suppose $B \neq A$. Then there exists a $\overline{\sigma}_S \in \overline{X}_S$ such that $\overline{\pi}(\overline{\sigma}_S, \overline{\tau}_{N\setminus S}) \subset B$ for all $\overline{\tau}_{N\setminus S} \in \overline{X}_{N\setminus S}$. Take $\overline{\tau}_{N\setminus S} \in \overline{X}_{N\setminus S}$ such that $\overline{\tau}_i = (\{i\}, A)$ for all $i \in N\setminus S$. Since $\overline{\pi}(\overline{\sigma}_S, \overline{\tau}_{N\setminus S}) \subset B$, one can find disjoint subsets $S_1, \ldots, S_k$ of S and $B_r \in \overline{E}(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $\overline{\pi}(\overline{\sigma}_S, \overline{\tau}_{N\setminus S}) = \bigcap_{r=1}^k B_r$. Since $\overline{E}$ is superadditive, we obtain $\bigcap_{r=1}^k B_r \in \overline{E}(\bigcup_{r=1}^k S_r)$. Since $\overline{E}$ is superadditive and hence N-monotonic, this implies $\bigcap_{r=1}^k B_r \in \overline{E}(S)$. Hence, since $\overline{E}$ is A-monotonic and $\bigcap_{r=1}^k B_r \subset B$ it follows that $B \in \overline{E}(S)$.

($\Rightarrow$) Suppose $\overline{E} = E_\alpha^{G(\overline{E})}$. Then $\overline{E}$ is A-monotonic since $E_\alpha^{G(\overline{E})}$ is A-monotonic.

(iii) ($\Leftarrow$) If E is superadditive, then $E = \overline{E}$ by Lemma 2.5. Using A-monotonicity (i) and (ii) imply that $E = E_\alpha^{G(E)}$.

($\Rightarrow$) Suppose $E = E_\alpha^{G(E)}$. By Lemma 2.5, $\overline{E} = E_\alpha^{G(E)} = E$. So, in particular, E is superadditive, and from (i) and (ii) it follows that E is A-monotonic. ■

Using Lemma 2.6 we now obtain

COROLLARY 5.7. *Let E: $2^N \to \mathcal{P}(2^A)$ be an upper cycle free effectivity function. Then $\overline{E} = E_\alpha^{G(\overline{E})}$ if and only if E is weakly A-monotonic.*

In the last part of this section we examine relations between solution concepts of effectivity functions at a profile R^N and solution concepts of the associated claim game correspondence at R^N. It is shown that, if the effectivity function is superadditive, the setcore exactly corresponds to the set of outcomes of the strong Nash equilibria.

THEOREM 5.8. *Let E: $2^N \to \mathcal{P}(2^A)$ be an upper cycle free effectivity function and R^N a profile on A. Let $G(E) := (X_1, \ldots, X_n, A, \pi)$ be the associated claim game correspondence. Then the following two assertions hold.*

(i) If σ_N is a strong Nash equilibrium of $G(E)$ at R^N, then $\pi(\sigma_N)$ is an element of the setcore of E at R^N.

(ii) If E is superadditive, then for each element B of the setcore of E at R^N there exists a strong Nash equilibrium σ_N of $G(E)$ at R^N such that $\pi(\sigma_N) = B$.

Proof. (i) Let $\sigma_N \in X_N$ be a strong Nash equilibrium of $G(E)$ at R^N. Define $B := \pi(\sigma_N)$. Suppose there are $S \in 2^N$ and $B' \in E(S)$ such that $B' \tilde{P}^S B$. Define $\tau_S \in X_S$ by $\tau_i := (S, B')$ for all $i \in S$. Then $\pi(\tau_S, \sigma_{N\setminus S}) \subset B'$ and therefore $B \setminus \pi(\tau_S, \sigma_{N\setminus S}) \neq \emptyset$. Moreover, we have $F(\tau_S, \sigma_{N\setminus S}) \setminus \{B'\} \subset F(\sigma_N)$. Hence, $\pi(\tau_S, \sigma_{N\setminus S}) = \cap\{D \mid D \in F(\tau_S, \sigma_{N\setminus S})\} \supset \cap\{D \mid D \in F(\sigma_N)\} \cap B' = B \cap B'$. Since $B \setminus \pi(\tau_S, \sigma_{N\setminus S}) \neq \emptyset$, $B \cap B' \subset \pi(\tau_S, \sigma_{N\setminus S}) \subset B'$, and $B' \tilde{P}^S B$, it follows by definition of $\tilde{P}^S$ that $\pi(\tau_S, \sigma_{N\setminus S}) \tilde{P}^S B$. This leads to a contradiction since σ_N is a strong Nash equilibrium of $G(E)$ at R^N.

