Modelling Interactive Behaviour, and Solution Concepts

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Proefschrift

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PROMOTOR:	prof.dr. P.E.M. Borm
COPROMOTORES:	dr. R.L.P. Hendrickx
	dr. J.H. Reijnierse

Nothing shocks me. I'm a scientist.

Indiana Jones, *Indiana Jones and the Temple of Doom* (1984)

Preface

Questions of science, science and progress
Do not speak as loud as my heart

Coldplay, *A Rush of Blood to the Head: The Scientist* (2002)

From (personal) experience I know that the Preface often gets the most, if not exclusive, attention from the reader. If you are such a reader, and the main reason for checking out this part of the thesis is that Prefaces are generally short, then you might want to skip this one. For a shorter section in this thesis I can suggest either the Samenvatting (summary in Dutch) (if you can read Dutch) or the Preliminaries (if you can read mathematics).

Since the first paragraph of this Preface did not scare you away you are currently reading my Ph. D. thesis "Modelling Interactive Behaviour, and Solution Concepts", which covers several topics within the field of game theory. It is the result of four years' work (although the university has found an inventive way, by introducing a Research Master ($\neq$ Master of Philosophy), to only pay me for three of those years) as a researcher at Tilburg University within the Department of Econometrics and Operations Research. So what brought me there?

My educational life started in my home town, Tholen, at elementary school C.N.S. De Regenboog in 1986. With the exception of a one-year excursion to De Klimroos, probably to master the skills of cutting, pasting and playing outdoors, I stayed there until the summer of 1995. Then, at the age of 12, I moved on to high school R.K. Gymnasium Juvenaat H. Hart in Bergen op Zoom, where they prepared me an additional six years for Tilburg University. The first of many thanks in this Preface goes out to all my teachers who gave it their best.

The first time I came into contact with game theory was in 2003 during the course Oriëntatie ME/EM in the second semester of my second year as a student Econometrics and Operations Research. (I thank Paul A. for bringing this study to my attention.) Lecturer of the course? Ruud. Guest lecturer? Peter B. A coincidence? Highly doubtful. I remember seeing prisoners' dilemma, battle of the sexes, Cournot and Bertrand duopolies, bargaining problems and two-player mini-poker. It is a pity the first part of this course deals with econometric methods, otherwise the Educational Board could really be onto something with it. Anyway, these topics tickled my interest in game theory and hence, when we were able to choose courses for our third year, the course Game theory was the first to make the cut. (Although it is fair to say that the name of the course was also appealing.) This turned out to be an excellent choice, because from the very first lecture I was enthusiastic about the discussed topics, something which happened very rarely.

Therefore, it seemed to make sense to contact the lecturer of this course, Peter B., at the moment a topic for a Bachelor's thesis had to be found. And together with Ruud he supervised me on "The Visiting Repairman Problem and Related Games" of which some indirect results can be found in Section 5.10 of this thesis. Based on the idea to never change a winning team I was back at Peter B.'s office a year later when the assignment was to write a Master's thesis. With Ruud replaced by Marieke the work "Fall Back Proof Equilibria" was completed. In the final stage of this project also Hans joined the team as part of the committee. The direct results of this work can be found in Sections 7.7 and 7.8. I guess it was also during this period that I was "recruited" as a Ph. D. student, which pretty much completes the answer to the first question posed in this Preface.

In the next part of this Preface I want to express my gratitude to the people who made it possible for me to write this thesis. Since the list is quite extensive I have decided to thank people by category. This implies that some will be mentioned more than once, but note that a name count is not a representative measure for one's influence on this thesis. I take off by thanking the "3 non-angy men[1]" that (besides myself of course) had the most direct influence on the outcome of this thesis: my supervisors Peter B., Hans and Ruud. All of them were extremely helpful and even

[1]See 12 Angry Men (1957).

if meetings did not lead up to anything to work with, they were never a waste of time, at least not to me.

Furthermore, I am grateful to Peter B. for getting me interested in game theory in the first place. If it was not for his enthusiastic lectures I might not have started to work within this field at all. I also really appreciate the way he supervised me for the last six years. Whenever I was out of ideas, Peter B. had many more. Whenever I got stuck in details, Peter B. made me see the big picture again, something which never seemed to get out of his sight. And most importantly, he has always let me do the research I wanted to do. Besides all that he was also there if I needed some advice on topics not directly related to my research. Is there no criticism at all? Well, it is a bit of a disappointment when you have to wait for comments on your written work for weeks or even months and then they turn out to be unreadable.

While Peter B. always looks at the big picture, Hans has an excellent eye for detail. As a result, he has thought me to be more precise, and I use his comment "My name is not 3, my name is Hans," as a reminder of this. Additionally, it is really amazing at what speed he is able to find a counter-example for almost anything you are unable to prove (or sometimes even for things you thought you could prove), which earned him the nickname "Counter-example" Hans. Above all, Hans is a pleasant man to cooperate with, which comes to light most prominently in the fact that to me he was always more of an experienced co-author than a supervisor. On top of that I thank Hans for bringing me to Bilbao for a few days in 2008 to work with Javier. Besides the start of the work that culminated in Chapter 3 of this thesis I really enjoyed the time there.

I thank Ruud (a.k.a. Rudy, Rudy, Rudy, Rudy) for many things and as a consequence, his name will show up some more during this Preface, but let me focus here on his influence as a copromotor. Whenever I had an idea, whether this was the sketch of a proof, a counter-example or a plan to structure a paper, I always ran it first by Ruud, who made the time to hear me out and discuss the idea critically. These discussions have been crucial for the content of this thesis. Next to this, Ruud also made a great contribution to the presentation of this thesis with many linguistic comments (What else can you expert from a punctuation expert in the English language?) and solutions to many of my LaTeX-related problems. (Hardly ever did I have to turn to LaTeX guru Henri.) Most importantly, however, Ruud made it possible for me to come to the university every day by lending me his spare bike for three years.

After thanking my supervisors I figured this to be a natural place to thank the rest of my committee as well, for their time and effort as committee member.

On top of that I thank Dolf Talman, in particular for his detailed comments on my manuscript. I already met Dolf in 2001 as lecturer of the course Micro-economie, and over the years I was lucky to have him as a teacher for many other courses (Applied public sector economics, Micro-econometrics, Micro II). Later when I became his teaching assistant for the course Micro I: Equilibrium theory I found out that he is not only an excellent teacher, but also a pleasant colleague. I guess I slowly move ahead, because, fortunately, during the last few months we have worked together again, both as lecturers for the course Mathematics I.

I think Dries Vermeulen has no idea how much he influenced this thesis. At some point our research on fall back equilibrium was kind of stuck. Therefore, Peter B. decided to bring in Dries for a meeting, I guess hoping he would be our "Wolf[2]". However, since cleaning up a mess is often easier than creating something beautiful this meeting did not seem to be part of the solution at all, but only five minutes afterwards I wrote down the crucial idea for the characterisation of fall back equilibrium by blocking games which is the basis for most results in Chapter 7. Dries is also the first, and up to now only, one to personally invite me as a speaker. That he himself did not make it to this presentation is therefore forgiven. I also want to thank his son Feodor for lightening up a rather stiff conference dinner in Madrid by throwing paper airplanes through the room. Excellent stuff!

I met Javier Arin in Bibao in May 2008. Hans was invited as a speaker and suggested that I would accompany him so that we could work with Javier on some new research. I was doubtful as to whether working together for only two or three days would be useful at all, but I certainly was not going to say "no" to a five-day trip to Bilbao. And I was right, as also due to Javier I had a great time in this city. But I was also very very wrong, because the two days work culminated into fifty-three pages of this thesis (Chapter 3).

This brings me to the final member of my committee, Peter Sudhölter. I have to say that I am more familiar with some of his work than with the man himself, although he visited my talk in Amsterdam last summer. Some time earlier that year Peter B. suggested him as a member of my committee, at which time I only knew him from his book "Introduction to the Theory of Cooperative Games" (in cooperation with Bezalel Peleg), which I consulted many times for Chapter 3 of this

[2] See Pulp Fiction (1994).

thesis. In fact, I borrowed it (in stages) for about one and a half years from the university library. It is a good thing they have two copies, although sometimes I borrowed the other copy as well, for Hans. Well, since Peter S. did not know me either I especially appreciate his willingness to join my committee.

Without a committee there is no approval of the thesis, but without co-authors there may have been no thesis at all. Therefore, I want to thank Ignacio, Gloria, Hans, Javier, Marieke, Peter B. and Ruud for cooperating on the research of this thesis.

For a while it seemed that Marieke would not receive such a credit, because although she supervised my Master's thesis "Fall Back Proof Equilibria" she was dropped bluntly as a co-author of the final paper when we decided to take quite a different approach (fall back equilibrium versus dependent fall back equilibrium). Luckily, after stealing some of her work out of an unpublished CentER discussion paper to improve the paper "Public congestion network situations and related games", also Chapter 4 of this thesis, she can be credited as a co-author as well. It is well deserved.

The friendly Spaniards Ignacio and Gloria seem to be able to come up with a paper out of thin air. Although I had to adjust to their Spanish way of creation and explanation (with definitions instead of examples) we were able to write the paper "Transfers, contracts and strategic games" (Chapter 6 of this thesis) in just a few days basically.

Due to a lack of results Pedro cannot be mentioned as one of my co-authors (although some of our joint work ended up in Section 5.10), but I want to thank him anyway as it was amusing working with him. I think I have never seen anybody with so much passion and sometimes even desperation for his work, at least not in game theory.

I also want to thank Yvonne for making the front cover of this thesis, which assures that at least the exterior of this dissertation is exceptional.

This ends the list of people with a direct influence on this thesis. There are, however, many more who contributed in an indirect way. Let me begin to thank the ones whom I had the pleasure of sharing an office with during these years, starting in the Research Master with Gerwald and Gijs. It was a good way to get going, as our room hosted several exciting Slime game matches and was the creative home to classic songs like "Woo ooh, Gijs Rennen, bam-a-lam!" and "There's only one Gijsje Rennen". I also thank our other roommate Tim for never showing up, giving me the possibility to store all my stuff on the extra desk.

As a Ph. D. student I moved across the hall to join Paul Ka. in K514. Unfortunately, this was only a short collaboration, as he quit the job after nearly two months with me. His departure and that of next-door neighbour Frans also brought an end to the game of Hearts, which was frequently played (online) during this brief period. Since Paul Ka. left me his speakers he still managed to have a great impact on my remaining years, as the net influence of the music (coming from these speakers) on my work has been positive.

After Paul Ka. left soon Qu Liu (a.k.a. William) accompanied me in the office. I have many great memories of this friendly guy, especially since he had no idea how to deal with the sleeping mode of his computer, which is strange as you would expect someone with such a name[3] to be sort of a gadget expert, and moreover since he was on sleeping mode himself quite often. Well, maybe it had to do with the "Trojan whore" on his computer. Those things can be a bitch to deal with.

For the last two and a half years my roommate has been Salima. It has never been a greater mess, but at the same time, it has never been more fun. I apologise to her for humming, tapping, clapping and singing along with the music on one or two occasions, for my temporary addiction to Radiohead's "Kid A" in 2008 and for many bad jokes. I especially thank her for pimping up our room with a couch, for bringing delicious food from mamma Salima and for laughing at some of these jokes.

Whenever I was not in my room, chances were that I was on some sort of work-related trip. Let me first thank Feico, who made these trips financially possible.

The first time my work at Tilburg University brought me to a foreign country was in the summer of 2007 when I went to Madrid for a conference with Gerwald and Ruud. Later also Hans, HENK and Marco S. joined. It was an excellent start as some legendary events during this trip include sea breezes in Madrid, Gerwald breaking Ruud's camera at the first picture moment, deer from the mountains of Toledo, joining (with gracious bathing caps) the elderly for some gymnastics in the hotel swimming pool, competing with Marco S. in running up and down the same pool, and Gerwald "decorating" first our hotel bathroom and later, on the way to the airport, also the subway.

[3]See, *e.g.*, Goldfinger (1964).

The return to Spain for the successful spring 2008 trip to Bilbao was already mentioned before. Later that same year I also visited a conference in Evanston with Gerwald, Ruud and Marco S. I will never forget nearly pushing John Nash out of equilibrium while entering the men's room, nor the view on the Chicago skyline (regardless of whether Oprah Winfrey had lived there or not), especially in combination with some perfectly timed fireworks. The best part of this journey was, however, the more than marvelous road-trip with Vincent through North-East America after the conference.

Also 2009 was an excellent year for travelling. In April Edwin L. and I accompanied twenty-three students on a sixteen-day study trip to Japan. I never imagined that one could survive with this futile amount of sleep, but Tokyo (a warm welcome by Panasonic and beers by touch screen), Nagoya (city walks by "Silly walks" and an extraordinary lunch at AkzoNobel) and Kyoto (karaoke and neatly swept gravel) were fun all the way.

In the summer Edwin L., Gerwald, Ruud and I teamed up for a conference in St. Petersburg. Despite a taxi driver using my bag as a tire cleaner, suddenly closing bridges, Edwin L.'s lost sunglasses and a creepy associated police station visit, too much wodka during the conference dinner for Gerwald (a.k.a. Gerardiño) and the multiple debates on the colour of cheese cake, also this trip was first-rate. I think Simon Tahamata would agree.

A non-work related weekend trip to CentreParcs in 2008 with the faculty Ph. D. students Alex G., Chris, Christian, Jan, Kenan, Kim, Marta and Martin, including a visit to Paparazzi and multiple-ball bowling, should also not be forgotten.

Of course there were many fun activities in the neighbourhood of my own home as well. I thank Cristian, Edwin L., Elleke, Gerwald, Gijs, Iris, Josine, Kim, Lisanne, Maaike, Marieke, Mark, Marloes G., Mikel, Mirjam, Oriol, Ralph, Romeo, Ruud, Salima and Soesja for several sub-departmental activities like Sinterklaas, laser-gaming, kart racing, bungee-soccer, bowling, wii-ing, board and card games (in particular Oranje Boven), and drinks (combined with "bittergarnituur") in Kadinsky.

Writing a thesis can be seen as quite an achievement, but also outside room K514 I accomplished a thing or two the last few years, *e.g.*, by winning both the Asset | Econometrics (a.k.a. TEV) football tournament and sports afternoon in 2009. I thank my teammates Cristian, Edwin D., Gerwald, HENK, Jacob, Marco D. and

Mohammed, and Cristian, Edwin L., Gerwald and Josine for making this possible. Also thanks to everyone, including Chris, Elleke, Geraldo (a.k.a. "The Star"), Gerard, Herbert, Nathanael, Pedro, Peter O., Ralph and Sander, for teaming up in other less successful football competitions.

You might almost forget it, but sometimes it was also time to do some work. Most often this involved tinkering at this thesis by myself, but during teaching I could fortunately cooperate with some of my colleagues. I thank Dolf, Edwin D., Edwin L., Elleke, HENK, Herbert, Gerwald, Jacob, Leo, Marieke, Romeo, Ruud and Thijs for their effort as (fellow) coordinator and/or fellow teacher. A special thanks goes out to Willem, just because he deserves it. And by the way, B.E.M. rules!

Over the last years sometimes the work has been made easier due to the help of the secretaries Jolande, Karin, Korine, Loes, Marloes V. and Nicole. In particular I thank Heidi for her enthusiasm, loud laughter and uplifting spirit. And there is also a special thanks for the former "semi-head of department" Annemiek. Somehow she made me feel at home from the very beginning, maybe even before that. I figured that this could very well be due to my familiarity with a smoking woman who meddles with people's affairs, but I am not sure.

Unfortunately, some (former) colleagues do not seem to fall within any of the above categories, but I feel that Alex S., Baris, Bart, David, Hanka, Patrick, Pim, Ramon, Roy and Tunga deserve their place in this Preface as well.

Although there is an extremely strong bias in this Preface towards colleagues, I certainly do not want to pass over the people who kept me connected to the real world during the past few years. As it seems rather stupid to thank certain people for anything in particular, I thank the "BoZ BoyZ" Bas, Bert, Kristel, Cornald, Ellen, Gregor, Marjolein S., Lars, Peter M., Karlijn, Remy, Lia, Vincent and Caroline, and Marjolein J. for everything. Their direct influence on the results of this thesis might be quite limited, but on the other hand, their indirect effects may have no limit at all. I hope that I can thank all of them again if I ever write another Preface.

I also thank the black and white gladiators of TSVV Merlijn. Over the last two and a half years they made it possible for me to worry about losing possession, defending opponents and missing chances instead of proving theorems, composing papers and writing meaningless (sub)sentences. Besides that, I especially want to thank the whole football club for giving me the feeling of being a student whenever needed. Without them I would probably never have gotten familiar with expressions like “Atten-John”, “Adtje schoen” and “Broek-uit-op-je-hoofd”.

Last but not least I want to express my gratitude to my family, including Marisca and Suzanne. In particular I thank Julian for countless discussions on movies and TV-series, often driving others crazy, Peter K. for countless discussions on Feyenoord, often driving others (sometimes including myself) crazy, and Dennis for not starting such discussions. Above all, I thank my parents, Janny and Paul Kl., for always supporting me in everything I do, without pushing me in any direction.

Finally, I want to conclude this Preface with some, to me relevant, quotes that put the work on this thesis into some perspective. Let me, however, first say that I hope you will not stop reading after that. Please check out the rest of this thesis as well, maybe you will find something you like.

Literature:

> He knows how to read. And he also knows that finishing an entire book doesn’t prove anything.
>
> ---
>
> George Costanza, *Seinfeld: The Van Buren boys* (1997)

> Who’s the more foolish, the fool or the fool who follows him?
>
> ---
>
> Obi-Wan Kenobi, *Star Wars* (1977)

> Never underestimate the predictability of stupidity.
>
> ---
>
> “Bullet tooth” Tony, *Snatch* (2000)

> Karma police, arrest this man
> He talks in maths
> He buzzes like a fridge
> He’s like a detuned radio
>
> ---
>
> Radiohead, *OK Computer: Karma police* (1997)

> Many of the truths we cling to depend greatly on our own point of view.
>
> ---
>
> Obi-Wan Kenobi, *Star Wars Episode VI: Return of the Jedi* (1983)

Writing a thesis:

> Everything in its right place
>
> ---
>
> Radiohead, *Kid A: Everything in its right place* (2000)

> You get confused, but you know it
>
> ---
>
> U2, *Pop: Discotheque* (1996)

> You're here on your own who you gonna find to blame?
>
> ---
>
> Oasis, *Definitely Maybe: Bring it on down* (1994)

> You've gotta give it up to get off sometimes
>
> ---
>
> Matchbox twenty, *Mad Season: Stop* (2000)

> You can try the best you can
> The best you can is good enough
>
> ---
>
> Radiohead, *Kid A: Optimistic* (2000)

> I still haven't found what I'm looking for
>
> ---
>
> U2, *The Joshua Tree: I still haven't found what I'm looking for* (1987)

To the readers of this thesis:

> Are you watching closely?
>
> ---
>
> Alfred Borden, *The Prestige* (2006)

> My mistakes were made for you
>
> ---
>
> The Last shadow puppets, *The Age of the Understatement: My mistakes were made for you* (2008)

Contents

Chapter 1

Introduction

Coming from a long line of travelling sales people on my *father's* side
I wasn't gonna buy just anyone's cockatoo
So why would I invite a complete stranger into my home?
Would you?

U2, *No Line on the Horizon: Breathe* (2009)

1.1 Introduction to game theory

Interaction between decision makers (players) can lead to cooperative or competitive behaviour. Game theory is the mathematical tool to study such behaviour. The foundation of game theory is laid in Von Neumann (1928) where the famous minimax theorem is proved, but it is through the book "Theory of Games and Economic Behavior" by Von Neumann and Morgenstern in 1944 that game theory developed into an important tool for mathematical modelling of cooperative behaviour and competition. Applications of game theory can be found in, *e.g.*, (evolutionary) biology, political science, international relations, computer science, philosophy and social sciences, especially (micro) economics, which illustrates the wide applicability of the subject.

Game theory is usually divided into two branches. In competitive, or *non-cooperative game theory*, players are considered to be individual utility maximisers playing a game against each other. The term *game* in this context is interpreted as any interactive situation in which a player's payoff depends on both his own actions and the actions of the opponents. The players may be able to negotiate about how to act but they cannot make binding agreements. Therefore, the focus of non-cooperative game theory is on individual incentives and formalising notions of rationality. The most important concept to determine a reasonable strategy combination in such games is the notion of Nash equilibrium (Nash (1951)). A *Nash equilibrium* is a combination of strategies such that unilateral deviation does not pay, *i.e.*, in a Nash equilibrium each player maximises his utility given the actions of his opponents.

Example 1.1.1 We illustrate the notion of a non-cooperative game, and in particular the concept of a Nash equilibrium, by means of the famous battle of the sexes. Imagine a couple that can either go to a football match (F) or to the opera (O). The husband prefers to go to the football match, while the wife prefers to go to the opera. However, both only enjoy the activity if they go to the same place together rather than to different ones. If they have to make the decision independently, where should each one of them go? This situation can be depicted by a non-cooperative game in strategic form.

$$\begin{array}{c c} & \begin{array}{cc} O^h & F^h \end{array} \\ \begin{array}{c} O^w \\ F^w \end{array} & \left[\begin{array}{cc} 3,2 & 0,0 \\ 0,0 & 2,3 \end{array}\right] \end{array}$$

In the above matrix the wife chooses a row (O^w or F^w) and the husband a column (O^h or F^h). For each combination of strategies the utility of the wife (husband) is the first (second) number in the corresponding cell. Hence, *e.g.*, if the wife chooses F^w and the husband F^h then they both go to the football match resulting in a utility of 2 for the wife and a utility of 3 for the husband.

Since both the wife and the husband want to go to the same place, the best choice of each one depends on the choice of the other. In particular, if the wife chooses O^w, then the husband's best choice is O^h, while his best choice is F^h if the wife chooses F^w. Since a Nash equilibrium is a combination of strategies such that each player maximises his/her utility given the actions of the others, this game has two (pure) Nash equilibria. These Nash equilibria are the strategy combinations (O^w, O^h) and (F^w, F^h), as for both strategy combinations unilateral deviation of either one of the players results in a decrease of utility for the deviator. One can easily check that the other two strategy combinations are not Nash equilibria, as there is at least one player (in fact both players) that can increase his/her utility by deviating to another strategy. $\triangleleft$

The second branch of game theory is *cooperative game theory*, which studies situations where players can cooperate in order to generate benefits (or reduce costs). Its main focus is on the study of fair allocations of the joint benefits by means of cooperation. The most commonly used model in this type of situations is that of *transferable utility games*. In a transferable utility (or TU) game each coalition of players is associated with a certain worth, which corresponds to the benefits this coalition can obtain without help from players outside the coalition. These coalitional worths can be used as a reference point for dividing the worth of the grand coalition (the coalition of all players).

The most fundamental *solution concept* for TU-games is the *core* (Gillies (1959)). An allocation is an element of the core if it satisfies two requirements. First of all it should be *efficient*, which means that the worth of the grand coalition should be divided among the players of the game. Secondly, it should be stable, which means that no coalition of players is better off by separating from the grand coalition and obtaining its coalitional worth.

In order to allocate the worth of the grand coalition of a TU-game also several *single-valued* solution concepts are introduced in the literature, each with its own appealing properties. In this introduction we would like to mention three of these solution concepts: the *Shapley value* (Shapley (1953)), the *(pre)nucleolus* (Schmeidler (1969)) and the *compromise value* (Tijs (1981)).

Example 1.1.2 We illustrate the idea of a TU-game, its core and an associated single-valued solution concept (the Shapley value) by means of a glove game. Suppose there are three players; player 1, player 2 and player 3. Player 1 is in possession of a right hand glove, while players 2 and 3 both have a left hand glove, but of different quality. Gloves can only be sold in pairs (one left and one right hand glove). Therefore, none of the players can obtain any benefits by himself and this also holds for the coalition of players 2 and 3. However, if players 1 and 2 cooperate they have a pair of gloves, which can be sold for 10. Also players 1 and 3 together have a pair of gloves, but due to the inferior quality of player 3's glove this pair is only worth 5. Note that if all players cooperate there is still only one pair of gloves, with a total worth of 10. This situation can be modelled by a TU-game (N, v), with $N = \{1, 2, 3\}$.

The worths $v(S)$ of all[1] coalitions $S \subseteq N$ are given in the table below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	10	5	0	10

Let us first of all consider the core of this game. By efficiency, all players together should receive exactly 10. Further, by the stability restriction players 1 and 2 together should receive at least 10 and player 3 at least 0. This implies that player 3 receives exactly 0. Moreover, since players 1 and 3 together should receive at least 5 this means that player 1 should receive something between 5 and 10 and that the remainder goes to player 2. Formally the core $C(N,v)$ of this game is given by $C(N,v) = \{x \in \mathbb{R}^N \mid 5 \leq x^1 \leq 10, x^2 = 10 - x^1, x^3 = 0\}$.

In order to determine the Shapley value of this game we first have to calculate the *marginal vector* for each *ordering* of the player set. Let the players "enter" the game in the order $(1,3,2)$. This means that player 1 joins the empty coalition. Since the worths of both coalition $\{1\}$ and the empty coalition are 0 the marginal contribution of player 1 is 0 for this ordering. Then player 3 joins coalition $\{1\}$, which increases the worth from 0 to 5, as due to player 3's presence a pair of inferior-quality gloves can be sold. Therefore, the marginal contribution of player 3 in this ordering is 5. Finally, player 2 joins coalition $\{1,3\}$ resulting in a marginal contribution of also 5, because now the high-quality pair of gloves can be sold. Hence, the marginal vector associated with ordering $(1,3,2)$ is given by $(0,5,5)$. We denote an ordering of the player set by π and the corresponding marginal vector by m_π. The table below gives for each ordering of the player set the corresponding marginal vector.

π	m_π^1	m_π^2	m_π^3
$(1,2,3)$	0	10	0
$(1,3,2)$	0	5	5
$(2,1,3)$	10	0	0
$(2,3,1)$	10	0	0
$(3,1,2)$	5	5	0
$(3,2,1)$	10	0	0

The Shapley value $\Phi(N,v)$ of a TU-game (N,v) is the average over all marginal vectors. Therefore, the Shapley value of this game is given by $\Phi(N,v) = (5\frac{5}{6}, 3\frac{1}{3}, \frac{5}{6})$.

[1]The worth of the empty coalition is defined to be 0 for all TU-games. Therefore, we omit the worth of this coalition.

Note that since player 3 receives a positive amount the Shapley value is not a core element of this game. $\triangleleft$

1.2 Overview

In the first part of this thesis we discuss topics within the field of cooperative game theory. In Chapter 3 we analyse solution concepts for TU-games. One of the most important single-valued solution concepts for TU-games is the (pre)nucleolus which is the unique element in the *(pre)imputation set* for which the maximal coalitional objection to it is minimised. In Chapter 3 we discuss the related *per capita (pre)nucleolus*, which is the unique element in the (pre)imputation set for which the maximal objection *per player* of a coalition to it is minimised. For the per capita prenucleolus and the per capita nucleolus we discuss several properties and their relations to other solution concepts for TU-games. Furthermore, for both concepts we define a *reduced game* and prove that they satisfy the corresponding *reduced game property*. Moreover, we characterise the concepts by the use of this reduced game property and the properties single-valuedness, *covariance* and *anonymity*.

We also introduce the *per capita (pre)kernel*, which is related to the per capita (pre)nucleolus in the same way as the *(pre)kernel* (Davis and Maschler (1965)) is related to the (pre)nucleolus. Our analysis of the per capita (pre)kernel is analogous to our analysis of the per capita (pre)nucleolus and includes a characterisation of the per capita prekernel. Moreover, we also provide a new characterisation of the core.

In Chapters 4 and 5 the starting point is not a TU-game, but an underlying cooperative situation. A *cooperative situation* typically involves a group of players that can choose from a set of alternatives, where each alternative results in a cost for the (group of) players. The set of alternatives and associated costs usually stem from a combinatorial operations research problem in which several players control parts (*e.g.*, vertices, edges, jobs, machines) of the underlying system. A cooperative situation gives rise to two main questions; which alternative should be realised and how should the costs of this alternative be divided? To answer the second question, the cooperative situation can be modelled as a TU-game. Well-known examples of cooperative situations are, *e.g.*, *travelling salesman problems* (Potters et al. (1992))

and *minimum cost spanning tree situations* (Claus and Kleitman (1973) and Bird (1976)). We refer to Borm et al. (2001) for a comprehensive survey of these type of problems, usually summarised under the heading of *operations research games*.

In Chapter 4 we discuss a specific class of cooperative situations, *public congestion network situations*. In congestion network situations a single source is considered to which all players have to be connected, and the cost of using an arc in order to achieve this depends on the number of its users. Quant et al. (2006) discuss this situation in the context of private arcs. We analyse congestion network situations with public arcs, which means that each coalition of players is allowed to use any arc of the *network*. We consider public congestion network situations with either *concave* or *convex* cost functions. We model the first type by the *direct congestion cost game* in which the coalitional costs are based upon the idea that the complementary set of players does not make use of any arcs. However, the main focus is on public congestion network situations with convex cost functions. Within this framework we present an algorithm to find an *optimal network* for each coalition of players. Furthermore, with the explicit use of transferable utility we argue that this type of situations should be modelled differently by the *marginal congestion cost game*, which is the dual of the direct congestion cost game. We show that the marginal congestion cost game is *concave*. As a consequence, cooperation is likely to occur and stable allocations exist. We also introduce a solution concept, based upon three equal treatment principles, that provides such a stable allocation. Finally, we extend these results to a divisible framework in which, contrary to the standard framework, each player can use several paths to get connected to the source.

In Chapter 5 we take a more general approach. The starting point is an arbitrary cooperative situation and the central question is how to divide the costs of the optimal alternative of this cooperative situation among the associated players. We assume that there is a general consensus that in principle total costs should be minimised, which means that TU-games can be used to solve this problem. However, in general it is not clear which TU-game best fits the situation. Therefore, we introduce a model that can be used as a guidance for finding a suitable TU-game. Our approach is based on the idea that a cooperative situation can be represented by a corresponding order problem. An *order problem* consists of three elements: the player set of the underlying cooperative situation, the set of orderings of the player set and an *individualised cost function* that describes for each ordering of the player set a corresponding cost to every player.

We discuss two types of order problems. In a *positive externality order problem* each group of players obtains the minimum cost for an ordering in which the group is "served" last. In a *negative externality order problem* it is the other way around and the minimum cost for a group of players is obtained for an ordering in which they are served first. We argue that each positive externality order problem is appropriately modelled by the so called *direct cost game* in which the players of a coalition are served first. Furthermore, we argue that each negative externality order problem is appropriately modelled by the dual of the direct cost game, called the *marginal cost game*. Consequently, if an order problem is a fair representation of the underlying cooperative situation the game by which this order problem is modelled seems a good fit for the cooperative situation itself.

The order problem framework is not only used to find suitable TU-games for cooperative situations, but also to obtain core elements of these games. We associate with each order problem a *generalised Bird solution* that is based upon *Bird's tree solution* (Bird (1976)) for the class of minimum cost spanning tree situations, in the sense that each player contributes his individual cost in the optimal order for the grand coalition. With the order problem framework and the associated generalised Bird solution in mind we discuss several classes of cooperative situations, among which *sequencing situations without initial order* (Klijn and Sánchez (2006)), minimum cost spanning tree situations, *permutation situations without initial allocation* (cf. Tijs et al. (1984)) and travelling salesman problems.

In somewhat more detail we discuss the class of *travelling repairman problems*. In a travelling repairman problem (Afrati et al. (1986)) the objective is to find a tour visiting a group of players such that the total waiting time of the players is minimised. We introduce the associated cost allocation problem and argue by the use of the order problem framework to model these situations by the associated marginal cost game of which we discuss several properties. Furthermore, we also consider two context-specific single-valued solution concepts for this class.

Chapter 6 forms a bridge between cooperative and non-cooperative game theory, as we investigate the role of allowing certain aspects of commitment and cooperation within the framework of non-cooperative games in strategic form. More in particular, we focus on the explicit strategic option of costless contracting on monetary transfer schemes with respect to particular outcomes.

The first part deals with the possibility of making a specific strategy combination individually stable by having a simple monetary transfer scheme contingent on whether the agreed strategy combination is actually realised. Under standard regularity conditions it turns out that the set of such individually stable strategy combinations, called *transfer equilibria*, coincides with the set of Nash equilibria. Transfer equilibria are especially analysed in finite games without randomisation in which they generalise Nash equilibria.

The second part models contracting on monetary transfers as an explicit strategic option within a two-stage extensive form setting. We obtain a full characterisation of all Nash and *virtual subgame perfect equilibria* (García-Jurado and González-Díaz (2006)) payoff vectors in the same spirit as the well-known Folk theorems in the context of repeated games.

In Chapter 7 we consider mixed extensions of finite non-cooperative games in strategic form. For such games the notion of Nash equilibrium is the fundamental concept, but since the set of Nash equilibria may be large and can contain counterintuitive outcomes several refinements of this solution concept, *e.g.*, *perfect* (Selten (1975)) and *proper* (Myerson (1978)) equilibrium have been introduced. In Chapter 7 we introduce a new equilibrium concept, called *fall back equilibrium*, in which the idea is that an equilibrium should be stable against pertubations in the strategies due to blocked actions. In the associated thought experiment each player anticipates the possibility of a blocked action by choosing beforehand a *back-up action*, which he plays whenever the action of his first choice is blocked.

We show that the set of fall back equilibria is a non-empty and closed subset of the set of Nash equilibria. We also analyse the relation between fall back equilibrium and other equilibrium concepts. We prove, *e.g.*, that each *robust equilibrium* (Okada (1983)) is a fall back equilibrium and that for bimatrix games each proper equilibrium is a fall back equilibrium. Similar to the way Okada (1984) refines perfectness to *strict perfectness*, we define the concept of *strictly fall back equilibrium.* It turns out that the sets of fall back and strictly fall back equilibria coincide for bimatrix games.

In the thought experiment underlying fall back equilibrium we assume that exactly one action of each player can be blocked. We also consider two modifications of this concept. We first of all analyse the equilibrium concept that emerges when we allow multiple actions of each player to be blocked. The first main result provided for this concept, called *complete fall back equilibrium*, is that the set of complete fall back equilibria is a non-empty and closed subset of the set of proper equilibria. Secondly, for bimatrix games the sets of complete fall back and proper equilibria coincide, which means that the concept of complete fall back equilibrium is a strategic characterisation of proper equilibrium.

In the second modification we consider there can only be one blocked action in total. For this concept, called *dependent fall back equilibrium*, we show that for 2×2 bimatrix games the sets of dependent fall back and perfect equilibria coincide, but for bimatrix games in general the intersection between the two sets can be empty.

We conclude Chapter 7 with a complete overview of fall back equilibrium in bimatrix games in which at least one of the players only has two pure strategies. Within this framework we geometrically characterise the sets of fall back, complete fall back and dependent fall back equilibria.

Chapter 2

Preliminaries

Words are meaningless and forgettable

Depeche mode, *Violator: Enjoy the silence* (1990)

In these preliminaries we introduce some basic notation, and fundamental concepts in cooperative game theory. In this thesis we also consider non-cooperative games, but since we discuss different types of strategic games the notation for these games is introduced in the corresponding chapters.

2.1 Basic notation

The set of all natural numbers is denoted by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, the set of non-negative reals by $\mathbb{R}_+$ and the set of positive reals by $\mathbb{R}_{++}$. For a finite set N the cardinality of N is denoted by $|N|$ or n and the set $N \cup \{0\}$ is denoted by N^0. We denote the power set of N, *i.e.*, the collection of all its subsets, by 2^N. By $\mathbb{R}^N$ we denote the set of elements of $\mathbb{R}^n$ whose entries are indexed by N, or equivalently, the set of all real-valued functions on N. An element of $\mathbb{R}^N$ is denoted by a vector $x = (x^i)_{i \in N}$. For $S \subseteq N, S \neq \emptyset$, we denote the restriction of x on S by $x_S = (x^i)_{i \in S}$ and we denote $\sum_{i \in S} x^i$ also by $x(S)$. For a finite set N and a subset $S \subseteq N$, we denote by e_S the vector in $\mathbb{R}^N$ defined by $e_S^i = 1$ for all $i \in S$ and $e_S^i = 0$ for all $i \in N \backslash S$.

An *ordering* of the elements in a finite (player) set N is a bijection $\pi : \{1, \ldots, n\} \to N$, where $\pi(t)$ denotes the player at position t. The set of all $n!$ orderings of N is denoted by Π. By Π_S we denote the set of all orderings $\pi \in \Pi$ such that the players in $S \subseteq N$ are placed on the first $|S|$ positions, *i.e.*, $\pi^{-1}(i) < \pi^{-1}(j)$ for all $i \in S, j \in N \backslash S$.

For a set $A \subseteq \mathbb{R}^m$ we denote by $\mathrm{cl}(A)$ the closure of A, by $\mathrm{relint}(A)$ its relative interior and by $\mathrm{conv}(A)$ its convex hull.

2.2 Cooperative game theory

A *cooperative game with transferable utility*, or TU-game, is a pair (N, v), where N denotes the finite set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function, assigning to every coalition $S \subseteq N$ of players a value, or worth, $v(S)$, representing the total benefits of this coalition of players when they cooperate. By convention, $v(\emptyset) = 0$.

In case a TU-game involves costs instead of revenues it is denoted by (N, c), with $c : 2^N \to \mathbb{R}$ the characteristic function, assigning to every coalition $S \subseteq N$ of players a cost, $c(S)$, representing the total cost of this coalition of players when they cooperate. By convention, $c(\emptyset) = 0$. In the remainder of these preliminaries we restrict our attention to TU-games with revenues. Note that for cost games the definitions are analogous, but often different with respect to signs.

Let (N, v) be a TU-game. Then $x \in \mathbb{R}^N$ is called an allocation. The *carrier* $\mathrm{Car}_v(x)$ of an allocation x with respect to (N, v) is given by

$$\mathrm{Car}_v(x) = \{i \in N \,|\, x^i > v(\{i\})\}.$$

The set of feasible payoff vectors $X^*(N, v)$ is given by

$$X^*(N, v) \quad = \quad \{x \in \mathbb{R}^N \,|\, x(N) \leq v(N)\}.$$

A *solution* σ on the set of all TU-games associates with each TU-game (N, v) a subset $\sigma(N, v)$ of $X^*(N, v)$. The *preimputation set* $X(N, v)$ is the set of all efficient allocation vectors and it is given by

$$X(N, v) \quad = \quad \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}.$$

An element of this set is called a *preimputation*. The *imputation set* $I(N, v)$ is the set of all preimpuations for which no player receives less than his individual worth.

This set is defined by

$$I(N, v) = \{x \in X(N, v) \,|\, x^i \geq v(\{i\}) \text{ for all } i \in N\}.$$

An element of the imputation set is called an *imputation*. The *core* $C(N, v)$ (Gillies (1959)) consists of those imputations for which no coalition would be better off if it would separate itself and get its coalitional worth. It is given by

$$C(N, v) = \{x \in \mathbb{R}^N \,|\, x(N) = v(N),\; x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

Let $x \in I(N, v)$. An objection of player i against player j with respect to x is a pair $(S, y) \in 2^N \times \mathbb{R}^S$ such that

(i) $i \in S, j \notin S$,

(ii) $y > x_S$,

(iii) $\sum_{k \in S} y^k = v(S)$.

A counterobjection of player j against i to the objection (S, y) is a pair $(T, z) \in 2^N \times \mathbb{R}^T$ such that

(i) $j \in T, i \notin T$,

(ii) $z \geq (x_{T \setminus S}, y_{T \cap S})$,

(iii) $\sum_{k \in T} z^k = v(T)$.

An objection is called justified if there is no counterobjection of any player against it. The *bargaining set* $BS(N, v)$ (Aumann and Maschler (1964)) is the set of imputations to which no justified objection of any player exists.

A TU-game (N, v) is called *additive* if for all coalitions $S, T \subseteq N$ such that $S \cap T = \emptyset$ we have

$$v(S) + v(T) = v(S \cup T).$$

Hence, in an additive TU-game two coalitions cannot create extra profit by cooperating.

A TU-game (N, v) is called *superadditive* if for all coalitions $S, T \subseteq N$ such that $S \cap T = \emptyset$ we have

$$v(S) + v(T) \leq v(S \cup T).$$

In a superadditive TU-game cooperation pays. A TU-game (N, v) is called *convex* if for all $S \subseteq T \subseteq N \backslash \{i\}$,

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

for all $i \in N$. Thus in a convex TU-game the marginal contribution of a player to a coalition is higher in a larger coalition. A TU-game (N, v) is called *monotonic* if for all coalitions $S \subseteq T \subseteq N$ we have

$$v(S) \leq v(T).$$

So, in monotonic TU-games, larger coalitions have a higher value.

The *marginal vector* $m_\pi(N, v)$ of a TU-game (N, v) corresponding to the ordering $\pi \in \Pi$ is defined by

$$m_\pi^{\pi(t)}(N, v) \quad = \quad v(\{\pi(1), \ldots, \pi(t)\}) - v(\{\pi(1), \ldots, \pi(t-1)\})$$

for all $t \in \{1, \ldots, n\}$.

The *Shapley value* $\Phi(N, v)$ (Shapley (1953)) of a TU-game (N, v) is defined as the average over the marginal vectors

$$\Phi(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi} m_\pi(N, v).$$

For each TU-game (N, v) we define the utopia payoff to player $i \in N$ by $M_v^i = v(N) - v(N \backslash \{i\})$. Furthermore, $m_v^i = \max_{S: i \in S} \{v(S) - M_v(S \backslash \{i\})\}$ is the minimal right of player i. TU-game (N, v) is called *compromise admissible* if $m_v \leq M_v$ and $m_v(N) \leq v(N) \leq M_v(N)$.

For a compromise admissible TU-game (N, v) the *compromise value* $\tau(N, v)$ (Tijs (1981)) is defined by

$$\tau(N, v) \quad = \quad \alpha M_v + (1 - \alpha) m_v,$$

where α is the unique element of $[0,1]$ such that $\sum_{i\in N}\tau^i(N,v) = v(N)$. Hence, the compromise value is the efficient weighted average between the utopia and minimal rights vector.

Let (N,v) be a TU-game. The gap function $g_v : 2^N \to \mathbb{R}$ is defined by $g_v(S) = M_v(S) - v(S)$ for all $S \subseteq N$. A compromise admissible TU-game (N,v) is called *strongly compromise admissible* if $g_v(N) \leq g_v(S)$ for all $S \subseteq N$. If (N,v) is strongly compromise admissible, then $C(N,v) = \text{conv}\{M_v - g_v(N)e_{\{i\}}\}_{i\in N} \neq \emptyset$ (Driessen and Tijs (1983)).

The *excess* of coalition $S \subseteq N$ for preimputation $x \in X(N,v)$ is defined by

$$e(S,x,v) = v(S) - x(S).$$

If x is proposed as an allocation vector, the excess of S measures to which extent S is satisfied with x: the lower the excess, the more pleased S is with the proposed allocation. The idea behind the (pre)nucleolus is to minimise the highest excesses in a hierarchical manner.

For $x, y \in \mathbb{R}^n$ we have $x \leq_L y$, *i.e.*, x is lexicographically smaller than (or equal to) y, if $x = y$ or if there exists a $j \in \{1,\ldots,n\}$ such that $x^i = y^i$ for all $i \in \{1,\ldots,j-1\}$ and $x^j < y^j$. For a TU-game (N,v) and $x \in X(N,v)$ the excess vector $\chi(x) \in \mathbb{R}^{2^N}$ has as its coordinates the excesses of all possible 2^N coalitions written down in a (weakly) decreasing order. So $\chi^k(x) \geq \chi^{k+1}(x)$ for all $k \in \{1,2,\ldots,2^n-1\}$.

Let (N,v) be a TU-game with a non-empty imputation set. The *nucleolus* $n(N,v)$ (Schmeidler (1969)) of (N,v) is the unique point in $I(N,v)$ for which the excesses are lexicographically minimal, *i.e*,

$$\chi(n(N,v)) \leq_L \chi(x)$$

for all $x \in I(N,v)$. For TU-game (N,v) the *prenucleolus* $pn(N,v)$ is the unique point in $X(N,v)$ for which the excesses are lexicographically minimal. If the prenucleolus is an element of the imputation set, then the prenucleolus and the nucleolus coincide.

Let (N,v) be a TU-game. If $i,j \in N$, $i \neq j$, then we denote $\mathfrak{T}^{ij} = \{S \subseteq N\backslash\{j\} \mid i \in S\}$. The maximum excess of i over j at $x \in \mathbb{R}^N$ (with respect to (N,v)) is given by $z^{ij}(x,v) = \max_{S\in\mathfrak{T}^{ij}} e(S,x,v)$.

The *prekernel* $PK(N, v)$ of TU-game (N, v) is given by

$$PK(N,v) \;\;=\;\; \{x \in X(N,v) \,|\, z^{ij}(x,v) = z^{ji}(x,v) \text{ for all } i,j \in N\},$$

while the *kernel* $K(N, v)$ (Davis and Maschler (1965)) of a TU-game (N, v) with a non-empty imputation set is given by

$$K(N,v) = \{x \in I(N,v) \,|\, z^{ij}(x,v) \geq z^{ji}(x,v) \text{ or } x^i = v(\{i\}) \text{ for all } i,j \in N, i \neq j\}.$$

Chapter 3

Per capita nucleolus

> A compromise is the art of dividing a cake in such a way that everyone believes he has the biggest piece.
>
> Ludwig Erhard (1897 - 1977)

3.1 Introduction

Two of the most important and studied single-valued solution concepts for cooperative games are the nucleolus (Schmeidler (1969)) and the closely related prenucleolus. The (pre)nucleolus is the unique element in the (pre)imputation set for which the maximal coalitional objection, called excess, to it is minimised. Schmeidler (1969) shows that for each cooperative game the nucleolus is single-valued, *continuous* and a core element (if the core is non-empty). The main contribution of Kohlberg (1971) is a characterisation of the nucleolus by the use of balanced collections. Later the prenucleolus (Sobolev (1975)) and the nucleolus (Snijders (1995)) are characterised by the axioms single-valuedness, covariance, anonymity and the (imputation saving) reduced game property.

Related to the (pre)nucleolus is the per capita (pre)nucleolus. The per capita (pre)nucleolus is the unique element in the (pre)imputation set for which the maximal objection *per player* of a coalition to it is minimised. The idea of the per capita nucleolus is first considered by Grotte (1970), who calls it the normalised nucleolus. He claims, but does not show, that Kohlberg (1971)'s characterisation based on balanced collections can also be applied for the per capita nucleolus. The actual result is shown, in a more general setting, by Potters and Tijs (1992). Over the years the per capita (pre)nucleolus has made its appearance in several papers, *e.g.*, Young et al. (1982), Zhou (1991) and Arin and Feltkamp (1997), but it has never been extensively studied. Therefore, this chapter tries to give a comprehensive and objective overview of both the per capita prenucleolus and the per capita nucleolus. We discuss several properties and their relations to other solution concepts for cooperative games. Furthermore, we define for both solution concepts a reduced game and prove that they satisfy the corresponding reduced game properties. Moreover, we characterise both the per capita prenucleolus and the per capita nucleolus by the use of these reduced game properties in a similar way as the characterisations of the prenucleolus (Sobolev (1975), as presented by Peleg and Sudhölter (2003)) and the nucleolus (Snijders (1995)).

Example 3.1.1 To illustrate a difference between the prenucleolus and the per capita prenucleolus we consider the following ten-player game (N, v). Let $N = \{1, \ldots, 10\}$, $T = \{1, 2\}$ and $U = N \backslash T$. The coalitional worths are defined by

$$v(S) = \begin{cases} 18 & \text{if} \quad T \subseteq S, \quad S \neq N, \\ 72 & \text{if} \quad U \subseteq S, \quad S \neq N, \\ 100 & \text{if} \quad S = N, \\ 0 & \text{else.} \end{cases}$$

The only interesting coalitions of this game are T and U, in which the average payoff is 9, and N, in which the average payoff is 10. If we consider a core selector that satisfies anonymity, which implies that all benefits of T (U) are equally distributed among the players in T (U), the only question is how to divide the additional benefits of 10 $(100 - 18 - 72)$ obtained by full cooperation among the players of coalitions T and U.

The idea behind the prenucleolus is that (the complaint of) every coalition is equally important. Consequently, since the cooperation of two disjoint coalitions is needed to obtain the additional benefits, both coalitions receive an equal amount of these benefits. However, due to the fact that coalition U contains more players than T, each player in U gets less than each player in T. The prenucleolus of this game, $pn(N, v)$, is given by $pn^i(N, v) = 11\frac{1}{2}$ if $i \in T$ and $pn^i(N, v) = 9\frac{5}{8}$ if $i \in U$.

The idea behind the per capita prenucleolus is that (the complaint of) each *player* in every coalition is equally important. Hence, since all players are needed to form the grand coalition and all players receive 9 if T and U do not cooperate, the benefits of 10 are equally divided into 10 parts such that each player receives 1. Therefore, the per capita prenucleolus of this game, $pcpn(N, v)$, is given by $pcpn^i(N, v) = 10$ for all $i \in N$. $\triangleleft$

Besides the per capita (pre)nucleolus we also introduce, analyse and discuss the related concepts of the per capita prekernel and the per capita kernel. The kernel (Davis and Maschler (1965)) and the prekernel are well-known solution concepts in cooperative game theory. The prekernel contains all preimputations for which the maximum excess of player i over player j is equal to to maximum excess of j over i for all players i and j. The maximum excess of a player i over player j with respect to preimputation x is defined as the maximal amount player i can gain without the cooperation of player j by withdrawing from the grand coalition under preimputation x, assuming that the other players in i's withdrawing coalition are satisfied with their payoffs under x. The maximum excess can be seen as a way to measure a player's bargaining power over another given a particular preimputation. Peleg and Sudhölter (2003) characterise the prekernel by the axioms *non-emptiness*, efficiency, covariance, the *equal treatment property*, the reduced game property and the converse reduced game property.

In the definition of the maximum excess one assumes that all players in a coalition S, except player i, are satisfied with their payoffs under x. Further, it is assumed that any player in S can use the difference between the worth of S, $v(S)$, and the payoff to S, $x(S)$, to express his bargaining power over a player outside S. If we express the bargaining power by the *per capita excess*, which is given by $v(S) - x(S)$ divided by the number of players in S, then the bargaining power denotes what each player in coalition S can additionally gain simultaneously given x. This idea leads to the notion of the per capita (pre)kernel. In this chapter we discuss several properties of both the per capita prekernel and the per capita kernel and relate them to other solution concepts. Furthermore, we characterise the per capita prekernel by the use of the reduced game property in a similar way as the characterisation of the prekernel (Peleg and Sudhölter (2003)).

This reduced game property is also used to obtain a new characterisation of the core (Gillies (1959)) in a similar way as the characterisation by Peleg (1986), who shows that the core can be characterised (on the set of all games with a non-empty core) by non-emptiness, *individual rationality*, *super-additivity* and the weak reduced game property.

The outline of this chapter is as follows. In Section 3.2 we introduce some notation and definitions needed in the remainder of this chapter. In Section 3.3 we discuss the per capita prenucleolus and in Section 3.4 we analyse the per capita prekernel. Both solution concepts are characterised. Then we switch to the solution concepts that only consider elements in the imputation set. In Section 3.5 we discuss and characterise the per capita nucleolus and in Section 3.6 the per capita kernel is considered. In Section 3.7 we provide a new characterisation of the core. We conclude with an overview of the discussed solution concepts and their properties in Section 3.8.

3.2 Preliminaries

Let $\mathcal{U}$ be a non-empty set of players. The set $\mathcal{U}$ is either finite or countable. A TU-game is a pair (N, v), where $N \subseteq \mathcal{U}$ denotes the finite set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function, assigning to every coalition $S \subseteq N$ of players a worth, $v(S)$. By convention, $v(\emptyset) = 0$. The set of all TU-games is denoted by Γ.

Let $|S|$ denote the cardinality of $S \subseteq N, S \neq \emptyset$. For a TU-game (N, v) we define the *per capita excess* $e^{pc}(S, x, v)$ of coalition $S \subseteq N, S \neq \emptyset$ with respect to $x \in \mathbb{R}^N$ by

$$e^{pc}(S, x, v) \quad = \quad \frac{v(S) - x(S)}{|S|}.$$

If no confusion can occur we use the notation $e^{pc}(S, x)$. The per capita excess $e^{pc}(S, x)$ measures the complaint or dissatisfaction per player in S with the proposed vector x.

For $(N, v) \in \Gamma$ and $x \in X(N, v)$ the excess vector $\theta(x) \in \mathbb{R}^{2^n-1}$ has as its coordinates the per capita excesses of all possible $2^n - 1$ coalitions ($S \subseteq N$, $S \neq \emptyset$) written down in a (weakly) decreasing order. So $\theta^k(x) \geq \theta^{k+1}(x)$ for all $k \in \{1, 2, \ldots, 2^n - 2\}$.

Given a TU-game $(N, v) \in \Gamma$ and allocation $x \in \mathbb{R}^N$ we define the set of all coalitions $S \subseteq N, S \neq \emptyset$ with a per capita excess of at least $t \in \mathbb{R}$ by $\mathcal{B}(x, v, t) = \{S \subseteq N, S \neq \emptyset \,|\, e^{pc}(S, x, v) \geq t\}$. If no confusion can occur we also use the notation $\mathcal{B}(x, t)$.

Several proofs and proof constructions in this chapter are based upon similar proofs for the (pre)nucleolus and the (pre)kernel. We are in particular inspired by the work of Schmeidler (1969), Kohlberg (1971), Driessen and Tijs (1983), Peleg (1986), Snijders (1995) and Peleg and Sudhölter (2003).

3.3 Per capita prenucleolus

In this section we thoroughly discuss the per capita prenucleolus. This solution concept, which is related to the prenucleolus (cf. Schmeidler (1969)) is the preimputation for which the maximal objection per player of a coalition to it is minimised.

After its definition we discuss several properties in Subsection 3.3.1. In particular, we introduce a reduced game and show that the prenucleolus satisfies the corresponding reduced game property. In Subsection 3.3.2 we consider the relations between the per capita prenucleolus and other solution concepts for cooperative games. Finally, in Subsection 3.3.3 we characterise the per capita prenucleolus by the use of the reduced game property.

Definition Let $(N, v) \in \Gamma$. The *per capita prenucleolus* is given by $pcpn(N, v) = \{x \in X(N, v) \,|\, \theta(x) \leq_L \theta(y) \text{ for all } y \in X(N, v)\}$.

3.3.1 Properties

In this subsection we consider which (well-known) properties are satisfied by the per capita prenucleolus. Let σ be a solution on Γ. Then σ satisfies *non-emptiness* if $\sigma(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma$.

Lemma 3.3.1 *The per capita prenucleolus satisfies non-emptiness.*

Proof: Consider the equal split solution ESS, with $ESS^i(N, v) = \frac{v(N)}{n}$ for all $i \in N$ and all $(N, v) \in \Gamma$. The set $\{x \in X(N, v) \,|\, \theta(x) \leq_L \theta(ESS(N, v))\}$ is compact. Since $\theta(\cdot)$ is a continuous function there exists a lexicographic minimum on this set and hence, the per capita prenucleolus is non-empty. $\square$

Let σ be a solution on Γ. Then σ satisfies *efficiency* if $\sigma(N, v) \subseteq X(N, v)$ for all $(N, v) \in \Gamma$. Efficiency implies that the worth of the grand coalition is exactly distributed over the players. Since $pcpn(N, v)$ is defined as a subset of $X(N, v)$ it follows immediately that the per capita prenucleolus is efficient.

Corollary 3.3.2 *The per capita prenucleous satisfies efficiency.*

Let $(N, v) \in \Gamma$. Let $(\mathcal{B}_1, \ldots, \mathcal{B}_p)$ be a sequence of sets whose elements are coalitions of N. This sequence is an *ordered partition* whenever every coalition $S \subseteq N, S \neq \emptyset$ is contained in exactly one of the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$. Let $\mathcal{B}$ be a collection of coalitions. Then $\mathcal{B}$ is called *balanced* if there exist weights $\lambda_S \in \mathbb{R}, S \in \mathcal{B}$, with $\sum_{S \in \mathcal{B}} \lambda_S e_S = e_N$ and $\lambda_S > 0$ for all $S \in \mathcal{B}$. An ordered partition $(\mathcal{B}_1, \ldots, \mathcal{B}_p)$ is called balanced if $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ is balanced for all $k \in \{1, \ldots, p\}$.

For $(N, v) \in \Gamma$ and payoff vector x, let $\mathcal{B}_1(x, v)$ be the set[1] of those coalitions $S \subseteq N, S \neq \emptyset$ for which $\max\{e^{pc}(S, x)\}$ is attained. Similarly, $\mathcal{B}_2(x)$ is the set of those $S \subseteq N, S \neq \emptyset$ where $\max\{e^{pc}(S, x) \mid S \notin \mathcal{B}_1(x)\}$ is attained, and so on. This procedure results in the ordered partition $(\mathcal{B}_1(x), \ldots, \mathcal{B}_p(x))$.

Let σ be a solution on Γ. Then σ satisfies *single-valuedness* if $|\sigma(N, v)| = 1$ for all $(N, v) \in \Gamma$.

Theorem 3.3.3 The per capita prenucleolus satisfies single-valuedness.

Proof: Let $(N, v) \in \Gamma$ and let $x, y \in pcpn(N, v)$, which implies that $\theta^t(x) = \theta^t(y)$ for all $t \in \{1, 2, \ldots, 2^n - 1\}$. Let $z = \frac{1}{2}(x + y)$. For all $S \subseteq N$ we have $e^{pc}(S, z) \leq \max\{e^{pc}(S, x), e^{pc}(S, y)\}$. Let $k \in \mathbb{N}$ be such that $\mathcal{B}_k(x) \neq \emptyset$ and $\mathcal{B}_\ell(x) = \mathcal{B}_\ell(y) = \mathcal{B}_\ell(z)$ for all $\ell \in \{1, \ldots, k-1\}$. Let $t = \sum_{\ell=1}^{k-1} |\mathcal{B}_\ell(x)| + 1$. For all $S \in \mathcal{B}_k(z)$,

$$
\begin{aligned}
\theta^t(x) &\leq \theta^t(z) \\
&= e^{pc}(S, z) \\
&\leq \max\{e^{pc}(S, x), e^{pc}(S, y)\} \\
&\leq \max\{\theta^t(x), \theta^t(y)\} \\
&= \theta^t(x),
\end{aligned}
$$

which implies that all inequalities are in fact equalities. Hence, $\mathcal{B}_k(z) \subseteq \mathcal{B}_k(x)$. However, since $x \in pcpn(N, v)$, $\mathcal{B}_k(x) \subseteq \mathcal{B}_k(z)$, which implies that $\mathcal{B}_k(z) = \mathcal{B}_k(x)$. Similarly, $\mathcal{B}_k(z) = \mathcal{B}_k(y)$. We conclude that $x = y = z$. □

[1] If no confusion can occur we use the notation $\mathcal{B}_1(x)$ instead.

Theorem 3.3.4 Preimputation $x \in X(N, v)$ is the per capita prenucleolus of $(N, v) \in \Gamma$ if and only if the ordered partition $(\mathcal{B}_1(x), \ldots, \mathcal{B}_p(x))$ is balanced.

The proof of this theorem follows from Theorem 5.(a) in Potters and Tijs (1992), where it is shown that each weighted (pre)nucleolus satisfies this type of condition. It has been considered first in Kohlberg (1971) for the nucleolus.

Let $(N, v), (N, w) \in \Gamma$ and let σ be a solution on Γ. Then σ satisfies *covariance* if whenever $\alpha > 0, \beta \in \mathbb{R}$ and $w = \alpha v + \beta$, then $\sigma(N, w) = \alpha\sigma(N, v) + \beta$. Covariance implies that if two games are strategically equivalent, then the solution sets are related by the same transformation of the utilities of the players.

Proposition 3.3.5 *The per capita prenucleolus satisfies covariance.*

Proof: Let $(N, v), (N, w) \in \Gamma$ such that $w = \alpha v + \beta$. Let $x = pcpn(N, v)$ and let $y = \alpha x + \beta$. Let $S \subseteq N, S \neq \emptyset$. Then $e^{pc}(S, x, v) = \frac{v(S)-x(S)}{|S|}$. Further,

$$\begin{aligned} e^{pc}(S, y, w) &= \frac{w(S) - y(S)}{|S|} \\ &= \frac{(\alpha v(S) + \beta(S)) - (\alpha x(S) + \beta(S))}{|S|} \\ &= \alpha \cdot e^{pc}(S, x, v). \end{aligned}$$

Since the ordered partition $(\mathcal{B}_1(x, v), \ldots, \mathcal{B}_p(x, v))$ is balanced (Theorem 3.3.4) the ordered partition $(\mathcal{B}_1(y, w), \ldots, \mathcal{B}_p(y, w))$ is also balanced and hence, again by Theorem 3.3.4, $y = pcpn(N, w)$. □

Let N be fixed and let $\mathcal{V}(N) = \{(N, v) \,|\, v : 2^N \to \mathbb{R}, v(\emptyset) = 0\}$. We now prove that the per capita prenucleolus is a continuous function on $\mathcal{V}(N)$.

Theorem 3.3.6 The per capita prenucleolus $pcpn(N, v) : \mathcal{V}(N) \to \mathbb{R}^N$ is continuous.

Proof: Let $\{(N, v_t)\}_{t\in\mathbb{N}}$ be a sequence of games converging to (N, v) and let $\{x_t\}_{t\in\mathbb{N}}$ be a sequence such that the ordered partitions $(\mathcal{B}_1^t(x_t, v_t), \ldots, \mathcal{B}_{p_t}^t(x_t, v_t))$ are balanced for all $t \in \mathbb{N}$. Note that by Theorem 3.3.4, x_t is the per capita prenucleolus of (N, v_t) for all $t \in \mathbb{N}$. There exists an $M \in \mathbb{N}$ with $\max_{t\in\mathbb{N}, S\subseteq N} |v_t(S)| \leq M$. Let for all $t \in \mathbb{N}$, ESS_t be the equal split solution of (N, v_t), hence $ESS_t^i = \frac{v_t(N)}{n}$ for all $i \in N$. We obtain $\max_{S \subseteq N} e^{pc}(S, ESS_t, v_t) = \frac{ESS_t(S) - v_t(S)}{|S|} \leq \frac{|v_t(N)| + M}{1} \leq 2M$ for all $t \in \mathbb{N}$. Hence, $\max_{S \subseteq N} e^{pc}(S, x_t, v_t) \leq 2M$ for all $t \in \mathbb{N}$, which implies that $v_t(\{i\}) - x_t^i \leq 2M$ for all $i \in N$ and all $t \in \mathbb{N}$. Consequently, $x_t^i \geq v_t(\{i\}) - 2M \geq -3M$. On the other hand, $x_t^i = v_t(N) - x_t(N \backslash \{i\}) \leq M + 3M(n-1)$. Hence, the sequence $\{x_t\}_{t \in \mathbb{N}}$ is bounded. Therefore, this sequence has a converging subsequence. This subsequence is denoted by $\{\bar{x}_t\}_{t \in \mathbb{N}}$, with $\{(N, \bar{v}_t)\}_{t \in \mathbb{N}}$ the corresponding subsequence of games converging to (N, v). Let x be the limit of the subsequence $\{\bar{x}_t\}_{t \in \mathbb{N}}$. The ordered partition corresponding to game (N, v) and allocation x is denoted by $(\mathcal{B}_1(x, v), \ldots, \mathcal{B}_p(x, v))$. It suffices to prove that $x = pcpn(N, v)$.

As the number of ordered partitions is finite, we may assume, without loss of generality, that the ordered partition of $((N, \bar{v}_t), \bar{x}_t)$ is the same for all $t \in \mathbb{N}$. Since all weak inequalities are preserved under limits, it follows that $(\mathcal{B}_1(x, v), \ldots, \mathcal{B}_p(x, v))$ is a coarsening of $(\mathcal{B}_1^t(\bar{x}_t, \bar{v}_t), \ldots, \mathcal{B}_{p_t}^t(\bar{x}_t, \bar{v}_t))$, *i.e.*, for all $k \leq p$, $\mathcal{B}_1(x, v) \cup \cdots \cup \mathcal{B}_k(x, v) = \mathcal{B}_1^t(\bar{x}_t, \bar{v}_t) \cup \cdots \cup \mathcal{B}_\ell^t(\bar{x}_t, \bar{v}_t)$ for some $\ell \leq p_t$. Hence, $(\mathcal{B}_1(x, v), \ldots, \mathcal{B}_p(x, v))$ is balanced. Consequently, $x = pcpn(N, v)$. □

Let $\zeta : N \to N$ be an injection. The game $(\zeta(N), \zeta v)$ is defined by $\zeta v(\zeta(S)) = v(S)$ for all $S \subseteq N$. Let $(N, v) \in \Gamma$ and let σ be a solution on Γ. Then σ satisfies *anonymity* if $\zeta : N \to \mathcal{U}$ is an injection and $(\zeta(N), \zeta v) \in \Gamma$, then $\sigma(\zeta(N), \zeta v) = \zeta(\sigma(N, v))$. Anonymity means that σ is independent of the names of the players. Since the result is obvious we provide the following proposition without proof.

Proposition 3.3.7 *The per capita prenucleolus satisfies anonymity.*

Let $(N, v) \in \Gamma$. A player $i \in N$ is said to be at least as desirable as player $j \in N$ with respect to (N, v), denoted by $i \succeq_v j$, if $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$, which implies that i is at least as desirable as j if the marginal contribution of player i (weakly) exceeds the marginal contribution of player j for each coalition they might join. If $i \succeq_v j$ and $j \succeq_v i$, then we write $i \sim_v j$. Let σ be a solution on Γ. Then σ satisfies *desirability* if $x^i \geq x^j$ for all $x \in \sigma(N, v)$ and all players $i, j \in N$ satisfying $i \succeq_v j$.

Proposition 3.3.8 *The per capita prenucleolus satisfies desirability.*

For the proof of this proposition we refer to Proposition 3.4.6 in which we show that the per capita prekernel satisfies desirability. Since the per capita prenucleolus is an element of the per capita prekernel for all $(N, v) \in \Gamma$ (Theorem 3.3.20) this is sufficient.

Let $(N, v) \in \Gamma$ and let σ be a solution on Γ. Then σ satisfies the *equal treatment property* if whenever $x \in \sigma(N, v)$, and $i, j \in N$ satisfy $i \sim_v j$, then $x^i = x^j$. Note that desirability implies the equal treatment property.

Corollary 3.3.9 *The per capita prenucleolus satisfies the equal treatment property.*

Let $(N, v) \in \Gamma$ and let $i, j \in N$. Player i is more desirable than player j, denoted by $i \succ_v j$, if $i \succeq_v j$, but not $j \succeq_v i$. Let σ be a solution on Γ. Then σ satisfies *strong desirability* if $x^i > x^j$ for all $x \in \sigma(N, v)$ and all players $i, j \in N$ satisfying $i \succ_v j$. The next example illustrates that core selection and strong desirability are not compatible.

Example 3.3.10 Consider the three-player game (N, v) depicted below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	1	0	0	4	4	4	6

The core is given by $C(N, v) = \{(2, 2, 2)\}$. However, $1 \succ_v 2$, which implies that this allocation does not satisfy strong desirability. $\triangleleft$

Since the per capita prenucleolus is a core selector (Theorem 3.3.19) the per capita prenucleolus does not satisfy strong desirability.

Let $(N, v) \in \Gamma$ and let σ be a solution on Γ. Then σ satisfies *individual rationality* if $x^i \geq v(\{i\})$ for all $i \in N$ and all $x \in \sigma(N, v)$. Hence, individual rationality implies that each player gets at least his individual worth. Furthermore, σ is

- *reasonable from above* if $x^i \leq \max_{S \subseteq N \setminus \{i\}}(v(S \cup \{i\}) - v(S))$ for all $x \in \sigma(N, v)$,
- *reasonable from below* if $x^i \geq \min_{S \subseteq N \setminus \{i\}}(v(S \cup \{i\}) - v(S))$ for all $x \in \sigma(N, v)$,
- *reasonable* if it is both reasonable from above and from below.

The property reasonable (from above) is due to Milnor (1952). The arguments that support both reasonableness from below and above are straightforward. It seems unreasonable to pay any player more than his maximal marginal contribution to any coalition, because that seems to be the strongest threat that he can employ against a particular coalition. On the other hand, he may refuse to join any coalition that offers him less than his minimal marginal contribution. Moreover, player i can demand $\min_{S\subseteq N\setminus\{i\}}(v(S\cup\{i\})-v(S))$ and nevertheless join any coalition without hurting its members by this demand. Note that individual rationality implies reasonableness from below. The following example illustrates that the per capita prenucleolus is not reasonable from below and therefore neither individually rational.

Example 3.3.11 Consider the four-player game[2] (N, v) given below.

S	1	2	3	4	1,2	1,3	1,4	2,3	2,4	3,4	1,2,3	1,2,4	1,3,4	2,3,4	N
$v(S)$	0	0	0	0	8	8	0	8	0	0	8	8	8	8	8

The per capita prenucleolus of this game is given by $x = (3, 3, 3, -1)$. Since $x^4 < 0 = \min_{S\subseteq N\setminus\{i\}}(v(S \cup \{i\}) - v(S))$ the per capita prenucleolus is not reasonable from below. ◁

Proposition 3.3.12 *The per capita prenucleolus is reasonable from above.*

For the proof of this proposition we refer to Proposition 3.4.8 in which we show that the per capita prekernel is reasonable from above.

Let $(N, v) \in \Gamma$. Player $i \in N$ is called a dummy player if $v(S\cup\{i\}) = v(S)+v(\{i\})$ for all $S \subseteq N\setminus\{i\}$. A solution σ satisfies the *dummy property* if $x^i(N, v) = v(\{i\})$ for all dummy players $i \in N$, all $x \in \sigma(N, v)$ and all $(N, v) \in \Gamma$. Consequently, the dummy property says that players that do not contribute anything (except their individual worths) should receive their individual worths. A solution σ satisfies the *adding dummies property* if for all games $(N_1, v_1), (N_2, v_2) \in \Gamma$, with $N_2 = N_1 \cup H$, $H \cap N_1 = \emptyset$, such that $v_2(S \cup Q) = v_1(S) + v_2(Q)$ for all $S \subseteq N$ and all $Q \subseteq H$, it holds that $\sigma(N_1, v_1) = \sigma_{N_1}(N_2, v_2)$. This property requires that the adding of dummy players to the player set does not influence the distribution of the worth of

[2]Due to space limitations we omit the braces around the player sets.

the grand coalition over the (original) players. Note that the adding dummies property implies the dummy property. The per capita prenucleolus does not satisfy the dummy property. This is illustrated by Example 3.3.11, where player 4 is a dummy player, but $x^4 = -1 \neq 0 = v(\{4\})$. Consequently, the per capita prenucleolus does not satisfy the adding dummies property either.

Let $(N, v), (N, w), (N, v+w) \in \Gamma$ and let σ be a single-valued solution on Γ. Then σ satisfies *additivity* if $\sigma(N, v) + \sigma(N, w) = \sigma(N, v + w)$. The per capita prenucleolus does not satisfy additivity, as the following example illustrates.

Example 3.3.13 Consider the games $(N, v), (N, w)$ and $(N, v+w)$ depicted below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	1	6	7	8
$w(S)$	0	0	0	1	0	0	2
$(v+w)(S)$	0	0	0	2	6	7	10

The per capita prenucleolus of (N, v) is given by $pcpn(N, v) = (\frac{1}{3}, 1\frac{1}{3}, 6\frac{1}{3})$, while $pcpn(N, w) = (\frac{5}{6}, 1\frac{5}{6}, \frac{1}{3})$. Moreover, $pcpn(N, v + w) = (1\frac{1}{3}, 2\frac{1}{3}, 6\frac{1}{3}) \neq (1\frac{1}{6}, 2\frac{1}{6}, 6\frac{2}{3}) = pcpn(N, v) + pcpn(N, w)$ and the per capita prenucleolus violates the additivity requirement. $\triangleleft$

Let $(N, v), (N, w) \in \Gamma$ and let σ be a single-valued solution on Γ. Then σ is *coalitionally monotonic* if whenever $v(S) \leq w(S)$ for some $S \subseteq N$ and $v(T) = w(T)$ for all $T \neq S$, then $\sigma^i(N, v) \leq \sigma^i(N, w)$ for all $i \in S$. This property states that if the worth of only one coalition increases all its members should be (weakly) better off. Since the per capita prenucleolus is a core selector (Theorem 3.3.19) and no core selector can be coalitionally monotonic (Young (1985)), the per capita prenucleolus is not coalitionally monotonic. It is, however, weakly coalitionally monotonic, which is a concept introduced by Zhou (1991). Let $(N, v), (N, w) \in \Gamma$ and let σ be a single-valued solution on Γ. Then σ is *weakly coalitionally monotonic* if whenever $v(S) \leq w(S)$ for some $S \subseteq N$ and $v(T) = w(T)$ for all $T \neq S$, then $\sum_{i \in S} \sigma^i(N, v) \leq \sum_{i \in S} \sigma^i(N, w)$. Hence, this property requires that if the worth of only one coalition increases its members should *on average* be better off. By Theorem 1 and Remark 3 of Zhou (1991) we obtain Proposition 3.3.14. Note that Zhou (1991) refers to the (per capita) prenucleolus as (per capita) nucleolus.

Proposition 3.3.14 *The per capita prenucleolus is weakly coalitionally monotonic.*

Let $(N,v),(N,w) \in \Gamma$ and let σ be a single-valued solution on Γ. Then σ satisfies *aggregate monotonicity* if whenever $v(S)=w(S)$ for all $S \subsetneq N$ and $v(N) < w(N)$, then $\sigma^i(N,v) \leq \sigma^i(N,w)$ for all $i \in N$. Furthermore, σ satisfies *strong aggregate monotonicity* if whenever $v(S)=w(S)$ for all $S \subsetneq N$ and $v(N) < w(N)$, then $\sigma^i(N,w) - \sigma^i(N,v) = \sigma^j(N,w) - \sigma^j(N,v) > 0$ for all $i,j \in N$. Aggregate monotonicity has the following interpretation. If the worth of the grand coalition is increased, while at the same time the worth of any proper subcoalition remains unchanged, then everybody should benefit from the increase of $v(N)$. Moreover, strong aggregate monotonicity requires that everyone should benefit by receiving an equal share of the additional benefits.

Proposition 3.3.15 *The per capita prenucleolus satisfies strong aggregate monotonicity.*

Proof: Let $(N,v),(N,w) \in \Gamma$ such that $v(S)=w(S)$ for all $S \subsetneq N$ and $v(N) < w(N)$. Let $x = pcpn(N,v)$ and define $y \in X(N,w)$ such that $y^i = x^i + \frac{w(N)-v(N)}{n}$ for all $i \in N$. Let $S \subseteq N, S \neq \emptyset$. Then

$$\begin{aligned} e^{pc}(S,y,w) &= \frac{w(S)-y(S)}{|S|} \\ &= \frac{w(S)-(x(S)+\frac{w(N)-v(N)}{n}\cdot|S|)}{|S|} \\ &= \frac{w(S)-x(S)}{|S|} - \frac{w(N)-v(N)}{n}. \end{aligned}$$

Since the ordered partition $(\mathcal{B}_1(x,v),\ldots,\mathcal{B}_p(x,v))$ is balanced (Theorem 3.3.4) the ordered partition $(\mathcal{B}_1(y,w),\ldots,\mathcal{B}_p(y,w))$ is balanced as well and hence, by Theorem 3.3.4, $y = pcpn(N,w)$. □

Several solution concepts, *e.g.*, the prenucleolus, satisfy a reduced game property. This is also the case for the per capita prenucleolus. We introduce the *reduced game* $(T, v_{T,x})$ with respect to coalition T and preimputation x by

$$v_{T,x}(S) = \begin{cases} 0 & \text{if } S=\emptyset, \\ v(N)-x(N\backslash T) & \text{if } S=T, \\ \max_{Q\subseteq N\backslash T} v(S\cup Q) - x(Q) - |Q|\cdot e^{pc}(S\cup Q,x,v) & \text{otherwise.} \end{cases}$$

Let $(N,v)\in\Gamma$ and let σ be a solution on Γ. Then σ satisfies the *reduced game property* if whenever $T\subseteq N, T\neq\emptyset$ and $x\in\sigma(N,v)$, then $(T,v_{T,x})\in\Gamma$ and $x_T\in\sigma(T,v_{T,x})$. The reduced game property is a condition of consistency. If $(N,v)\in\Gamma$ and $x\in\sigma(N,v)$, then for all $T\subseteq N, T\neq\emptyset$ the proposal x_T solves $(T,v_{T,x})$ and therefore, it is consistent with the expectations of the members of T as reflected by the reduced game $(T,v_{T,x})$. The per capita prenucleolus satisfies the reduced game property, but in order to prove this result we first need some preliminary lemmas.

Lemma 3.3.16 *Let $(N,v)\in\Gamma$. For all $S\subsetneqq T\subseteq N$ and all $x\in X(N,v)$ we have*

$$e^{pc}(S,x_T,v_{T,x}) \;=\; \max_{Q\subseteq N\setminus T} e^{pc}(S\cup Q,x,v).$$

Proof: Let $S\subsetneqq T\subseteq N$. Then

$$\begin{aligned}
e^{pc}(S,x_T,v_{T,x}) &= \frac{1}{|S|}\cdot\Big(v_{T,x}(S)-x(S)\Big)\\
&= \frac{1}{|S|}\cdot\Big(\max_{Q\subseteq N\setminus T}(v(S\cup Q)-x(Q)-|Q|\cdot e^{pc}(S\cup Q,x,v))-x(S)\Big)\\
&= \frac{1}{|S|}\cdot\Big(\max_{Q\subseteq N\setminus T} v(S\cup Q)-x(S\cup Q)-\frac{|Q|\cdot(v(S\cup Q)-x(S\cup Q))}{|S\cup Q|}\Big)\\
&= \frac{1}{|S|}\cdot\Big(\max_{Q\subseteq N\setminus T}\frac{|S|\cdot(v(S\cup Q)-x(S\cup Q))}{|S\cup Q|}\Big)\\
&= \max_{Q\subseteq N\setminus T}\frac{v(S\cup Q)-x(S\cup Q)}{|S\cup Q|}\\
&= \max_{Q\subseteq N\setminus T} e^{pc}(S\cup Q,x,v).
\end{aligned}$$

□

Lemma 3.3.17 *Let $(N,v)\in\Gamma$ and let $T\subseteq N$. Then $\mathcal{B}(x_T,v_{T,x},t)=\{S\cap T\,|\,S\in\mathcal{B}(x,v,t)\}$.*

Proof: We first prove that $\{S\cap T\,|\,S\in\mathcal{B}(x,v,t)\}\subseteq\mathcal{B}(x_T,v_{T,x},t)$. Let $S\subseteq\mathcal{B}(x,v,t)$. Then

$$\begin{aligned}
e^{pc}(S\cap T,x_T,v_{T,x}) &= \max_{Q\subseteq N\setminus T} e^{pc}((S\cap T)\cup Q,x,v)\\
&\geq e^{pc}(S,x,v)\\
&\geq t,
\end{aligned}$$

where the equality follows from Lemma 3.3.16. Hence, $(S \cap T) \in \mathcal{B}(x_T, v_{T,x}, t)$.

Secondly, we show that $\mathcal{B}(x_T, v_{T,x}, t) \subseteq \{S \cap T \mid S \in \mathcal{B}(x,t)\}$. Let $U \in \mathcal{B}(x_T, v_{T,x}, t)$. Let $R \in \arg\max_{Q \subseteq N \setminus T} e^{pc}(U \cup Q, x, v)$ and let $S = U \cup R$, which implies that $U = S \cap T$. Then

$$\begin{aligned} e^{pc}(S, x, v) &= e^{pc}(U, x_T, v_{T,x}) \\ &\geq t, \end{aligned}$$

where the equality follows from Lemma 3.3.16. This implies that $S \in \mathcal{B}(x, v, t)$. $\square$

Theorem 3.3.18 The per capita prenucleolus satisfies the reduced game property.

Proof: Let $(N, v) \in \Gamma$ and let $x = pcpn(N, v)$. Let $T \subseteq N$, with reduced game $(T, v_{T,x})$. Further, let $t \in \mathbb{R}$ be such that $\mathcal{B}(x, v, t) \neq \emptyset$ and let $\{\lambda_S\}_{S \in \mathcal{B}(x,v,t)}$ be a set of balancing weights for $\mathcal{B}(x, v, t)$, *i.e.*, for all $i \in N$ we have

$$\sum_{S \in \mathcal{B}(x,v,t): i \in S} \lambda_S = 1.$$

Define for all $Q \in \mathcal{B}(x_T, v_{T,x}, t)$

$$\mu_Q = \sum_{S \in \mathcal{B}(x,v,t): S \cap T = Q} \lambda_S.$$

Then $\{\mu_Q\}_{Q \in \mathcal{B}(x_T, v_{T,x}, t)}$ is a set of balancing weights for $\mathcal{B}(x_T, v_{T,x}, t)$, because for all $i \in T$

$$\begin{aligned} \sum_{Q \in \mathcal{B}(x_T, v_{T,x}, t): i \in Q} \mu_Q &= \sum_{Q \in \{R \cap T \mid R \in \mathcal{B}(x,v,t)\}: i \in Q} \mu_Q \\ &= \sum_{Q \in \{R \cap T \mid R \in \mathcal{B}(x,v,t)\}: i \in Q} \; \sum_{S \in \mathcal{B}(x,v,t): S \cap T = Q} \lambda_S \\ &= \sum_{S \in \mathcal{B}(x,v,t): i \in S} \lambda_S \\ &= 1, \end{aligned}$$

where the first equality follows from Lemma 3.3.17. We conclude that the ordered partition $(\mathcal{B}_1(x_T, v_{T,x}), \ldots, \mathcal{B}_p(x_T, v_{T,x}))$ is balanced and therefore, by Theorem 3.3.4, x_T is the per capita prenucleolus of $(T, v_{T,x})$. $\square$

If $(N,v) \in \Gamma$, then we denote $P(N) = \left\{ T \subseteq N \,\middle|\, |T| = 2 \right\}$. A solution σ on a set Γ of games satisfies the *converse reduced game property* if whenever $n \geq 2$, $x \in X(N,v)$, $(T, v_{T,x}) \in \Gamma$, and $x_T \in \sigma(T, v_{T,x})$ for every $T \in P(N)$, then $x \in \sigma(N,v)$. This property implies that if x is a solution for any pair of players, then x is a solution for the whole group of players. The per capita prenucleolus does not satisfy the converse reduced game property, which follows from Theorem 3.4.13 in Section 3.4.

3.3.2 Relations to other solution concepts

In this subsection we discuss the relation of the per capita prenucleolus to other solution concepts for cooperative games. The first theorem states that the per capita prenucleolus is, just as the prenucleolus, an element of the core, whenever the core is non-empty. By $\Gamma^C \subseteq \Gamma$ we denote the set of all games with a non-empty core.

Theorem 3.3.19 Let $(N,v) \in \Gamma^C$. Then $pcpn(N,v) \in C(N,v)$.

Proof: Let $x \in C(N,v)$. Then $x(N) = v(N)$ and $x(S) \geq v(S)$ for all $S \subseteq N$. Therefore, $e^{pc}(S,x) \leq 0$ for all $S \subseteq N$. Hence, $\theta(x) \leq_L 0$ and $\theta^1(x) = 0$, as $e^{pc}(N,x) = 0$. By definition $\theta(pcpn(N,v)) \leq_L \theta(x)$. Consequently, $\max_{S \subseteq N} e^{pc}(S, pcpn(N,v), v) = \theta^1(pcpn(N,v)) \leq \theta^1(x) = 0$, which implies $pcpn(N,v) \in C(N,v)$. □

If $i, j \in N$, $i \neq j$, then we denote $\mathfrak{T}^{ij} = \{S \subseteq N \backslash \{j\} \,|\, i \in S\}$. The maximum per capita excess of i over j at $x \in \mathbb{R}^N$ (with respect to (N,v)) is given by[3] $s^{ij}(x,v) = \max_{S \in \mathfrak{T}^{ij}} e^{pc}(S,x,v)$. We define the per capita prekernel[4] in a similar way as the prekernel.