(ii) Let E be superadditive and let B be an element of the setcore of E at R^N. Define $\sigma_N \in X_N$ by $\sigma_i := (N, B)$ for all $i \in N$. Then $\pi(\sigma_N) = B$. Suppose that σ_N is not a strong Nash equilibrium of $G(E)$ at R^N. Then there exist $S \in 2^N$ and $\tau_S \in X_S$ such that $\pi(\tau_S, \sigma_{N\setminus S}) \tilde{P}^S \pi(\sigma_N)$. Hence, $\pi(\tau_S, \sigma_{N\setminus S}) \neq A$ and there are disjoint subsets $S_1, \ldots S_k$ of S and $B_r \in E(S_r)$ for all $r \in \{1, \ldots, k\}$ such that $\tau_j = (S_r, B_r)$ for all $j \in S_r$ and all $r \in \{1, \ldots, k\}$, and $\pi(\tau_S, \sigma_{N\setminus S}) = \cap_{r=1}^k B_r$. Since E is superadditive, we have $\cap_{r=1}^k B_r \in E(\cup_{r=1}^k S_r) \subset E(S)$. This leads to a contradiction since B is an element of the setcore of E at R^N. ■

6. TOWARD GAME FORMS

In Definition 5.1 every upper cycle free effectivity function E is associated with a claim game correspondence $G(E) = (X_1, \ldots, X_n, A, \pi)$. Here π is a correspondence from X_N to A. In this section we derive a claim game *form* $H(E) = (X_1, \ldots, X_n, A, \rho)$ from $G(E)$, where ρ is a surjective function from X_N to A. Moreover, this is done in such a way that $E_\alpha^{G(E)} = E_\alpha^{H(E)}$.

Moulin (1983) already describes a process for going from a game correspondence G to a game form H such that $E_\alpha^G = E_\alpha^H$, but his construction only works for a finite set of alternatives, while the construction described in this section can also be applied to an infinite set of alternatives. As a result this section yields an alternative proof of

THEOREM 6.1 (Moulin, 1983). *Let E be an effectivity function. Then there exists a game form H such that $E_\alpha^H = E$ if and only if E is superadditive and A-monotonic.*

An obvious way to go from a game correspondence to a game form is by means of a choice function.

LEMMA 6.2. *Let C: $2^A \to A$ be a "choice function," i.e., $C(B) \in B$ for all $B \in 2^A$. Let $G = (X_1, \ldots, X_n, A, \pi)$ be a game correspondence. Define $\rho = C \circ \pi$. Then*

(i) $H = (X_1, \ldots, X_n, A, \rho)$ is a game form;
(ii) $E_\alpha^G \subset E_\alpha^H$ and $E_\beta^G \subset E_\beta^H$.

Proof. (i) The surjectiveness of ρ follows from the non-imposedness of π.

(ii) Let $S \in 2^N$ and $B \in 2^A$ and suppose $B \in E_\alpha^G(S)$. Then there exists a strategy $\sigma_S \in X_S$ such that $\pi(\sigma_S, \tau_{N\setminus S}) \subset B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. But then $C \circ \pi(\sigma_S, \tau_{N\setminus S}) \in B$ for all $\tau_{N\setminus S} \in X_{N\setminus S}$. Hence $B \in E_\alpha^H(S)$.