Definition Let $(N,v) \in \Gamma$. The *per capita prekernel* of (N,v) is given by $PCPK(N,v) = \{x \in X(N,v) \,|\, s^{ij}(x) = s^{ji}(x) \text{ for all } i,j \in N\}$.

The next theorem shows that the per capita prenucleolus is an element of the per capita prekernel for any cooperative game. Note that this also implies that the per capita prekernel is a non-empty set.

[3]If no confusion can occur we use the notation $s^{ij}(x)$ instead of $s^{ij}(x,v)$.

[4]For an elaborate discussion of the per capita prekernel we refer to Section 3.4.

Theorem 3.3.20 Let $(N, v) \in \Gamma$. Then $pcpn(N, v) \in PCPK(N, v)$.

Proof: Let $x \in X(N, v) \backslash PCPK(N, v)$. Hence, $s^{ij}(x) > s^{ji}(x)$ for some $i, j \in N$. Take $t = s^{ij}(x)$. Then the collection $\mathcal{B}(x, v, t)$ contains a coalition $S \in \mathfrak{T}^{ij}$, but no coalition $T \in \mathfrak{T}^{ji}$. Consequently, $\mathcal{B}(x, v, t)$ cannot be balanced and by the use of Theorem 3.3.4 $x \neq pcpn(N, v)$. The fact that the per capita prenucleolus is non-empty (Lemma 3.3.1) completes the proof. $\square$

Recall that the bargaining set, introduced by Aumann and Maschler (1964), is the non-empty set of imputations to which no player has a justified objection. The per capita prenucleolus is not a bargaining set selector, as the following example illustrates.

Example 3.3.21 Consider the four-player game (N, v).

S	1	2	3	4	1,2	1,3	1,4	2,3	2,4	3,4	1,2,3	1,2,4	1,3,4	2,3,4	N
$v(S)$	0	0	0	0	8	0	0	0	0	0	0	0	8	8	8

The per capita prenucleolus of this game is given by $pcpn(N, v) = (3, 3, 1, 1)$. However, $(\{2, 3, 4\}, (5\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3}))$ is a justified objection of player 4 to player 1 with respect to $pcpn(N, v)$. Hence, $pcpn(N, v) \notin BS(N, v)$. $\triangleleft$

Note that since $C(N, v) \subseteq BS(N, v)$ for all $(N, v) \in \Gamma^C$, $pcpn(N, v) \in BS(N, v)$ for all $(N, v) \in \Gamma^C$.

Driessen and Tijs (1983) show that both the prenucleolus and the nucleolus are equal to the compromise value (Tijs (1981)) for strongly compromise admissible games. We obtain the same result for the per capita prenucleolus, which implies that for this class of games also the (pre)nucleolus and the per capita prenucleolus coincide. By $\Gamma^{SCA} \subseteq \Gamma$ we denote the set of strongly compromise admissible games.

Theorem 3.3.22 Let $(N, v) \in \Gamma^{SCA}$. Then $pcpn(N, v) = \tau(N, v)$.

Proof: Since $(N, v) \in \Gamma^{SCA}$ the compromise value is given by $\tau(N, v) = M_v - \frac{1}{n}(g_v(N), \ldots, g_v(N))$, with $g_v(S) = M_v(S) - v(S)$ for all $S \subseteq N$, $S \neq \emptyset$.

Therefore,

$$\begin{aligned} e^{pc}(S,\tau(N,v),v) &= \frac{v(S)-\tau(S)}{|S|} \\ &= \frac{v(S)-M_v(S)-\frac{|S|}{n}\cdot g_v(N)}{|S|} \\ &= \frac{1}{n}\cdot g_v(N)-\frac{g_v(S)}{|S|} \end{aligned}$$

for all $S \subseteq N$, $S \neq \emptyset$. We note that

(i) $e^{pc}(N,\tau(N,v),v)=0$.

(ii) For all $i \in N$,

$$\begin{aligned} e^{pc}(N\backslash\{i\},\tau(N,v),v) &= \frac{1}{n}\cdot g_v(N)-\frac{g_v(N\backslash\{i\})}{n-1} \\ &= \frac{1}{n}\cdot g_v(N)-\frac{g_v(N)}{n-1} \\ &= \frac{-1}{n(n-1)}\cdot g_v(N) \\ &\leq 0, \end{aligned}$$

where the inequality follows from the fact that $(N,v) \in \Gamma^{SCA}$.

(iii) For all $S \subseteq N$ with $2 \leq |S| \leq n-1$ we have

$$\begin{aligned} \frac{1}{n}\cdot g_v(N)-\frac{g_v(S)}{|S|} &\leq \frac{1}{n}\cdot g_v(N)-\frac{g_v(N)}{|S|} \\ &\leq \frac{1}{n}\cdot g_v(N)-\frac{g_v(N)}{n-1} \\ &= \frac{-1}{n(n-1)}\cdot g_v(N), \end{aligned}$$

where the first inequality follows from the fact that $(N,v) \in \Gamma^{SCA}$.

Hence, $\mathcal{B}_1(\tau(N,v))$ of the ordered partition $(\mathcal{B}_1(\tau(N,v)),\ldots,\mathcal{B}_p(\tau(N,v)))$ contains $\left\{S \,\middle|\, |S|=n-1\right\}$, which implies that $(\mathcal{B}_1(\tau(N,v)),\ldots,\mathcal{B}_p(\tau(N,v)))$ is balanced. Consequently, $\tau(N,v)=pcpn(N,v)$. $\square$

3.3.3 Characterisation

In this subsection we characterise the per capita prenucleolus. We first introduce some definitions, notation and preliminary lemmas.

Definition A *coalitional family* is a pair $\mathbf{B} = (N, \mathcal{B})$ where N is a finite non-empty set of players and $\mathcal{B} \subseteq 2^N$ is a non-empty collection of coalitions.

We denote coalitions by italic characters, collections of coalitions by calligraphic characters, and coalitional families by boldfaced characters.

Definition Let $\mathbf{B} = (N, \mathcal{B})$ be a coalitional family. A permutation π of N is a *symmetry* of $\mathbf{B}$ if for every $S \in \mathcal{B}$ we have[5] $\pi(S) \in \mathcal{B}$. $\mathbf{B}$ is *transitive* if for every pair of players (i, j) there exists a symmetry π of $\mathbf{B}$ such that $\pi(i) = j$.

If $(N, \mathcal{B})$ is a coalitional family and $i \in N$, then we denote $\mathcal{B}^i = \{S \in \mathcal{B} \,|\, i \in S\}$. Note that for all $S \in \mathcal{B}$ we have $S \in \bigcap_{i \in S} \mathcal{B}^i$.

Example 3.3.23 Let coalitional family $\mathbf{B} = (N, \mathcal{B})$ be given by $N = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{B} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}\}$. A symmetry that maps player 1 to player 5 is given by $(\pi(1), \pi(2), \pi(3), \pi(4), \pi(5), \pi(6)) = (5, 6, 3, 4, 1, 2)$. By Lemma 3.3.24, $(N, \mathcal{B})$ is transitive (take $m = 2$, $r = 1$). $\triangleleft$

Lemma 3.3.24 *Let $\boldsymbol{B} = (N, \mathcal{B})$ be a coalitional family and let $m, r \in \mathbb{N}$. If, for every subcollection $\mathcal{D} \subseteq \mathcal{B}$ of cardinality m, the number of players $i \in N$ with $\mathcal{D} = \mathcal{B}^i$ equals r, and if $|\mathcal{B}^i| = m$ for all $i \in N$, then $\boldsymbol{B}$ is transitive.*

Proof: Let $\mathbf{B} = (N, \mathcal{B})$ be a coalitional family such that for every subcollection $\mathcal{D} \subseteq \mathcal{B}$ of cardinality m, the number of players $i \in N$ with $\mathcal{D} = \mathcal{B}^i$ equals r, and $|\mathcal{B}^i| = m$ for all $i \in N$. Denote $|N| = n$ and $|\mathcal{B}| = b$. There are n/r subcollections of $\mathcal{B}$ of cardinality m, so $n = r\binom{b}{m}$. Every player $i \in N$ is a member of m coalitions in $\mathcal{B}$. Each coalition $S \in \mathcal{B}$ is contained in $\binom{b-1}{m-1}$ subcollections of $\mathcal{B}$ of cardinality m. Each of these subcollections corresponds with r players, all members of S, so $|S| = r\binom{b-1}{m-1} = \dfrac{nm}{b}$.

Let $k, \ell \in N$. We are going to find a symmetric permutation π of N with $\pi(k) = \ell$.

[5] Formally, $\pi(S)$ should be denoted by $\bigcup_{i \in S}\{\pi(i)\}$.

If $\mathcal{B}^k = \mathcal{B}^\ell$, simply take $\pi(k) = \ell$, $\pi(\ell) = k$, and $\pi(j) = j$ for all $j \in N \backslash \{k, \ell\}$. Then $\pi(S) = S$ for all $S \in \mathcal{B}$, and hence $\pi(S) \in \mathcal{B}$ for all $S \in \mathcal{B}$.

So, we assume from here that $\mathcal{B}^k \neq \mathcal{B}^\ell$. We define $\pi^{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ as a coalitional permutation, which means that $\pi^{\mathcal{B}}$ permutates coalitions in $\mathcal{B}$. Let $\pi^{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ be a coalitional permutation with $\pi^{\mathcal{B}}(\mathcal{B}^k) = \mathcal{B}^\ell$. (Note that there may be more than one coalitional permutation with this property.) Take a permutation π of N with $\mathcal{B}^{\pi(i)} = \pi^{\mathcal{B}}(\mathcal{B}^i)$ for all $i \in N$ and $\pi(k) = \ell$. (If $r > 1$ there also may be many of these permutations given $\pi^{\mathcal{B}}$.) The proof is completed by showing that π is symmetric, *i.e.*, $\pi(S) \in \mathcal{B}$ for all $S \in \mathcal{B}$.

Let $S \in \mathcal{B}$. For all $i \in N$, the following four statements are equivalent:

- $i \in S$,
- $S \in \mathcal{B}^i$,
- $\pi^{\mathcal{B}}(S) \in \mathcal{B}^{\pi(i)}$,
- $\pi(i) \in \pi^{\mathcal{B}}(S)$.

This explains the second equality in

$$\begin{aligned}
\bigcap_{j \in \pi^{\mathcal{B}}(S)} \mathcal{B}^j &= \bigcap_{i \in N:\ \pi(i) \in \pi^{\mathcal{B}}(S)} \mathcal{B}^{\pi(i)} \\
&= \bigcap_{i \in S} \mathcal{B}^{\pi(i)} \\
&= \bigcap_{j \in \pi(S)} \mathcal{B}^j.
\end{aligned}$$

Since $\pi^{\mathcal{B}}(S) \in \bigcap_{j \in \pi^{\mathcal{B}}(S)} \mathcal{B}^j$, the above equality gives $\pi^{\mathcal{B}}(S) \in \bigcap_{j \in \pi(S)} \mathcal{B}^j$. Hence, for all $j \in \pi(S)$, $\pi^{\mathcal{B}}(S) \in \mathcal{B}^j$, which implies that $j \in \pi^{\mathcal{B}}(S)$. Consequently, $\pi(S) \subseteq \pi^{\mathcal{B}}(S)$, and since $|\pi^{\mathcal{B}}(S)| = |\pi(S)| = |S|$, $\pi(S) = \pi^{\mathcal{B}}(S)$. Therefore, as $\pi^{\mathcal{B}}(S) \in \mathcal{B}$, also $\pi(S) \in \mathcal{B}$. $\square$

Definition Let $\mathbf{B} = (N, \mathcal{B})$ be a coalitional family. $\mathbf{B}$ is *balanced* if $\mathcal{B}$ is balanced. $\mathbf{B}$ is an *embedding* of coalitional family $(N^*, \mathcal{B}^*)$ if $N \subseteq N^*$ and $\mathcal{B} = \{N \cap S^* \,:\, S^* \in \mathcal{B}^*\}$.

Lemma 3.3.25 *Every balanced coalitional family is embedded in a transitive coalitional family.*

Proof: Let $(N, \mathcal{B})$ be a balanced coalitional family. We call players i and j equivalent if $\mathcal{B}^i = \mathcal{B}^j$. Let $r = \max_{S \subseteq N}\{|S| : \text{all pairs of players in } S \text{ are equivalent}\}$. Since $\mathcal{B}$ is balanced, there exist natural numbers λ_S, $S \in \mathcal{B}$, and m such that $\sum_{S \in \mathcal{B}^i} \lambda_S = m$ for all $i \in N$. Let $b = \sum_{S \in \mathcal{B}} \lambda_S$ and $n^* = r\binom{b}{m}$. Let $S_1, \ldots, S_b$ be a sequence of elements of $\mathcal{B}$ such that each $S \in \mathcal{B}$ occurs λ_S times in the sequence. Then $|\{u \in \{1, \ldots, b\} : \ i \in S_u\}| = m$ for all $i \in N$. We add $r\binom{b}{m} - n$ new players to N by the following procedure.

1. Initialise $t = 1$.
2. Set $N^*(t-1) = N$ and $S^*_u(t-1) = S_u$ for all $u \in \{1, \ldots, b\}$.
3. Take $M(t) \subseteq \{1, \ldots, b\}$ such that $|M(t)| = m$ and $M(t) \neq M(v)$ for all $v \in \{1, \ldots, t-1\}$.
4. Let $S_{M(t)}$ be a set of $r - |\bigcap_{u \in M(t)} S^*_u(t)|$ new players, *i.e.*, $S_{M(t)} \cap N^*(t-1) = \emptyset$.
5. Add $S_{M(t)}$ to $N^*(t-1)$ and to each coalition with its index in $M(t)$, *i.e.*, $N^*(t) = N^*(t-1) \cup S_M(t)$, $S^*_u(t) = S^*_u(t-1) \cup S_{M(t)}$ if $u \in M(t)$ and $S^*_u(t) = S^*_u(t-1)$ if $u \notin M(t)$.
6. If $t < \binom{b}{k}$, then set $t = t+1$ and return to Step 3. Otherwise, set $N^* = N^*(t)$ and $S^*_u = S^*_u(t)$ for all $u \in \{1, \ldots, b\}$.

Once this procedure has been finished, all coalitions in the sequence are different. Define $\mathcal{B}^* = \{S^*_u : \ u \in \{1, \ldots, b\}\}$. Then $(N^*, \mathcal{B}^*)$ satisfies the requirements of Lemma 3.3.24, with $m = \sum_{S \in \mathcal{B}^i} \lambda_S$ and $r = \max_{S \subseteq N}\{|S| : \text{all pairs of players in } S$ are equivalent$\}$. □

Example 3.3.26 Consider the balanced coalitional family $\mathbf{B} = (N, \mathcal{B})$ with $N = \{1, 2, 3, 4\}$ and $\mathcal{B} = \{\{1\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. No pair of players are equivalent, so $r = 1$. We can take $\lambda_S = \dfrac{2}{|S|}$ for all $S \in \mathcal{B}$, making $b = 5$ and $m = 2$. Hence, the number of players in N^* in coalitional family $(N^*, \mathcal{B}^*)$ equals $n^* = r\binom{b}{k} = 1 \cdot \binom{5}{2} = 10$. Initially, $(S_1, \ldots, S_5) = (\{1\}, \{1\}, \{2, 3\}, \{2, 4\}, \{3, 4\})$. We must add new players such that eventually for every pair (since $m = 2$) $\{u_1, u_2\} \subseteq \{1,2,3,4,5\}$, there is one (since $r = 1$) player i with $\mathcal{B}^i = \{S_{u_1}, S_{u_2}\}$. There are 10 pairs in $\{1,2,3,4,5\}$ and initially we have 4 players, so 6 new players have to be added. With $M(1) = \{1,2\}, M(2) = \{1,3\}, \ldots, M(10) = \{4,5\}$ and new players added in order $5, 6, \ldots, 10$ the procedure of Lemma 3.3.25 results in

$$\mathcal{B}^* \quad = \quad \{\{1,5,6,7\}, \{1,8,9,10\}, \{2,3,5,8\}, \{2,4,6,9\}, \{3,4,7,10\}\}.$$

Hence, for any two ($m = 2$) coalitions of $\mathcal{B}^*$ there is one player ($r = 1$) that is an element of both of them. Furthermore, $|\mathcal{B}^i| = 2$ for all $i \in N$. $\triangleleft$

Lemma 3.3.27 *Let σ be a solution on Γ. If σ satisfies single-valuedness, covariance and the reduced game property, then σ is also efficient.*

Proof: Let σ be a solution on Γ that satisfies single-valuedness, covariance and the reduced game property. Let $(\{i\}, v) \in \Gamma$ be a one-player game. If $v(\{i\}) = 0$, then, by covariance $\sigma(\{i\}, 0) = \sigma(\{i\}, 2 \cdot 0) = 2 \cdot \sigma(\{i\}), 0)$. Hence, $\sigma(\{i\}, 0) = \{0\}$. Again by covariance, $\sigma(\{i\}, v) = \sigma(\{i\}, 0 + v) = \sigma(\{i\}, 0) + v(\{i\}) = \{v(\{i\}\}$ and σ is efficient.

Now let (N, v) be an n-player game, with $n \geq 2$. Let $x \in \sigma(N, v)$ and $i \in N$. The reduced game $(\{i\}, v_{\{i\},x})$ is a one-player game. By the reduced game property, $x^i \in \sigma(\{i\}, v_{\{i\},x})$. Hence, by the definition of the reduced game $x^i = v_{\{i\},x}(\{i\}) = v(N) - x(N \backslash \{i\})$, which results in $x(N) = v(N)$ and therefore in the efficiency of σ. $\square$

Theorem 3.3.28 Let $\mathcal{U}$ be infinitely countable and let $\Gamma_{\mathcal{U}}$ be the set of all games whose player set is contained in $\mathcal{U}$. The per capita prenucleolus is the unique solution on $\Gamma_{\mathcal{U}}$ that satisfies single-valuedness, covariance, anonymity, and the reduced game property.

Proof: Step 1. By Theorem 3.3.3, Proposition 3.3.5, Proposition 3.3.7 and Theorem 3.3.18 we obtain that the per capita prenucleolus satisfies single-valuedness, covariance, anonymity and the reduced game property, respectively. Thus it remains to prove that the per capita prenucleolus is the only solution concept satisfying these four properties.

Step 2. Let σ be a solution on $\Gamma_{\mathcal{U}}$ that satisfies single-valuedness, covariance, anonymity and the reduced game property, let $(N,v) \in \Gamma_{\mathcal{U}}$ and let $x = pcpn(N,v)$. We have to prove that $\sigma(N,v) = \{x\}$. Define $(N,w) \in \Gamma_{\mathcal{U}}$ by $w(S) = v(S) - x(S)$ for every $S \subseteq N$. By covariance of the per capita prenucleolus, $pcpn(N,w) = 0$. By covariance of σ, it suffices to prove that $\sigma(N,w) = \{0\}$. Thus, we consider the game (N,w).

Step 3. Let $\left\{ \frac{w(S)}{|S|} \,\middle|\, \emptyset \neq S \subseteq N \right\} = \{\mu_1, \ldots, \mu_p\}$, where $\mu_1 > \cdots > \mu_p$. Let $P = \{1, \ldots, p\}$. We denote

$$\begin{aligned} \mathcal{B}_h &= \mathcal{B}(0, w, \mu_h) \\ &= \{\emptyset \neq S \subsetneqq N \mid \frac{w(S)}{|S|} \geq \mu_h\} \end{aligned}$$

for all $h \in P$. By Theorem 3.3.4, $\mathcal{B}_h$ is a balanced collection on N for all $h \in P$. Let, for every $h \in P$, $(N_h^*, \mathcal{B}_h^*)$ be a transitive coalitional family that embeds $(N, \mathcal{B}_h)$.

Define $\hat{N} = N_1^* \times N_2^* \times \cdots \times N_p^*$ and let $(\hat{N}, \hat{w})$ be the game defined by

$$\hat{w}(\hat{S}) = \begin{cases} 0 & \text{if } \hat{S} \in \{\emptyset, \hat{N}\}, \\ |\hat{S}| \cdot \mu_h & \text{if } \hat{S} = N_1^* \times \cdots \times N_{h-1}^* \times T^* \times N_{h+1}^* \times \cdots \times N_p^* \text{ for some } T^* \in \mathcal{B}_h^*, \\ |\hat{S}| \cdot \mu_p & \text{else.} \end{cases}$$

Step 4. We show that $(\hat{N}, \hat{w})$ is symmetric, *i.e.*, for every pair $\{\hat{\imath}, \hat{\jmath}\} \subseteq \hat{N}$, there exist a permutation π of $\hat{N}$ such that $\pi(\hat{\imath}) = \hat{\jmath}$ and $\hat{w}(\pi(\hat{S})) = \hat{w}(\hat{S})$ for all $\hat{S} \subseteq \hat{N}$, which means that every player is interchangeable with every other player. Let $\hat{\imath} = (\hat{\imath}_1, \ldots, \hat{\imath}_p)$ and $\hat{\jmath} = (\hat{\jmath}_1, \ldots, \hat{\jmath}_p)$ be players in $\hat{N}$. For all $h \in P$ coalitional family $(N_h^*, \mathcal{B}_h^*)$ is transitive, so there exists a permutation π_h of N_h^* with $\pi_h(\hat{\imath}_h) = \hat{\jmath}_h$ and $\pi_h(T^*) \in \mathcal{B}_h^*$ for all $T^* \in \mathcal{B}_h^*$. Define π by $\pi(\hat{k}) = (\pi_1(\hat{k}_1), \ldots, \pi_p(\hat{k}_p))$ for all $\hat{k} = (\hat{k}_1, \ldots, \hat{k}_p) \in \hat{N}$. It is straightforward to verify that π qualifies, which means that $(\hat{N}, \hat{w})$ is symmetric.

Step 5. From the game $(\hat{N}, \hat{w})$ we construct the game $(\bar{N}, \bar{w})$. Since $\mathcal{U}$ is an infinite set of players, there exists a player set $\bar{N} \subseteq \mathcal{U}$ with $N \subseteq \bar{N}$ and $|\hat{N}| = |\bar{N}|$.

Each player in $\hat{N}$ consists of p coordinates, but we take $\bar{N}$ such that each player in $\bar{N}$ consist of one coordinate. Let $f : \bar{N} \to \hat{N}$ be a bijection satisfying $f(i) = (i, \ldots, i)$ for all $i \in N$. Define $(\bar{N}, \bar{w})$ by $\bar{w}(\bar{S}) = \hat{w}(f(\bar{S}))$ for all $\bar{S} \subseteq \bar{N}$. The game $(\bar{N}, \bar{w})$ inherits the symmetry of $(\hat{N}, \hat{w})$. Due to this symmetry and the fact that $\bar{w}(\bar{N}) = 0$ we obtain by single-valuedness, anonymity and efficiency of σ that $\sigma(\bar{w}) = \{0\}$. We complete the proof by showing that $(N, \bar{w}_{N,0}) = (N, w)$.

Step 6. Let $S \subsetneq N$ and let $k \in P$ with $w(S) = |S| \cdot \mu_k$. Since $(N_k^*, \mathcal{B}_k^*)$ embeds $(N, \mathcal{B}_k)$, there exists a $T^* \in \mathcal{B}_k^*$ with $T^* \cap N = S$. Let $\hat{S} = N_1^* \times \cdots \times N_{k-1}^* \times T^* \times N_{k+1}^* \times \cdots \times N_p^*$. Note that $f(S) = \{(i, \ldots, i) \,|\, i \in S\}$. Since $f(S) \subseteq \hat{S}$, we have

$$\begin{aligned}
\bar{w}_{N,0}(S) &= \max_{Q \subseteq \bar{N} \setminus N} \bar{w}(S \cup Q) - |Q| \cdot \frac{\bar{w}(S \cup Q)}{|S \cup Q|} \\
&= \max_{Q \subseteq \bar{N} \setminus N} \hat{w}(f(S \cup Q)) - |Q| \cdot \frac{\hat{w}(f(S \cup Q))}{|f(S \cup Q|} \\
&= \max_{Q \subseteq \hat{N} \setminus f(N)} \hat{w}(f(S) \cup Q) - |Q| \cdot \frac{\hat{w}(f(S) \cup Q)}{|f(S) \cup Q|} \\
&\geq \hat{w}(\hat{S}) - |\hat{S} \setminus f(S)| \cdot \frac{\hat{w}(\hat{S})}{|\hat{S}|} \\
&= |\hat{S}| \cdot \mu_k - |\hat{S} \setminus f(S)| \cdot \mu_k \\
&= |f(S)| \cdot \mu_k \\
&= w(S).
\end{aligned}$$

In order to show that $\bar{w}_{N,0}(S) \leq w(S)$ let $Q \in \arg\max_{Q \subseteq \bar{N} \setminus N} (\bar{w}(S \cup Q) - |Q| \cdot \frac{\bar{w}(S \cup Q)}{|S \cup Q|})$ and let $\hat{S} = f(S \cup Q)$. Let $h \in P$ such that $\hat{w}(\hat{S}) = |\hat{S}| \cdot \mu_h$. We have

$$\begin{aligned}
\bar{w}_{N,0}(S) &= \bar{w}(S \cup Q) - |Q| \cdot \frac{\bar{w}(S \cup Q)}{|S \cup Q|} \\
&= \hat{w}(\hat{S}) - |Q| \cdot \frac{\hat{w}(\hat{S})}{|\hat{S}|} \\
&= |\hat{S}| \cdot \mu_h - |Q| \cdot \mu_h \\
&= |S| \cdot \mu_h.
\end{aligned}$$

If $h = p$, then the proof is completed by

$$\begin{aligned}\bar{w}_{N,0}(S) &= |S| \cdot \mu_p \\ &\leq |S| \cdot \mu_k \\ &\leq w(S).\end{aligned}$$

If $h < p$, then there exists a $T^* \in \mathcal{B}_h^*$ such that $\hat{S} = N_1^* \times \cdots \times N_{h-1}^* \times T^* \times N_{h+1}^* \times \cdots \times N_p^*$. Since $(i, \ldots, i) \in \hat{S}$ if and only if $i \in S$, it must be that $S = T^* \cap N$, so $S \in B_h$. This gives

$$\begin{aligned}\bar{w}_{N,0}(S) &= |S| \cdot \mu_h \\ &\leq w(S).\end{aligned}$$

□

In the remainder of this subsection we show that the axioms single-valuedness, covariance, anonymity and the reduced game property are independent. We show in Section 3.4 that the per capita prekernel satisfies covariance (Proposition 3.4.4), anonymity (Proposition 3.4.5) and the reduced game property (Theorem 3.4.10), but violates single-valuedness (Example 3.4.3).

The Shapley value (Shapley (1953)) satisfies single-valuedness, covariance and anonymity. However, the Shapley value also satisfies additivity, which implies that it does not coincide with the per capita prenucleolus (Example 3.3.13). Consequently, by Theorem 3.3.28 it does not satisfy the reduced game property.

Furthermore, the equal split solution, given by $ESS^i(N, v) = \frac{v(N)}{n}$ for all $i \in N$, satisfies single-valuedness, anonymity and the reduced game property, but violates covariance.

In the remainder of this subsection we show that anonymity is independent of the other three properties. We first define the positive per capita precore. The expression "positive core", as defined in relation to the prenucleolus, is due to Maschler (see Orshan (1994)). Let $(y)_+ = \max\{0, y\}$.

Definition Let $(N, v) \in \Gamma$. The *positive per capita precore* of (N, v) is given by $C_+^{pc}(N, v) = \{x \in X(N, v) \,|\, e^{pc}(S, x, v) \leq (e^{pc}(S, pcpn(N, v), v))_+ \text{ for all } S \subseteq N\}$.

Note that the positive per capita precore is a polytope. We provide two preliminary lemmas.

Lemma 3.3.29 *Let $(N, v) \in \Gamma^C$. Then $C(N, v) = C_+^{pc}(N, v)$.*

Proof: Since $C(N, v) \neq \emptyset$, $e^{pc}(S, pcpn(N, v), v) \leq 0$ for all $S \subseteq N$. Hence, $(e^{pc}(S, pcpn(N, v), v))_+ = 0$ for all $S \subseteq N$. Consequently, $x \in C_+^{pc}(N, v)$ if and only if $x \in C(N, v)$. □

The next lemma is straightforward and therefore, no proof is given.

Lemma 3.3.30 *Let $(N, v) \in \Gamma$. Then $pcpn(N, v) \in C_+^{pc}(N, v)$.*

A solution σ on Γ satisfies the *reconfirmation property* if the following condition is satisfied for all $(N, v) \in \Gamma$, all $x \in \sigma(N, v)$ and every $T \subseteq N, T \neq \emptyset$: if $(T, v_{T,x}) \in \Gamma$ and $y_T \in \sigma(T, v_{T,x})$, then $(y_T, x_{N \setminus T}) \in \sigma(N, v)$. The reconfirmation property in relation to the reduced game of the prenucleolus occurs first in Balinsky and Young (1982) as one condition inside a property. The reconfirmation property can be seen as a stability property. Any allocation that is part of the solution of the reduced game when combined with $x_{N \setminus T}$, the payoff vector of the passive players, yields an allocation in the solution set $\sigma(N, v)$. Hence, σ is stable for behaviour in the reduced games which is specified by σ itself.

Lemma 3.3.31 *The positive per capita precore on Γ satisfies non-emptiness, anonymity, covariance, the reduced game property, and the reconfirmation property.*

Proof: Non-emptiness follows from Lemma 3.3.30. Both anonymity and covariance follow from the fact that the per capita prenucleolus satisfies these properties (Propositions 3.3.7 and 3.3.5, respectively).

Next we show that the positive per capita precore satisfies the reduced game property. Let $(N, v) \in \Gamma$. If $C(N, v) \neq \emptyset$, then we know by Lemma 3.3.29 that $C(N, v) = C_+^{pc}(N, v)$ and in Section 3.7 we show that the core satisfies the reduced game property (Theorem 3.7.1). Hence, let us assume that $C(N, v) = \emptyset$, which means that $\theta^1(pcpn(N, v)) > 0$. Since $\theta(pcpn(N, v)) \leq_L \theta(x)$ for all $x \in X(N, v)$, for all $x \in C_+^{pc}(N, v)$, $e^{pc}(S, x, v) = e^{pc}(S, pcpn(N, v), v)$ for all $S \subseteq N$ with $e^{pc}(S, pcpn(N, v), v) > 0$. Hence, $x \in C_+^{pc}(N, v)$ if and only if $\mathcal{B}(x, v, t)$ is balanced for all $t > 0$. To prove that $\mathcal{B}(x, v, t)$ is balanced for all $t > 0$ is analogous to the proof that the per capita prenucleolus satisfies the reduced game property

(Theorem 3.3.18).

Finally, we prove that the positive per capita precore satisfies the reconfirmation property. Let $(N,v) \in \Gamma$ and $T \subseteq N, T \neq \emptyset$. Further, let $x \in C_+^{pc}(N,v)$, $y \in C_+^{pc}(T, v_{T,x})$ and $z = (y, x_{N\backslash T})$. Let $S \subseteq N, S \neq \emptyset$. We show that $(e^{pc}(S,x,v))_+ = (e^{pc}(S,z,v))_+$, which completes the proof. Note that $x = (x_{S\cap T}, x_{T\backslash S}, x_{N\backslash T})$ and $z = (y_{S\cap T}, y_{T\backslash S}, x_{N\backslash T})$, which implies that the only relevant part to consider is $y_{S\cap T}$ compared to $x_{S\cap T}$. Hence, if $S \cap T = \emptyset$ or if $T \subseteq S$, then $y_{S\cap T} = x_{S\cap T}$, and the proof is complete.

Next assume that $\emptyset \neq S \cap T \neq T$. Take $Q \subseteq N\backslash T$ such that

$$v_{T,x}(S\cap T) = v((S\cap T)\cup Q) - x(Q) - |Q| \cdot e^{pc}((S\cap T)\cup Q, x, v).$$

If $e^{pc}(S\cap T, x_T, v_{T,x}) > 0$, then since $x_T \in C_+^{pc}(T, v_{T,x})$ (by the reduced game property) we know that $e^{pc}(S\cap T, q, v_{T,x}) = e^{pc}(S \cap T, pcpn(T, v_{T,x}), v_{T,x})$ for all $q \in C_+^{pc}(T, v_{T,x})$, which implies that $y_{S\cap T} = x_{S\cap T}$. If $e^{pc}(S\cap T, x_T, v_{T,x}) \leq 0$, then since $x_T \in C_+^{pc}(T, v_{T,x})$ (by the reduced game property), $e^{pc}(S\cap T, pcpn(T, v_{T,x}), v_{T,x}) \leq 0$. Therefore, $e^{pc}(S\cap T, y, v_{T,x}) \leq 0$. As a consequence,

$$\begin{aligned} e^{pc}(S,z,v) &\leq \max_{Q\subseteq N\backslash T} e^{pc}((S\cap T)\cup Q), x, v) \\ &= e^{pc}(S\cap T, x_T, v_{T,x}) \\ &\leq 0, \end{aligned}$$

where the equality follows from Lemma 3.3.16. Similarly, $e^{pc}(S,x,v) \leq 0$, which completes the proof. $\square$

Assume $|\mathcal{U}| \geq 2$. Take an injection $\pi : \mathcal{U} \to \mathbb{N}$. Define $\xi(N,v) = \{x \in C_+^{pc}(N,v) \,|\, x \succeq_L y \text{ for all } y \in C_+^{pc}(N,v)\}$, where $x \succeq_L y$ if there exists a $k \in N$ such that $x^i = y^i$ for all $i \in N$ with $\pi(i) < \pi(k)$ and $x^k > y^k$.

Lemma 3.3.32 *The solution ξ on Γ satisfies single-valuedness, covariance and the reduced game property.*

Proof: By Lemma 3.3.30 we know that $C_+^{pc}(N,v) \neq \emptyset$ and clearly there is a unique element in this set that is preferred over all others. By Lemma 3.3.31 we know that the positive per capita precore is covariant under strategic equivalence. Hence, $\xi(N,v)$ also satisfies covariance.

Finally, we prove that $\xi(N,v)$ satisfies the reduced game property. Let (N,v) and $T \subseteq N, T \neq \emptyset$. Let $\{x\} = \xi(N,v) \in C_+^{pc}(N,v)$. By the reduced game property of the positive per capita precore $x_T \in C_+^{pc}(T, v_{T,x})$. Let $z \in C_+^{pc}(T, v_{T,x})$. By the reconfirmation property of the positive per capita precore $(z, x_{N\backslash T}) \in C_+^{pc}(N,v)$. Suppose that $z \succeq_L x_T$. Then $(z, x_{N\backslash T}) \succeq_L x$ and consequently, $x \notin \xi(N,v)$, which is a contradiction. Hence, $x_T \succeq_L y$ for all $y \in C_+^{pc}(N,v)$. □

Finally, we show by means of an example that ξ does not satisfy anonymity.

Example 3.3.33 Let $(N,v) \in \Gamma$ be given by the table below.

S	$\{1\}$	$\{2\}$	N
$v(S)$	0	0	1

The positive per capita precore is given by $C_+^{pc}(N,v) = \text{conv}\{(1,0),(0,1)\}$ and hence, $\xi(N,v) = \{(1,0)\}$ and ξ does not satisfy anonymity. ◁

3.4 Per capita prekernel

In this section we introduce, analyse and discuss the per capita prekernel, which is related to the per capita prenucleolus in the same way the prekernel (cf. Davis and Maschler (1965)) is related to the prenucleolus. In Subsection 3.4.1 we discuss several properties of the per capita prekernel. In Subsection 3.4.2 we relate the per capita prekernel to other solution concepts and in Subsection 3.4.3 we characterise the per capita prekernel. The definition of the per capita prekernel is already given in Section 3.3, but for the sake of completeness we repeat its definition here.

Definition Let $(N,v) \in \Gamma$. The *per capita prekernel* of (N,v) is given by $PCPK(N,v) = \{x \in X(N,v) \mid s^{ij}(x) = s^{ji}(x) \text{ for all } i,j \in N\}$.

3.4.1 Properties

In this subsection properties of the per capita prekernel are discussed. By Theorem 3.3.20 we know that the per capita prenucleolus is an element of the per capita prekernel. Consequently, this set is non-empty.

Corollary 3.4.1 *The per capita prekernel satisfies non-emptiness.*

Furthermore, by definition the per capita prekernel satisfies efficiency.

Corollary 3.4.2 *The per capita prekernel is efficient.*

The following example shows that the per capita prekernel is not single-valued.

Example 3.4.3 Consider the five-player game (N, v), with $v(S) = \min\{|S \cap \{1,2\}|, |S \cap \{3,4,5\}|\}$, which is the $(2,3)$-glove game. Then $C(N,v) = (1,1,0,0,0)$, which implies by Theorem 3.3.19 that $pcpn(N,v) = (1,1,0,0,0)$, and by Theorem 3.3.20, $pcpn(N,v) \in PCPK(N,v)$. However, also $x = (0,0,\frac{2}{3},\frac{2}{3},\frac{2}{3}) \in PCPK(N,v)$, with $s^{ij}(x) = \frac{1}{6}$ for all $i,j \in N$, which implies that the per capita prekernel is not single-valued.

Note that this example also implies that the per capita prekernel is not a subset of the core and that the per capita prenucleolus and the per capita prekernel are not equivalent on Γ. $\triangleleft$

Proposition 3.4.4 *The per capita prekernel satisfies covariance.*

The proof of this proposition is similar to the proof of Proposition 3.3.5 in which we show that the per capita prenucleolus satisfies covariance. The next proposition is given without proof.

Proposition 3.4.5 *The per capita prekernel satisfies anonymity.*

Proposition 3.4.6 *The per capita prekernel satisfies desirability.*

Proof: Let $(N, v) \in \Gamma$, with $i \succeq_v j$ for some $i, j \in N$. Assume on the contrary that $x \in PCPK(N, v)$, with $x^j > x^i$. Choose $S \in \mathfrak{T}^{ji}$ such that $s^{ji}(x) = e^{pc}(S, x)$. Let $T = (S \backslash \{j\} \cup \{i\})$. Then $v(T) \geq v(S)$, because $i \succeq_v j$. Therefore, $e^{pc}(T, x) > e^{pc}(S, x)$. Thus,

$$\begin{aligned} s^{ij}(x) &\geq e^{pc}(T, x) \\ &> e^{pc}(S, x) \\ &= s^{ji}(x), \end{aligned}$$

which contradicts the assumption that $x \in PCPK(N, v)$. □

Note that Proposition 3.4.6 implies that any solution concept that is always an element of the per capita prekernel, like the per capita prenucleous (Theorem 3.3.19), also satisfies desirability. Furthermore, since the per capita prekernel satisfies desirability it also satisfies the equal treatment property.

Corollary 3.4.7 *The per capita prekernel satisfies the equal treatment property.*

Since the per capita prenucleolus does not satisfy individual rationality, strong desirability, reasonableness from below, the dummy and the adding dummies property it follows that the per capita prekernel does not satisfy these properties either. The per capita prekernel is, just as the per capita prenucleolus, reasonable from above, which we show in the next proposition. Let $(N, v) \in \Gamma$ and $x \in X(N, v)$. Recall that $\mathcal{B}_1(x) = \{R \subseteq N, R \neq \emptyset \,|\, e^{pc}(R, x) \geq e^{pc}(T, x) \text{ for all } T \subseteq N, T \neq \emptyset\}$.

Proposition 3.4.8 *The per capita prekernel is reasonable from above.*

Proof: Let $(N, v) \in \Gamma$. If $x \in C(N, v)$, then x is reasonable from above, since the core is reasonable (Peleg and Sudhölter (2003)). So let us assume $x \in PCPK(N, v) \backslash C(N, v)$. Furthermore, assume on the contrary $x^i > \max_{S \subseteq N \backslash \{i\}} v(S \cup \{i\}) - v(S)$ for some $i \in N$. Let $S \subseteq N \backslash \{i\}$ be such that $(S \cup \{i\}) \in \mathcal{B}_1(x)$. Note that there must exist such a coalition $S \subseteq N \backslash \{i\}$, as otherwise $s^{ij}(x) < s^{ji}(x)$ for some $j \in N$, which contradicts that $x \in PCPK(N, v)$.

Then

$$\begin{aligned} e^{pc}(S \cup \{i\}, x) &= \frac{v(S \cup \{i\}) - x(S \cup \{i\})}{|S|+1} \\ &< \frac{v(S \cup \{i\}) - x(S \cup \{i\})}{|S|} \\ &< \frac{v(S) - x(S)}{|S|} \\ &= e^{pc}(S, x), \end{aligned}$$

where the first inequality follows from the fact that $v(S \cup \{i\}) - x(S \cup \{i\}) > 0$, as $x \notin C(N, v)$ and $(S \cup \{i\}) \in \mathcal{B}_1(x)$. Consequently, $(S \cup \{i\}) \notin \mathcal{B}_1(x)$, which is a contradiction. □

The property additivity for single-valued solution concepts is extended to the property super-additivity for set solutions. Let $(N, v), (N, w), (N, v + w) \in \Gamma$ and let σ be a solution on Γ. Then σ is *super-additive* if $\sigma(N, v) + \sigma(N, w) \subseteq \sigma(N, v + w)$. The per capita prekernel does not satisfy super-additivity as the following example illustrates.

Example 3.4.9 Let $(N, v) \in \Gamma$, with $n = 4$ be the unanimity game on $\{1, 2, 3\}$. Clearly, $PCPK(N, v) = \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)\}$. Let (N, w), with $n = 4$ be the unanimity game on $\{2, 3, 4\}$. Then $PCPK(N, w) = \{(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$. However, $PCPK(N, v) + PCPK(N, w) = \{(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3})\} \nsubseteq PCPK(N, v + w)$, because $s^{24}(PCPK(N, v) + PCPK(N, w), v + w) = -\frac{2}{9} > -\frac{1}{3} = s^{42}(PCPK(N, v) + PCPK(N, w), v + w)$. ◁

In Section 3.3 we have introduced a reduced game and the corresponding reduced game property. The per capita prekernel satisfies this reduced game property. Let $(N, v) \in \Gamma$, and $T \subseteq N$. Then we define $\mathfrak{T}_T^{ij} = \{S \subseteq T \backslash \{j\} \,|\, i \in S\}$.

Theorem 3.4.10 The per capita prekernel satisfies the reduced game property.

Proof: Let $(N, v) \in \Gamma$, $x \in PCPK(N, v)$ and $T \subseteq N$. By means of Lemma 3.3.16 we obtain

$$\begin{aligned}
s^{ij}(x_T, v_{T,x}) &= \max_{S \in \mathfrak{T}_T^{ij}} e^{pc}(S, x_T, v_{T,x}) \\
&= \max_{S \in \mathfrak{T}_T^{ij}} \max_{Q \subseteq N \setminus T} e^{pc}(S \cup Q, x, v) \\
&= \max_{S \in \mathfrak{T}^{ij}} e^{pc}(S, x, v) \\
&= s^{ij}(x, v) \\
&= s^{ji}(x, v) \\
&= \max_{S \in \mathfrak{T}^{ji}} e^{pc}(S, x, v) \\
&= \max_{S \in \mathfrak{T}_T^{ji}} \max_{Q \subseteq N \setminus T} e^{pc}(S \cup Q, x, v) \\
&= \max_{S \in \mathfrak{T}_T^{ji}} e^{pc}(S, x_T, v_{T,x}) \\
&= s^{ji}(x_T, v_{T,x}).
\end{aligned}$$

□

Since the proof of Theorem 3.4.10 only contains equalities we can equivalently show that $s^{ij}(x, v) = s^{ji}(x, v)$ by the fact that $s^{ij}(x_T, v_{T,x}) = s^{ji}(x_T, v_{T,x})$. This observation results in the next proposition.

Proposition 3.4.11 *The per capita prekernel satisfies the converse reduced game property.*

3.4.2 Relations to other solution concepts

In this subsection we discuss the relations of the per capita prekernel to other solution concepts for cooperative games. In Section 3.3 we already establish that the per capita prenucleolus is an element of the per capita prekernel (Theorem 3.3.20). Further, we reconsider Example 3.4.3. In this example we show that the per capita prekernel is not equivalent to the per capita prenucleolus. Moreover, the per capita prekernel is not a subset of the core. Finally, since the per capita prenucleolus is not a bargaining set selector (Example 3.3.21) the per capita prekernel is not a subset of the bargaining set either.

3.4.3 Characterisation

In this subsection we characterise the per capita prekernel by the use of some properties discussed in Subsection 3.4.1. We first restate Lemma 5.4.3 of Peleg and Sudhölter (2003).

Lemma 3.4.12 *Let σ be a solution Γ that satisfies non-emptiness, efficiency, covariance and the equal treatment property on the class of all two-player games. Then for every two-player game $(N,v) \in \Gamma$, $\sigma(N,v)$ is the* standard solution *of (N,v), i.e.,*

$$\sigma^i(N,v) \quad = \quad v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{2}$$

for all $i \in N$.

Theorem 3.4.13 The per capita prekernel is the unique solution on Γ that satisfies non-emptiness, efficiency, covariance, the equal treatment property, the reduced game property and the converse reduced game property.

Proof: By Corollary 3.4.1, Corollary 3.4.2, Proposition 3.4.4, Corollary 3.4.7, Theorem 3.4.10 and Proposition 3.4.11 we obtain that the per capita prekernel satisfies non-emptiness, efficiency, covariance, the equal treatment property, the reduced game property and the converse reduced game property, respectively. Hence, it suffices to show that the per capita prekernel is the unique solution concept that satisfies these axioms.

Let σ be a solution on Γ that satisfies the foregoing six properties and let $(N,v) \in \Gamma$ be an n-player game. If $n = 1$, then $\sigma(N,v) = PCPK(N,v)$ by non-emptiness and efficiency. For the case $n = 2$, Lemma 3.4.12 shows that $\sigma(N,v) = PCPK(N,v)$. Next we assume that $n \geq 3$. If $x \in \sigma(N,v)$, then by the reduced game property, $x_T \in \sigma(T, v_{T,x})$ for all $T \in P(N)$. Hence, $x_T \in PCPK(T, v_{T,x})$ for all $T \in P(N)$. As the per capita prekernel satisfies the converse reduced game property, $x \in PCPK(N,v)$. Conversely, let $x \in PCPK(N,v)$. Then $x_T \in PCPK(T, v_{T,x})$ for all $T \in P(N)$, because the per capita prekernel satisfies the reduced game property. Hence, $x_T \in \sigma(T, v_{T,x})$ for all $T \in P(N)$. Thus, by the converse reduced game property, $x \in \sigma(N,v)$. □

In the previous section we have mentioned that the per capita prenucleolus does not satisfy the converse reduced game property. This follows from the fact that the per capita prenucleolus satisfies non-emptiness (Lemma 3.3.1), efficiency (Corollary 3.3.2), covariance (Proposition 3.3.5), the equal treatment property (Corollary 3.3.9) and the reduced game property (Theorem 3.3.18). Since the per capita prenucleolus is unequal to the per capita prekernel (Example 3.4.3), it can therefore, by Theorem 3.4.13, not satisfy the converse reduced game property.

In the remainder of this subsection we show that the six properties that characterise the per capita prekernel are independent. We discuss some solution concepts that satisfy five of these six properties. If it is clear that a solution concept satisfies the other five properties we only indicate which axiom is not satisfied.

As mentioned above, the per capita prenucleolus does not satisfy the converse reduced game property.

The empty solution is given by $\sigma(N,v) = \emptyset$ for all $(N,v) \in \Gamma$. The empty solution does not satisfy non-emptiness.

For $(N,v) \in \Gamma$ let $\sigma(N,v) = \{x \in X^*(N,v) \,|\, s^{ij}(x) = s^{ji}(x) \text{ for all } i \neq j\}$. By the corresponding proofs for the per capita prekernel it is easy to verify that this solution concept satisfies all six properties except efficiency.

The equal split solution, introduced in Section 3.3, satisfies all properties except covariance.

The preimputation set $X(N,v)$, with $(N,v) \in \Gamma$, satisfies all properties except the equal treatment property.

We finally show that the reduced game property is independent of the other properties. Let $(N,v) \in \Gamma$ and $i,j \in N$. Players i and j are *equivalent*, denoted by $i \cong_v j$, if $v(S \cup \{i\}) - v(\{i\}) = v(S \cup \{j\}) - v(\{j\})$ for all $S \subseteq N \backslash \{i,j\}$. Now, let

$$\sigma(N,v) \quad = \quad \{x \in X(N,v) \,|\, x^i - v(\{i\}) = x^j - v(\{j\}) \text{ if } i \cong_v j,\ i,j \in N\}.$$

It is straightforward that σ satisfies non-emptiness, efficiency, covariance and the equal treatment property. Consequently, by Lemma 3.4.12, $\sigma(N,v) = PCPK(N,v)$ if $n = 2$. We prove that σ satisfies the converse reduced game property. Let $x \in X(N,v)$ such that $x_T \in \sigma(T, v_{T,x})$ for all $T \in P(N)$. By the foregoing remarks, $x_T \in PCPK(T, v_{T,x})$ for all $T \in P(N)$. Since the per capita prekernel satisfies the converse reduced game property, $x \in PCPK(N,v)$. Now, as the per capita prekernel satisfies covariance and the equal treatment property we obtain that $PCPK(N,v) \subseteq$

$\sigma(N, v)$. Thus, $x \in \sigma(N, v)$, which means that σ satisfies the converse reduced game property. Finally, if $|\mathcal{U}| > 2$, then σ is unequal to the per capita prekernel and therefore, by Theorem 3.4.13, σ does not satisfy the reduced game property.

3.5 Per capita nucleolus

The per capita prenucleolus, discussed in Section 3.3, is the unique element in the preimputation set for which the maximal objection per player of a coalition to it is minimised. In this section we discuss the per capita *nucleolus*, which is the unique element in the *imputation* set for which the maximal objection per player of a coalition to it is minimised. Hence, the difference between the per capita nucleolus and the per capita prenucleolus is that by using the first solution concept it is ensured that each player gets at least his individual worth.

The setup of this section is similar to that of Section 3.3. Hence, we first discuss properties of the per capita nucleolus in Subsection 3.5.1. In Subsection 3.5.2 we discuss the relations to other solution concepts and in Subsection 3.5.3 we characterise the per capita nucleolus.

The set of all games with a non-empty imputation set is denoted by Γ^I.

Definition Let $(N, v) \in \Gamma^I$. The *per capita nucleolus* of (N, v) is given by $pcn(N, v) = \{x \in I(N, v) \,|\, \theta(x) \leq_L \theta(y) \text{ for all } y \in I(N, v)\}$.

3.5.1 Properties

In this subsection we discuss a number of properties of the per capita nucleolus. By the same reasoning as for the per capita prenucleolus we obtain that the per capita nucleolus is non-empty.

Lemma 3.5.1 *The per capita nucleolus satisfies non-emptiness.*

Since $pcn(N, v) \subseteq I(N, v)$ we obtain that the per capita nucleolus is efficient.

Corollary 3.5.2 *The per capita nucleolus satisfies efficiency.*

Theorem 3.5.3 The per capita nucleolus satisfies single-valuedness.

The proof of this theorem is similar to the proof of Theorem 3.3.3 in which we show that the per capita prenucleolus is single-valued. An ordered partition $(\mathcal{B}_1(x), \ldots, \mathcal{B}_p(x))$ is called balanced on $\mathrm{Car}_v(x)$ if $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ is balanced on $\mathrm{Car}_v(x)$ for all $k = 1, \ldots, p$.

Theorem 3.5.4 Imputation $x \in I(N, v)$ is the per capita nucleolus of $(N, v) \in \Gamma^I$ if and only if the ordered partition $(\mathcal{B}_1(x), \ldots, \mathcal{B}_p(x))$ is balanced on $\mathrm{Car}_v(x)$.

The proof of this theorem follows from Theorem 5.(a) in Potters and Tijs (1992). It follows from the proof of Proposition 3.3.5 that the per capita nucleolus satisfies covariance.

Proposition 3.5.5 *The per capita nucleolus satisfies covariance.*

Let N be fixed and let $\mathcal{V}^I(N) = \{(N, v) \in \Gamma^I \mid v : 2^N \to \mathbb{R}, v(\emptyset) = 0\}$.

Theorem 3.5.6 The per capita nucleolus $pcpn(N, v) : \mathcal{V}^I(N) \to \mathbb{R}^N$ is continuous.

Also the proof of this theorem is analogous to the proof of the corresponding theorem (Theorem 3.3.6) for the per capita prenucleolus. Without proof we provide the following proposition.

Proposition 3.5.7 *The per capita nucleolus satisfies anonymity.*

Proposition 3.5.8 *The per capita nucleolus satisfies desirability.*

For the proof of this proposition we refer to Proposition 3.6.4 which states that the per capita kernel satisfies desirability. Since the per capita nucleolus is an element of the per capita kernel (Theorem 3.5.21) this is sufficient.

Since the per capita nucleolus is a core selector (Theorem 3.5.20) it does not satisfy strong desirability. The per capita nucleolus satisfies individual rationality, which follows directly from the definition. Consequently, the per capita nucleolus is reasonable from below. Furthermore, the per capita nucleolus is also reasonable from above.

Proposition 3.5.9 *The per capita nucleolus is reasonable from above.*

For a proof of this proposition we refer to Proposition 3.6.5 which states that the per capita kernel is reasonable from above.

Corollary 3.5.10 *The per capita nucleolus is reasonable.*

Note that any solution concept that is reasonable satisfies the dummy property, as the upper- and lower-bound of the payoff to each dummy player $i \in N$ equal $v(\{i\})$.

Corollary 3.5.11 *The per capita nucleolus satisfies the dummy property.*

The per capita nucleolus does not satisfy the adding dummies property, which is shown in the following example.

Example 3.5.12 Consider the game (N, v) given in the next table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	0	8	8	9

The per capita nucleolus of this game is given by $pcn(N, v) = (\frac{1}{3}, \frac{1}{3}, 8\frac{1}{3})$. Now we add dummy player 4 to this game, which gives the game $(N \cup \{4\}, w)$, depicted below.

S	1	2	3	4	1,2	1,3	1,4	2,3	2,4	3,4	1,2,3	1,2,4	1,3,4	2,3,4	N
$w(S)$	0	0	0	0	0	8	0	8	0	0	9	0	8	8	9

The per capita nucleolus of $(N \cup \{4\}, w)$ is given by $pcn(N \cup \{4\}, w) = (\frac{1}{5}, \frac{1}{5}, 8\frac{1}{5}, 0)$. Hence, the way $v(N)$ is divided is not invariant to the adding of dummy players. $\triangleleft$

The per capita nucleolus does not satisfy additivity. This result follows from the game of Example 3.3.13, where in each game the per capita nucleolus equals the per capita prenucleolus. Since the per capita nucleolus is a core selector (Theorem 3.5.20) and no core selector can be coalitionally monotonic, the per capita nucleolus is not coalitionally monotonic. However, the per capita nucleolus is weak coalitionally monotonic.

Proposition 3.5.13 *The per capita nucleolus is weak coalitionally monotonic.*

This result follows with a minor adjustment[6] in its proof by Theorem 3 (and Remark 1) in Zhou (1991) in which it is shown that the (per capita) prenucleolus[7] satisfies weak coalitional monotonicity. The per capita nucleolus is not aggregate monotonic, as is shown in the next example, and therefore also not strongly aggregate monotonic.

Example 3.5.14 Consider the five-player game (N, v), which is given by

$$v(S) = \begin{cases} 14 & \text{if} \quad S = \{1,3\} \text{ or } \{2,3\}, \\ 25 & \text{if} \quad S = \{1,2,3\}, \\ 5 & \text{if} \quad S = \{1,4,5\} \text{ or } \{2,4,5\}, \\ 24 & \text{if} \quad S = N, \\ 0 & \text{otherwise.} \end{cases}$$

The per capita nucleolus of this game is given by $pcn(N, v) = (8, 8, 8, 0, 0)$. The game (N, w) is defined such that $w(N) = 25$ and $w(S) = v(S)$ for all $S \subsetneq N$. The per capita nucleolus of (N, w) is given by $pcn(N, w) = (8\frac{3}{5}, 8\frac{3}{5}, 7\frac{4}{5}, 0, 0)$. Hence, although only the worth of the grand coalition increases player 3 is worse off under (N, w) than under (N, v). $\triangleleft$

In Section 3.3 we introduce the reduced game, which is used to characterise both the per capita prenucleolus and the per capita prekernel. Similar to the imputation saving reduced game introduced for the nucleolus (Snijders (1995)) we introduce an imputation saving reduced game for the per capita nucleolus. The *imputation saving reduced game* $(T, \tilde{v}_{T,x})$ with respect to coalition $T \subseteq N$ and preimputation x is defined as follows. If $|T| = 1$, then $(T, \tilde{v}_{T,x}) = (T, v_{T,x})$. If $|T| \geq 2$, then

$$\tilde{v}_{T,x}(S) = \begin{cases} v_{T,x}(S) & \text{if } S \subseteq T, |S| \neq 1, \\ \min\{x^i, v_{T,x}(\{i\})\} & \text{if } S = \{i\}, i \in T. \end{cases}$$

In this way it is ensured that $x_T \in I(T, \tilde{v}_{T,x})$ for all $T \subseteq N$, whenever $x \in I(N, v)$. Let $(N, v) \in \Gamma$ and let σ be a solution on Γ. Then σ satisfies the *imputation saving reduced game property* whenever $T \subseteq N, T \neq \emptyset$ and $x \in \sigma(N, v)$, then $(T, \tilde{v}_{T,x}) \in \Gamma$ and $x_T \in \sigma(T, \tilde{v}_{T,x})$. We prove that the per capita nucleolus satisfies this imputation saving reduced game property. We start by introducing some additional notation and preliminary lemmas.

[6] In the proof of Theorem 3 in Zhou (1991) it is claimed that "... since y is the *prenucleolus* of w, $\theta^w(y)$ is lexicographically smaller than $\theta^w(x)$." If we consider the nucleolus and not the prenucleolus, then this result is not immediate, but it can be shown to be true nevertheless.

[7] Note that Zhou (1991) refers to the (per capita) prenucleolus as (per capita) nucleolus.

Lemma 3.5.15 *Let* $(N,v) \in \Gamma^I$, $x \in X(N,v)$ *and* $T \subseteq N$. *Then,* $Car_{\tilde{v}_{T,x}}(x_T) \subseteq (Car_v(x) \cap T)$.

Proof:

$$\begin{aligned} i \in \mathrm{Car}_{\tilde{v}_{T,x}}(x_T) &\Leftrightarrow x^i > \tilde{v}_{T,x}(\{i\}) \text{ and } i \in T \\ &\Leftrightarrow x^i > v_{T,x}(\{i\}) \text{ and } i \in T \\ &\Rightarrow x^i > v(\{i\}) \text{ and } i \in T \\ &\Leftrightarrow i \in \mathrm{Car}_v(x) \text{ and } i \in T \\ &\Leftrightarrow i \in (\mathrm{Car}_v(x) \cap T). \end{aligned}$$

□

Lemma 3.5.16 *Let* $(N,v) \in \Gamma^I$, $S', T \subseteq N, S', T \neq \emptyset$, $t \in \mathbb{R}$, *then we obtain the following two properties.*

1. $S \in \mathcal{B}(x_T, \tilde{v}_{T,x}, t)$ *if and only if* $S \in \{S' \cap T \mid S' \in \mathcal{B}(x,v,t)\}$ *for all* $S \subseteq T$, $S \neq \emptyset$ *with* $\tilde{v}_{T,x}(S) = v_{T,x}(S)$.
2. *If* $\{i\} \in \mathcal{B}(x_T, \tilde{v}_{T,x}, t)$, *then* $\{i\} \in \{S' \cap T \mid S' \in \mathcal{B}(x,v,t)\}$ *for all* $i \in N$ *such that* $\tilde{v}_{T,x}(\{i\}) = x^i$.

Proof: For the second part note that $\tilde{v}_{T,x}(\{i\}) \leq v_{T,x}(\{i\})$ for all $i \in N$, which implies that $\mathcal{B}(x_T, \tilde{v}_{T,x}, t) \subseteq \mathcal{B}(x_T, v_{T,x}, t)$. The remainder of the proof follows by the proof of Lemma 3.3.17 in which $\mathcal{B}(x_T, v_{T,x}, t) = \{S \cap T \mid S \in \mathcal{B}(x,v,t)\}$ is shown.

□

Theorem 3.5.17 The per capita nucleolus satisfies the imputation saving reduced game property.

Proof: Let $(N,v) \in \Gamma^I$ and $T \subseteq N, T \neq \emptyset$. For $|T| = 1$ the result follows immediately. So let us assume that $|T| > 1$. Let $x = pcn(N,v)$ and let $t \in \mathbb{R}$. By Theorem 3.5.4 this implies that $\mathcal{B}(x,v,t)$ is balanced on $\mathrm{Car}_v(x)$. Furthermore, $x \in I(N,v)$.

Since $(\mathrm{Car}_v(x) \cap T) \subseteq \mathrm{Car}_v(x)$ this implies that $\mathcal{B}(x,v,t)$ is balanced on $(\mathrm{Car}_v(x) \cap T)$.

Also, since $\tilde{v}_{T,x}(\{i\}) \leq x^i$ we obtain that $x_T \in I(T, \tilde{v}_{T,x})$. Due to Lemma 3.5.15 $\mathrm{Car}_{\tilde{v}}(x_T) \subseteq (\mathrm{Car}_v(x) \cap T)$ and consequently, $\mathcal{B}(x, v, t)$ is balanced on $\mathrm{Car}_{\tilde{v}}(x_T)$. Finally, by the first part of Lemma 3.5.16 this implies that $\mathcal{B}(x_T, \tilde{v}_{T,x}, t)$ is balanced on $\mathrm{Car}_{\tilde{v}}(x_T)$. Combining this with the fact that $x_T \in I(T, \tilde{v}_{T,x})$ results in $x_T = pcn(T, \tilde{v}_{T,x})$. □

Let $(N, v) \in \Gamma^I$ and let σ be a solution on Γ^I. Then σ satisfies the *converse imputation saving reduced game property* if whenever $n \geq 2$, $x \in I(N, v)$, $(T, \tilde{v}_{T,x}) \in \Gamma^I$, and $x_T \in \sigma(T, \tilde{v}_{T,x})$ for all $T \in P(N)$, then $x \in \sigma(N, v)$. The per capita nucleolus does not satisfy the converse imputation saving reduced game property, as the following example illustrates.

Example 3.5.18 Consider the game (N, v), which is due to Sudhölter and Potters (2001), depicted below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$u(S)$	-1	-1	-1	1	0	0	0

Let $x = (0, 0, 0)$. Then in each imputation saving reduced game, with $n = 2$, all coalition worths are zero, which implies that the per capita nucleolus of each game is $(0, 0)$. Hence, $x_T = pcn(T, \tilde{v}_{T,x})$ for all $T \in P(N)$. However, $x = (0, 0, 0) \neq (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}) = pcn(N, v)$. Consequently, the per capita nucleolus does not satisfy the converse imputation saving reduced game property. ◁

3.5.2 Relations to other solution concepts

In this subsection we discuss the relation of the per capita nucleolus to other solution concepts for cooperative games. The first theorem states that the per capita nucleolus coincides with the per capita prenucleolus whenever the core is non-empty.

Theorem 3.5.19 Let $(N, v) \in \Gamma^C$. Then the per capita nucleolus and the per capita prenucleolus coincide.

Proof: Let $(N, v) \in \Gamma^C$. Since $pcpn(N, v) \in C(N, v)$ (Theorem 3.3.19) and $C(N, v) \subseteq I(N, v)$, $pcn(N, v) = pcpn(N, v)$. □

As a direct result we obtain also the following theorem.

Theorem 3.5.20 Let $(N, v) \in \Gamma^C$. Then $pcn(N, v) \in C(N, v)$.

Analogous to the per capita prekernel, introduced in Section 3.3 and discussed in Section 3.4, we also introduce the per capita kernel[8].

Definition Let $(N, v) \in \Gamma^I$. The *per capita kernel* of (N, v) is given by $PCK(N, v) = \{x \in I(N, v) \,|\, s^{ij}(x) \geq s^{ji}(x) \text{ or } x^i = v(\{i\}) \text{ for all } i, j \in N, i \neq j\}$.

Theorem 3.5.21 Let $(N, v) \in \Gamma^I$. Then $pcn(N, v) \in PCK(N, v)$.

Proof: Let $x \in I(N, v)$, and let $i, j \in N$ such that $s^{ij}(x) < s^{ji}(x)$ and $x^i > v(\{i\})$. Hence, $x \notin PCK(N,)$. We show that $x \neq pcn(N, v)$, which gives by the non-emptiness of the per capita nucleolus (Corollary 3.5.1) the desired result. Take $t = s^{ji}(x)$. Then collection $\mathcal{B}(x, t)$ contains a coalition $S \in \mathfrak{T}^{ji}$, but no coalition $T \in \mathfrak{T}^{ij}$. Since $i \in \text{Car}_v(x)$ we obtain that $(\mathcal{B}_1(x), \ldots, \mathcal{B}_p(x))$ is not balanced on $\text{Car}_v(x)$, which proves $x \neq pcn(N, v)$. □

The per capita nucleolus is not a bargaining set selector. This result follows from the game of Example 3.3.21, where the per capita nucleolus is equal to the per capita prenucleolus. Note however that since $pcn(N, v) = pcpn(N, v)$ for all $(N, v) \in \Gamma^C$ we obtain that $pcn(N, v) \in BS(N, v)$ for all $(N, v) \in \Gamma^C$.

Each strongly compromise admissible game has a non-empty core (Driessen and Tijs (1983)). Therefore, $pcn(N, v) = pcpn(N, v)$ for all $(N, v) \in \Gamma^{SCA}$. Consequently, we obtain the following proposition.

Proposition 3.5.22 *Let $(N, v) \in \Gamma^{SCA}$. Then $pcn(N, v) = \tau(N, v)$.*

3.5.3 Characterisation

In this subsection we characterise the per capita nucleolus.

[8] For an elaborate analysis of the per capita kernel we refer to Section 3.6.

Theorem 3.5.23 Let $\mathcal{U}$ be infinitely countable and let $\Gamma_{\mathcal{U}}$ be the set of all games whose player set is contained in $\mathcal{U}$. The per capita nucleolus is the unique solution on $\Gamma_{\mathcal{U}}$ that satisfies single-valuedness, covariance, anonymity, and the imputation saving reduced game property.

Proof: By Theorem 3.5.3, Proposition 3.5.5, Proposition 3.5.7 and Theorem 3.5.17 the per capita nucleolus satisfies single-valuedness, covariance, anonymity and the imputation saving reduced game property. Hence, it suffices to show that the per capita nucleolus is the only solution concept satisfying these four axioms.

The remainder of the proof is closely related to the proof of Theorem 3.3.28 in which we characterise the per capita prenucleolus. Therefore, we will only indicate where this proof differs from the proof of Theorem 3.3.28.

In Step 2, x now has to be the per capita nucleolus of (N, v).

In Step 3, the sets $\mathcal{B}_h$ are defined for all $h \in P$. Let us alternatively denote these sets by $\mathcal{B}'_h$ for $h \in P$. Then, for every $h \in P$ there exists a set $\mathcal{E}_h \subseteq \{\{i\} \mid i \in N, x^i = w(\{i\})\}$ such that $\mathcal{B}_h = \mathcal{B}'(0, w, \mu_h) \cup \mathcal{E}_h$ is balanced. In the remainder of the proof we use these sets $\mathcal{B}_h$, $h \in P$.

In Step 6, let $(N, \tilde{w}_{N,0})$ be the imputation saving reduced game with respect to coalition $N \subseteq \bar{N}$ and imputation 0 given game $(\bar{N}, \bar{w})$. It follows that $\tilde{w}_{N,0}(S) = w(S)$ for all $S \subseteq N$, with $|S| \neq 1$, because the imputation saving reduced game may differ from the reduced game only for one-person coalitions. Furthermore, since $0 = pcn(N, w)$, $w(\{i\}) \leq 0$ for all $i \in N$. Consequently, $\tilde{w}_{N,0}(\{i\}) = w(\{i\})$ for all $i \in N$. $\square$

In the remainder of this section we show that the axioms single-valuedness, covariance, anonymity and the imputation saving reduced game property are independent. We show in Section 3.6 that the per capita kernel satisfies covariance (Proposition 3.6.2), anonymity (Proposition 3.6.3) and the imputation saving reduced game property (Theorem 3.6.9), but violates single-valuedness (Example 3.6.1).

It is known that the Shapley value satisfies single-valuedness, covariance and anonymity. However, since it also satisfies additivity it does not coincide with the per capita nucleolus, which implies that it does not satisfy the imputation saving reduced game property.

The equal split solution satisfies single-valuedness, anonymity and the imputation saving reduced game property, but violates covariance.

It remains to be shown that anonymity is independent of the other three properties. Therefore, we introduce the positive per capita core.

Definition Let $(N, v) \in \Gamma^I$. The *positive per capita core* of (N, v) is given by $\bar{C}_+^{pc}(N, v) = \{x \in I(N, v) \,|\, e^{pc}(S, x, v) \leq (e^{pc}(S, pcpn(N, v), v))_+ \text{ for all } S \subseteq N\}$.

A solution σ on Γ satisfies the *imputation saving reconfirmation property* if the following condition is satisfied for all $(N, v) \in \Gamma^I$, all $x \in \sigma(N, v)$ and every $T \subseteq N, T \neq \emptyset$: if $(T, \tilde{v}_{T,x}) \in \Gamma^I$ and $y_T \in \sigma(T, \tilde{v}_{T,x})$, then $(y_T, x_{N \setminus T}) \in \sigma(N, v)$.

Lemma 3.5.24 *The positive per capita core on Γ^I satisfies non-emptiness, anonymity, covariance, the imputation saving reduced game property, and the imputation saving reconfirmation property.*

Proof: The properties non-emptiness, anonymity, covariance, and the imputation saving reduced game property can be shown analogously to the proof of Lemma 3.3.31 in which it is shown that the positive per capita precore satisfies non-emptiness, anonymity, covariance, the reduced game property, and the reconfirmation property.

Also the proof to show that the positive per capita core satisfies the imputation saving reconfirmation property is mainly equivalent to its counterpart in the proof of Lemma 3.3.31. Only if $S \cap T = \{i\}$ for some $i \in N$ and $\tilde{v}_{T,x}(\{i\}) = x^i$ we have to provide an alternative proof. In that case we know that $y^i \geq x^i$. Hence, $e^{pc}(\{i\}, x, v) \geq e^{pc}(\{i\}, z, v)$, which implies that $z \in \bar{C}_+^{pc}(N, v)$, as $x \in \bar{C}_+^{pc}(N, v)$. This completes the proof. $\square$

Assume $|\mathcal{U}| \geq 2$. Take an injection $\pi : \mathcal{U} \to \mathbb{N}$. Define $\bar{\xi}(N, v) = \{x \in \bar{C}_+^{pc}(N, v) \,|\, x \succeq_L y \text{ for all } y \in \bar{C}_+^{pc}(N, v)\}$, where $x \succeq_L y$ if there exists a $k \in N$ such that $x^i = y^i$ for all $i \in N$ with $\pi(i) < \pi(k)$ and $x^k > y^k$.

Lemma 3.5.25 *The solution $\bar{\xi}$ satisfies single-valuedness, covariance and the imputation saving reduced game property.*

The proof of this lemma is analogous to the proof of Lemma 3.3.32 in which we show that ξ satisfies single-valuedness, covariance and the reduced game property.

Finally, we show by means of an example that $\bar{\xi}$ does not satisfy anonymity.

Example 3.5.26 Reconsider the game (N, v) of Example 3.3.33. The positive per capita core of (N, v) is given by $\bar{C}_{+}^{pc}(N, v) = \text{conv}\{(1, 0), (0, 1)\}$ and hence, $\bar{\xi}(N, v) = \{(1, 0)\}$ and $\bar{\xi}$ does not satisfy anonymity. $\triangleleft$

3.6 Per capita kernel

In this section we analyse the per capita kernel, which, contrary to the per capita prekernel, only considers elements of the imputation set of a game. The idea of the per capita kernel is that the maximum excess of player i over player j must equal the maximal excess of j over i, except if one of the players receives his individual worth. In that case he is considered to be immune for the additional power another player might have over him. This section is divided into two subsections; in Subsection 3.6.1 we discuss several properties of the per capita kernel, and in Subsection 3.6.2 we discuss the relation between the per capita kernel and other solution concepts for cooperative games.

The definition of the per capita kernel is already given in Section 3.5, but is, for the sake of completeness, repeated below.

Definition Let $(N, v) \in \Gamma^I$. The *per capita kernel* of (N, v) is given by $PCK(N, v) = \{x \in I(N, v) \,|\, s^{ij}(x) \geq s^{ji}(x) \text{ or } x^i = v(\{i\}) \text{ for all } i, j \in N, i \neq j\}$.

3.6.1 Properties

In this subsection we discuss some properties of the per capita kernel. By Theorem 3.5.21 we know that the per capita nucleolus is an element of the per capita kernel for games with a non-empty imputation set. Consequently, the per capita kernel satisfies non-emptiness on Γ^I. Furthermore, by definition, we obtain that the per capita kernel satisfies efficiency. The per capita prekernel is not single-valued as the following example illustrates.

Example 3.6.1 Reconsider the five-player game (N, v) of Example 3.4.3. By Theorem 3.5.20, $pcn(N, v) = (1, 1, 0, 0, 0)$ and by Theorem 3.5.21, $pcn(N, v) \in PCK(N, v)$. Furthermore, $x = (0, 0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in PCK(N, v)$. Consequently, the per capita kernel is not single-valued.

Note that this example also implies that the per capita kernel is not a subset of the core and that the per capita nucleolus and the per capita kernel are not equivalent on Γ. $\triangleleft$

Proposition 3.6.2 *The per capita kernel satisfies covariance.*

The proof of Proposition 3.6.2 is equivalent to the proof of Proposition 3.3.5 in which we show that the per capita prenucleolus satisfies covariance. Without proof we provide the following proposition.

Proposition 3.6.3 *The per capita kernel satisfies anonymity.*

Proposition 3.6.4 *The per capita kernel satisfies desirability.*

Proof: Let $(N, v) \in \Gamma^I$, with $i \succeq_v j$ for some $i, j \in N$. Assume on the contrary $x \in PCK(N, v)$, with $x^j > x^i$. Let us first assume that $x^j = v(\{j\})$. Then by the fact that we assume $x^j > x^i$ and that $x \in I(N, v)$ we obtain

$$\begin{aligned} v(\{j\}) &= x^j \\ &> x^i \\ &\geq v(\{i\}), \end{aligned}$$

which contradicts the assumption that $i \succeq_v j$. Hence, $x^j > v(\{j\})$ and the proof proceeds as the proof of Proposition 3.4.6 in which we show that the per capita prekernel satisfies desirability. $\square$

Since the per capita nucleolus does not satisfy strong desirability, the per capita kernel does not satisfy this property either. By definition the per capita kernel is individually rational, and consequently, the per capita kernel is reasonable from below. The next proposition shows that the per capita kernel is also reasonable from above.

Proposition 3.6.5 *The per capita kernel is reasonable from above.*

Proof: Let $(N, v) \in \Gamma^I$ and $x \in PCK(N, v)$. We have to show that

$$x^i \leq \max_{S \subseteq N \setminus \{i\}} v(S \cup \{i\}) - v(S)$$

for all $i \in N$. If $x^i = v(\{i\})$, then this result follows immediately. Otherwise the proof of Proposition 3.4.8, in which we show that the per capita prekernel is reasonable from above, can be followed. $\square$

Corollary 3.6.6 *The per capita kernel is reasonable.*

This result also leads to the following corollary.

Corollary 3.6.7 *The per capita kernel satisfies the dummy property.*

The following example shows that the per capita kernel does not satisfy the adding dummies property.

Example 3.6.8 Reconsider Example 3.5.12, with $pcn(N, v) = (\frac{1}{3}, \frac{1}{3}, 8\frac{1}{3})$. Consider the game $(N \cup \{4\}, w)$ and allocation $x = (\frac{1}{3}, \frac{1}{3}, 8\frac{1}{3}, 0)$. Then $s^{13}(x, w) = -\frac{1}{6} > -\frac{2}{9} = s^{31}(x, w)$, which implies that $x \notin PCK(N \cup \{4\}, w)$. Consequently, the per capita kernel does not satisfy the adding dummies property. $\triangleleft$

The per capita kernel does not satisfy super-additivity, which follows from the game of Example 3.4.9 in which the per capita kernel equals the per capita prekernel.

Theorem 3.6.9 The per capita kernel satisfies the imputation saving reduced game property.

Proof: Let $(N,v) \in \Gamma^I$ and let $x \in PCK(N,v)$. Let $T \subseteq N, T \neq \emptyset$. We have to show that $x_T \in PCK(T, \tilde{v}_{T,x})$, or equivalently that $s^{ij}(x, \tilde{v}_{T,x}) \geq s^{ji}(x, \tilde{v}_{T,x})$ or $x^i = \tilde{v}_{T,x}(\{i\})$ for all $i,j \in N$, $i \neq j$. Let $i \in N$. If $x_T^i = \tilde{v}_{T,x}(\{i\})$, then the above condition is fulfilled. Hence, we assume $x_T^i > \tilde{v}_{T,x}(\{i\})$. Note that this implies that $x^i > v(\{i\})$. It remains to be shown that $s^{ij}(x_T, \tilde{v}_{T,x}) \geq s^{ji}(x_T, \tilde{v}_{T,x})$ for all $j \in N\backslash\{i\}$. Let $j \in N\backslash\{i\}$. Since $x_T^i > \tilde{v}_{T,x}(\{i\})$ we know that $\max_{S \in \mathfrak{T}_T^{ij}} e^{pc}(S, x_T, \tilde{v}_{T,x}) = \max_{S \in \mathfrak{T}_T^{ij}} e^{pc}(S, x_T, v_{T,x})$. Hence,

$$\begin{aligned}
s^{ij}(x_T, v_{T,x}) &= \max_{S \in \mathfrak{T}_T^{ij}} e^{pc}(S, x_T, \tilde{v}_{T,x}) \\
&= \max_{S \in \mathfrak{T}_T^{ij}} e^{pc}(S, x_T, v_{T,x}) \\
&= \max_{S \in \mathfrak{T}_T^{ij}} \max_{Q \subseteq N\backslash T} e^{pc}(S \cup Q, x, v) \\
&= \max_{S \in \mathfrak{T}^{ij}} e^{pc}(S, x, v) \\
&= s^{ij}(x, v) \\
&\geq s^{ji}(x, v) \\
&= \max_{S \in \mathfrak{T}^{ji}} e^{pc}(S, x, v) \\
&= \max_{S \in \mathfrak{T}_T^{ji}} \max_{Q \subseteq N\backslash T} e^{pc}(S \cup Q, x, v) \\
&= \max_{S \in \mathfrak{T}_T^{ji}} e^{pc}(S, x_T, v_{T,x}) \\
&\geq \max_{S \in \mathfrak{T}_T^{ji}} e^{pc}(S, x_T, \tilde{v}_{T,x}) \\
&= s^{ji}(x_T, v_{T,x}),
\end{aligned}$$

where the first inequality follows from the fact that $x \in PCK(N,v)$ and $x^i > v(\{i\})$. The second inequality follows from the fact that $v_{T,x}(S) \geq \tilde{v}_{T,x}(S)$ for all $S \subseteq N$. $\square$

The per capita kernel does not satisfy the converse imputation saving reduced game property, as is illustrated by the following example.

Example 3.6.10 Consider the game (N,v) of Example 3.5.18. Once more, let $x = (0,0,0)$. Then allocation $(0,0) \in PCK(T, \tilde{v}_{T,x})$ for all $T \in P(N)$. However, $s^{13}(x,v) = \frac{1}{2} > 0 = s^{31}(x,v)$, while $v(\{3\}) \neq x^3$, which implies that $x \notin PCK(N,v)$. Hence, the per capita kernel does not satisfy the converse imputation saving reduced game property. $\triangleleft$

Since the per capita kernel does not satisfy the converse imputation saving reduced game property we are not able to characterise the per capita kernel in a similar way as the per capita prekernel.

3.6.2 Relations to other solution concepts

In this subsection we discuss the relations of the per capita kernel to other solution concepts for cooperative games. In Section 3.5 we already establish that the per capita nucleolus is an element of the per capita kernel (Theorem 3.5.21). Further, we reconsider Example 3.6.1. In this example we show that the per capita prekernel is not equivalent to the per capita prenucleolus. Moreover, the per capita prekernel is not a subset of the core. Also, since the per capita nucleolus is not a bargaining set selector, which follows from Example 3.3.21 in which the per capita prenucleolus and the per capita nucleolus coincide, the per capita prekernel is not a subset of the bargaining set either.

3.7 Core

Peleg and Sudhölter (2003) show that the core can be characterised by the use of the weak reduced game property, corresponding to the reduced game of the prenucleolus. The main result of this section is that in this characterisation we can replace this property by the weak reduced game property, corresponding to the reduced game of the per capita prenucleolus, as defined in Section 3.3. For a discussion of the properties of the core we refer to Peleg and Sudhölter (2003).

Let $(N, v) \in \Gamma$ and let σ be a solution on Γ. Then σ satisfies the *weak reduced game property* if whenever $T \subseteq N$, $1 \leq |T| \leq 2$, and $x \in \sigma(N, v)$, then $(T, v_{T,x}) \in \Gamma$ and $x_T \in \sigma(T, v_{T,x})$. Clearly, if some solution concept satisfies the reduced game property, then it also satisfies the weak reduced game property. We start this section by showing that the core satisfies the (weak) reduced game property.

Theorem 3.7.1 The core satisfies the reduced game property.

Proof: Let $(N, v) \in \Gamma$ and $x \in C(N, v)$. By Lemma 3.3.16 we obtain

$$e^{pc}(S, x_T, v_{T,x}) = \max_{Q \subseteq N \setminus T} \frac{v(S \cup Q) - x(S \cup Q)}{|S \cup Q|}$$

for all $S \subsetneq T \subseteq N$. Let $T \subseteq N, T \neq \emptyset$. Since $x \in C(N,v)$ the right hand side of the equality is non-positive for all $S \subsetneq T$. Hence, $x_T \in C(T, v_{T,x})$. $\square$

Corollary 3.7.2 *The core satisfies the weak reduced game property.*

Before we obtain our main result we first have to provide some prelimary lemmas. In the remainder of this section we assume that the universe $\mathcal{U}$ of players contains at least three members.

Lemma 3.7.3 *Let σ be a solution on Γ. If σ satisfies individual rationality and the weak reduced game property, then σ satisfies efficiency.*

Proof: This proof is by contradiction. Hence, assume σ satisfies individual rationality and the weak reduced game property, and there exists a game $(N,v) \in \Gamma$ and an allocation $x \in \sigma(N,v)$ such that $x(N) < v(N)$. Let $i \in N$. By the weak reduced game property, $(\{i\}, v_{\{i\},x}) \in \Gamma$ and $x^i \in \sigma(\{i\}, v_{\{i\},x})$. By individual rationality $x^i \geq v_{\{i\},x}(\{i\})$. On the other hand, $v_{\{i\},x}(\{i\}) = v(N) - x(N\backslash\{i\}) > x^i$. Thus, the desired contradiction has been obtained. $\square$

Lemma 3.7.4 *The core satisfies the converse reduced game property.*

Proof: Let $(N,v) \in \Gamma$ and $x \in X(N,v)$. Assume that $(T, v_{T,x}) \in \Gamma$ and $x_T \in C(T, v_{T,x})$ for all $T \in P(N)$. Let $S \subsetneq N$, $S \neq \emptyset$. Choose $i \in S$ and $j \in N\backslash S$, and let $T = \{i,j\}$. The fact that $x_T \in C(T, v_{T,x})$ implies that

$$
\begin{aligned}
0 &\geq v_{T,x}(\{i\}) - x^i \\
&= \max_{Q \subseteq N\backslash T} (v(\{i\} \cup Q) - x(Q) - |Q| \cdot e^{pc}(\{i\} \cup Q, x, v)) - x^i \\
&\geq v(S) - x(S) - (|S|-1) \cdot e^{pc}(S,x,v) \\
&= \frac{1}{|S|}(v(S) - x(S)) \\
&= e^{pc}(S,x,v).
\end{aligned}
$$

Consequently, $x \in C(N,v)$. $\square$

Lemma 3.7.5 *Let σ be a solution on Γ. If σ satisfies individual rationality and the weak reduced game property, then $\sigma(N,v) \subseteq C(N,v)$ for every $(N,v) \in \Gamma$.*

Proof: Let $(N,v) \in \Gamma$. If $n = 1$, then $\sigma(N,v) \subseteq C(N,v)$ by individual rationality. By Lemma 3.7.3 σ satisfies efficiency. Hence, if $n = 2$, then $\sigma(N,v) \subseteq \{x \in X(N,v) \,|\, x^i \geq v(\{i\}) \text{ for all } i \in N\} = C(N,v)$.