The proof of the second assertion is similar. ∎

However, in general the inclusions in Lemma 6.2(ii) need not be equalities, not even if G is a claim game correspondence. The following example shows that there are claim game correspondences such that for every choice function these inclusions are not equalities.

EXAMPLE 6.3. Let $N = \{1, 2\}$, $A = \{a, b\}$, and define the effectivity function E: $2^N \to \mathcal{P}(2^A)$ as follows. $E(\{1\}) = E(\{2\}) = \{A\}$, and $E(N) = 2^A$. Then $G(E) = (X_1, X_2, A, \pi)$, where for all $i \in N$, $X_i = \{(\{i\}, A)\} \cup \{(N, B) \mid B \in 2^A\}$ and for all $\sigma_N \in X_N$

$$\pi(\sigma_N) = \begin{cases} \{a\} & \text{if } \sigma_1 = \sigma_2 = (N, \{a\}) \\ \{b\} & \text{if } \sigma_1 = \sigma_2 = (N, \{b\}) \\ A & \text{otherwise.} \end{cases}$$

Let C: $2^A \to A$ be a choice function. Suppose, w.l.o.g., that $C(A) = \{a\}$. Let $\rho := C \circ \pi$ and $H(E) := (X_1, X_2, A, \rho)$. Then

$$\{A\} = E_\beta^{G(E)}(\{i\}) = E_\alpha^{G(E)}(\{i\}) \subsetneq E_\alpha^{H(E)}(\{i\})$$
$$= E_\beta^{H(E)}(\{i\}) = \{\{a\}, A\} \qquad \text{for all } i \in N.$$

In order to establish an equality between $E_\alpha^{G(E)}$ and $E_\alpha^{H(E)}$ for some game form $H(E)$ derived from $G(E)$, we must go beyond the scope of choice functions. In particular, $\rho(\sigma_N)$ will have to depend more directly on σ_N itself, not only on $\pi(\sigma_N)$.

DEFINITION 6.4. Let C: $2^A \to A$ be a choice function. For each $B \in 2^A$ define a surjection h_B from A to B such that

$$h_B^C(b) := \begin{cases} b & \text{if } b \in B \\ C(B) & \text{if } b \in A \backslash B. \end{cases}$$

Let $\overline{\Delta}$: $2^A \times 2^A \to 2^A$ be a binary operation on the non-empty subsets of A defined for all $B, D \in 2^A$ as follows.

$$B \overline{\Delta} D := \begin{cases} B & \text{if } B = D \\ (B \backslash D) \cup (D \backslash B) & \text{if } B \neq D. \end{cases}$$

If $B \neq D$, then $B\overline{\Delta}D$ is the symmetric difference between B and D. Let $B, D \in 2^A$ with $B \neq D$. Note that $(B\overline{\Delta}B)\overline{\Delta}D = B\overline{\Delta}D$ and that $B\overline{\Delta}(B\overline{\Delta}D) = D$. So, $\overline{\Delta}$ is not associative. In order to avoid parentheses it is necessary to define the order in which a sequence of $\overline{\Delta}$ operations must be evaluated. Let $D_1, \ldots, D_k$ be elements of 2^A. Then by $D_1\overline{\Delta}D_2\overline{\Delta}D_3$ we mean $(D_1\overline{\Delta}D_2)\overline{\Delta}D_3$ and for all $3 < t \leq k$ by $D_1\overline{\Delta}D_2 \ldots \overline{\Delta}D_t$ we mean $(D_1 \ldots \overline{\Delta}D_{t-1})\overline{\Delta}D_t$. So, the evaluation of $D_1\overline{\Delta}D_2 \ldots \overline{\Delta}D_t$ is from left to right.

PROPOSITION 6.5. *Let* $2 \leq k$ *and* $1 \leq t \leq k$. *Let* $D_1, \ldots, D_k$ *be elements of* 2^A. *Then* $\{D_1 \ldots \overline{\Delta}D_{t-1}\overline{\Delta}B\overline{\Delta}D_{t+1} \ldots \overline{\Delta}D_k \mid B \in 2^A\} = 2^A$.