If $n \geq 3$ and $x \in \sigma(N,v)$, then the weak reduced game property implies $x_T \in \sigma(T, v_{T,x})$ for all $T \in P(N)$. Consequently, $x_T \in C(T, v_{T,x})$ for all $T \in P(N)$. Hence, by the fact that the core satisfies the converse reduced game property (Lemma 3.7.4), $x \in C(N,v)$. $\square$

Corollary 3.7.6 *Let σ be a solution on Γ^C that satisfies non-emptiness, individual rationality and the weak reduced game property. If the core of a game (N,v) consists of a unique point, then $\sigma(N,v) = C(N,v)$.*

Since for all $x \in C(N,v)$, $x^i \geq v(\{i\})$ for all $i \in N$ and all $(N,v) \in \Gamma^C$, it follows that the core is individually rational.

Corollary 3.7.7 *The core satisfies individual rationality.*

Furthermore, one can easily verify that the core is super-additive.

Lemma 3.7.8 *The core satisfies super-additivity.*

The above results lead us to the main theorem of this section.

Theorem 3.7.9 The core is the unique solution on Γ^C that satisfies non-emptiness, individual rationality, the weak reduced game property and super-additivity.

Proof: The core on Γ^C satisfies non-emptiness (by definition), individual rationality (Corollary 3.7.7), super-additivity (Lemma 3.7.8) and the weak reduced game property (Corollary 3.7.2). Hence, it suffices to prove uniqueness.

Let σ be a solution on Γ^C that satisfies non-emptiness, individual rationality, super-additivity and the weak reduced game property, and let $(N, v) \in \Gamma^C$ be an n-player game. By Lemma 3.7.5, $\sigma(N, v) \subseteq C(N, v)$. Thus we only have to show $C(N, v) \subseteq \sigma(N, v)$. Let $x \in C(N, v)$. We distinguish between three cases; $n = 1$, $n \geq 3$ and $n = 2$. If $n = 1$, then $x \in \sigma(N, v)$ by non-emptiness and individual rationality.

Secondly, we consider the situation $n \geq 3$. Define the game (N, w) by $w(\{i\}) = v(\{i\})$ for all $i \in N$ and $w(S) = x(S)$ for all $S \subseteq N$ with $|S| \neq 1$. As $n \geq 3$, $C(N, w) = \{x\}$. Hence, by Corollary 3.7.6, $\sigma(N, w) = \{x\}$. Let $u = v - w$. Then $u(\{i\}) = 0$ for all $i \in N$, $u(N) = 0$, and $u(S) \leq 0$ for all $S \subseteq N$. Therefore, $C(N, u) = \{0\}$ and, again by Corollary 3.7.6, $\sigma(N, u) = \{0\}$. Hence, by super-additivity, $\{x\} = \sigma(N, u) + \sigma(N, w) \subseteq \sigma(N, v)$. We conclude that $x \in \sigma(N, v)$, and thus $C(N, v) \subseteq \sigma(N, v)$.

Finally, we consider the case $n = 2$. Let $N = \{i, j\}$ and let $k \in \mathcal{U} \backslash N$. We define the game (M, u), with $M = \{i, j, k\}$ and u given by the table below, with A such that $u_{N,y} = v$.

S	$\{i\}$	$\{j\}$	$\{k\}$	$\{i,j\}$	$\{i,k\}$	$\{j,k\}$	M
$u(S)$	$v(\{i\})$	$v(\{j\})$	0	$v(\{i\}) + v(\{j\})$	A	A	$v(N)$

Then $y \in \mathbb{R}^M$, defined by $y^k = 0$ and $y_N = x$, is an element of $C(M, u)$. As $|M| = 3$, $C(M, u) \subseteq \sigma(M, u)$. Thus $y \in \sigma(M, u)$. Since $u_{N,y} = v$ the weak reduced game property gives $x \in \sigma(N, v)$, and thus $C(N, v) \subseteq \sigma(N, v)$. $\square$

We proceed by showing that the properties non-emptiness, individual rationality, super-additivity and the weak reduced game property are independent. The empty solution satisfies individual rationality, super-additivity and the (weak) reduced game property, but violates non-emptiness.

Further, let the solution σ on Γ^C be defined by $\sigma(N, v) = C(N, v)$ if $n \geq 2$ and by $\sigma(\{i\}, v) = X^*(\{i\}, v)$ for all one-player games $(\{i\}, v)$. Then σ satisfies non-emptiness, super-additivity and the (weak) reduced game property, but on one-player games it violates individual rationality.

Define solution σ on Γ^C by $\sigma(N, v) = \{x \in X(N, v) \,|\, x^i \geq v(\{i\}) \text{ for all } i \in N\}$. Then σ satisfies non-emptiness, individual rationality and super-additivity, but by

Lemma 3.7.5 σ violates the weak reduced game property, as $\sigma(N,v)$ is not a subset of the core for all $(N,v)\in\Gamma$.

Finally, let us consider the per capita prenucleolus on Γ^C. We have shown that the per capita prenucleolus satisfies non-emptiness (Lemma 3.3.1) and the reduced game property (Theorem 3.3.18) and since it is an element of the core (Theorem 3.3.19) it also satisfies individual rationality. However, by Example 3.3.13 we also know that it does not satisfy (super-)additivity on Γ^C.

We conclude this section by considering the relation between the core and the imputation saving reduced game, defined in Section 3.5.

Proposition 3.7.10 *The core satisfies the imputation saving reduced game property.*

Proof: Let $(N,v)\in\Gamma$ and $x\in C(N,v)$. Let $(T,\tilde{v}_{T,x})$ be the imputation saving reduced game with respect to $T\subseteq N, T\neq\emptyset$ and x. Firstly, let $S\subsetneqq T$ such that $\tilde{v}_{T,x}(S)=v_{T,x}(S)$. Then it follows by Lemma 3.3.16 that $e^{pc}(S,x_T,v_{T,x})=\max_{Q\subseteq N\setminus T}\frac{v(S\cup Q)-x(S\cup Q)}{|S\cup Q|}$. Since $x\in C(N,v)$ the right-hand side of the equality is non-positive. Secondly, if $\tilde{v}_{T,x}(S)\neq v_{T,x}(S)$, then $\tilde{v}_{T,x}(\{i\})=x^i$ for some $i\in N$, which implies that $e^{pc}(\{i\},x_T,\tilde{v}_{T,x})=0$. Consequently, $e^{pc}(S,x_T,\tilde{v}_{T,x})\leq 0$ for all $S\subseteq T$, which means that $x_T\in C(T,\tilde{v}_{T,x})$. □

The following example shows that the core does not satisfy the converse imputation saving reduced game property.

Example 3.7.11 Consider the game depicted below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	1	1	0	1

Consider preimputation $x = (0, 5, -4)$ and the corresponding two-player imputation saving reduced games.

S	$\{1\}$	$\{2\}$	$\{1,2\}$
$\tilde{v}_{\{1,2\},x}(S)$	0	3.5	5

S	$\{1\}$	$\{3\}$	$\{1,3\}$
$\tilde{v}_{\{1,3\},x}(S)$	0	-4	-4

S	$\{2\}$	$\{3\}$	$\{2,3\}$
$\tilde{v}_{\{2,3\},x}(S)$	3	-4	1

Hence, for each $T \in P(N)$, $x_T \in C(T, \tilde{v}_{T,x})$. However, $x \notin C(N, v)$. $\triangleleft$

Since the converse reduced game property is used (Lemma 3.7.4) for proving that the core can be characterised by non-emptiness, individual rationality, the weak reduced game property and super-additivity, a similar characterisation with the weak imputation saving reduced game property instead of the weak reduced game property cannot be obtained in an analogous way.

3.8 Overview

We conclude this chapter with an overview of the analysed solution concepts and their properties in Table 3.8.1. For comparison we have also added the prenucleolus (pn) and the prekernel (PK) to this table.

Note that we consider the reduced game properties with respect to the reduced games introduced in this chapter. Further, if a box is empty, then the property is either not defined for the corresponding solution concept (*e.g.*, some properties are only defined for single-valued solution concepts) or irrelevant (*e.g.*, our reduced game property for the prenucleolus).

Properties / Solution concepts	*pcpn*	*PCPK*	*pcn*	*PCK*	*core*	*pn*	*PK*
non-emptiness	$\surd$	$\surd$	$\surd^+$	$\surd^+$	$\times$	$\surd$	$\surd$
efficiency	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
single-valuedness	$\surd$	$\times$	$\surd$	$\times$	$\times$	$\surd$	$\times$
covariance	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
anonymity	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
desirability	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\surd$
equal treatment property	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\surd$
strong desirability	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
individual rationality	$\times$	$\times$	$\surd$	$\surd$	$\surd$	$\times$	$\times$
reasonableness from above	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
reasonableness from below	$\times$	$\times$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
reasonableness	$\times$	$\times$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
dummy property	$\times$	$\times$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
adding dummies property	$\times$	$\times$	$\times$	$\times$	$\surd$	$\surd$	$\surd$
(super-)additivity	$\times$	$\times$	$\times$	$\times$	$\surd$	$\times$	$\times$
continuity	$\surd$		$\surd$			$\surd$	
coalitional monotonicity	$\times$		$\times$			$\times$	
weak coalitional monotonicity	$\surd$		$\surd$			$\surd$	
aggregate monotonicity	$\surd$		$\times$			$\times$	
strong aggregate monotonicity	$\surd$		$\times$			$\times$	
reduced game property	$\surd$	$\surd$			$\surd$		
converse reduced game property	$\times$	$\surd$			$\surd$		
imputation saving reduced game property			$\surd$	$\surd$	$\surd$		
converse imputation reduced game property			$\times$	$\times$	$\times$		
core selector	$\surd^*$	$\times$	$\surd^*$	$\times$		$\surd^*$	$\times$
bargaining set selector	$\times$	$\times$	$\times$	$\times$	$\surd^*$	$\surd$	$\times$

$+$ If the imputation set is non-empty.

$*$ If the core is non-empty.

Table 3.8.1: Properties for solution concepts

Chapter 4

Public congestion network situations and related games

Nobody goes there; it's too crowded.

Yogi Berra (1925 -)

4.1 Introduction

This chapter, which is based on Kleppe et al. (2009), analyses congestion on network situations from a cooperative game theoretic perspective. Economic congestion situations arise when a group of players uses facilities from a common pool and the cost of using a certain facility depends on the number of its users. If players decide to minimise the joint costs of using the facility a congestion situation leads to interaction between players and the analysis of a joint cost allocation problem.

In their work Monderer and Shapley (1996) underline the importance of the analysis of congestion situations from an economic viewpoint. And there is a rich literature concerning congestion in a non-cooperative game theoretical setting, starting from Braess's paradox (Braess (1968)) and Rosenthal (1973), in which the existence of pure Nash equilibria in such types of games is shown. For an overview of recent results on congestion in a non-cooperative setting we refer to Vöcking (2006).

In general, a Nash equilibrium of a congestion game does not result in a social optimum in the sense that it does not minimise the aggregated costs of all players involved. Although there are several papers concerning the gap between selfish (Nash) solutions and the social optimum, *e.g.*, Johari and Tsitsiklis (2004), the social optimum itself has been studied much less frequent. Examples are Milchtaich

(2004), who studies the allocation of the cost associated with the social optimum in a setup with a continuum of players, and Matsubayashi et al. (2005) in which hub-spoke network systems with congestion effects are discussed using cooperative games.

The main inspiration for this chapter comes, however, from a third example of research on congestion within a cooperative setting; Quant et al. (2006) study congestion network situations, which generalise the well-known minimum cost spanning tree problems (Claus and Kleitman (1973) and Bird (1976)).

In congestion network situations a single source is considered to which all players have to be connected, and the cost of using an arc in order to achieve this depends on the number of its users. In Quant et al. (2006) all arcs are private, *i.e.*, a coalition is only allowed to use the arcs between the players of the coalition and the source in order to establish their connection. In this chapter, however, we consider congestion network situations with public arcs, which means that each coalition of players is allowed to use any arc of the network.

For applications of public congestion network situations one could think, *e.g.*, of computer networks with one main server, a communication network with a unique information provider, or a single distribution center with several suppliers on a publicly available road network.

We discuss congestion network situations with the underlying idea that players have to get from their initial nodes to the source. This describes, *e.g.*, a situation in which suppliers supply a single distribution center. Note however that we could equivalently think of a setup in which players start at the source and have to get to their final nodes, which would model a situation where suppliers supply *from* a single distribution center.

In case the players have the intention to cooperate, it is natural that (in principle) an optimal network is constructed. In order to find an appropriate allocation of the involved joint costs, two transferable utility games are introduced. In the so-called *direct cost game* a coalition that is formed must construct a network that connects all of its members, assuming that the other players do *not* make use of any arcs. This is in our opinion a convenient model for situations with concave congestion costs. In that situation the presence of other players would decrease marginal costs of construction and it is reasonable to assume that a coalition formed does not benefit from the presence of other players. In case of convex costs, however, it is reasonable to assume that the coalitional costs are determined from a situation in which the non-members have been connected already. This is modelled by the so-called *marginal cost game*, which is the dual of the direct cost game. We elaborate on this in Section 4.4.

Quant et al. (2006) show that if arcs are private and costs are concave, there exists an optimal network which is a tree. Since for the grand coalition it does not matter whether arcs are private or public, this result still stands in our model. Furthermore, Example 4.1 in Quant et al. (2006) of a private congestion network situation with concave costs of which the corresponding transferable utility cost game is not balanced, gives the same result for the direct cost game in the case of public arcs. For linear costs it is easy to verify that the direct cost game is additive (and hence coincides with the marginal cost game). For these reasons we will mainly focus on public congestion network situations with convex costs.

Within this framework we first present an algorithm to find an optimal network for each coalition of players. Our second main result is that the marginal cost game of a convex congestion network situation is concave. As a consequence, cooperation is likely to occur and stable allocations exist. Our third main result is the introduction of a solution concept that provides such a stable allocation. Finally, we extend these results to a framework with divisibility in which we drop the restriction that players have to use a single path to the source.

The structure of this chapter is as follows. Section 4.2 settles notation for public congestion network situations, while Section 4.3 deals with the problem of finding an optimal network in case of convex costs. We start the analysis of the cost allocation problem for public congestion network situations with convex costs in Section 4.4, where we introduce the marginal cost game and prove that this game is concave. The analysis of cost allocation is continued in Section 4.5, where we refine the core of the marginal cost game by three equal treatment principles. In Section 4.6 we extend the results of Sections 4.4 and 4.5 to a framework with divisibility.

4.2 Public congestion network situations

A *public congestion network situation*, or congestion network situation as we call it from here, is given by a triple $G = (N, 0, \gamma)$, where N is the finite set of players that has to be connected to the source 0. By A_S we denote the set of all arcs between pairs in $S \subseteq N^0$, *i.e.*, (S, A_S) is the complete digraph on S. For each arc $a \in A_{N^0}$ the function $\gamma_a : \{0, 1, \ldots, n\} \to \mathbb{R}_+$ is a non-negative (weakly) increasing cost function. Hence, γ_a associates each number of users of arc a with a corresponding cost. We assume that $\gamma_a(0) = 0$ for all $a \in A_{N^0}$. Elements of A_{N^0} are denoted by a or by (i, j), where $i, j \in N^0$. The arc (i, j) denotes the connection between i and j in the direction from i to j. If $a = (i, j)$, then a^{-1} denotes the arc in the opposite direction, *i.e.*, $a^{-1} = (j, i)$. The cost function of an arc (i, j) is denoted by $\gamma_{i,j}$. A congestion network situation is called symmetric if $\gamma_{i,j} = \gamma_{j,i}$ for all $i, j \in N^0$.

In a congestion network situation each player chooses a path from his initial node to the source. A path between any two nodes i and j is denoted by $P(i, j)$ and is a sequence of arcs $((i_0, i_1), (i_1, i_2), \ldots, (i_{p-1}, i_p))$, such that $i_0 = i$, $i_p = j$ and $i_r \neq i_s$ for all $r, s \in \{0, \ldots, p-1\}$, $r \neq s$. A non-empty path $P(i, i)$ is called a circuit. Furthermore, instead of $P(i, 0)$ we also write P_i.

Let $G = (N, 0, \gamma)$ be a congestion network situation. A *network* is defined by an integer valued function $f : A_{N^0} \to \{0, 1, \ldots, n\}$, such that f assigns to each arc a number of users. The indegree for a node $i \in N^0$ with respect to network f is defined by $\text{indegree}^f(i) = \sum_{j \in N^0 \setminus \{i\}} f(j, i)$. Analogously, the outdegree is defined by $\text{outdegree}^f(i) = \sum_{j \in N^0 \setminus \{i\}} f(i, j)$. Combining these two concepts results in $\text{netdegree}^f(i) = \text{outdegree}^f(i) - \text{indegree}^f(i)$. For a coalition $S \subseteq N$ the collection of all feasible networks connecting the members of S to the source is given by

$$F_S = \Big\{ f : A_{N^0} \to \{0, \ldots, n\} \,\Big|\, \begin{array}{l} \text{netdegree}^f(i) = 1 \text{ for all } i \in S, \\ \text{netdegree}^f(i) = 0 \text{ for all } i \in N \setminus S, \\ f(a) \in \{0, \ldots, |S|\} \text{ for all } a \in A_{N^0} \Big\}. \end{array}$$

Note that in a feasible network for S each player of S is connected to the source by some path. However, as all arcs are publicly available these paths may consist of arcs between any two nodes in N^0. Each network f induces a digraph (N^0, A_f), where A_f consists of all used arcs:

$$A_f = \{a \in A_{N^0} \mid f(a) > 0\}.$$

The cost of a network f is defined by

$$\gamma(f) = \sum_{a \in A_f} \gamma_a(f(a)).$$

With each congestion network situation $G = (N, 0, \gamma)$ one can associate a *direct cost game* (N, c_d^G), in which $c_d^G(S)$ denotes the minimum cost of a network connecting all players of S to the source in case no players outside coalition S make use of the arcs of the network:

$$c_d^G(S) = \min_{f \in F_S} \gamma(f)$$

for all $S \subseteq N$. We omit the superscript G if no confusion can occur. As discussed in Section 4.1, this game only models the situation properly for concave congestion network situations. A *concave congestion network situation* $G = (N, 0, \gamma)$ is a congestion network situation in which all γ_a are concave. A cost function γ_a, $a \in A_{N^0}$, is concave if for all $r \in \{1, \ldots, n-1\}$

$$\gamma_a(r+1) - \gamma_a(r) \leq \gamma_a(r) - \gamma_a(r-1).$$

In Section 4.1 we argue that for concave cost functions all results derived by Quant et al. (2006) for congestion network situations with private arcs are almost directly applicable for the setup with public arcs. Since this is not the case for congestion network situations with convex cost functions, we only consider this type of situations in the next three sections. A *convex congestion network situation* $G = (N, 0, \gamma)$ is a congestion network situation in which all γ_a are convex. A cost function γ_a, $a \in A_{N^0}$, is convex if for all $r \in \{1, \ldots, n-1\}$

$$\gamma_a(r+1) - \gamma_a(r) \geq \gamma_a(r) - \gamma_a(r-1).$$

Note that especially within the context of road or computer networks this convexity assumption is plausible. Furthermore, the requirement of convex cost functions is a similar, but stronger, assumption than the one often used in literature, *e.g.*, Milchtaich (2004), where the cost per user of each arc is set to be increasing in the number of its users.

Example 4.2.1 An example of a symmetric convex congestion network situation is given in Figure 4.2.1. In this situation there are three players, which are denoted by 1, 2 and 3, and the source, which is given by 0. The numbers on the arcs represent the total usage costs for each number of users. $\triangleleft$

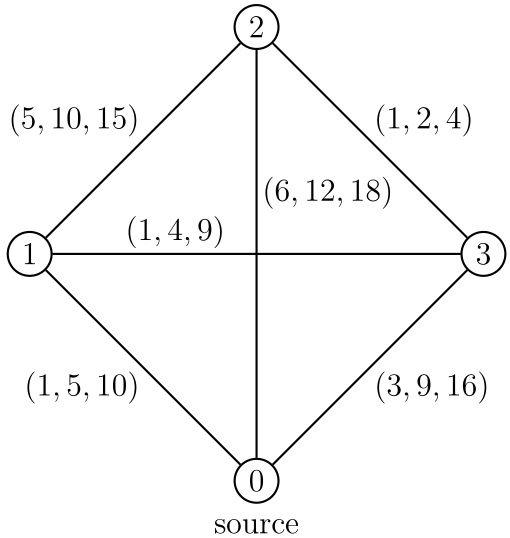

Figure 4.2.1: A symmetric convex congestion network situation

4.3 Optimal networks

This section focusses on finding an optimal network for a coalition $S \subseteq N$ for a convex congestion network situation. After introducing for each network f the length function ℓ_f, we characterise the optimality of a network f by the use of the corresponding length function. We use this result to construct an algorithm to find an optimal network for each coalition of players.

Let $G = (N, 0, \gamma)$ be a convex congestion network situation and consider a feasible network f. Recall that a circuit is a non-empty path $P(i, i)$, with $i \in N^0$. In the remainder of this chapter we assume that A_f contains no circuits, because the network arising from f by decreasing the number of users of the arcs in a circuit by one yields a feasible network at least as cheap as f. The fact that we exclude circuits in particular implies that if $a \in A_f$, then $a^{-1} \notin A_f$.

Given network f, we define the length function ℓ_f on the complete digraph (N^0, A_{N^0}) as follows:

$$\ell_f(a) = \begin{cases} \infty & \text{if} \quad f(a) = n, \\ \gamma_a(f(a)+1) - \gamma_a(f(a)) & \text{if} \quad f(a^{-1}) = 0 \text{ and } f(a) < n, \\ \gamma_{a^{-1}}(f(a^{-1})-1) - \gamma_{a^{-1}}(f(a^{-1})) & \text{if} \quad f(a^{-1}) > 0. \end{cases}$$

This function can be interpreted as the marginal cost of an extra user of an arc. Note that if the opposite a^{-1} of an arc is used in the current network, an extra user of arc a should be interpreted as the reduction of the number of users of a^{-1} by one. A circuit C is called negative with respect to length function ℓ_f if $\sum_{a\in C}\ell_f(a) < 0$.

Let f^1 and f^2 be networks. The sum $f^1 \oplus f^2$ is defined by

$$f^1 \oplus f^2(a) = \max\{f^1(a) + f^2(a) - f^1(a^{-1}) - f^2(a^{-1}), 0\}$$

for all $a \in A_{N^0}$. This operation takes into account that the usage of two oppositely directed arcs cannot be beneficial. If there is two-way traffic between nodes, the numbers of users are subtracted instead of added.

Similarly, we define the network $f^1 \ominus f^2$ that measures the difference between f^1 and f^2 by

$$f^1 \ominus f^2(a) \quad = \quad \max\{f^1(a) - f^2(a) + f^2(a^{-1}) - f^1(a^{-1}), 0\}$$

for all $a \in A_{N^0}$. It assigns a positive number of users to an arc $a \in A_{N^0}$ if the arc is used more in f^1 than in f^2 and if the arc in opposite direction is used more in f^2 than in f^1.

Example 4.3.1 Consider the convex congestion network situation of Figure 4.2.1. Let f^1 be given by $f^1(1,0) = 2$, $f^1(2,0) = 1$ and $f^1(3,1) = 1$ and let f^2 be given by $f^2(1,0) = 1$, $f^2(1,3) = 1$, $f^2(2,0) = 1$ and $f^2(3,0) = 1$. Then $f^{\oplus} = f^1 \oplus f^2$ is given by $f^{\oplus}(1,0) = 3$, $f^{\oplus}(2,0) = 2$ and $f^{\oplus}(3,0) = 1$ and network $f^{\ominus} = f^1 \ominus f^2$ is given by $f^{\ominus}(0,3) = 1$, $f^{\ominus}(1,0) = 1$ and $f^{\ominus}(3,1) = 2$. $\triangleleft$

Proposition 4.3.2 *Let $G = (N, 0, \gamma)$ be a convex congestion network situation, and let $S \subseteq N$. Then $f \in F_S$ is an optimal network for S if and only if A_{N^0} contains no negative circuit with respect to the length function ℓ_f.*

Proof: We first show the "only if" part. Since f is optimal, $\gamma(f) \leq \gamma(f) + \sum_{a\in C}\ell_f(a)$ for all circuits C. Hence, there cannot be a negative circuit with respect to length function ℓ_f.

We now show the "if" part. Let $f, \bar{f} \in F_S$ and assume that f is not optimal for S, but $\bar{f}$ is. Consider the network $\bar{f} \ominus f$. Since both $\bar{f}$ and f are feasible for S, and

$\bar{f} \ominus f$ measures the difference between $\bar{f}$ and f, $\text{netdegree}^{\bar{f}\ominus f}(i) = 0$ for all $i \in N^0$. This implies that $A_{\bar{f}\ominus f}$ contains a circuit. We show that it contains a negative circuit with respect to ℓ_f, which completes the proof.

Let C be a circuit in $A_{\bar{f}\ominus f}$ and let $a \in C$. In the following table the five possibilities of the presence of a and a^{-1} in $A_{\bar{f}}$ and A_f are illustrated. In this table $\times$ indicates that neither a nor a^{-1} are present.

	C_1	C_2	C_3	C_4	C_5
$A_{\bar{f}}$	a	a	a	$\times$	a^{-1}
A_f	a	$\times$	a^{-1}	a^{-1}	a^{-1}

The last column, *e.g.*, indicates that $0 < \bar{f}(a^{-1}) < f(a^{-1})$, because $a \in A_{\bar{f}\ominus f}$ and in both networks a is used in opposite direction. Note that the above are the only five possibilities. Hence, the set C can be partitioned into the five sets $C_1, \ldots, C_5$.

The length of C with respect to ℓ_f equals

$$\begin{aligned}
\sum_{a\in C}\ell_f(a) &= \sum_{a\in C_1\cup C_2}(\gamma_a(f(a)+1)-\gamma_a(f(a)))+\\
&\quad+\sum_{a\in C_3\cup C_4\cup C_5}(\gamma_{a^{-1}}(f(a^{-1})-1)-\gamma_{a^{-1}}(f(a^{-1})))\\
&\le \sum_{a\in C_1\cup C_2}(\gamma_a(\bar f(a))-\gamma_a(\bar f(a)-1))+\\
&\quad+\sum_{a\in C_4\cup C_5}(\gamma_{a^{-1}}(\bar f(a^{-1}))-\gamma_{a^{-1}}(\bar f(a^{-1})+1))+\\
&\quad+\sum_{a\in C_3}(\gamma_{a^{-1}}(f(a^{-1})-1)-\gamma_{a^{-1}}(f(a^{-1})))\\
&\le \sum_{a\in C_1\cup C_2}(\gamma_a(\bar f(a))-\gamma_a(\bar f(a)-1))+\\
&\quad+\sum_{a\in C_4\cup C_5}(\gamma_{a^{-1}}(\bar f(a^{-1}))-\gamma_{a^{-1}}(\bar f(a^{-1})+1))+\\
&\quad+\sum_{a\in C_3}(\gamma_a(\bar f(a))-\gamma_a(\bar f(a)-1))\\
&= -[\sum_{a\in C_1\cup C_2\cup C_3}(\gamma_a(\bar f(a)-1)-\gamma_a(\bar f(a)))+\\
&\quad+\sum_{a\in C_4\cup C_5}(\gamma_{a^{-1}}(\bar f(a^{-1})+1)-\gamma_{a^{-1}}(\bar f(a^{-1})))]\\
&= -\sum_{a^{-1}\in C^{-1}}\ell_{\bar f}(a^{-1})\\
&\le 0. \qquad (4.1)
\end{aligned}$$

The first inequality follows from the convexity of the functions γ_a and the fact that $\bar f(a)\ge f(a)+1$ if $a\in C_1\cup C_2$ and $f(a^{-1})\ge \bar f(a^{-1})+1$ if $a\in C_4\cup C_5$. Inequality (4.1) follows from the optimality of $\bar f$. Consequently, C is a (weakly) negative circuit with respect to the length function ℓ_f.

If inequality (4.1) is strict, then a negative circuit has been found. In case inequality (4.1) is tight, so $\sum_{a^{-1}\in C^{-1}}\ell_{\bar f}(a^{-1})=0$, we proceed by changing the network $\bar f$ to $\bar f_1=\bar f\ominus f_C$. Network $\bar f_1$ is feasible and costs $\gamma(\bar f_1)=\gamma(\bar f)+\sum_{a^{-1}\in C^{-1}}\ell_{\bar f}(a^{-1})=\gamma(\bar f)$ and is therefore also optimal. One can measure the difference between f and $\bar f_1$ in a similar way as the difference between f and $\bar f$. In comparison to $\bar f\ominus f$, for all arcs $a\in A_{N^0}$, $\bar f_1\ominus f(a)=\bar f\ominus f(a)$ if $a\notin C$ and $\bar f_1\ominus f(a)=\bar f\ominus f(a)-1$ if $a\in C$. The set of arcs $A_{\bar{f}_1 \ominus f}$ also contains a circuit and one can follow the lines of the proof above. If inequality (4.1) is again tight, one must define networks $\bar{f}_2$, $\bar{f}_3$ and $\bar{f}_2 \ominus f$, $\bar{f}_3 \ominus f$ and so on, until a strict inequality arises.

Since the number of feasible networks is finite, eventually a strict inequality will arise. The values of $\bar{f}_k \ominus f(a)$ are decreasing in k for all $a \in A_{N^0}$. Furthermore, $\bar{f}_k \ominus f$ is the zero network if and only if $f(a) = \bar{f}_k(a)$ for all $a \in A_{N^0}$. Since f is not optimal, $\bar{f}_k \ominus f$ cannot be the zero network for any k, and hence there is a k for which inequality (4.1) is strict. □

The existence of negative circuits can be detected by a shortest path algorithm, *e.g.*, the Floyd-Warshall algorithm (Cormen et al. (1990)). If the algorithm finds that the "shortest path" (cheapest way) to go from some node to itself has negative costs, then there must be a negative circuit containing this node.

By Proposition 4.3.2 we know whether a network is optimal by its length function. We use this relation to construct an algorithm that provides an optimal network for a coalition $S \subseteq N$. Such a network is denoted by f_S^*. The idea of the algorithm is to connect players sequentially to the source in such a way that each player minimises the length of his path to the source, given the length function corresponding to the network constructed by his predecessors. We denote by $P_{S,i}^*$ the shortest path in (N^0, A_{N^0}) from $i \in N$ to 0, given length function $\ell_{f_S^*}$. It is important to note that, based on the fact that two-way traffic between nodes cannot be beneficial, paths constructed in earlier stages of the algorithm can be altered later on. We denote the network corresponding to a path P by f_P.

Algorithm 4.3.3

Input: *a convex congestion network situation* $G = (N, 0, \gamma)$, *and an ordering* $\pi \in \Pi_S$.

Output: *an optimal network* f_S^* *for coalition* $S \subseteq N$.

1. *Initialise* $V = \emptyset$ *and* $t = 1$.
2. *Find* $P_{V,\pi(t)}^*$.
3. *Set* $f_{V \cup \{\pi(t)\}}^* = f_V^* \oplus f_{P_{V,\pi(t)}^*}$.
4. *If* $t < |S|$, *set* $t = t + 1$, $V = V \cup \{\pi(t)\}$ *and return to step 2.*

The complexity of the Floyd-Warshall algorithm to find shortest paths is of order $\mathcal{O}(n^3)$. Since we have to go through the steps of the algorithm at most n times the complete algorithm has a complexity of order $\mathcal{O}(n^4)$.

Theorem 4.3.4 Let $G = (N, 0, \gamma)$ be a convex congestion network situation, let $S \subseteq N$ and $\pi \in \Pi_S$. The output f_S^* of Algorithm 4.3.3 is an optimal network for coalition S.

Proof: Assume without loss of generality that π is the identity, hence $\pi(i) = i$ for all $i \in N$. We first show that f_S^* is feasible for S. At iteration t of the algorithm the netdegree of node 0 decreases from $1 - t$ to $-t$, the netdegree of the nodes $1, \ldots, t-1$ remains 1, the netdegree of node t increases from 0 to 1 and the netdegree of the nodes $t+1, \ldots, n$ remains 0. Hence, $\text{netdegree}^{f_S^*}(i) = 1$ for all $i \in S$ and $\text{netdegree}^{f_S^*}(i) = 0$ for all $i \in N \backslash S$, which implies that f_S^* is feasible. To prove optimality of f_S^* we use an induction argument.

If $V = \emptyset$, then $\ell_{f_V^*}(a) = \gamma_a(1) \geq 0$ for all $a \in A_{N^0}$. Path $P_{\emptyset,1}^*$ is the shortest path from 1 to 0 and trivially $f_{\{1\}}^* = P_{\emptyset,1}^*$ is an optimal network for $\{1\}$.

Let $V = \{1, \ldots, t\} \subsetneqq S$. Assume that the network f_V^* is optimal for coalition V. There exists a shortest path $P = P_{V,t+1}^*$ from player $t+1$ to 0 with respect to length function $\ell_{f_V^*}$. We have to prove that $f_{V\cup\{t+1\}}^* = f_V^* \oplus f_P$ is an optimal network for coalition $V \cup \{t+1\}$. According to Proposition 4.3.2 this can be done by showing that there is no negative circuit with respect to $\ell_{f_{V\cup\{t+1\}}^*}$. We prove this by contradiction.

Assume that C is a negative circuit with respect to $\ell_{f_{V\cup\{t+1\}}^*}$. By means of C and P we find a network feasible for coalition V that costs less than f_V^*, which yields a contradiction. Recall $f_{V\cup\{t+1\}}^* = f_V^* \oplus f_P$. Let $\hat{f}$ be the network arising from $f_{V\cup\{t+1\}}^*$ by adding the circuit C. Hence, $\hat{f} = f_{V\cup\{t+1\}}^* \oplus f_C$ and this network can also be written as $\hat{f} = f_V^* \oplus h$, with $h = f_P \oplus f_C$. Consequently, for all $a \in A_{N^0}$

$$h(a) = \begin{cases} 2 & \text{if } a \text{ is used by both } P \text{ and } C, \\ 1 & \text{if } a^{-1} \text{ is used by neither } P \text{ nor } C \text{ and } a \text{ is used by either } P \text{ or } C, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, $\text{netdegree}^h(t+1) = 1$, $\text{netdegree}^h(0) = -1$ and $\text{netdegree}^h(i) = 0$ for all

$i \notin \{0, t+1\}$. Hence, there is a path $\bar{P} \subseteq A_h$ from $t+1$ to 0. Notice that $\bar{P}$ is not necessarily the same path as P. We decompose network h into three 0,1-networks $f_{\bar{P}}$, h_1 and h_2 such that $h = f_{\bar{P}} + h_1 + h_2$[1] for all $a \in A_{N^0}$. After path $\bar{P}$ has been chosen, h_1 and h_2 are given by

$$\begin{aligned} h_1(a) &= \begin{cases} 1 & \text{if } h(a) - f_{\bar{P}}(a) > 0, \\ 0 & \text{otherwise,} \end{cases} \\ h_2(a) &= \begin{cases} 1 & \text{if } h(a) - f_{\bar{P}}(a) = 2, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

for all $a \in A_{N^0}$. Note that $h_1 + h_2$ is a network in which all nodes have netdegree zero.

The costs to obtain network $\hat{f}$ from network f_V^* can be computed in two ways:

$$\gamma(\hat{f}) - \gamma(f_V^*) = \sum_{a \in P} \ell_{f_V^*}(a) + \sum_{a \in C} \ell_{f_{V,t+1}^*}(a), \tag{4.2}$$

and

$$\gamma(\hat{f}) - \gamma(f_V^*) = \sum_{a \in \bar{P}} \ell_{f_V^*}(a) + \sum_{a \in A_{h_1}} \ell_{f_V^* \oplus f_{\bar{P}}}(a) + \sum_{a \in A_{h_2}} \ell_{f_V^* \oplus f_{\bar{P}} \oplus h_1}(a). \tag{4.3}$$

Since P is the shortest path from $t+1$ to 0, $\sum_{a \in P} \ell_{f_V^*}(a) \leq \sum_{a \in \bar{P}} \ell_{f_V^*}(a)$. Consequently, combining the fact that C is a negative circuit with respect to $\ell_{f^*_{V \cup \{t+1\}}}$ with equations (4.2) and (4.3) leads to

$$\sum_{a \in A_{h_1}} \ell_{f_V^* \oplus f_{\bar{P}}}(a) + \sum_{a \in A_{h_2}} \ell_{f_V^* \oplus f_{\bar{P}} \oplus h_1}(a) < 0.$$

Note that $a^{-1} \notin \bar{P}$ for all $a \in A_{h_1}$. Consequently, by the convexity of γ_a we obtain that

$$\ell_{f_V^*}(a) \leq \ell_{f_V^* \oplus f_{\bar{P}}}(a)$$

for all $a \in A_{h_1}$. Furthermore, since $\bar{P} \cap A_{h_2} = \emptyset$,

$$\ell_{f_V^* \oplus f_{\bar{P}} \oplus h_1}(a) = \ell_{f_V^* \oplus h_1}(a)$$

for all $a \in A_{h_2}$.

[1] Note that here the regular operation $+$ is used.

As a result,

$$\begin{aligned}\gamma(f_V^* \oplus (h_1+h_2)) &= \gamma(f_V^*) + \sum_{a\in A_{h_1}} \ell_{f_V^*}(a) + \sum_{a\in A_{h_2}} \ell_{f_V^*\oplus h_1}(a) \\ &\leq \gamma(f_V^*) + \sum_{a\in A_{h_1}} \ell_{f_V^*\oplus f_{\bar{P}}}(a) + \sum_{a\in A_{h_2}} \ell_{f_V^*\oplus f_{\bar{P}}\oplus h_1}(a) \\ &< \gamma(f_V^*). \end{aligned} \tag{4.4}$$

Since $h_1 + h_2$ is a network in which all nodes have netdegree zero, the network $f_V^* \oplus (h_1 + h_2)$ is feasible for coalition V. Consequently, inequality (4.4) contradicts the assumption that f_V^* is optimal for V. □

Example 4.3.5 We illustrate the use of Algorithm 4.3.3 by the convex congestion network situation of Figure 4.2.1. We determine an optimal network for coalition $\{1,3\}$ by the use of ordering $\pi = \{3,1,2\}$. First of all, path P_3^* is given by $((3,1),(1,0))$. The shortest path from player 1 to the source given $\ell_{f^*_{\{3\}}}$ is $P^*_{\{3\},1} = ((1,3),(3,0))$. Adding this path to $f^*_{\{3\}}$ results in network $f^*_{\{1,3\}}$ with $f^*_{\{1,3\}}(1,0) = f^*_{\{1,3\}}(3,0) = 1$. Hence, the cost of an optimal network for coalition $\{1,3\}$ is 4. Note that the cost of an optimal network for a coalition is by definition equal to the cost of this coalition in the direct cost game (N, c_d), and hence $c_d(\{1,3\}) = 4$.

The optimal network for coalition $\{1,3\}$ for this situation is unique, but this is not a general result. For this convex congestion network situation there are, *e.g.*, two optimal networks for the grand coalition, being f_N^*, given by $f_N^*(2,3) = f_N^*(3,1) = f_N^*(3,0) = 1$ and $f_N^*(1,0) = 2$, and $\bar{f}_N^*$ with $\bar{f}_N^*(1,0) = \bar{f}_N^*(2,0) = \bar{f}_N^*(3,0) = 1$. We come back to the issue of multiple optimal networks in Section 4.6. ◁

4.4 The marginal cost game

In this section we introduce the marginal cost game, which we use to analyse convex congestion network situations. In order to determine the cost of coalition $S \subseteq N$ one generally adopts a pessimistic viewpoint. Therefore, for convex congestion network situations one has to take account of the possibility that the players in $N \backslash S$ are already using the network. However, this still leaves many options open. The most pessimistic view is that the remaining players will try to frustrate S as much as possible. However, we think it is a fair reference point to assume that $N\backslash S$ minimises its own cost. Note that this viewpoint implicitly assumes cooperation between the players in $N\backslash S$, but since full cooperation is to be expected in the first place, this is not an unreasonable assumption in order to determine the cost of coalition S. Since it is the objective of $N\backslash S$ to minimise its own cost a reasonable approach is to assume that the members of $N\backslash S$ are willing to change their paths to the source, as long as S compensates them for the additional costs. Note that this approach incorporates the transferability of utility in the definition of the cost game. Note also that if an allocation is stable under this approach, it is stable under every less optimistic approach as well. This reasoning boils down to the idea that if coalition S forms, it constructs a network that is feasible and optimal for the grand coalition, and that the complementary coalition $N\backslash S$ pays $c_d(N\backslash S)$ to make use of the network. This idea is formalised by the *marginal cost game* (N, c_m^G) (or (N, c_m) if no confusion can occur), which is given by

$$c_m^G(S) = c_d^G(N) - c_d^G(N\backslash S)$$

for all $S \subseteq N$. Note that the idea of duality as used here is well-imbedded in the game theoretic literature; think, *e.g.*, of bankruptcy games (O'Neill (1982) and Thomson (2003)). Furthermore, the notion of a dual game is closely related to the economic concept of subcontracting, as, *e.g.*, discussed by Kamien et al. (1989) and Spiegel (1993).

Example 4.4.1 We illustrate the idea of the marginal cost game by means of the convex congestion network situation of Figure 4.2.1. We first consider this situation without player 2. Hence, only the arcs between players 1 and 3 and the source are present, and $N = \{1, 3\}$. Suppose we would like to determine the cost of coalition $\{1\}$ in this situation. We assume that player 3 has been optimally connected to the source already, by the path $((3, 1), (1, 0))$ with cost 2. Then player 1 could use the direct link, resulting in a cost of 4, or connect himself via node 3, which also costs 4. In our setup coalition $\{1\}$ has a third option. He can ask player 3 to link himself directly to the source, making it possible for player 1 to form a less expensive direct connection. This results in a cost of 1 for himself and a cost of 1 to compensate player 3. It is straightforward to verify that this leads to a cost for coalition $\{1\}$ of $c_d(\{1, 3\}) - c_d(\{3\}) = 4 - 2 = 2$. Note that Algorithm 4.3.3 also follows this procedure to obtain an optimal network.

Let us now consider the entire situation of Figure 4.2.1, hence including player 2. All coalitional costs for the direct and marginal cost game associated with this congestion network situation are given in the next table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c_d(S)$	1	3	2	5	4	6	10
$c_m(S)$	4	6	5	8	7	9	10

We can determine the direct cost for each coalition by Algorithm 4.3.3. From those costs the marginal cost game follows immediately. $\triangleleft$

Theorem 4.4.2 links the convexity of the congestion network situation to the concavity of the corresponding marginal cost game (N, c_m). A cost game (N, c) is *concave* if $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$ for all $S \subseteq T \subseteq N \backslash \{i\}$. We introduce an alternative way to denote a path from node i to j. A path from i to j can also be given by a sequence of nodes, $Q(i,j) = (i_0, i_1, \ldots, i_p)$, with $i_0 = i$, $i_p = j$ and $i_r \neq i_s$ for all $r, s \in \{0, \ldots, p-1\}$, $r \neq s$. By $i \prec_Q j$ we denote that node i is a predecessor of node j on path Q.

Theorem 4.4.2 Let $G = (N, 0, \gamma)$ be a convex congestion network situation. Then the corresponding marginal cost game (N, c_m) is concave.

Proof: We prove that (N, c_m) is a concave game by showing that the game (N, c_d) is convex. Let $S \subseteq N$, $|S| \leq n-2$. Algorithm 4.3.3 finds an optimal network with corresponding costs for a particular coalition by putting players one by one onto the network. If we assume that the players of coalition S constructed f_S^*, the next player $i \in N \backslash S$ adds a cost which equals the length of path $P_{S,i}^*$ derived by the algorithm.

Let f_S^* be an optimal network for S. Let $i, j \in N \backslash S$ and let $T = S \cup \{i\}$, with optimal network f_T^*. By $\ell_f(P) = \sum_{a \in P} \ell_f(a)$ we denote the length of a path P, given network f. We construct a path $\bar{P}$ from j to 0 with the property that $\ell_{f_T^*}(P_{T,j}^*) \geq \ell_{f_S^*}(\bar{P})$. As $\ell_{f_T^*}(P_{T,j}^*) = c_d(T \cup \{j\}) - c_d(T)$ and $\ell_{f_S^*}(\bar{P}) \geq \ell_{f_S^*}(P_{S,j}^*) = c_d(S \cup \{j\}) - c_d(S)$, this result implies that $c_d(T \cup \{j\}) - c_d(T) \geq c_d(S \cup \{j\}) - c_d(S)$ for every $S \subseteq T \subseteq N \backslash \{j\}$, and hence that (N, c_d) is convex.

Let us first assume that there does not exist an arc $a \in P^*_{T,j}$ such that $a^{-1} \in P^*_{S,i}$. Then $f^*_T(a) \geq f^*_S(a)$ for all $a \in P^*_{T,j}$, which implies by the convexity of γ_a that $\ell_{f^*_T}(P^*_{T,j}) \geq \ell_{f^*_S}(P^*_{T,j})$. Hence, we can choose $\bar{P}$ equal to $P^*_{T,j}$.

Secondly, we assume that there does exist an arc $a \in P^*_{T,j}$ such that $a^{-1} \in P^*_{S,i}$. Let (k_1, ℓ_1) be the unique arc on $P^*_{T,j}$ with the property that the inverse arc $(\ell_1, k_1) \in P^*_{S,i}$ and $a^{-1} \notin P^*_{S,i}$ for all $a \prec_{P^*_{T,j}} (k_1, \ell_1)$. We define $m_1 \in Q^*_{T,j} \cap Q^*_{S,i}$ as the first node on path $Q^*_{T,j}$ such that $k_1 \prec_{P^*_{T,j}} m_1$ and $k_1 \prec_{P^*_{S,i}} m_1$.

Take m_1 as the new starting node. If there exist arcs on $P^*_{T,j}$ beyond node m_1 used by $P^*_{S,i}$ in the opposite direction, the arc (k_2, ℓ_2) is defined similar to the way we defined arc (k_1, ℓ_1). Moreover, all nodes k_r and m_r, $r \in \{1, \ldots, R\}$, are sequentially defined analogously to the definitions of k_1 and m_1. Note that node m_r may coincide with k_{r+1} and that m_R may be equal to the source, 0. To get an idea how $P^*_{S,i}$ and $P^*_{T,j}$ may relate, see Figure 4.4.1.

Define the generalised path[2]

$$\bar{P} \;=\; (P^*_{T,j}(j, k_1), P^*_{S,i}(k_1, m_1), P^*_{T,j}(m_1, k_2), \ldots, P^*_{S,i}(k_R, m_R), P^*_{T,j}(m_R, 0)),$$

where $P(x, y)$ here specifically denotes the subpath of P from x to y. Let $m_0 = j$, $k_{R+1} = 0$, and let $r \in \{0, \ldots, R\}$. As none of the arcs on $P^*_{T,j}(m_r, k_{r+1})$ are taken by player i in the reverse direction, $f^*_T(a) \geq f^*_S(a)$. Consequently, by the convexity of γ_a and the fact that $\bar{P}(m_r, k_{r+1}) = P^*_{T,j}(m_r, k_{r+1})$,

$$\ell_{f^*_T}(P^*_{T,j}(m_r, k_{r+1})) \;\geq\; \ell_{f^*_S}(\bar{P}(m_r, k_{r+1})). \tag{4.5}$$

Now let $r \in \{1, \ldots, R\}$. We show that $\ell_{f^*_T}(P^*_{T,j}(k_r, m_r)) \geq \ell_{f^*_S}(\bar{P}(k_r, m_r))$. None of the arcs of $P^*_{T,j}(k_r, m_r)$, nor their reversed arcs, are used by player i when traversing path $P^*_{S,i}$ *after* he arrives at node k_r, *i.e.*, $\{a, a^{-1}\} \cap P^*_{S,i}(k_r, 0) = \emptyset$ for all $a \in P^*_{T,j}(k_r, m_r)$. Hence, if player i would have used $P^*_{T,j}(k_r, m_r)$ to travel from k_r to m_r, then the costs would have been exactly $\ell_{f^*_T}(P^*_{T,j}(k_r, m_r))$. However, the fact that i does not choose this option, but uses $P^*_{S,i}(k_r, m_r)$, which coincides with $\bar{P}(k_r, m_r)$, implies that

$$\ell_{f^*_T}(P^*_{T,j}(k_r, m_r)) \;\geq\; \ell_{f^*_S}(\bar{P}(k_r, m_r)). \tag{4.6}$$

[2] A *generalised* path may, contrary to a path, contain a circuit.

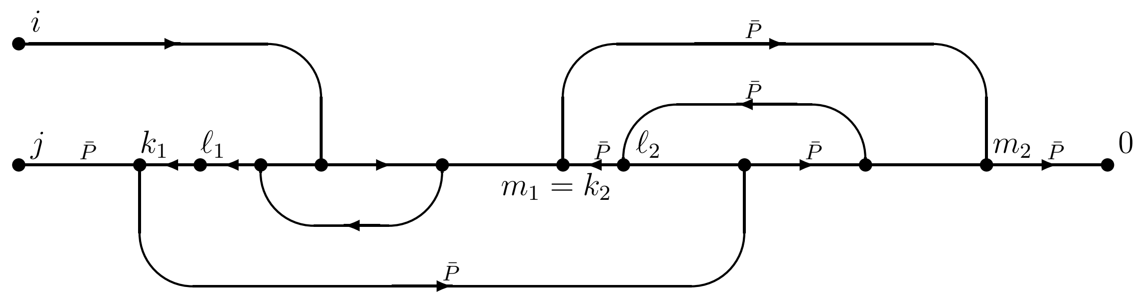

Figure 4.4.1: Paths $P^*_{S,i}$ (following the arrows) and $P^*_{T,j}$ (going in a straight line from j to 0). Arcs belonging to path $\bar{P}$ are labelled accordingly.

Combining (4.5) and (4.6) leads to

$$\ell_{f^*_T}(P^*_{T,j}) \quad \geq \quad \ell_{f^*_S}(\bar{P}),$$

which completes the proof. □

4.5 Cost allocation

Since the marginal cost game (N, c_m) is concave it has a non-empty core. In this section we construct and analyse a core element in which players pay for each arc proportionally to their average usage of the arc in an optimal network.

Let $G = (N, 0, \gamma)$ be a convex congestion network situation and let $S \subseteq N$ with optimal network f^*_S. Let $D_S = \{D_S(i)\}_{i \in S}$ be a *decomposition* of f^*_S into $|S|$ paths. A decomposition of an optimal network f^*_N is denoted by D.

Example 4.5.1 Consider the convex congestion network situation of Figure 4.2.1, with optimal network f^*_N given by $f^*_N(2,3) = f^*_N(3,1) = f^*_N(3,0) = 1$ and $f^*_N(1,0) = 2$. Then $D(1) = ((1,0))$, $D(2) = ((2,3),(3,0))$, and $D(3) = ((3,1),(1,0))$ forms a decomposition of f^*_N. ◁

The solution concept we introduce in this section is based upon the following equal treatment principles:

- every player should only contribute to the cost of the arcs he uses,

- two players whose paths share some arc should contribute an equal part of the cost of this arc,
- if there are several path decompositions possible for an optimal network, the average over all decompositions should be used to allocate the total cost.

The first two principles naturally lead to the idea that the contribution of each player should only depend on the arcs used by him in an optimal network, and furthermore that his contribution to the total cost of each arc is proportional to his usage of the arc. These ideas result in the following cost allocation, given decomposition D of optimal network f_N^*:

$$\psi_D^i(N, 0, \gamma; f_N^*) = \sum_{a \in A_{f_N^*}} \frac{f_{D(i)}(a)}{f_N^*(a)} \gamma_a(f_N^*(a)) \tag{4.7}$$

for all $i \in N$. In the remainder we denote this allocation by $\psi_D^i(f_N^*)$.

Example 4.5.2 For decomposition D given in Example 4.5.1, $\psi_D(f_N^*) = (2\frac{1}{2}, 4, 3\frac{1}{2})$. ◁

Lemma 4.5.3 *Let $G = (N, 0, \gamma)$ be a convex congestion network situation with corresponding marginal cost game (N, c_m). Then $\psi_D(f_N^*) \in C(N, c_m)$.*

Proof: It follows from the definition that $\psi_D(f_N^*)$ is efficient. Let us therefore focus on the stability constraints of the core. Let $S \subseteq N$ and let $\bar{f} = \sum_{i \in S} f_{D(i)}$. Since $\bar{f}$ is feasible for S we obtain

$$\begin{aligned} c_d(S) &\leq \gamma(\bar{f}) \\ &= \sum_{a \in A_{\bar{f}}} \gamma_a\Big(\sum_{i \in S} f_{D(i)}(a)\Big) \\ &\leq \sum_{a \in A_{\bar{f}}} \frac{\sum_{i \in S} f_{D(i)}(a)}{f_N^*(a)} \gamma_a(f_N^*(a)) \\ &= \sum_{i \in S} \sum_{a \in A_{f_N^*}} \frac{f_{D(i)}(a)}{f_N^*(a)} \gamma_a(f_N^*(a)) \\ &= \sum_{i \in S} \psi_D^i(f_N^*). \end{aligned}$$

The second inequality follows from the fact that $\bar{f}$ is a feasible network for coalition S and the convexity of the cost functions γ_a. The concluding argument is

$$\begin{aligned}\sum_{i\in S}\psi^i_D(f^*_N) &= \sum_{i\in N}\psi^i_D(f^*_N) - \sum_{i\in N\backslash S}\psi^i_D(f^*_N)\\ &\leq c_d(N) - c_d(N\backslash S)\\ &= c_m(S).\end{aligned}$$

□

Hence, a decomposition of an optimal network gives rise to a core element of the marginal cost game. However, given an optimal network f^*_S, the decomposition D_S need not be unique, in the sense that we cannot distinguish which arcs are used by which players. This is only the case when digraph $(N^0, A_{f^*_S})$ contains a cycle[3].

Example 4.5.4 Let us reconsider the convex congestion network situation of Figure 4.2.1. Optimal network f^*_N does not have a unique decomposition, because this network can be decomposed into both D, which has been considered already in Example 4.5.1, and D', with $D'(1) = ((1,0))$, $D'(2) = ((2,3),(3,1),(1,0))$, and $D'(3) = ((3,0))$, resulting in allocation $\psi_{D'}(f^*_N) = (2\frac{1}{2}, 4\frac{1}{2}, 3)$. Note that D and D' are the only possible decompositions. ◁

The fact that allocation $\psi_D(f^*_N)$ depends on the decomposition chosen gives it a flavour of arbitrariness. In order to overcome this drawback we use the third equal treatment principle and introduce, given an optimal network f^*_N, the allocation $\psi(N,0,\gamma;f^*_N)$ as the average over all allocations $\psi_D(f^*_N)$ that follow from each of the possible decompositions:

$$\psi(N,0,\gamma;f^*_N) = \frac{1}{|\mathcal{D}|}\sum_{D\in\mathcal{D}}\psi_D(f^*_N), \tag{4.8}$$

with $\mathcal{D}$ the set of all path decompositions of an optimal network f^*_N. In the remainder we denote this allocation by $\psi(f^*_N)$.

Theorem 4.5.5 Let $G = (N,0,\gamma)$ be a convex congestion network situation with corresponding marginal cost game (N, c_m). Then $\psi(f^*_N) \in C(N, c_m)$.

[3]We say that the digraph (N, A) contains a cycle if the non-directed graph $(N, E(A))$, with $E(A) = \{\{i,j\} \mid (i,j) \in A \text{ or } (j,i) \in A\}$ contains a cycle.

Proof: As each allocation $\psi_D(f^*_N)$ is an element of the core (Lemma 4.5.3), the convexity of the core yields that $\psi(f^*_N)$ is a core element as well. □

Each optimal network f^*_N gives rise to a unique cost allocation $\psi(f^*_N)$. However, if a convex congestion network situation has multiple optimal networks the correspon-ding cost allocations need not be the same.

Example 4.5.6 Consider the convex congestion network situation of Figure 4.2.1. For optimal network f^*_N cost allocation $\psi(f^*_N)$ is given by $\psi(f^*_N) = \frac{1}{2}(2\frac{1}{2}, 4, 3\frac{1}{2}) + \frac{1}{2}(2\frac{1}{2}, 4\frac{1}{2}, 3) = (2\frac{1}{2}, 4\frac{1}{4}, 3\frac{1}{4})$, and for optimal network $\bar{f}^*_N$ we obtain that $\psi(\bar{f}^*_N) = (1, 6, 3)$. ◁

Since the number of decompositions of an optimal network can grow exponentially, $\psi(f^*_N)$ is not polynomially computable by means of its definition. However, $\psi(f^*_N)$ can be computed by the following polynomial algorithm.

Algorithm 4.5.7

Input: *a convex congestion network situation* $G = (N, 0, \gamma)$, *and an optimal network* f^*_N.

Output: *cost allocation* $\psi(f^*_N)$.

1. *Initialise* $K = N$ *and for all* $i \in N$, $g(i, i) = 1$.
2. *Find a node* $k \in K$ *such that* $\displaystyle\sum_{i \in K \setminus \{k\}} f^*_N(i, k) = 0$.
3. *Define for all* $i \in N \setminus K$: $\displaystyle g(i, k) = \sum_{j:(j,k) \in A_{f^*_N}} \frac{g(i, j) \cdot f^*_N(j, k)}{\mathit{outdegree}^{f^*_N}(j)}$ *for all* $k \in K \setminus \{i\}$.
4. *If* $K \neq \emptyset$, *set* $K = K \setminus \{j\}$ *and return to step 2.*
5. *For all* $i \in N$, *set* $\displaystyle\psi^i(f^*_N) = \sum_{(j,k) \in A_{f^*_N}} \frac{g(i, j) \cdot \gamma_{j,k}(f^*_N(j, k))}{\mathit{outdegree}^{f^*_N}(j)}$.

The interpretation of $g(i, k)$ is that it denotes the fraction of all decompositions of f^*_N in which agent i visits node k on his way to the root. The complexity of the third and fifth step of the algorithm is of order $\mathcal{O}(n^2)$. The complexity of the other steps is of order $\mathcal{O}(n)$. Since steps 2 - 4 are repeated n times, the complexity of the complete algorithm is of order $\mathcal{O}(n^3)$.

Example 4.5.8 We illustrate Algorithm 4.5.7 by the convex congestion network situation of Figure 4.2.1 and optimal network $f_N^*(2,3) = f_N^*(3,1) = f_N^*(3,0) = 1$ and $f_N^*(1,0) = 2$.

We start by finding a node k with $\text{indegree}^{f_N^*}(k) = 0$. The only node satisfying this condition is node 2. The first time that the algorithm visits step 3, this step is void, as $N \backslash K = \emptyset$. We remove node 2 from K. Among all nodes in K node 3 is the only node k such that $\sum_{i \in K \backslash \{k\}} f_N^*(i,k) = 0$. This time step 2 sets $g(2,3)$ to $\frac{g(2,2) f_N^*(2,3)}{\text{outdegree}^{f_N^*}(2)} = \frac{1 \cdot 1}{1} = 1$. Indeed, in all (both) decompositions of f_N^* player 2 visits node 3. Now, node 3 is removed from K and the final visit of step 2 yields $g(2,1) = g(3,1) = \frac{1 \cdot 1}{2} = \frac{1}{2}$, which equals the fraction of decompositions in which player 2 (3) visits node 1. Step 5 gives

$$\begin{aligned}
\psi^1(f_N^*) &= \frac{g(1,1) \cdot \gamma_{1,0}(f_N^*(1,0))}{\text{outdegree}^{f_N^*}(1)} \\
&= \frac{1 \cdot 5}{2}, \\
\psi^2(f_N^*) &= \frac{g(2,2) \cdot \gamma_{2,3}(f_N^*(2,3))}{\text{outdegree}^{f_N^*}(2)} + \frac{g(2,3) \cdot \gamma_{3,1}(f_N^*(3,1))}{\text{outdegree}^{f_N^*}(3)} + \\
&\quad + \frac{g(2,3) \cdot \gamma_{3,0}(f_N^*(3,0))}{\text{outdegree}^{f_N^*}(3)} + \frac{g(2,1) \cdot \gamma_{1,0}(f_N^*(1,0))}{\text{outdegree}^{f_N^*}(1)} \\
&= \frac{1 \cdot 1}{1} + \frac{1 \cdot 1}{2} + \frac{1 \cdot 3}{2} + \frac{\frac{1}{2} \cdot 5}{2}, \\
\psi^3(f_N^*) &= \frac{g(3,3) \cdot \gamma_{3,0}(f_N^*(3,0))}{\text{outdegree}^{f_N^*}(3)} + \frac{g(3,3) \cdot \gamma_{3,1}(f_N^*(3,1))}{\text{outdegree}^{f_N^*}(3)} + \frac{g(3,1) \cdot \gamma_{1,0}(f_N^*(1,0))}{\text{outdegree}^{f_N^*}(1)} \\
&= \frac{1 \cdot 3}{2} + \frac{1 \cdot 1}{2} + \frac{\frac{1}{2} \cdot 5}{2},
\end{aligned}$$

which results in $\psi(f_N^*) = (2\frac{1}{2}, 4\frac{1}{4}, 3\frac{1}{4})$. Note that this cost allocation coincides with the allocation $\psi(f_N^*)$ derived from the definition of equation (4.8), which is a result formalised in the following proposition. $\triangleleft$

Proposition 4.5.9 *Let $G = (N, 0, \gamma)$ be a convex congestion network situation with optimal network f_N^*. Then the output of Algorithm 4.5.7 is $\psi(f_N^*)$.*

Proof: Since f_N^* is circuit-free, we can rename the players such that $N = \{1, \ldots, n\}$ and, for all $i, k \in N$, $f_N^*(i,k) > 0$ implies that $i < k$. Note that a decomposition

D of f_N^* can be seen as a description telling at each node which player uses which arc. Consider the description to be decentralised in the sense that at each node j a decision is made of the following form:

The lowest ranked player using node j is told to continue via some arc (j, k_1) with $f_N^*(j, k_1) > 0$, the second lowest ranked player is told to continue via arc (j, k_2), and so on. This is done in such a way that $f_N^*(j, k)$ ranks of players are pointed to arc (j, k) for all $k \in \{0, j+1, \ldots, n\}$.

Since decisions in other nodes determine which players visit node j it is clear that who will be sent from node j whereto does not only depend on decisions in node j. Nevertheless, all local decisions can in the above way be made independently. As a result each (global) decomposition determines a unique profile of local decisions and vice versa. Therefore, the fraction of decompositions for which a player uses arc (j, k) equals the fraction of decompositions that he visits node j times the fraction of the visitors of node j that leave j via arc (j, k). Hence, by the third step of the algorithm we obtain

$$\begin{aligned} g(i,k) &= \sum_{j:(j,k)\in A_{f_N^*}} \frac{|\{D \,|\, i \text{ uses arc } (j,k)\}|}{|\mathcal{D}|} \\ &= \sum_{j:(j,k)\in A_{f_N^*}} g(i,j) \cdot \frac{f_N^*(j,k)}{\text{outdegree}^{f_N^*}(j)}. \end{aligned}$$

By step 5 it follows that

$$\begin{aligned} \psi^i(f_N^*) &= \frac{1}{|\mathcal{D}|} \sum_{D\in\mathcal{D}} \psi_D(f_N^*) \\ &= \sum_{(j,k)\in A_{f_N^*}} \frac{|\{D \,|\, i \text{ uses arc } (j,k)\}|}{|\mathcal{D}|} \cdot \{\text{average costs of } (j,k)\} \\ &= \sum_{(j,k)\in A_{f_N^*}} g(i,j) \cdot \frac{f_N^*(j,k)}{\text{outdegree}^{f_N^*}(j)} \cdot \frac{\gamma_{j,k}(f_N^*(j,k))}{f_N^*(j,k)} \\ &= \sum_{(j,k)\in A_{f_N^*}} g(i,j) \cdot \frac{\gamma_{j,k}(f_N^*(j,k))}{\text{outdegree}^{f_N^*}(j)}. \end{aligned}$$

$\square$

4.6 Divisible congestion network situations

In the previous sections we have considered network situations in which players have to be connected to the source by a single path. However, if we think in the context of continuous streams of traffic (*e.g.*, data traffic from terminals to a mainframe, or road traffic from suppliers to a distribution center) a player can divide his unit of traffic and use several paths to the source. As a consequence, the capacity and usage of an arc need no longer be integer, and therefore we switch from discrete to continuous cost functions.

Congestion network situations arising from this relaxed setting are called *divisible congestion network situations* and are given by $\mathcal{G} = (N, 0, \gamma)$ in which $\gamma_a : [0, n] \to \mathbb{R}_+$ is a (weakly) increasing cost function for all $a \in A_{N^0}$, with $\gamma_a(0) = 0$. The set of all feasible networks for a coalition $S \subseteq N$ is given by

$$F_S = \Big\{ f : A_{N^0} \to [0, n] \;\Big|\; \begin{array}{l} \text{netdegree}^f(i) = 1 \text{ for all } i \in S, \\ \text{netdegree}^f(i) = 0 \text{ for all } i \in N \backslash S, \\ f(a) \in [0, |S|] \text{ for all } a \in A_{N^0} \Big\}. \end{array}$$

Note that although each player has the possibility to use several paths the netdegree of his node is still one in case he is connected to the source, and zero otherwise. The corresponding direct divisible network cost game is denoted by $(N, c_d^{\mathcal{G}})$, the marginal cost game by $(N, c_m^{\mathcal{G}})$.

In the remainder of this section we call the congestion network situations in which players cannot divide their unit of traffic, as discussed in the previous sections, indivisible congestion network situations. Given a divisible congestion network situation $\mathcal{G} = (N, 0, \gamma)$, one can find the related indivisible congestion network situation by restricting the function γ_a to the domain $\{0, 1, \ldots, n\}$. The congestion network situation derived in this way is denoted by $G(\mathcal{G})$.

Proposition 4.6.1 *Let $\mathcal{G} = (N, 0, \gamma)$ be a divisible concave congestion network situation with corresponding indivisible concave congestion network situation $G(\mathcal{G})$. Then $(N, c_d^{\mathcal{G}}) = (N, c_d^{G(\mathcal{G})})$.*

Proof: This result follows from Theorem 4.1 of Quant et al. (2006), stating that for a indivisible concave congestion network situation[4] there exists an optimal network that is a tree. □

For convex congestion network situations the games $(N, c_d^{\mathcal{G}})$ and $(N, c_d^{G(\mathcal{G})})$ do not coincide, as is shown by the following example.

Example 4.6.2 Consider the symmetric divisible convex congestion network situation $\mathcal{G}$ of Figure 4.6.1, where x denotes the usage of an arc. The corresponding

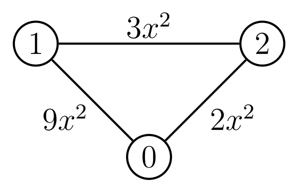

Figure 4.6.1: A divisible convex congestion network situation

optimal network is given by $f_N^*(1,0) = f_N^*(1,2) = \frac{1}{2}$ and $f_N^*(2,0) = \frac{3}{2}$, with a total cost of $7\frac{1}{2}$. However, an optimal network for $G(\mathcal{G})$ is given by $f_N^*(1,0) = f_N^*(2,0) = 1$, with a total cost of 11. ◁

Since a divisible convex congestion network situation and the related indivisible convex congestion network situation may have different optimal networks, we discuss divisible convex congestion network situations in more detail and focus in particular on the two main results of Sections 4.4 and 4.5.

First of all, we consider the solution concept discussed in Section 4.5. Analogous to the indivisible setup, for the divisible framework $\psi_D(f_N^*)$ can be defined for each decomposition by (4.7) of optimal network f_N^*, with $\psi(f_N^*)$ as the average over all (extreme) decompositions. However, once an optimal network f_N^* has been found we can directly calculate $\psi(f_N^*)$ by means of Algorithm 4.5.7 as well.

Example 4.6.3 For the divisible convex congestion network situation of Figure 4.6.1 $\psi(f_N^*)$ is given by $\psi(f_N^*) = (4\frac{1}{2}, 3)$. ◁

[4]Quant et al. (2006) refer to divisible congestion network situations as relaxed congestion network situations.

Theorem 4.6.4 Let $\mathcal{G} = (N, 0, \gamma)$ be a divisible convex congestion network situation with corresponding marginal cost game $(N, c_m^{\mathcal{G}})$. Then $\psi(f_N^*) \in C(N, c_m^{\mathcal{G}})$.

Proof: The proof of Theorem 4.5.5, stating that $\psi(f_N^*) \in C(N, c^G)$, is also valid to prove that $\psi(f_N^*) \in C(N, c^{\mathcal{G}})$. Consequently, the allocation $\psi(f_N^*)$ is situated in the core of $(N, c_m^{\mathcal{G}})$ for divisible convex congestion network situations. $\square$

Note that Theorem 4.6.4 demonstrates the non-emptiness of $C(N, c_m^{\mathcal{G}})$. A drawback of the indivisible setup is that a situation may have multiple optimal networks, *e.g.*, the indivisible convex congestion network situation of Figure 4.2.1 has two optimal networks. As a consequence $\psi(f_N^*)$ need not be unique, which implies that an indivisible congestion network situation gives in general rise to a set of allocations $\psi(f_N^*)$. However, each divisible congestion network situation with strictly convex cost functions is a strictly convex optimisation problem and therefore has a single optimal network. As a result $\psi(f_N^*)$ is unique.

The content of the following theorem is that the marginal cost game associated with a divisible convex congestion network situation is concave. Its proof is constructive in the sense that it provides an algorithm for approximating an optimal network for divisible convex congestion network situations with arbitrary precision. The idea is to approximate the situation by an indivisible situation where a player can send his unit of traffic into z packages of size $\frac{1}{z}$, in which z is considered to be a large natural number. In order to translate the situation into the settings of Section 4.4, we regard a player as a group of z agents, each having a whole unit of traffic, and scale the cost functions accordingly. Theorem 4.4.2 can be applied to show the concavity of the approximated marginal cost game. By letting z go to infinity we obtain a divisible convex congestion network situation with a corresponding marginal cost game, which is also concave.

Lemma 4.6.5 *Let (N, c) be a concave game. Define $(\bar{N}, \bar{c})$, with $\bar{N} = (N \backslash K) \cup \{*\}$, for some $K \subseteq N$, and*

$$\bar{c}(S) = \begin{cases} c(S) & \text{if} \quad * \notin S, \\ c((S \backslash \{*\}) \cup K) & \text{if} \quad * \in S. \end{cases}$$

Then $(\bar{N}, \bar{c})$ is concave.

Proof: Note that $(\bar{N}, \bar{c})$ is derived from (N, c) by assuming that coalition K acts as a single player. The concavity of the game (N, c) implies the concavity of the game $(\bar{N}, \bar{c})$ as for the latter property only a subset of the constraints of the first has to be satisfied. □

Theorem 4.6.6 Let $\mathcal{G} = (N, 0, \gamma)$ be a divisible convex congestion network situation. Then the corresponding marginal cost game $(N, c_m^{\mathcal{G}})$ is concave.

Proof: The marginal cost game corresponding to $\mathcal{G} = (N, 0, \gamma)$ is given by $(N, c_m^{\mathcal{G}})$. Based on this divisible convex congestion network situation we define a special type of indivisible convex congestion situation: $G_z = (N_z, 0, \gamma^z)$, with $N_z = \{1, \dots, |N_z|\}$ the finite player set. In this situation there are n *regular* nodes, forming the set N, and for each regular node $i \in N$ we add $z - 1$ nodes, which we call *friends* of i. We define $Z_i = \{i\} \cup \{j \mid j \text{ is a friend of } i\}$. Furthermore, for fixed z, cost function γ_a^z is defined such that $\gamma_{i,j}^z(r) = \gamma_{i,j}(\frac{r}{z})$, with $i, j \in N^0$, $r \in \{0, \dots, |N_z|\}$. For each friend j of a regular node i we define $\gamma_{i,j}^z(r) = \gamma_{j,i}^z(r) = 0$, and $\gamma_{j,k}^z(r) = \gamma_{k,j}^z(r) = r \cdot M$ for all $r \in \{1, \dots, |N_z|\}$ and all $k \in N_z^0 \backslash \{i, j\}$, with M sufficiently large.

Due to this construction friend j of regular node i uses a path towards the source that starts with arc (j, i). From node i, each friend may use a different path, but note that such a path only visits regular nodes. Let $(N_z, c_m^{G_z})$ be the marginal cost game corresponding to G_z. By Theorem 4.4.2, for fixed z the game $(N_z, c_m^{G_z})$ is concave.

By Lemma 4.6.5, the game (N, c_m^z) defined by $c_m^z(S) = c^{G_z}(\bigcup_{i \in S} Z_i)$, $S \subseteq N$, is also concave. Finally, for every $S \subseteq N$, $c_m^{\mathcal{G}}(S) = \lim_{z \to \infty} c_m^z(S)$, which implies that $(N, c_m^{\mathcal{G}})$ is concave as well. □

Chapter 5

Cooperative situations: games and cost allocations

There's no right, there's no wrong,
there's only popular opinion.

Jeffrey Goines, *Twelve Monkeys* (1995)

5.1 Introduction

In this chapter we discuss several classes of cooperative situations. A cooperative situation typically involves a group of players that can choose from a set of alternatives, where each alternative results in a cost for the (group of) players. A cooperative situation gives rise to two main questions; which alternative should be realised and how should the costs of this alternative be divided? In this chapter we focus on the latter question. We assume that there is a general consensus that in principle total costs should be minimised and therefore, transferable utility games might help to solve this question. However, in general it is not obvious which transferable utility game best fits the situation. It is clear what the cost of the grand coalition should be; simply the total costs of the cheapest alternative. The worth of a proper subcoalition should somehow reflect its costs when its members would separate from the grand coalition and decide not to cooperate with players outside the coalition. But how will the other players (re)act? Should they be treated as if they were not there? Should the subcoalition fear the worst-case scenario in which the outsiders act as unfavorably as possible? Or can the subcoalition expect that the complementary coalition just ignores them and simply minimises their own costs?

In this chapter we present a general model that can be used as a guidance to obtain an appropriate transferable utility game for several classes of cooperative situations. This approach is based upon the idea that a cooperative situation can be represented by a corresponding order problem. An order problem consists of three elements: the player set of the underlying cooperative situation, the set of all possible orderings of the player set and an individualised cost function that describes for each ordering of the player set the corresponding cost for every player.

We discuss two types of order problems. In a positive externality order problem the minimum cost for each group of players is obtained for an ordering in which the group is "served" last. In an order problem we say that a player is "served" if it is his turn to act, *e.g.*, make a connection or choose a machine. In a negative externality order problem the minimum cost for each group of players obtained for an ordering in which they are served first. We argue that each positive externality order problem is appropriately modelled by the so called direct cost game in which the players of a coalition are served first. Furthermore, we argue that each negative externality order problem is appropriately modelled by the dual of the direct cost game, called the marginal cost game.

Hence, whenever the descriptions of a cooperative situation and the corresponding order problem are closely related, the game by which this order problem is appropriately modelled seems a good fit for the cooperative situation itself. Since the representation of a cooperative situation is a modelling decision, it may depend on one's personal view whether the cooperative situation and the corresponding order problem essentially describe the same cost allocation problem. We provide examples of several classes of cooperative situations in which we think our model is adequate. We also discuss why for some classes of cooperative situations the order problem framework does not give an appropriate transferable utility game.

Besides presenting a model that can be used to find appropriate transferable utility games for cooperative situations, we also focus on finding core elements of these transferable utility games. For this we also associate each order problem with a minimal and a maximal cost game. In the *minimal cost game* the cost of a coalition equals the cost corresponding to the cheapest ordering for the players of that coalition according to the individualised cost function of the associated order problem. The *maximal cost game* is the dual of the minimal cost game. Furthermore, we associate with each order problem a generalised Bird solution that is inspired by Bird's tree solution (Bird (1976)) for the class of minimum cost spanning tree situations (Claus and Kleitman (1973) and Bird (1976)), in the sense that each player contributes his individual cost in the optimal ordering for the grand coalition. We show that for each order problem the associated generalised Bird solution is an element of the core of the associated maximal cost game. This result is useful in several instances, *e.g.*, for providing an alternative proof of the result that the *P-value* (Branzei et al. (2004)) is an element of the core of the cost game proposed by Bird (1976) for the class of minimum cost spanning tree situations.

Moreover, it can be used to show that for each negative externality order problem the associated generalised Bird solution is in the core of the associated marginal cost game. And for each positive externality order problem that satisfies *predecessor order independence*, which means that individual costs only depend on the set of predecessors and not on their exact ordering, the associated generalised Bird solution is an element of the core of the associated direct cost game.

We also show that a cooperative situation cannot only be represented by an order problem, but also by an *alternative problem*. An alternative problem consists of three elements: the player set of the underlying cooperative situation, a set of alternatives and an individualised cost function that describes for each alternative the corresponding cost for every player. The alternative problem framework is not used to find appropriate transferable utility games for the underlying cooperative situation, but only to generalise the minimal and maximal cost game, as well as the generalised Bird solution to this framework. We show that for each alternative problem the associated generalised Bird solution is an element of the core of the associated maximal cost game. This result is useful, *e.g.*, for providing an alternative proof of the result that Bird's tree solution is in the core of the cost game proposed by Bird for the class of minimum cost spanning tree situations.

We apply our findings to several classes of cooperative situations. We first discuss the class of sequencing situations without initial order (Klijn and Sánchez (2006)). By the order problem framework we introduce an appropriate transferable utility game for such situations. Moreover, we show that the generalised Bird solution is a core element of this game. We also show that the core of our transferable utility game is a subset of the core of the two games proposed by Klijn and Sánchez (2006).

After that we consider the class of minimum cost spanning tree situations. We provide alternative proofs of the results that Bird's tree solution and the P-value are in the core of the associated cost game proposed by Bird. Moreover, the order problem framework supports the use of this game to model minimum cost spanning tree situations.

Then we introduce and discuss the class of permutation situations without initial order. Permutation situations are introduced by Tijs et al. (1984), who assume that there is an initial allocation of machines to players and that a coalition of players can interchange their machines between them. We assume on the contrary that there is no initial allocation of machinery. For this class of cooperative situations we propose a suitable transferable utility game and show that the generalised Bird solution is an element in the core of this game.

At that point we reconsider the public congestion network situations, discussed in Chapter 4. In Chapter 4 we associated with each congestion network situation a direct and a marginal cost game. To distinguish between these two games and the notions of direct and marginal cost games we discuss in this chapter, the direct and marginal cost game associated with a congestion network situation are in this chapter called the direct *congestion* cost game and the marginal *congestion* cost game. For public congestion network situations with convex cost functions we provide an alternative proof to show that the solution concept $\psi(f_N^*)$ as defined in Chapter 4 is an element of the core of the associated marginal congestion cost game. Secondly, we use the order problem framework to show that the Shapley value (Shapley (1953)) is also in the core of this cost game. Finally, it is seen that the order problem framework supports the use of the marginal congestion cost game for convex congestion network situations.

For the class of public congestion network situations with concave cost functions we argue that there is no order problem that suits this situation. Consequently, the order problem framework does not help to find an appropriate transferable utility game for this class of cooperative situations. A different type of modelling problem arises for the class of travelling salesman problems (Potters et al. (1992)), where potentially suitable order problems are shown to be neither positive nor negative externality order problems. The same problem arises even more prominently for the closely related class of *shared taxi problems*, a new class of cooperative situations introduced in this chapter.

Finally, we discuss the class of travelling repairman problems. These problems are considered by Afrati et al. (1986), but we introduce the associated cost allocation problem. In a travelling repairman problem the objective is to find a tour visiting all players such that the total waiting time of the players is minimised. We argue by the use of the order problem framework to model these situations by the associated marginal cost game of which we discuss several properties. Furthermore, we also introduce two context-specific single-valued solution concepts and discuss their properties.

The structure of this chapter is as follows. In Section 5.2 we present the order problem framework that can be used as a guidance to obtain appropriate transferable utility games for classes of cooperative situations. For this we associate with each order problem a direct and a marginal cost game. Moreover, we introduce for each order and alternative problem the generalised Bird solution and show that it is an element of the core of the associated maximal cost game, the dual of the minimal cost game. In the remainder of this chapter we apply both the order problem framework and the results with respect to the generalised Bird solution to several classes of cooperative situations, being sequencing situations without initial order (Section 5.3), minimum cost spanning tree situations (Section 5.4), permutation situations without initial allocation (Section 5.5), convex public congestion network situations (Section 5.6), concave public congestion network situations (Section 5.7), travelling salesman problems (Section 5.8), shared taxi problems (Section 5.9) and travelling repairman problems (Section 5.10).

5.2 A general model

5.2.1 Appropriate TU-games

In this section we present a model that can be used as a guidance to find appropriate TU-games for several classes of cooperative situations. Generally speaking, a cooperative situation consists of an operations research problem in which various decision makers (players) are involved in a joint cost minimisation problem. Well-known examples of cooperative situations are, *e.g.*, a travelling salesman problem or a minimum cost spanning tree situation.

Such a cooperative situation, which we in general denote by Υ, can often be represented by an order problem. An *order problem* Ω^{Υ} representing a cooperative situation Υ is given by $\Omega^{\Upsilon} = (N, \Pi, k)$, with N the finite player set (of the cooperative situation Υ), and Π the set of orderings of the player set, with $\pi : \{1, \ldots, n\} \to N$

an ordering. Further, $k : \Pi \to \mathbb{R}^N$ is an *individualised cost function*, denoting for each player $i \in N$ an individual cost $k^i(\pi)$ for each ordering $\pi \in \Pi$.

The idea is to represent Υ in such a way by Ω^Υ that the underlying costs are reflected by cost function k, while this cost function is based upon orderings of the player set. Note that an order problem Ω^Υ is a stylised representation of a cooperative situation Υ and that choosing this representation is a modelling decision.

The choice of the individualised cost function plays a fundamental role in this representation. In theory this function does not have to fulfill any requirements, but in this thesis we only define individualised cost functions for which the sum of the individualised costs associated with an ordering equals the total cost of the resulting alternative.

Furthermore, our idea is that an order problem can only adequately represent the underlying cooperative situation if the individualised cost function does not involve cooperation or reallocation of costs. By this we mean that whenever it is a player's turn to act he, first of all, cannot change anything "created" by his predecessors, and secondly, he should act like he has no followers, which implies that he must create a feasible solution for the group of players consisting of himself and his predecessors. Consequently, each player acts as an individual.

Of course it depends on the cooperative situation under consideration what the exact implication of this idea is. In the context of minimum cost spanning tree this implies, *e.g.*, that a player cannot change the tree created by his predecessors, but also that he has to connect himself to the source (possibly using the already created tree), as he cannot act on followers eventually connecting him to the source.

Recall that $\Pi_S \subseteq \Pi$ denotes the set of all orderings $\pi \in \Pi$ such that the players in $S \subseteq N$ are placed on the first $|S|$ positions. For order problem $\Omega^\Upsilon = (N, \Pi, k)$ we define the associated *direct cost game* (N, c_d^Υ) by

$$c_d^\Upsilon(S) = \min_{\pi \in \Pi_S} \sum_{i \in S} k^i(\pi) \tag{5.1}$$

for all $S \subseteq N$. Hence, in the direct cost game a coalition that is formed is "served" first. It can optimise their sequence in the first $|S|$ positions of the ordering, but cannot use the positions in the last part of the ordering. Furthermore, we define the associated *marginal cost game* (N, c_m^Υ) as the dual of the direct cost game, *i.e.*,

$$c_m^\Upsilon(S) = c_d^\Upsilon(N) - c_d^\Upsilon(N \backslash S) \tag{5.2}$$

for all $S \subseteq N$. The cost of coalition S in the marginal cost game reflects the additional or marginal cost it causes the grand coalition by its presence.