Proof. Let $D, D' \in 2^A$. Then there is a $B \in 2^A$ such that $D\overline{\Delta}B = D'$: if $D = D'$, then take $B = D$, else take $B = D\Delta D'$. So, $\{D\overline{\Delta}B \mid B \in 2^A\} = \{B\overline{\Delta}D \mid B \in 2^A\} = 2^A$. Consequently, $\{D_1 \ldots \overline{\Delta}D_{t-1}\overline{\Delta}B \mid B \in 2^A\} = 2^A$. Hence, $\{D_1 \ldots \overline{\Delta}D_{t-1}\overline{\Delta}B\overline{\Delta}D_{t+1} \mid B \in 2^A\} = \{P\overline{\Delta}D_{t+1} \mid P \in 2^A\} = 2^A$. Hence, $\{D_1 \ldots \overline{\Delta}D_{t-1}\overline{\Delta}B\overline{\Delta}D_{t+1}\overline{\Delta}D_{t+2} \mid B \in 2^A\} = \{P\overline{\Delta}D_{t+2} \mid P \in 2^A\} = 2^A$. Repetition of this argument yields $\{D_1 \ldots \overline{\Delta}D_{t-1}\overline{\Delta}B\overline{\Delta}D_{t+1} \ldots \overline{\Delta}D_k \mid B \in 2^A\} = 2^A$. ■

Now we are able to define a claim game form derived from a claim game correspondence.

DEFINITION 6.6. Let $G(E) = (X_1, \ldots, X_n, A, \pi)$ be a claim game correspondence associated with an upper cycle free effectivity function E. Let $C: 2^A \to A$ be a choice function and let $\{h_B^C \mid B \in 2^A\}$ be as in Definition 6.4.

Define $f_E: X_N \to 2^A$ as follows: For $\sigma_N = (S_i, B_i)_{i \in N} \in X_N$, $f_E(\sigma_N) := B_1 \ldots \overline{\Delta} B_n$. Then the *claim game form* $H(E) := (X_1, \ldots, X_n, A, \rho)$ *corresponding to* $G(E)$ *and* C is defined by

$$\rho(\sigma_N) := h^C_{\pi(\sigma_N)}(C \circ f_E(\sigma_N))$$

for all $\sigma_N \in X_N$. Clearly, $\rho(\sigma_N) \in \pi(\sigma_N)$ for all $\sigma_N \in X_N$. So by non-imposedness of π it follows that ρ is surjective.

We now show that the α-effectivity functions of $G(E)$ and $H(E)$ coincide for every upper cycle free effectivity function E.

THEOREM 6.7. *Let* $E: 2^N \to \mathcal{P}(2^A)$ *be an upper cycle free effectivity function. Let* $G(E) = (X_1, \ldots, X_n, A, \pi)$ *be the claim game correspondence associated with* E, *and let* $H(E) = (X_1, \ldots, X_n, A, \rho)$ *be the claim game form corresponding to* $G(E)$ *and a choice function* C. *Then*

$$E_\alpha^{G(E)} = E_\alpha^{H(E)}.$$

Proof. As in the proof of Lemma 6.2(ii) one can show that $E_\alpha^{G(E)} \subset E_\alpha^{H(E)}$. It remains to prove that $E_\alpha^{G(E)}(S) \supset E_\alpha^{H(E)}(S)$ for all $S \in 2^N$. Let $S \in 2^N$ and $B \in E_\alpha^{H(E)}(S)$. We have to prove that $B \in E_\alpha^{G(E)}(S)$. This is obvious if $B = A$ or if $S = N$. Therefore, suppose $B \neq A$ and $S \neq N$. Since $B \in E_\alpha^{H(E)}(S)$, there is a strategy $\sigma_S \in X_S$ such that for all $\tau_{N \setminus S} \in X_{N \setminus S}$ we have $\rho(\sigma_S, \tau_{N \setminus S}) \in B$.

CLAIM. *For each* $D \in 2^A$ *there is an* $i \in S$ *such that* $\sigma_i \neq (N, D)$.