Given a cooperative situation Υ the coalitional costs of an associated TU-game should be based on what a group of players, $S \subseteq N$, can *guarantee* itself when its members would separate from the grand coalition and decide not to cooperate with players outside the coalition, $N \backslash S$. In this section we discuss two classes of order problems and argue that one class is appropriately modelled by the direct cost game, while the other is suited for the marginal cost game.

Given an order problem $\Omega^{\Upsilon} = (N, \Pi, k)$ we denote by $\pi^*_S \in \arg\min_{\pi \in \Pi} \sum_{i \in S} k^i(\pi)$ a cheapest or optimal ordering[1] for coalition $S \subseteq N$.

Definition Order problem $\Omega^{\Upsilon} = (N, \Pi, k)$ is a *negative externality order problem (neop)* if for all $S \subseteq N$ there exists a

$$\pi^*_S \in \Pi_S. \tag{5.3}$$

Order problem $\Omega^{\Upsilon} = (N, \Pi, k)$ is a *positive externality order problem (peop)* if for all $S \subseteq N$ there exists a

$$\pi^*_S \in \Pi_{N \backslash S}. \tag{5.4}$$

Hence, in a *neop* each group of players prefers to be served first, while for a *peop* each group of players wants to be served last.

Since in a *peop* each group of players prefers to be served last, the cost of coalition S is minimal for an order in which all players in $N \backslash S$ are served before all players in S. As it is reasonable to assume that a coalition formed does not benefit from the presence of other players we suggest to model any *peop* by the direct cost game, because in that game the players in S are served first.

Next we consider how to define the cost of coalition $S \subseteq N$ for a *neop*. In that case each group of players, and hence also $N \backslash S$, prefers to be served first. Therefore, the direct cost game, in which the players in S are served first, is in our opinion not

[1] Given an order problem $\Omega^{\Upsilon} = (N, \Pi, k)$ an optimal ordering π^*_S for coalition $S \subseteq N$ need not be unique. However, for expositional purposes we assume in this chapter that π^*_S is unique unless mentioned otherwise. All results are valid if optimal orderings are not unique.

suitable for such order problems, as it does not reflect what S can guarantee itself. It even leads to the minimal attainable cost for S. Hence, for a *neop* one has to take account of the possibility that the players in $N \backslash S$ are served first. However, this still leaves many options open. The most pessimistic view is that $N \backslash S$ will try to frustrate S as much as possible. However, we think it is a fair reference point to assume that $N \backslash S$ minimises its own cost. Note that this viewpoint implicitly assumes cooperation between the players in $N \backslash S$, but since full cooperation is to be expected in the first place, this is not an unreasonable assumption in order to determine the cost of coalition S. The notion that the objective of $N \backslash S$ is to minimise its cost further leads to the idea that the players in S can alter the ordering preferred by $N \backslash S$ as long as S compensates $N \backslash S$ for additional costs incurred due to the deviation from their optimal ordering. Based on this insight coalition S has to decide on an ordering that minimises the sum of the cost of its own players plus the compensation for the players in $N \backslash S$. This idea boils down to the marginal cost game.

Note that a TU-game is used as a reference point for dividing the worth of the grand coalition among the players of the game. Therefore, we claim that the explicit use of transferable utility in the definition of a TU-game itself is justified.

5.2.2 Core elements

Order problems

For the purpose of finding core elements of the appropriate TU-games of a cooperative situation we introduce two additional TU-games. For order problem $\Omega^\Upsilon = (N, \Pi, k)$ we define the associated *minimal cost game* (N, c_-^Υ) by

$$c_-^\Upsilon(S) \;=\; \min_{\pi \in \Pi} \sum_{i \in S} k^i(\pi) \tag{5.5}$$

for all $S \subseteq N$. Hence, for each coalition, $S \subseteq N$, the minimal cost game denotes the sum of the individual cost of the players in S in the cheapest ordering $\pi \in \Pi$ for S. Therefore, $c_-^\Upsilon(S) = \sum_{i \in S} k^i(\pi_S^*)$ for all $S \subseteq N$. For order problem $\Omega^\Upsilon = (N, \Pi, k)$ the associated *maximal cost game* (N, c_+^Υ) is defined as the dual of the minimal cost game, *i.e.*,

$$c_+^\Upsilon(S) \;=\; c_-^\Upsilon(N) - c_-^\Upsilon(N \backslash S) \tag{5.6}$$

for all $S \subseteq N$.

Remark 5.2.1 *Let* $\Omega^\Upsilon = (N, \Pi, k)$ *be a* neop. *Then the associated direct cost game* (N, c_d^Υ) *and minimal cost game* (N, c_-^Υ) *coincide. As a consequence, also the associated marginal cost game* (N, c_m^Υ) *and maximal cost game* (N, c_+^Υ) *coincide.*

We define the *generalised Bird solution* β associated with order problem Ω^Υ such that each player contributes his individual cost according to the optimal alternative for the grand coalition. Hence, this solution is defined by

$$\beta^i \quad = \quad k^i(\pi_N^*) \tag{5.7}$$

for all $i \in N$. The solution β is called the generalised Bird solution, because it is a generalisation of the Bird solution (Bird (1976)) as defined for minimal cost spanning tree situations. These situations are extensively discussed in Section 5.4.

Theorem 5.2.2 Let $\Omega^\Upsilon = (N, \Pi, k)$ be an order problem with associated maximal cost game (N, c_+^Υ). Then $\beta \in C(N, c_+^\Upsilon)$.

Proof: By definition β is efficient. Furthermore,

$$\begin{aligned} c_+^\Upsilon(S) &= c_-^\Upsilon(N) - c_-^\Upsilon(N \backslash S) \\ &= \sum_{i \in N} k^i(\pi_N^*) - \sum_{i \in N \backslash S} k^i(\pi_{N\backslash S}^*) \\ &\geq \sum_{i \in N} k^i(\pi_N^*) - \sum_{i \in N \backslash S} k^i(\pi_N^*) \\ &= \sum_{i \in S} k^i(\pi_N^*) \\ &= \sum_{i \in S} \beta^i. \end{aligned}$$

□

By Remark 5.2.1, we immediately obtain the next result.

Corollary 5.2.3 *Let* $\Omega^\Upsilon = (N, \Pi, k)$ *be a* neop *with associated marginal cost game* (N, c_m^Υ). *Then* $\beta \in C(N, c_m^\Upsilon)$.

Hence, for each *neop* the associated generalised Bird solution is an element of the core of the associated marginal cost game. Since we propose to model *neops* by the marginal cost game, β is a reasonable single-valued solution concept for such order problems. A similar result, placing the generalised Bird solution in the core of the direct cost game associated with a *peop*, requires an additional condition. This condition, which is called predecessor order independence, boils down to the idea that the *ordering* of the predecessors of each player is irrelevant for his individual cost. Let $\Omega^\Upsilon = (N, \Pi, k)$ be an order problem and let $V^i(\pi) = \{j \in N \,|\, \pi^{-1}(j) < \pi^{-1}(i)\}$ denote the set of predecessors of player i given ordering $\pi \in \Pi$. Then $\Omega^\Upsilon = (N, \Pi, k)$ satisfies *predecessor order independence* (*poi*) if, for all $i \in N$,

$$k^i(\pi) \;=\; k^i(\bar{\pi}) \tag{5.8}$$

for all $\pi, \bar{\pi} \in \Pi$ such that $V^i(\pi) = V^i(\bar{\pi})$.[2]

Theorem 5.2.4 Let $\Omega^\Upsilon = (N, \Pi, k)$ be a *peop* that satisfies *poi* and let (N, c_d^Υ) be the associated direct cost game. Then $\beta \in C(N, c_d^\Upsilon)$.

Proof: From the definition it follows that β is efficient. Let $S \subseteq N$ and let $\hat{\pi}_S^* = \arg\min_{\pi \in \Pi_S} \sum_{i \in S} k^i(\pi)$. We define $\tilde{\pi}_N$ such that $\tilde{\pi}_N(t) = \hat{\pi}_S^*(t)$ for all $t \in \{1, \dots, |S|\}$ and $\tilde{\pi}_N(t) = \pi^*_{N\setminus S}(t)$ for all $t \in \{|S|+1, \dots, n\}$. Note that since $\Omega^\Upsilon = (N, \Pi, k)$ is a *peop* that satisfies *poi* such an ordering exists. Then,

$$\begin{aligned}
c_d^\Upsilon(S) &= \sum_{i \in S} k^i(\hat{\pi}_S^*) \\
&\geq \sum_{i \in S} k^i(\hat{\pi}_S^*) + \sum_{i \in N\setminus S} k^i(\pi^*_{N\setminus S}) - \sum_{i \in N\setminus S} k^i(\pi_N^*) \\
&= \sum_{i \in N} k^i(\tilde{\pi}_N) - \sum_{i \in N\setminus S} k^i(\pi_N^*) \\
&\geq \sum_{i \in N} k^i(\pi_N^*) - \sum_{i \in N\setminus S} k^i(\pi_N^*) \\
&= \sum_{i \in S} k^i(\pi_N^*) \\
&= \sum_{i \in S} \beta^i.
\end{aligned}$$

□

[2] Note that predecessor order independence implies that the cost of a player is also not influenced by the order of his *followers*. Although this is no formal requirement of an individualised cost function in this thesis we only use individualised cost functions for which this is satisfied. Therefore, the focus of this property is on the independence of the individualised costs on the ordering of the predecessors of a player.

Alternative problems

Let Υ be a cooperative situation. An *alternative problem* Λ^Υ representing cooperative situation Υ is given by $\Lambda^\Upsilon = (N, \mathcal{A}, k)$, with N the finite player set (of the underlying cooperative situation), $\mathcal{A}$ a finite set of alternatives and $k : \mathcal{A} \to \mathbb{R}^N$ an individualised cost function denoting for each alternative $\alpha \in \mathcal{A}$ a corresponding cost $k^i(\alpha)$ for every player $i \in N$. Note that with $\mathcal{A} = \Pi$ we return to the order problem framework. The main difference between these two types of problems is that for the order problem framework an alternative is the *result* of a sequence of individual choices, while in the alternative problem framework an alternative is immediately *chosen*. The definitions of a *neop* and *peop*, as well as the definition of the direct and marginal cost game are based upon orderings of the player set and can therefore not be used in this framework. Consequently, we do not use this framework to find appropriate TU-games. However, the minimal and maximal cost game, as well as the generalised Bird solution are straightforwardly generalised to the alternative problem framework, which implies that this framework can be used to find core elements of (these) TU-games.

For alternative problem $\Lambda^\Upsilon = (N, \mathcal{A}, k)$ we define the associated *minimal cost game* (N, c_-^Υ) by

$$c_-^\Upsilon(S) = \min_{\alpha \in \mathcal{A}} \sum_{i \in S} k^i(\alpha) \tag{5.9}$$

for all $S \subseteq N$. The associated *maximal cost game* (N, c_+^Υ) is defined as the dual of the minimal cost game, *i.e.*,

$$c_+^\Upsilon(S) = c_-^\Upsilon(N) - c_-^\Upsilon(N \backslash S) \tag{5.10}$$

for all $S \subseteq N$. By $\alpha_S^* \in \arg\min_{\alpha \in \mathcal{A}} \sum_{i \in S} k^i(\pi)$ we denote a cheapest or optimal alternative[3] for coalition $S \subseteq N$. Hence, the generalised Bird solution is defined by

$$\beta^i = k^i(\alpha_N^*) \tag{5.11}$$

for all $i \in N$.

[3]Given an alternative problem $\Lambda^\Upsilon = (N, \mathcal{A}, k)$ an optimal alternative α_S^* for coalition $S \subseteq N$ need not be unique. However, for expositional purposes we assume in this chapter that α_S^* is unique unless mentioned otherwise. All results are valid if optimal alternatives are not unique.

Theorem 5.2.5 Let $\Lambda^{\Upsilon} = (N, \mathcal{A}, k)$ be an alternative problem with associated maximal cost game (N, c_+^{Υ}). Then $\beta \in C(N, c_+^{\Upsilon})$.

Proof: By definition β is efficient. Furthermore,

$$\begin{aligned}
c_+^{\Upsilon}(S) &= c_-^{\Upsilon}(N) - c_-^{\Upsilon}(N\backslash S) \\
&= \sum_{i\in N} k^i(\alpha_N^*) - \sum_{i\in N\backslash S} k^i(\alpha_{N\backslash S}^*) \\
&\geq \sum_{i\in N} k^i(\alpha_N^*) - \sum_{i\in N\backslash S} k^i(\alpha_N^*) \\
&= \sum_{i\in S} k^i(\alpha_N^*) \\
&= \sum_{i\in S} \beta^i.
\end{aligned}$$

□

5.3 Sequencing situations without initial order

The first class of cooperative situations we consider in this chapter is the class of sequencing situations without initial order. Sequencing situations are, *e.g.*, discussed in Smith (1956). The cost allocation problem associated with sequencing situations (with initial order) has been introduced by Curiel et al. (1989), but the cost allocation problem of the current class is first discussed in a more recent paper by Klijn and Sánchez (2006). We provide a natural way to represent a sequencing situation without initial order by a corresponding order problem, which is in our opinion a good representation of the underlying class of cooperative situations. We show that this order problem is a *neop* and use this result to argue to model this class of cooperative situations by the marginal cost game associated with the order problem. Furthermore, we observe by the use of Corollary 5.2.3 that the generalised Bird solution associated with the order problem is an element of the core of this game, and compare the marginal cost game with the cost games proposed by Klijn and Sánchez (2006).

A *sequencing situation without initial order* is given by a triple[4] $Q = (N, p, \delta)$,

[4] Klijn and Sánchez (2006) use the notation (N, p, α).

with N the finite set of players. Each player $i \in N$ owns one job that has to be processed on a single machine. The job of player i is with slight abuse of notation also denoted by i. The processing times of the jobs are given by $p = (p^i)_{i \in N}$ with $p^i > 0$ for all $i \in N$. Furthermore, each player has a cost function $c^i : [0, \infty) \to \mathbb{R}$ given by $c^i(t) = \delta^i t$, $t \in [0, \infty)$, where $\delta^i > 0$. The expression $c^i(t)$ is interpreted as the cost incurred by agent i if his job is completed at time t. The optimal ordering of jobs for the grand coalition is obtained by putting them in non-decreasing order of their urgency indices (Smith (1956)), which are defined as $u^i = \frac{\delta^i}{p^i}$ for all $i \in N$.

Next we represent a sequencing situation without initial order by a corresponding order problem. Under the assumption that only active schedules are considered, meaning that the jobs are processed without any breaks in between, the cost of player i is completely determined by an ordering of the player set $\pi \in \Pi$. Therefore, the natural cost function c can alternatively be given by $c^i(\pi) = \delta^i \sum_{j \in V^i(\pi) \cup \{i\}} p^j$ for all $i \in N$ and all $\pi \in \Pi$. Consequently, a sequencing situation without initial order $Q = (N, p, \delta)$ can be represented by order problem $\Omega^Q = (N, \Pi, c)$, with N the finite player set, Π the set of orderings of the player set and $c : \Pi \to \mathbb{R}^N$, with $c^i(\pi) = \delta^i \sum_{j \in V^i(\pi) \cup \{i\}} p^j$ for all $i \in N$ and all $\pi \in \Pi$. Note that all information contained in cooperative situation $Q = (N, p, \delta)$ is also contained in order problem $\Omega^Q = (N, \Pi, c)$, and that they describe the same cost allocation problem. The direct (N, c_d^Q) and marginal cost game (N, c_m^Q), as well generalised Bird solution β are straightforwardly given by (5.1), (5.2) and (5.7), respectively. Since each coalition has the lowest cost when served first we obtain the following proposition.

Proposition 5.3.1 *Order problem* $\Omega^Q = (N, \Pi, c)$ *is a* neop.

Hence, order problem $\Omega^Q = (N, \Pi, c)$ is appropriately modelled by the associated marginal cost game (N, c_m^Q). And since this order problem suits a sequencing situations without initial order $Q = (N, p, \delta)$, we suggest to model this class of cooperative situations by this marginal cost game. Since Ω^Q is a *neop* (Proposition 5.3.1) the following result follows by Corollary 5.2.3.

Proposition 5.3.2 *Let* $\Omega^Q = (N, \Pi, c)$ *be the order problem corresponding to sequencing situation without initial order* $Q = (N, p, \delta)$ *and let* (N, c_m^Q) *be the associated marginal cost game. Then* $\beta \in C(N, c_m^Q)$.

Next we compare the marginal cost game to the cost games proposed by Klijn and Sánchez (2006) to model sequencing situations without initial order. In the tail game (N, c^Q_{tail}) it is assumed that the players in $N\backslash S$ are served first and then the players in S can optimise their sequence in the tail of the ordering. This game is, given our formulation, defined by

$$c^Q_{tail}(S) = \min_{\pi \in \Pi_{N\backslash S}} \sum_{i \in S} c^i(\pi)$$

for all $S \subseteq N$.

Proposition 5.3.3 *Let $Q = (N, p, \delta)$ be a sequencing situation without initial order, with corresponding tail game (N, c^Q_{tail}). Let $\Omega^Q = (N, \Pi, c)$ be the corresponding order problem with associated marginal cost game (N, c^Q_m). Then $C(N, c^Q_m) \subseteq C(N, c^Q_{tail})$.*

Proof: Since a coalition $S \subseteq N$ has the option to also use the first part of the sequence in the marginal cost game, it follows that $c^Q_m(S) \leq c^Q_{tail}(S)$ for all $S \subseteq N$, with equality for $S = N$, which shows that $C(N, c^Q_m) \subseteq C(N, c^Q_{tail})$. $\square$

Klijn and Sánchez (2006) also introduce a pessimistic game (N, c^Q_{pess}) of which they show in Proposition 3.3 of their paper that $C(N, c^Q_{tail}) \subseteq C(N, c^Q_{pess})$ for each sequencing situation without initial order. Therefore, the core of the marginal cost game is also a subset of the core of the pessimistic game. This implies that any allocation stable under the marginal cost game approach is also stable under the two approaches of Klijn and Sánchez (2006). Moreover, since the generalised Bird solution is an element of the core of the marginal cost game (Proposition 5.3.2), such a stable allocation exists.

5.4 Minimum cost spanning tree situations

In this section we consider minimum cost spanning tree, or *mcst*, situations (Claus and Kleitman (1973) and Bird (1976)). By the use of the order and alternative problem formulation and Theorems 5.2.2, 5.2.4 and 5.2.5 we provide alternative proofs of the results that the Bird solution and the P-value are in the core of the cost game proposed by Bird. Furthermore, the order problem formulation supports the use of this game for *mcst* situations.

Formally, a *minimum cost spanning tree situation* is a triple[5] $M = (N, 0, \gamma)$, with N the finite player set, 0 the source, and $\gamma : E_{N^0} \to \mathbb{R}_+$ a non-negative cost function specifying the cost to construct edge $e \in E_{N^0}$. An edge is also denoted by (j, ℓ), with $j, \ell \in N^0$. Bird (1976) associates with each *mcst* situation $M = (N, 0, \gamma)$ cooperative cost game (N, c^M), where $c^M(S)$, $S \subseteq N$, represents the minimal cost of a tree on $S \cup \{0\}$:

$$c^M(S) = \min\{\sum_{e \in R} \gamma(e) \mid R \subseteq E_{S\cup\{0\}} \text{ and } (S \cup \{0\}, R) \text{ is a tree}\}$$

for all $S \subseteq N$.

Let the minimum cost spanning tree for the grand coalition N be given by $(N \cup \{0\}, R^*)$ and let e^i, for all $i \in N$, be the first edge on the unique path in $(N \cup \{0\}, R^*)$ from player i to the source. Then *Bird's tree solution*[6] $\dot{\beta}$ is obtained by assigning to each player $i \in N$ the cost of e^i, hence

$$\dot{\beta}^i \quad = \quad \gamma(e^i)$$

for all $i \in N$.

5.4.1 Alternative problem

Since orderings of the player set are not explicitly or implicitly included in the description of an *mcst* situation it is not straightforward to represent an *mcst* situation by an order problem. More natural is it to represent *mcst* situation $M = (N, 0, \gamma)$ by alternative problem $\Lambda^M = (N, \mathcal{A}, k)$, with N the finite player set and $\mathcal{A}$ the set of all spanning trees on N^0. Further, we define the individualised cost function $k : \mathcal{A} \to \mathbb{R}^N$ by $k^i(\alpha) = \gamma(e^i(\alpha))$, where $e^i(\alpha)$ denotes the first edge on the unique path in spanning tree $\alpha \in \mathcal{A}$ from player i to the source.

The minimal (N, c_-^M) and maximal cost game (N, c_+^M), as well as the generalised Bird solution β are straightforwardly given by (5.9), (5.10) and (5.11), respectively. A careful inspection leads to the conclusion that the coalitional costs for $S \subseteq N$ according to the minimal cost game equal the additional costs for coalition S to build a tree on N^0, given that the players in $N \backslash S$ constructed a tree on $N^0 \backslash S$.

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[5] Usually, the notation $(N, 0, t)$ is used.

[6] We denote this solution by $\dot{\beta}$ instead of β to distinguish between this solution concept and the generalised Bird solution defined in Section 5.2.

Therefore (N, c_-^M) is equivalent to the 'optimistic TU-game' of Bergantiños and Vidal-Puga (2007).

Lemma 5.4.1 *Let* $M = (N, 0, \gamma)$ *be an* mcst *situation with corresponding alternative problem* $\Lambda^M = (N, \mathcal{A}, k)$*. Then* $\dot{\beta} = \beta$*.*

Proof: Let α_N^* be a minimum cost spanning tree. Then

$$\begin{aligned} \dot{\beta}^i &= \gamma^i(e^i(\alpha_N^*)) \\ &= k^i(\alpha_N^*) \\ &= \beta^i \end{aligned}$$

for all $i \in N$. □

Note that both β and $\dot{\beta}$ depend on the minimum cost spanning tree under consideration.

Example 5.4.2 Consider the *mcst* situation of Figure 5.4.1.

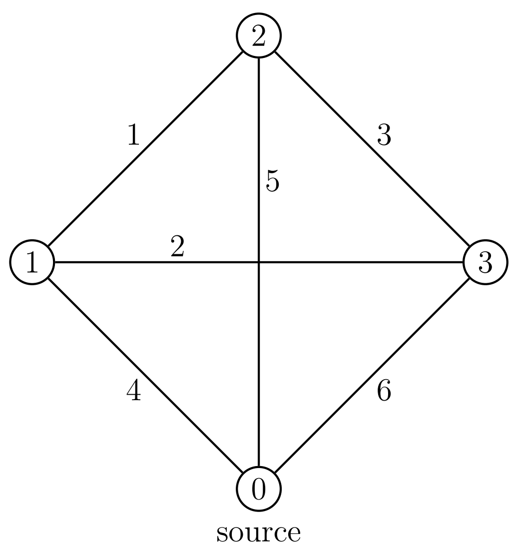


Figure 5.4.1: A minimal cost spanning tree situation

The coalitional costs for the games (N, c^M), (N, c_-^M) and (N, c_+^M) are given in the table below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c^M(S)$	4	5	6	5	6	8	7
$c_-^M(S)$	1	1	2	3	3	3	7
$c_+^M(S)$	4	4	4	5	6	6	7

Note once more that $c^M(S)$ denotes the cost to form a tree on $S \cup \{0\}$. Hence, $c^M(\{1,2\}) = 5$, because coalition $\{1,2\}$ can use the edges $(1,0)$ and $(2,1)$ to form a tree on $\{0,1,2\}$. The cost for coalition S according to the minimal cost game reflects the cost to build a tree on N^0, given that there is a tree on $N^0 \backslash S$. Hence, coalition $\{1,2\}$ only has a cost of 3 to connect itself to the nodes of the set $\{0,3\}$, as players 1 and 2 can use the edges $(2,1)$ and $(1,3)$ for this. The individual costs for the corresponding tree $\alpha \in \mathcal{A}$ are given by $k(\alpha) = (2,1,6)$.

The minimum cost spanning tree, $\alpha_N^* \in \mathcal{A}$, consists of the edges $(1,0)$, $(2,1)$ and $(3,1)$, which leads to $\beta = (4,1,2)$. $\triangleleft$

Lemma 5.4.3 *Let $M = (N, 0, \gamma)$ be an* mcst *situation with associated cost game (N, c^M). Let $\Lambda^M = (N, \mathcal{A}, k)$ be the corresponding alternative problem with associated maximal cost game (N, c_+^M). Then $C(N, c_+^M) \subseteq C(N, c^M)$.*

Proof: Since $\sum_{i \in N} k^i(\alpha_N^*)$ is the cost of the minimum cost spanning tree it is clear that $c^M(N) = c_+^M(N)$. Hence, it remains to be shown that $c_+^M(S) \leq c^M(S)$ for all $S \subseteq N$. Let $S \subseteq N$. Then

$$\begin{aligned}
c_+^M(S) &= c_-^M(N) - c_-^M(N \backslash S) \\
&\leq \min_{\alpha \in \mathcal{A}} \{ \sum_{i \in N} k^i(\alpha) \,|\, (S \cup \{0\}, \bigcup_{i \in S} e^i(\alpha)) \text{ is a tree} \} - c_-^M(N \backslash S) \\
&= c^M(S) + c_-^M(N \backslash S) - c_-^M(N \backslash S) \\
&= c^M(S),
\end{aligned}$$

where the second equality follows from the fact that $\min_{\alpha \in \mathcal{A}} \{ \sum_{i \in N} k^i(\alpha) \,|\, (S \cup \{0\}, \bigcup_{i \in S} e^i(\alpha)) \text{ is a tree}\}$ equals the minimum cost to build a tree on $S \cup \{0\}$ $(c^M(S))$ plus the minimum cost to build a tree on N^0 given that there is a tree on $S \cup \{0\}$ $(c_-^M(N \backslash S))$. $\square$

We are now ready to provide an alternative proof of the result that Bird's tree solution is in the core of the game (N, c^M). The original result is due to Bird (1976).

Theorem 5.4.4 Let $M = (N, 0, \gamma)$ be an *mcst* situation with associated cost game (N, c^M). Then $\dot{\beta} \in C(N, c^M)$.

Proof: Let $\Lambda^M = (N, \mathcal{A}, k)$ be the alternative problem corresponding to $M = (N, 0, \gamma)$. By Lemma 5.4.1, $\dot{\beta} = \beta$. Further, by Theorem 5.2.5, $\dot{\beta} \in C(N, c_+^M)$. Finally, by Lemma 5.4.3, $\dot{\beta} \in C(N, c^M)$. □

5.4.2 Order problem 1

Since the cost function of an alternative problem is not based upon orderings of the player set conditions (5.3) and (5.4) cannot be used to obtain an appropriate TU-game to model alternative problem Λ^M, and therefore also no suggestion can be given for the use of a TU-game for the underlying *mcst* situation. For this we require an order problem formulation. Since orderings are not part of the underlying cooperative situation an algorithm based upon such orderings is necessary for this representation.

Let $M = (N, 0, \gamma)$ be an *mcst* situation. We represent $M = (N, 0, \gamma)$ by order problem $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$, with N the finite player set, Π the set of orderings of N and $\tilde{k} : \Pi \to \mathbb{R}^N$ such that it represents the minimal cost for a player to connect his node to the source (possibly via other nodes) given the tree constructed by his predecessors. Formally, $\tilde{k}^i(\pi) = \min_{(i,j) \in E_{N^0}} \left\{ \gamma(i,j) \,\middle|\, j \in \{V^i(\pi) \cup \{0\}\} \right\}$ for all $i \in N$ and $\pi \in \Pi$. Note that this procedure is based upon *Prim's algorithm* (Prim (1957)) to obtain a minimum cost spanning tree.

The associated direct $(N, \tilde{c}_d^M)$, minimal $(N, \tilde{c}_-^M)$ and maximal cost game $(N, \tilde{c}_+^M)$, as well as the generalised Bird solution $\tilde{\beta}$ are given by (5.1), (5.5), (5.6) and (5.7), respectively.

Lemma 5.4.5 *Let $M = (N, 0, \gamma)$ be an* mcst *situation with corresponding order problem $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$. Then $\dot{\beta} = \tilde{\beta}$.*

Proof: Let $\Lambda^M = (N, \mathcal{A}, k)$ be the alternative problem corresponding to $M = (N, 0, \gamma)$.

Then

$$\begin{aligned}\beta^i &= k^i(\alpha^*_N)\\ &= \tilde{k}^i(\pi^*_N)\\ &= \tilde{\beta}^i\end{aligned}$$

for all $i \in N$. Hence, $\beta = \tilde{\beta}$. By Lemma 5.4.1, $\dot{\beta} = \beta$, which completes the proof. $\square$

Lemma 5.4.6 *Order problem* $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$ *satisfies* poi.

Proof: Let $i \in N$ and $\pi \in \Pi$. Since $\tilde{k}^i(\pi)$ is only based upon the set of predecessors of player i, $V^i(\pi)$, and not on their ordering, $\tilde{k}$ satisfies equality (5.8). Hence, $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$ satisfies *poi*. $\square$

Proposition 5.4.7 *Order problem* $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$ *is a* peop.

Proof: Let $i \in N$. Since $\tilde{k}^i(\pi) = \min_{(i,j)\in E_{N0}} \left\{\gamma(i,j) \,\middle|\, j \in \{V^i(\pi) \cup \{0\}\right\}$ for all $\pi \in \Pi$ it follows that the later player i is served, the more choice he has and therefore, the lower his individual cost. Combined with Lemma 5.4.6 gives that condition (5.4) is satisfied, which implies that $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$ is a *peop*. $\square$

In our opinion order problem $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$ gives an appropriate description of *mcst* situation $M = (N, 0, \gamma)$. In particular, individualised cost function $\tilde{k}$ is a fair representation of individual costs. Hence, since this order problem is a *peop* we suggest to model *mcst* situations by the associated direct cost game $(N, \tilde{c}^M_d)$. This game coincides with the TU-game (N, c^M) proposed by Bird.

Proposition 5.4.8 *Let* $M = (N, 0, \gamma)$ *be an* mcst *situation with associated cost game* (N, c^M)*. Let* $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$ *be the corresponding order problem with associated direct cost game* $(N, \tilde{c}^M_d)$*. Then* $(N, c^M) = (N, \tilde{c}^M_d)$*.*

Proof: Let $S \subseteq N$. The coalitional cost for S according to (N, c^M) denotes the cost of the cheapest tree on $S \cup \{0\}$. Consider Prim's algorithm to obtain such a minimum cost spanning tree on $S \cup \{0\}$ and let π be the ordering in which the players make their connection according to this algorithm. Then $c^M(S) = \sum_{i \in S} \tilde{k}^i(\pi) = \tilde{c}_d^M(S)$. $\square$

Proposition 5.4.8, combined with the fact that $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$ is a *peop*, supports the use of TU-game (N, c^M) to model *mcst* situations. Next we provide a second alternative proof of the result that Bird's tree solution $\dot{\beta}$ is in the core of this game. The following theorem is originally due to Bird (1976).

Theorem 5.4.9 Let $M = (N, 0, \gamma)$ be an *mcst* situation with associated cost game (N, c^M). Then $\dot{\beta} \in C(N, c^M)$.

Proof: Let $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$ be the order problem corresponding to *mcst* situation $M = (N, 0, \gamma)$, with associated direct cost game $(N, \tilde{c}_d^M)$ and generalised Bird solution $\tilde{\beta}$. Since $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$ is a *peop* (Proposition 5.4.7) and $\tilde{\Omega}^M = (N, \Pi, \tilde{k})$ satisfies *poi* (Lemma 5.4.6) we obtain by Theorem 5.2.4 that $\tilde{\beta} \in C(N, \tilde{c}_d^M)$. By Lemma 5.4.5, $\dot{\beta} \in C(N, \tilde{c}_d^M)$. Finally, by Proposition 5.4.8, $\dot{\beta} \in C(N, c^M)$. $\square$

5.4.3 Order problem 2

The representation of *mcst* situation $M = (N, 0, \gamma)$ by an order problem can be done in yet another way. By using a different underlying algorithm based upon orderings of the player set we obtain a different order problem. We illustrate this by representing *mcst* situation $M = (N, 0, \gamma)$ by order problem $\bar{\Omega}^M = (N, \Pi, \bar{k})$, with N the finite player set and Π the set of orderings of N. A component of player $i \in N$ is defined as the maximal set of players in N^0, which includes i, that forms a tree. The individualised cost function $\bar{k} : \Pi \to \mathbb{R}^N$ is based upon the following algorithm. Let $\pi \in \Pi$. At his turn in π player i has to connect his component in the cheapest way to another component. The component of player i given that the first t players of ordering π made a connection is denoted by $T_t^i(\pi)$. Hence, the individualised cost function $\bar{k} : \Pi \to \mathbb{R}^N$ is formally defined by $\bar{k}^i(\pi) = \min_{(j,\ell) \in E_{N^0}} \left\{ \gamma(j, \ell) \middle| \, |T^i_{\pi^{-1}(i)}(\pi)| > |T^i_{\pi^{-1}(i)-1}(\pi)| \right\}$ for all $i \in N$ and all $\pi \in \Pi$. Since a player might depend on his followers to get connectd to the source this order problem is not an adequate representation of an *mcst* situation. However, we use it to give an alternative proof of the result that the P-value is in the core of the game (N, c^M).

The associated direct $(N, \bar{c}_d^M)$, marginal $(N, \bar{c}_m^M)$, minimal $(N, \bar{c}_-^M)$ and maximal cost game $(N, \bar{c}_+^M)$, as well as the generalised Bird solution $\bar{\beta}$ are given by (5.1), (5.2), (5.5), (5.6) and (5.7), respectively.

Example 5.4.10 We illustrate the described procedure by the *mcst* situation of Figure 5.4.1. For $\pi = (1,2,3)$ player 1 starts by connecting $T_0^1(\pi)$ with $T_0^2(\pi)$. After that, player 2 connects $T_1^2(\pi)$ with $T_1^3(\pi)$, and finally, player 3 uses edge $(1,0)$ to connect $T_2^3(\pi)$ with the source. The corresponding costs are given by $\bar{k}(\pi) = (1,2,4)$. Note that $\bar{k}(\pi)$ is efficient, which is a general result we obtain in Proposition 5.4.11. The table below provides all the coalitional costs according to the associated direct and marginal cost game.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$\bar{c}_d^M(S)$	1	1	2	3	3	3	7
$\bar{c}_m^M(S)$	4	4	4	5	6	6	7

The cost for coalition $\{1,2\}$ in the direct cost game is determined by ordering $\pi = (1,2,3)$ and is therefore equal to $\bar{k}^1(\pi) + \bar{k}^2(\pi) = 3$. $\triangleleft$

The procedure described above to define cost function $\bar{k}$ is equal to the *V-algorithm* of Çiftçi and Tijs (2007). In Theorem 3.1 of that paper it is shown that for each ordering $\pi \in \Pi$ this algorithm leads to a minimum cost spanning tree. We restate this theorem.

Proposition 5.4.11 *Let $M = (N, 0, \gamma)$ be an* mcst *situation with associated cost game (N, c^M). Let $\bar{\Omega}^M = (N, \Pi, \bar{k})$ be the corresponding order problem. Then $\sum_{i \in N} \bar{k}^i(\pi) = c^M(N)$ for all $\pi \in \Pi$.*

Since each ordering $\pi \in \Pi$ results in an optimal allocation a more interesting solution concept than $\bar{\beta}$ itself would therefore be to take the average over all these allocations. We define $\bar{\beta}^*$ such that $\bar{\beta}^* = \frac{1}{n!} \sum_{\pi \in \Pi} \bar{k}(\pi)$. It follows immediately that $\bar{\beta}^*$ equals the *V-value*, V, of Çiftçi and Tijs (2007).

Proposition 5.4.12 *Let $M = (N, 0, \gamma)$ be an* mcst *situation with corresponding order problem $\bar{\Omega}^M = (N, \Pi, \bar{k})$. Then $\bar{\beta}^* = V$.*

Çiftçi and Tijs (2007) show in Theorem 4.1 of their paper that the V-value coincides with the P-value (Branzei et al. (2004)), which in its turn equals the *ERO-value* of Feltkamp et al. (1994). The P-value is known to be an element of the core of (N, c^M). In the remainder of this section we provide an alternative proof of this result, which makes use of the framework of Section 5.2.

Lemma 5.4.13 *Order problem* $\bar{\Omega}^M = (N, \Pi, \bar{k})$ *satisfies* poi.

Proof: Let $\hat{\pi} \in \Pi$ with $\hat{\pi}^{-1}(j) = t$, $\hat{\pi}^{-1}(\ell) = t+1$ and $\hat{\pi}^{-1}(i) \geq t+2$ and define $\bar{\pi}$ such that $\bar{\pi}^{-1}(r) = \hat{\pi}^{-1}(r)$ for all $r \in N \backslash \{j, \ell\}$, $\bar{\pi}^{-1}(j) = \hat{\pi}^{-1}(\ell)$ and $\bar{\pi}^{-1}(\ell) = \hat{\pi}^{-1}(j)$. We show that $T^r_{t+1}(\hat{\pi}) = T^r_{t+1}(\bar{\pi})$ for all $r \in N$, which completes the proof.

Note first of all that $|T^r_t(\pi)| = |\{s \in T^r_t(\pi) : \pi^{-1}(s) \leq t\}| + 1$ for all $r \in N$. Therefore, $T^j_{t-1}(\bar{\pi}) = T^j_{t-1}(\hat{\pi}) \neq T^\ell_{t-1}(\hat{\pi}) = T^\ell_{t-1}(\bar{\pi})$.

We first assume that in $\hat{\pi}$ player j does not connect $T^j_{t-1}(\hat{\pi})$ with $T^\ell_{t-1}(\hat{\pi})$. This implies that player ℓ's choices according to $\bar{\pi}$ are the same as under $\hat{\pi}$. Consequently, player ℓ will do the same as under $\hat{\pi}$. The choices of player j according $\bar{\pi}$ are then the same as under $\hat{\pi}$ with the possible exception of connecting his component to that of player ℓ, but since this was not his choice under $\hat{\pi}$ this has no affect. Consequently, also player j makes the same choice under $\hat{\pi}$ as under $\bar{\pi}$, which implies that $T^r_{t+1}(\hat{\pi}) = T^r_{t+1}(\bar{\pi})$ for all $r \in N$.

Secondly we assume that in $\hat{\pi}$ player j connects $T^j_{t-1}(\hat{\pi})$ with $T^\ell_{t-1}(\hat{\pi})$. This implies that player ℓ's choices according to $\bar{\pi}$ are extended with the possibility to connect $T^j_{t-1}(\bar{\pi})$ with $T^\ell_{t-1}(\bar{\pi})$, but no longer contain connecting $T^j_{t-1}(\bar{\pi})$ to a component unequal to $T^\ell_{t-1}(\pi)$. If player ℓ connects $T^j_{t-1}(\bar{\pi})$ with $T^\ell_{t-1}(\bar{\pi})$ then player j's choices under $\hat{\pi}$ are the same as player ℓ's choices under $\bar{\pi}$, *i.e.*, $T^j_t(\hat{\pi}) = T^\ell_t(\hat{\pi}) = T^j_t(\bar{\pi}) = T^\ell_t(\bar{\pi})$, which implies that player j will do under $\bar{\pi}$ exactly what player ℓ does under $\hat{\pi}$. As a consequence, $T^r_{t+1}(\hat{\pi}) = T^r_{t+1}(\bar{\pi})$ for all $r \in N$.

If player ℓ does not connect $T^j_{t-1}(\bar{\pi})$ with $T^\ell_{t-1}(\bar{\pi})$ then he makes the same choice under $\bar{\pi}$ as under $\hat{\pi}$, because connecting $T^j_{t-1}(\bar{\pi})$ with $T^\ell_{t-1}(\bar{\pi})$ is cheaper than any other connection from $T^j_{t-1}(\bar{\pi})$. (This follows from the fact that player j connects $T^j_{t-1}(\hat{\pi})$ with $T^\ell_{t-1}(\hat{\pi})$). As player ℓ makes the same choice under $\bar{\pi}$ as under $\hat{\pi}$ player j

connects $T_t^j(\bar{\pi})$ with $T_t^\ell(\bar{\pi})$, which leads to the result that $T_{t+1}^r(\hat{\pi}) = T_{t+1}^r(\bar{\pi})$ for all $r \in N$. $\square$

Lemma 5.4.14 *Order problem $\bar{\Omega}^M = (N, \Pi, \bar{k})$ is a* neop.

Proof: Let $\hat{\pi} \in \Pi$ with $\hat{\pi}^{-1}(j) = t$, $\hat{\pi}^{-1}(\ell) = t+1$ and define $\bar{\pi}$ such that $\bar{\pi}^{-1}(r) = \hat{\pi}^{-1}(r)$ for all $r \in N \backslash \{j, \ell\}$, $\bar{\pi}^{-1}(j) = \hat{\pi}^{-1}(\ell)$ and $\bar{\pi}^{-1}(\ell) = \hat{\pi}^{-1}(j)$. We show that $\bar{k}^\ell(\bar{\pi}) \leq \bar{k}^\ell(\hat{\pi})$, which completes the proof.

Note first of all that $|T_t^r(\pi)| = |\{s \in T_t^r(\pi) : \pi^{-1}(s) \leq t\}| + 1$ for all $r \in N$. Therefore, $T_{t-1}^j(\bar{\pi}) = T_{t-1}^j(\hat{\pi}) \neq T_{t-1}^\ell(\hat{\pi}) = T_{t-1}^\ell(\bar{\pi})$.

We first assume that in $\hat{\pi}$ player j does not connect $T_{t-1}^j(\hat{\pi})$ with $T_{t-1}^\ell(\hat{\pi})$. This implies that player ℓ's choices according to $\bar{\pi}$ are the same as under $\hat{\pi}$. Consequently, $\bar{k}^\ell(\bar{\pi}) = \bar{k}^\ell(\hat{\pi})$.