Proof of the claim. Let $D \in 2^A$. Suppose for all $i \in S$, $\sigma_i = (N, D)$. Take $a \in A \setminus B$. By Proposition 6.5 it follows that there are $D_j \in 2^A$, $j \in N \setminus S$, such that $D_1 \overline{\Delta} D_2 \ldots \overline{\Delta} D_n = \{a\}$, where for ease of notation $D_i = D$ if $i \in S$. Let $\tau_j = (N, D_j)$ for all $j \in N \setminus S$. Then

$$\rho(\sigma_S, \tau_{N \setminus S}) = h^C_{\pi(\sigma_S, \tau_{N \setminus S})}(C \circ f_E(\sigma_S, \tau_{N \setminus S})) = h^C_{\pi(\sigma_S, \tau_{N \setminus S})}(C(\{a\})) = h^C_{\pi(\sigma_S, \tau_{N \setminus S})}(a).$$

If $D_j = D$ for all $j \in N \setminus S$, then $\{a\} = D \overline{\Delta} D \ldots \overline{\Delta} D = D$ and $\pi(\tau_S, \sigma_{N \setminus S}) = D$. However, this would imply that $\rho(\sigma_S, \tau_{N \setminus S}) = a \notin B$. So, there is a $j \in N \backslash S$ such that $D_j \neq D$. Hence, $\pi(\sigma_S, \tau_{N\backslash S}) = A$ and again $\rho(\sigma_S, \tau_{N\backslash S}) = a \notin B$. So, there is no $D \in 2^A$ such that for all $i \in S$ we have $\sigma_i = (N, D)$, and this proves the claim.

Fix $i \in N \backslash S$. For each $D \in 2^A$, consider the strategy vector $\tau^D_{N\backslash S} \in X_{N\backslash S}$ defined by $\tau^D_i = (N, D)$ and $\tau^D_j = (N, A)$ for all $j \in N\backslash(S \cup \{i\})$. Then it follows that there exists a $Z \in 2^A$ such that for all $D \in 2^A$, $\pi(\sigma_S, \tau^D_{N\backslash S}) = Z$. By definition $\rho(\sigma_S, \tau^D_{N\backslash S}) = h^C_Z(C \circ f_E(\sigma_S, \tau^D_{N\backslash S}))$. By Proposition 6.5 we have $\{f_E(\sigma_S, \tau^D_{N\backslash S}) \mid D \in 2^A\} = 2^A$ and therefore, $\{\rho(\sigma_S, \tau^D_{N\backslash S}) \mid D \in 2^A\} = Z$. Since $\rho(\sigma_S, \tau^D_{N\backslash S}) \in B$ for all $D \in 2^A$, we have $\pi(\sigma_S, \tau^D_{N\backslash S}) = Z \subset B$ for all $D \in 2^A$. But then it readily follows that $\pi(\sigma_S, \tau_{N\backslash S}) \subset B$ for all $\tau_{N\backslash S} \in X_{N\backslash S}$. Hence, $B \in E^{G(E)}_\alpha(S)$. ■

COROLLARY 6.8. *Let E: $2^N \to \mathcal{P}(2^A)$ be an upper cycle free effectivity function. Then there exists a game form H such that $E^H_\alpha = E$ if and only E is superadditive and A-monotonic.*

Proof. Combine Theorems 5.6 and 6.7. ■

EXAMPLE 6.9. Again consider Example 6.3. Applying Theorem 6.7 yields $\rho(\sigma_N) = b$ if $\sigma_1 = \sigma_2 = (N, \{b\})$ or if $\sigma_1 = (N, \{a\})$ and $\sigma_2 \in \{(N, A), (\{2\}, A)\}$ or if $\sigma_2 = (N, \{a\})$ and $\sigma_1 \in \{(N, A), (\{1\}, A)\}$ and in all other cases $\rho(\sigma_N) = a$. Now it follows that $E^{H(E)}_\beta(\{1\}) = E^{H(E)}_\beta(\{2\}) = \{\{a\}, A\}$. Moreover, since E is not maximal (cf. Moulin and Peleg, 1982), it follows that there is no game form H such that $E^H_\beta = E$.

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