Secondly we assume that in $\hat{\pi}$ player j connects $T_{t-1}^j(\hat{\pi})$ with $T_{t-1}^\ell(\hat{\pi})$. This implies that player ℓ's choices according to $\bar{\pi}$ are extended with the possibility to connect $T_{t-1}^j(\bar{\pi})$ with $T_{t-1}^\ell(\bar{\pi})$, but no longer contain connecting $T_{t-1}^j(\bar{\pi})$ to a component unequal to $T_{t-1}^\ell(\pi)$. However, since player j chose to connect $T_{t-1}^j(\hat{\pi})$ with $T_{t-1}^\ell(\hat{\pi})$ the latter options are worse than connecting $T_{t-1}^j(\bar{\pi})$ with $T_{t-1}^\ell(\bar{\pi})$ and hence, $\bar{k}^\ell(\bar{\pi}) \leq \bar{k}^\ell(\hat{\pi})$. $\square$

Proposition 5.4.15 *Let $\Lambda^M = (N, \mathcal{A}, k)$ and $\bar{\Omega}^M = (N, \Pi, \bar{k})$ be the alternative and order problem corresponding to* mcst *situation $M = (N, 0, \gamma)$ with associated minimal cost games (N, c_-^M) and $(N, \bar{c}_-^M)$. Then $(N, c_-^M) = (N, \bar{c}_-^M)$*

Proof: Let $S \subseteq N$ and let $(N, \bar{c}_d^M)$ be the direct cost game associated with order problem $\bar{\Omega}^M = (N, \Pi, \bar{k})$. By Lemma 5.4.14 and Remark 5.2.1, $\bar{c}_-^M(S) = \bar{c}_d^M(S) = \sum_{i \in S} \bar{k}^i(\pi)$ for some $\pi \in \Pi_S$. By this ordering π we connect in each stage $t \in \{0, \ldots, |S|-1\}$ the component of player $\pi(t) \in S$ to another component in the cheapest way possible. Note that only players of coalition S are allowed to connect components, and therefore $|T_{|S|}^i(\pi)| = |T_{|S|}^i(\pi) \cap S| + 1$ for all $i \in S$. Consequently, after player $\pi(|S|)$ has made his connection we obtain the cheapest network consisting of $|S|$ edges out of E_{N^0} such that each component $T_{|S|}^i(\pi)$, with $i \in S$, is connected to the set $N^0 \backslash S$ with a *single* edge. Hence, this equals the cheapest way to build a tree on N^0, given a tree on $N^0 \backslash S$, which corresponds to the cost $c_-^M(S)$. □

We are now ready to provide the alternative proof of the result that the P-value, P, is an element of the core of the TU-game (N, c^M). The original result is due to Branzei et al. (2004).

Theorem 5.4.16 Let $M = (N, 0, \gamma)$ be an *mcst* situation. Then $P \in C(N, c^M)$.

Proof: Let $\Lambda^M = (N, \mathcal{A}, k)$ and $\bar{\Omega}^M = (N, \Pi, \bar{k})$ be the alternative and order problem corresponding to *mcst* situation $M = (N, 0, \gamma)$ with associated maximal cost games (N, c_+^M) and $(N, \bar{c}_+^M)$.

By Theorem 5.2.2, $\bar{\beta}^* \in C(N, \bar{c}_+^M)$. Then by Proposition 5.4.15, $\bar{\beta}^* \in C(N, c_+^M)$ and by Lemma 5.4.3, $\bar{\beta}^* \in C(N, c^M)$. Finally, by the equivalence of $\bar{\beta}^*$ to the V-value (Proposition 5.4.12) and the equivalence of the V-value to the P-value (Theorem 4.1 in Çiftçi and Tijs (2007)) we obtain the desired result. □

5.5 Permutation situations without initial allocation

In this section we introduce the class of permutation situations without initial allocation. Permutation situations are introduced by Tijs et al. (1984). A *permutation situation without initial allocation* is given by a triple $P = (N, \Theta, \Gamma)$, with N the finite set of players, $\Theta = \{1, \ldots, n\}$ the set of machines and Γ an $n \times n$ cost matrix. Element Γ_{ij} of this matrix denotes the cost for the use of machine $j \in \Theta$ by player $i \in N$. The objective is to allocate each machine to a different player with minimal total cost. This problem can be solved by the *Hungarian method* (Kuhn (1955)). In the paper by Tijs et al. (1984) it is assumed that there is an initial allocation of machines to players and that a coalition of players can interchange their machines between them. On the contrary we assume here that there is no initial allocation.

We represent a permutation situation without initial allocation $P = (N, \Theta, \Gamma)$ by order problem $\Omega^P = (N, \Pi, k)$, with N the finite player set and Π the set of orderings of N. We define the individualised cost function $k : \Pi \to \mathbb{R}^N$ such that

$k^i(\pi)$ denotes the cost of the cheapest possible machine player i can choose at his turn in π. Hence, let $\Theta_t(\pi)$ denote the set of available machines given that the first t players of π have made their choice. Then $k^i(\pi) = \min_{j \in \Theta_{\pi^{-1}(i)-1}(\pi)} \Gamma_{ij}$ for all $i \in N$ and $\pi \in \Pi$. Since there are more options for any coalition that is served first it follows that order problem $\Omega^P = (N, \Pi, k)$ is a *neop*

Proposition 5.5.1 *Order problem* $\Omega^P = (N, \Pi, k)$ *is a* neop.

Based on the fact that order problem $\Omega^P = (N, \Pi, k)$ adequately represents a permutation situation without initial order and is a *neop* we suggest to model this class of cooperative situations by the marginal cost game (N, c_m^P), which is defined following (5.1) and (5.2). The associated generalised Bird solution is defined by (5.7).

Proposition 5.5.2 *Let* $P = (N, \Theta, \Gamma)$ *be a permutation situation without initial allocation. Let* (N, c_m^P) *be the marginal cost game associated with the corresponding order problem* $\Omega^P = (N, \Pi, k)$. *Then* $\beta \in C(N, c_m^P)$.

Proof: By Proposition 5.5.1, $\Omega^\Upsilon = (N, \Pi, k)$ is a *neop*. Hence, by Corollary 5.2.3, $\beta \in C(N, c_m^P)$ □

5.6 Convex public congestion network situations

In this section we consider public congestion network situations with convex cost functions, as discussed in Chapter 4. We focus on indivisible networks. In the remainder of this section we call these situations, just as in (the main part of) Chapter 4, convex congestion network situations. In Chapter 4 we associate with each congestion network situation a direct and a marginal cost game. To distinguish between these two games and the games defined in Section 5.2 of this chapter, the direct and marginal cost game associated with a convex congestion network situation G are in this chapter called the direct congestion cost game, denoted by (N, c_{dG}^G), and the marginal congestion cost game, denoted by (N, c_{mG}^G). Note that the idea behind the direct and marginal cost game and the direct congestion and marginal congestion cost game is similar, and also the argumentation for the use of one or the other is closely related. However, the translation of the underlying idea into a TU-game is different. In the direct congestion cost game it is assumed that a coalition $S \subseteq N$ constructs a network while the non-members are absent, while in the direct cost game $N \backslash S$ is present, but served after coalition S.

In this section we provide by the order and alternative problem framework and Corollary 5.2.3 and Theorem 5.2.5 an (alternative) proof of the results that cost allocation $\psi(f_N^*)$, defined in Chapter 4, and the Shapley value of the marginal congestion cost game are in the core of the marginal congestion cost game associated with a convex congestion network situation. Furthermore, the order problem framework supports the use of this game to model convex congestion network situations.

5.6.1 Alternative problem

Since orderings of the player set are not explicitly or implicitly included in the description of a convex congestion network situation, representing such a situation by an order problem is not straightforward. More natural is it to represent a convex congestion network situation $G = (N, 0, \gamma)$ by an alternative problem $\Lambda^G = (N, \mathcal{A}, k)$, with N the finite player set, $\mathcal{A}$ the set of all feasible networks on N^0 and individualised cost function $k : \mathcal{A} \rightarrow N$ such that $k^i(\alpha) = \psi^i(\alpha)$ for all $i \in N$ given feasible network $\alpha \in \mathcal{A}$. Note that in Chapter 4 we only define cost allocation ψ for an *optimal* network. However, we could equivalently define this allocation for any feasible network, with $\psi(f_N^*) = \psi(\alpha_N^*)$ the originally defined allocation.

The associated minimal (N, c_-^G) and maximal cost game (N, c_+^G), as well as the generalised Bird solution β are straightforwardly given by (5.9), (5.10) and (5.11), respectively.

Lemma 5.6.1 *Let $G = (N, 0, \gamma)$ be a convex congestion network situation with corresponding alternative problem $\Lambda^G = (N, \mathcal{A}, k)$. Then $\beta = \psi(f_N^*)$.*

Lemma 5.6.2 *Let $G = (N, 0, \gamma)$ be a convex congestion network situation with associated direct congestion cost game (N, c_{dG}^G). Let $\Lambda^G = (N, \mathcal{A}, k)$ be the corresponding alternative problem with associated minimal cost game (N, c_-^G). Then $(N, c_{dG}^G) = (N, c_-^G)$.*

Proof: Let $S \subseteq N$. The coalitional cost for S according to the direct congestion cost game corresponds to the cost of the cheapest feasible network for S in absence of the players in $N \backslash S$. This network is denoted by f^*_S. Since we only consider complete, publicly available networks there always exists a feasible network for N, denoted by f_N, such that $f_N(a) = f^*_S(a)$ for all $a \in A_{N^0}$ with $f^*_S(a) > 0$. Due to the convexity of the cost function γ such an alternative f_N is optimal for S, hence $f_N = \alpha^*_S$. Then, since $c^G_-(S) = \sum_{i \in S} k^i(\alpha^*_S)$ the direct congestion cost game and minimal cost game coincide. $\square$

We now provide an alternative proof of Theorem 4.5.5 of Chapter 4.

Proposition 5.6.3 *Let $G = (N, 0, \gamma)$ be a convex congestion network situation with associated marginal congestion cost game (N, c^G_{mG}). Then $\psi(f^*_N) \in C(N, c^G_{mG})$.*

Proof: Let (N, c^G_+) be the maximal cost game associated with alternative problem $\Lambda^G = (N, \mathcal{A}, k)$ corresponding to convex congestion network situation $G = (N, 0, \gamma)$. By Theorem 5.2.5, $\beta \in C(N, c^G_+)$. By Lemma 5.6.2, $\beta \in C(N, c^G_{mG})$. Finally, by Lemma 5.6.1 $\psi(f^*_N) \in C(N, c^G_{mG})$. $\square$

5.6.2 Order problem 1

In order to represent a convex congestion network situation by an order problem we need an algorithm based upon orderings of the player set. In Chapter 4 such an algorithm is provided, Algorithm 4.3.3. Let $G = (N, 0, \gamma)$ be a convex congestion network situation. We represent $G = (N, 0, \gamma)$ by order problem $\bar{\Omega}^G = (N, \Pi, \bar{k})$, with N the finite player set and Π the set of orderings of N. The individualised cost function $\bar{k} : \Pi \to \mathbb{R}^N$ is given by $\bar{k}^i(\pi) = \sum_{a \in P^*_{V^i(\pi),i}} \ell_{f^*_{V^i(\pi)}}(a)$, which means that $\bar{k}^i(\pi)$ denotes the cost induced by player i according to Algorithm 4.3.3 given an ordering $\pi \in \Pi$. Since this algorithm involves cooperation with predecessors in the sense that links can be redirected, this order problem is not an adequate representation of a convex congestion network situation.

The associated direct $(N, \bar{c}^G_d)$, marginal $(N, \bar{c}^G_d)$, minimal $(N, \bar{c}^G_-)$ and maximal cost game $(N, \bar{c}^G_+)$, as well as the generalised Bird solution $\bar{\beta}$ are straightforwardly given by (5.1), (5.2), (5.5), (5.6) and (5.7), respectively.

By Theorem 4.3.4 we obtain that $\bar{k}$ is efficient for each ordering $\pi \in \Pi$. Therefore, we define, equivalent to the concept used for the class of *mcst* situations (see Section 5.4), $\bar{\beta}^* = \frac{1}{n!} \sum_{\pi \in \Pi} \bar{k}(\pi)$ as the average over all possible outcomes $\bar{\beta}$.

Example 5.6.4 Reconsider the convex congestion network situation of Figure 4.2.1. The next table gives for each ordering $\pi \in \Pi$ the corresponding cost allocation $\bar{k}(\pi) = \bar{\beta}$.

π	$\bar{k}^1(\pi)$	$\bar{k}^2(\pi)$	$\bar{k}^3(\pi)$
$(1,2,3)$	1	4	5
$(1,3,2)$	1	6	3
$(2,1,3)$	2	3	5
$(2,3,1)$	4	3	3
$(3,1,2)$	2	6	2
$(3,2,1)$	4	4	2

This results in cost allocation $\bar{\beta}^* = (2\frac{1}{3}, 4\frac{1}{3}, 3\frac{1}{3})$. ◁

From the definition of cost function k we immediately obtain the next lemma.

Lemma 5.6.5 *Let $G = (N, 0, \gamma)$ be a convex congestion network situation with associated direct congestion cost game (N, c^G_{dG}). Let $\bar{\Omega}^G = (N, \Pi, \bar{k})$ be the corresponding order problem with associated direct cost game $(N, \bar{c}^G_d)$. Then $(N, c^G_{dG}) = (N, \bar{c}^G_d)$.*

The Shapley value of a cost game (N, c) is denoted by $\Phi(N, c)$.

Proposition 5.6.6 *Let $G = (N, 0, \gamma)$ be a convex congestion network situation with associated marginal congestion cost game (N, c^G_{mG}). Let $\bar{\Omega}^G = (N, \Pi, \bar{k})$ be the corresponding order problem. Then $\bar{\beta}^* = \Phi(N, c^G_{mG})$.*

Proof: Note that Algorithm 4.3.3 provides an optimal network for any coalition $\{\pi(1), \ldots, \pi(t)\}$, with $t \in \{1, \ldots, n\}$ given $\pi \in \Pi$. Therefore, each $\bar{k}(\pi)$ corresponds to a marginal vector of the direct cost game $(N, \bar{c}^G_d)$. Hence, $\bar{\beta}^* = \Phi(N, \bar{c}^G_d)$ and by Lemma 5.6.5, $\bar{\beta}^* = \Phi(N, c^G_{dG})$. Since the Shapley value of a game is equal to the Shapley value of its dual game, $\bar{\beta}^* = \Phi(N, c^G_{mG})$. □

Proposition 5.6.7 *Order problem $\bar{\Omega}^G = (N, \Pi, \bar{k})$ is a* neop.

Proof: It follows from (the proof of) Theorem 4.4.2 in Chapter 4 that condition (5.3) is satisfied. Consequently, $\bar{\Omega}^G = (N, \Pi, \bar{k})$ is a *neop*. □

Since the marginal congestion cost game is concave (Theorem 4.4.2) the Shapley value of this game is an element of its core. However, by the order problem formulation we also obtain an alternative way to prove this result.

Proposition 5.6.8 *Let $G = (N, 0, \gamma)$ be a convex congestion network situation with associated marginal congestion cost game (N, c^G_{mG}). Then $\Phi(N, c^G_{mG}) \in C(N, c^G_{mG})$.*

Proof: Let $\bar{\Omega}^G = (N, \Pi, \bar{k})$ be the corresponding order problem. Since $\bar{\Omega}^G = (N, \Pi, \bar{k})$ is a *neop* (Proposition 5.6.7) we obtain by Corollary 5.2.3 that $\bar{\beta}^* \in C(N, c^G_{mG})$. Hence, by Proposition 5.6.6, $\Phi(N, c^G_{mG}) \in C(N, c^G_{mG})$. □

5.6.3 Order problem 2

In the previous two subsections we used the alternative and order problem formulation to show that $\psi(f^*_N)$ and $\Phi(N, c^G_{mg})$ are elements of the core of the marginal congestion cost game associated with a convex congestion network situation. In this subsection we use the order problem framework to support the use of the marginal congestion cost game to model convex congestion network situations.

We argue that the individualised cost functions k, associated with alternative problem $\Lambda^G = (N\mathcal{A}, k)$, and $\bar{k}$, associated with order problem $\bar{\Omega}^G = (N, \Pi, \bar{k})$, cannot be seen as representative or fair individualised cost functions for convex congestion network situations, as both involve cooperation and reallocation of costs. As a result the cost allocation problem of a convex congestion network situation G is not appropriately described by either alternative problem Λ^G or order problem $\bar{\Omega}^G$.

Therefore, we now consider a third approach. Let $G = (N, 0, \gamma)$ be a convex congestion network situation. We represent $G = (N, 0, \gamma)$ by order problem $\tilde{\Omega}^G = (N, \Pi, \tilde{k})$, with N the finite player set and Π the set of orderings of N. We denote by $\mathcal{P}^i$ the set of all paths from player i to the source. Let f^t be a feasible network constructed by players $\pi(1), \ldots, \pi(t)$ given $\pi \in \Pi$. We define individual costs by $\tilde{k}^i(\pi) = \min_{P^i \in \mathcal{P}^i} \sum_{a \in P^i} \gamma_a(f^{\pi^{-1}(i)}(a)) - \gamma_a(f^{\pi^{-1}(i)-1}(a))$ for all $i \in N$ and all $\pi \in \Pi$, *i.e.*, the individual cost denotes the minimum additional cost of player i when he enters the situation given the network constructed by his predecessors. In particular, player i is not allowed to alter paths constructed by his predecessors, like in the procedure of Algorithm 4.3.3. This means that the order in which the players enter the situation determines the final network constructed and consequently, not all orderings lead to an optimal network.

The associated direct $(N, \tilde{c}_d^G)$, marginal $(N, \tilde{c}_m^G)$, minimal $(N, \tilde{c}_-^G)$ and maximal cost game $(N, \tilde{c}_+^G)$, as well as the generalised Bird solution $\tilde{\beta}$ are straightforwardly given by (5.1), (5.2), (5.5), (5.6) and (5.7), respectively.

Example 5.6.9 Reconsider the convex congestion network situation of Figure 4.2.1. The next table[7] gives for each ordering $\pi \in \Pi$ the corresponding cost allocation $\tilde{k}(\pi)$.

π	$\tilde{k}^1(\pi)$	$\tilde{k}^2(\pi)$	$\tilde{k}^3(\pi)$
$(1,2,3)$	1	4	5
$(1,3,2)$	1	6	3
$(2,1,3)$	4	3	3
$(2,3,1)$	4	3	3
$(3,1,2)$	4	4	2
$(3,2,1)$	4	4	2

Note that the cost allocations $\tilde{k}(\pi)$ differ from the cost allocations $\bar{k}(\pi)$ of Example 5.6.4. In this example $\tilde{k}(\pi)$ is efficient for all $\pi \in \Pi$, *i.e.*, each ordering leads to an optimal network. This is, however, generally not the case. Consider, *e.g.*, the situation without player 2, which means that only the arcs between the players 1 and 3 and source are present and $N = \{1,3\}$. In that case ordering $\pi = (3,1)$ leads to a network f with $f(1,0) = 2$ and $f(3,1) = 1$, with a cost of 6, which is not optimal. ◁

Convexity of the congestion network situation implies the following proposition.

Proposition 5.6.10 *Order problem* $\tilde{\Omega}^G = (N, \Pi, \tilde{k})$ *is a* neop.

Given the discussion in Section 5.2 order problem $\tilde{\Omega}^G = (N, \Pi, \tilde{k})$ is appropriately modelled by the associated marginal cost game. Furthermore, we think of $\tilde{\Omega}^G$ as an adequate representation of a convex congestion network situation. We show in Proposition 5.6.11 that the marginal cost game $(N, \tilde{c}_m^G)$ coincides with the marginal congestion cost game (N, c_{mG}^G), used in Chapter 4 to model these situations.

[7] The cost allocations for the orderings $(2,1,3)$ and $(3,1,2)$ are not uniquely defined in this situation. Depending on the choice of player 1 these cost allocations can also be given by $(4,3,6)$ and $(4,6,2)$, respectively.

Proposition 5.6.11 *Let $G = (N, 0, \gamma)$ be a convex public congestion network situation with associated marginal congestion cost game (N, c^G_{mG}). Let $\tilde{\Omega}^G = (N, \Pi, \tilde{k})$ be the corresponding order problem with associated marginal cost game $(N, \tilde{c}^G_m)$. Then $(N, c^G_{mG}) = (N, \tilde{c}^G_m)$.*

Proof: The direct congestion cost game associated with $G = (N, 0, \gamma)$ is given by (N, c^G_{dG}). The direct cost game associated with $\tilde{\Omega}^G = (N, \Pi, \tilde{k})$ is given by $(N, \tilde{c}^G_d)$. We show that $(N, c^G_{dG}) = (N, \tilde{c}^G_d)$, which completes the proof.

Let $S \subseteq N$. It is clear that $\tilde{c}^G_d(S) \geq c^G_{dG}(S)$. Hence, it suffices to prove that there exists a $\pi \in \Pi_S$ such that $\sum_{i \in S} \tilde{k}^i(\pi) = c^G_{dG}(S)$. Let f^*_S be an optimal network obtained by Algorithm 4.3.3. Initially, let j be the last added player. Then $f^*_S = f^*_{S\backslash\{j\}} \oplus f_{P^*_{S\backslash\{j\},j}}$.

Define player $\ell \in S$ such that:

- $f^*_S(r, s) > f^*_{S\backslash\{j\}}(r, s)$ for all $r, s \in N^0$ such that $\ell \prec_{Q^*_{S\backslash\{j\},j}} s$ and $(r, s) \in P^*_{S\backslash\{j\},j}$,
- $f^*_S(b, \ell) \not> f^*_{S\backslash\{j\}}(b, \ell)$ for some $b \in Q^*_{S\backslash\{j\},j}$.

Note that such a player ℓ exists and that it could be player j.

Define $P'_\ell = ((i_0, i_1), (i_1, i_2), \ldots, (i_{p-1}, i_p))$, with $i_0 = \ell$, $i_p = 0$ and $(i_r, i_{r+1}) \in P^*_{S\backslash\{j\},j}$ for all $r \in \{0, \ldots, p-1\}$. Hence, P'_ℓ is the last part of the path $P^*_{S\backslash\{j\},j}$ starting from ℓ. Then $f^*_S = f^*_{S\backslash\{\ell\}} \oplus f_{P'_\ell}$ and $f^*_S(a) > f^*_{S\backslash\{\ell\}}(a)$ for all $a \in P'_\ell$. Consequently, $P'_\ell = P^*_{S\backslash\{\ell\},\ell}$. Let $\pi(|S|) = \ell$.

Proceed with $P^*_{S\backslash\{\ell\},\ell}$ to find the player on position $\pi(|S| - 1)$. If $\ell \neq j$ proceed as above. If $\ell = j$, then consider as the new player j the player added before player j according to Algorithm 4.3.3. In the end we obtain the complete ordering $\pi \in \Pi_S$ which gives $\tilde{c}^G_d(S) = c^G_{dG}(S)$. □

Note that this proof is constructive in the sense that it shows for each $S \subseteq N$ how to obtain an ordering $\pi \in \Pi_S$ such that $\sum_{i \in S} \tilde{k}^i(\pi) = \tilde{c}^G_d(S)$.

Proposition 5.6.12 *Let $G = (N, 0, \gamma)$ be a convex public congestion network situation with associated marginal congestion cost game (N, c^G_{mG}). Then $\tilde{\beta} \in C(N, c^G_{mG})$.*

Proof: Let $\tilde{\Omega}^G = (N, \Pi, \tilde{k})$ be the corresponding order problem with associated marginal cost game $(N, \tilde{c}_m^G)$. By Proposition 5.6.11, $(N, c_{mG}^G) = (N, \tilde{c}_m^G)$. Hence, since $\tilde{\Omega} = (N, \Pi, \tilde{k})$ is a *neop* (Proposition 5.6.10), by Corollary 5.2.3, $\tilde{\beta} \in C(N, c_{mG}^G)$. □

5.7 Concave public congestion network situations

In this section we consider public congestion network situations with concave cost functions, as briefly discussed in Chapter 4. We focus on indivisible networks. In this section we call these situations, just as in Chapter 4, concave congestion network situations. In Chapter 4 we associate with each congestion network situation a direct and a marginal cost game. To distinguish between these two games and the games defined in Section 5.2 of this chapter, the direct and marginal cost game associated with a concave[8] congestion network situation G' are in this chapter called the direct congestion cost game, denoted by $(N, c_{dG'}^{G'})$, and the marginal congestion cost game, denoted by $(N, c_{mG'}^{G'})$. In this section we illustrate that the order problem framework cannot always be used to obtain an appropriate TU-game for a class of cooperative situations.

Let $G' = (N, 0, \gamma)$ be a concave congestion network situation. We represent G' by order problem $\Omega^{G'} = (N, \Pi, k)$, with N the finite player set, Π the set of orderings of N, and $k : \Pi \to \mathbb{R}^N$ defined by $k^i(\pi) = \min_{P^i \in \mathcal{P}^i} \sum_{a \in P^i} \gamma_a(f^{\pi^{-1}(i)}(a)) - \gamma_a(f^{\pi^{-1}(i)-1}(a))$ for all $i \in N$ and all $\pi \in \Pi$. Consequently, the individual costs in this setup denote the additional cost of player i when he enters the situation given the network constructed by his predecessors. Note that order problem $\Omega^{G'} = (N, \Pi, k)$ is similar to the order problem $\tilde{\Omega}^G = (N, \Pi, \tilde{k})$ for the class of *convex* congestion network situations. By concavity of the congestion network situation we obtain the next proposition.

Proposition 5.7.1 *Order problem $\Omega^{G'} = (N, \Pi, k)$ is a* peop*.*

The representation of concave congestion network situation $G' = (N, 0, \gamma)$ by order problem $\Omega^{G'} = (N, \Pi, k)$ seems straightforward and since k does not require coope-

[8] We denote concave congestion network situations by G' to distinguish between concave and convex congestion network situations that are denoted by G.

ration or the reallocation of costs one might think of k as a fair individualised cost function. However, k is not efficient, in the sense that for some concave congestion network situations there exists no ordering $\pi \in \Pi$ such that $\sum_{i \in N} k^i(\pi) = c_{dG'}^{G'}(N)$. This is illustrated in the next example.

Example 5.7.2 Consider the symmetric concave congestion network situation $G' = (N, 0, \gamma)$ of Figure 5.7.1. The optimal network f_N^* is given by $f_N^*(1,2) = 1$ and

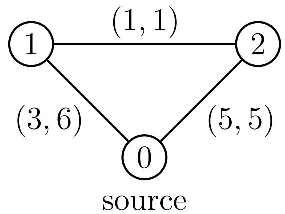


Figure 5.7.1: A concave congestion network situation

$f_N^*(2,0) = 2$, with a cost of 6. However, both orderings $(1,2)$ and $(2,1)$ result in network f_N given by $f_N(2,1) = 1$ and $f_N(1,0) = 2$, with a cost of 7. $\triangleleft$

Since cost function k is not efficient, concave congestion network situation $G' = (N, 0, \gamma)$ and corresponding order problem $\Omega^{G'} = (N, \Pi, k)$ do not describe the same cost allocation problem. Hence, although a similar representation results for convex congestion network situations in an appropriate order problem, the representation of $G' = (N, 0, \gamma)$ by $\Omega^{G'} = (N, \Pi, k)$ does not help to find a reasonable TU-game for the class of concave congestion network situations. Moreover, since any other order problem would involve an individualised cost function that would require cooperation and/or reallocation of costs, the order problem framework does not seem to fit the class of concave congestion network situations at all.

5.8 Travelling salesman problems

In this section we consider the well-known class of travelling salesman problems (Potters et al. (1992)). In a travelling salesman problem, or *tsp*, a single salesman has to visit a group of players. Formally, a *travelling salesman problem* can be given by a triple[9] $L = (N, 0, \gamma)$, with N the finite set of players that has to be visited by

[9]Potters et al. (1992) use N^0 to indicate the player set including home. Furthermore, they denote the $(n+1) \times (n+1)$ cost matrix by K.

the salesman. He starts at node 0, called *home*. The function $\gamma : E_{N^0} \to \mathbb{R}_+$ is a non-negative cost function, which can be viewed as the travel time of the salesman from one node to another. The objective is to find the shortest tour starting and ending in the node of the salesman in such a way that all players are visited. It is well-known that this problem is NP-hard. In one of the TU-games introduced by Potters et al. (1992) to analyse the associated cost allocation problem the cost of a coalition $S \subseteq N$ is defined as the cost of the optimal tour on the nodes of the players in S and 0, hence the cost of the optimal tour in absence of the players in $N \backslash S$. This game, denoted by (N, c^L), has become the standard for modelling *tsp*s. Note that Tamir (1989) shows that the core of this game can be empty if $n \geq 6$.

Consider the *tsp* of Figure 5.8.1 with three players, denoted by 1, 2 and 3, and home, given by 0. The numbers on the edges denote the travelling time for the salesman from one node to another. The optimal tour is given by either $(2, 3, 1)$ or

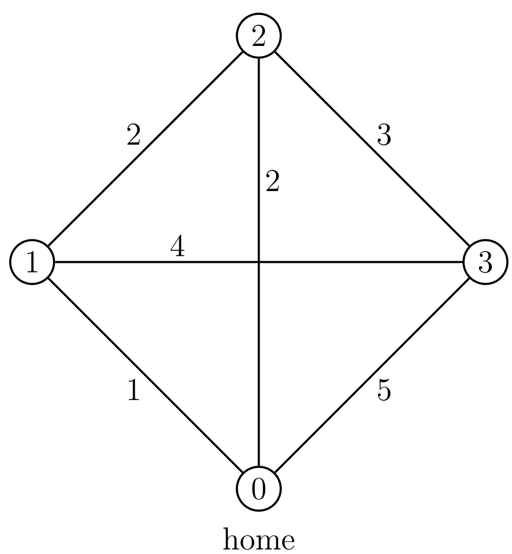


Figure 5.8.1: A travelling salesman problem

$(1, 3, 2)$ leading to a cost of 10 for the salesman.

We mainly use this particular *tsp* to illustrate the problem of representing a *tsp* $L = (N, 0, \gamma)$ by an appropriate order problem $\Omega^L = (N, \Pi, k)$. Since the salesman has to make a tour visiting all players, orderings of the player set are implicitly included in the description of a *tsp*. The definition of an individualised cost function is however far from obvious, because the costs of a tour are essentially incurred by the salesman and not by the players. Furthermore, the cost of $n + 1$ edges must be divided over n players.

Consider, *e.g.*, the tour that visits the players in order $(1, 2, 3)$. We could argue that each player should pay the cost of the edge used to travel to him, which would lead to player 1 contributing 1, player 2 contributing 2 and player 3 contributing 3. In that case, however, the cost of the edge used by the salesman to return home is not paid for by anyone. Options for appointing this cost are to transfer it either to the final player or to divide it equally among all players. Note that the latter requires the reallocation of costs and is therefore not suitable as an individualised cost function of an adequate order problem. Hence, we discard this option from the outset.

We consider the first option instead. We represent a *tsp* $L = (N, 0, \gamma)$ by order problem $\Omega^L = (N, \Pi, k)$, with N the finite player set, Π the set of orderings of N and $k^i(\pi) = \gamma(\pi^{-1}(i) - 1, i)$ for all $i \neq \pi(n)$ and $k^i(\pi) = \gamma(\pi^{-1}(i) - 1, i) + \gamma(i, 0)$ for $i = \pi(n)$. The next example illustrates that this order problem is neither a *neop* nor a *peop*.

Example 5.8.1 The individual costs for a given ordering according order problem $\Omega^L = (N, \Pi, k)$ corresponding to the *tsp* of Figure 5.8.1 are given in the next table.

π	$k^1(\pi)$	$k^2(\pi)$	$k^3(\pi)$
$(1, 2, 3)$	1	2	8
$(1, 3, 2)$	1	5	4
$(2, 1, 3)$	2	2	9
$(2, 3, 1)$	5	2	3
$(3, 1, 2)$	4	4	5
$(3, 2, 1)$	3	3	5

The costs for ordering $(1, 2, 3)$ result from the fact that both players 1 and 2 pay for the cost of the edge used by the salesman to reach them, while player 3 pays this cost plus the cost of the salesman's return home. Since $\pi^*_{\{3\}} = (2, 3, 1)$ we obtain that $\Omega^L = (N, \Pi, k)$ is neither a *peop* nor a *neop*. $\triangleleft$

Another reasonable possibility to define an individualised cost function given a tour is to let each player pay for the additional costs incurred by him given the actions of his predecessors in establishing a subtour. In that case tour $(1, 2, 3)$ leads to a cost for player 1 of 2, because if there are no other players present, then the salesman should only visit him, but return home as well. The additional cost player 2 incurs equals 5 (the cost of the tour $(1,2)$) minus 2 (the cost of a tour on player 1), which makes 3. Finally, player 3 contributes 6. This cost function requires neither cooperation nor reallocation of costs and is efficient. Formally, we can represent *tsp* $L = (N, 0, \gamma)$ by $\bar{\Omega}^L = (N, \Pi, \bar{k})$, with N the finite player set and Π the set of orderings of N. The individualised cost function $\bar{k} : \Pi \to \mathbb{R}^N$ is given by $\bar{k}^i(\pi) = \gamma(\pi^{-1}(i)-1, i) + \gamma(i, 0) - \gamma(\pi^{-1}(i)-1, 0)$ for all $i \in N$ and all $\pi \in \Pi$. The associated direct cost game $(N, \bar{c}_d^L)$ is given by (5.1).

Proposition 5.8.2 *Let* $L = (N, 0, \gamma)$ *be a* tsp *with associated cost game* (N, c^L). *Let* $\bar{\Omega}^L = (N, \Pi, \bar{k})$ *be the corresponding order problem with associated direct cost game* $(N, \bar{c}_d^L)$. *Then* $(N, c^L) = (N, \bar{c}_d^L)$.

Proof: This result follows since $\bar{c}_d^L(S)$ denotes for all $S \subseteq N$ the cost of an optimal tour on $S \cup \{0\}$. □

The next two examples show that order problem $\bar{\Omega}^L = (N, \Pi, \bar{k})$ is neither a *neop* nor a *peop*.

Example 5.8.3 The individual costs according to $\bar{k}$ for a given ordering according to order problem $\bar{\Omega}^L = (N, \Pi, \bar{k})$ corresponding to the *tsp* $L = (N, 0, \gamma)$ of Figure 5.8.1 are given in the table below.

π	$\bar{k}^1(\pi)$	$\bar{k}^2(\pi)$	$\bar{k}^3(\pi)$
$(1,2,3)$	2	3	6
$(1,3,2)$	2	0	8
$(2,1,3)$	1	4	6
$(2,3,1)$	0	4	6
$(3,1,2)$	0	1	10
$(3,2,1)$	1	0	10

Since $\pi^*_{\{1\}} = (2,3,1)$ or $\pi^*_{\{1\}} = (3,1,2)$ it follows that $\bar{\Omega}^L = (N, \Pi, \bar{k})$ is not a *neop*. Since it is (due to the triangle inequality) never optimal to be the first player and costs only depend on a player's direct predecessor we need a four-player example to show that $\bar{\Omega}^L = (N, \Pi, \bar{k})$ is a *peop* neither. ◁

Example 5.8.4 Consider the four-player *tsp* of Figure 5.8.2. One finds that $\pi^*_{\{1,3\}} = (2,1,4,3)$, with $k^1(2,1,4,3) + k^3(2,1,4,3) = 0$, while in any ordering in which

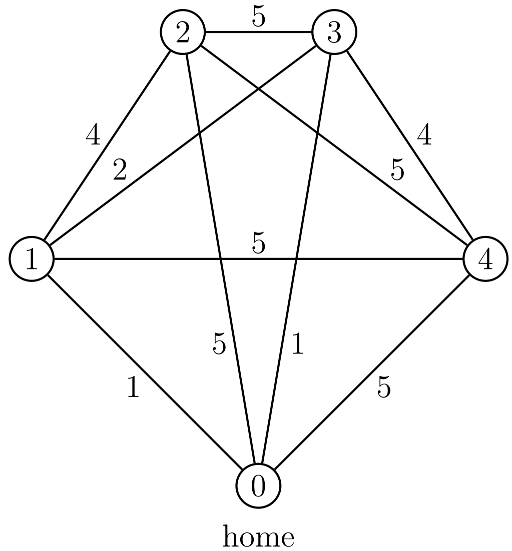


Figure 5.8.2: A travelling repairman problem

players 1 and 3 are on the last two positions their total costs are at least 2. Hence, $\bar{\Omega}^L = (N, \Pi, \bar{k})$ is not a *peop*. ◁

Since both discussed order problems are neither a *neop* nor a *peop* there is no clear-cut TU-game that fits these order problems. Moreover, it is unclear which order problem is an appropriate substitute for the original problem, if any. Consequently, the order problem framework does not result in an appropriate TU-game for the class of *tsp*s. In particular, our framework does not support the use of the game (N, c^L), proposed by Potters et al. (1992).

5.9 Shared taxi problems

The class of cooperative situations we consider in this section is not previously discussed in literature, but is closely related to the class of *tsp*s. Consider a situation in which a group of players at a particular location wants to share a taxi in order to reach their individual destinations. The objective is to find the cheapest way to bring everyone to their individual destinations, while the cost of the taxi only depends on the distance from the starting point to the final destination. We call this problem a *shared taxi problem*, or *stp*, and it is given by a triple $H = (N, 0, \gamma)$, where N is the finite set of players that shares the taxi, which starts at node 0, called *home*. The function $\gamma : E_{N^0} \to \mathbb{R}_+$ is a non-negative cost function, which can be viewed as the taxi cost from one node to another. An edge $e \in E_{N^0}$ is alternatively denoted by $(i,j) \in E_{N^0}$, with $i,j \in N^0$.

In an *stp* the single taxi has to drop off all players. We consider publicly available networks, which implies that the taxi is able to use the edge between any two nodes of the network. As a result, we can impose without loss of generality that the function γ satisfies the triangle inequality, *i.e.*, $\gamma(i,j) + \gamma(j,k) \geq \gamma(i,k)$ for all $i,j,k \in N^0$. Consequently, we assume that the taxi makes a tour[10] in which he drops off all players in some ordering $\pi \in \Pi$. The total cost for a group of players $S \subseteq N$ to use the taxi equals $\gamma(0,\pi(1))) + \sum_{t=2}^{\pi(|S|)} \gamma(\pi(t-1),\pi(t))$, with $\pi \in \Pi_S$. The objective is to find the tour $\pi \in \Pi$ that minimises the total cost. Note that the only difference with a *tsp* is that for an *stp* the link from the last player back to the source is excluded.

We analyse the associated cost allocation problem by the use of the order problem framework. Note that orderings of the player set are implicitly part of an *stp*. Furthermore, since costs do not depend on the return of the taxi home the definition of an adequate individualised cost function seems straightforward. Let $H = (N,0,\gamma)$ be an *stp*. The corresponding order problem Ω^H is given by $\Omega^H = (N,\Pi,k)$, with N the finite player set and Π the set of orderings of N. We define the individual cost of a player as the cost to travel from the node of his direct predecessor to his own node, hence $k^i(\pi) = \gamma(\pi^{-1}(i) - 1, \pi^{-1}(i))$ for all $i \in N$ and $\pi \in \Pi$.

Example 5.9.1 Reconsider Figure 5.8.1, to illustrate an *stp*. The individual costs for a given order according to order problem $\Omega^H = (N,\Pi,k)$ corresponding to the *stp* $H = (N,0,\gamma)$ of Figure 5.8.1 are given in the next table.

π	$k^1(\pi)$	$k^2(\pi)$	$k^3(\pi)$
$(1,2,3)$	1	2	3
$(1,3,2)$	1	3	4
$(2,1,3)$	2	2	4
$(2,3,1)$	4	2	3
$(3,1,2)$	4	2	5
$(3,2,1)$	2	3	5

Each player pays for the edge directly leading to him. Therefore, $\pi = (1,2,3)$ leads to

[10] In the context of *stp*s we use the term *tour* for a path from node 0 to a node $i \in N$ such that all players are visited exactly once.

the individualised costs $k^1(\pi) = 1$, $k^2(\pi) = 2$ and $k^3(\pi) = 3$. Since $\pi^*_{\{3\}} = (1, 2, 3)$ or $\pi^*_{\{3\}} = (2, 3, 1)$ order problem $\Omega^H = (N, \Pi, k)$ is no *neop*. Further, as $\pi^*_{\{1\}} = (1, 2, 3)$ or $\pi^*_{\{1\}} = (1, 3, 2)$ order problem is a *peop* neither. ◁

Example 5.9.1 demonstrates that order problem $\Omega^H = (N, \Pi, k)$ is neither a *neop* nor a *peop*, which implies that we cannot argue in favour of either the marginal or direct cost game for this order problem and the underlying *stp*. Note that this shows that the fact that the order problem framework does not seem to fit the class of *tsp*s is not (only) due to the fact that the costs of $n + 1$ edges are to be divided over n players.

5.10 Travelling repairman problems

5.10.1 Introduction

In this final section we consider situations in which several players need to be visited by a repairman. These players, as well as the single repairman, are not located at the same place and therefore, the repairman has to decide on a specific tour to visit all players. The cost of each player depends on the time he has to wait for the arrival of the repairman. As a result we obtain the problem of finding a tour that minimises the total waiting time of the players. This operations research problem is known as a travelling repairman problem (Afrati et al. (1986)).

A natural example of the above situation is to think of the players as factories with broken machinery that needs to be repaired. In this case costs reflect opportunity costs of production. Equivalently we could think of the following situation. Consider several players at one location who use a single vehicle of transportation to jointly bring all of them to their individual destinations. If we assume that each player's objective is to get to his individual destination as soon as possible we also end up with the problem of finding a tour that visits all players in such a way the total waiting time of the players is minimised.

The class of travelling repairman problems is related to other classes of cooperative situations. First of all, a travelling repairman problem can be seen as a special type of sequencing situation (see Section 5.3) in which the processing time of a player depends on his predecessor, *e.g.*, due to changeover costs. Furthermore, the class of travelling repairman problems is also related to the class of travelling salesman problems, discussed in Section 5.8 and even closer to the class of shared taxi problems, discussed in Section 5.9. However, in those two classes the objective is to minimise the travel time of the salesman or taxi, while in the class of travelling repairman problems the players' waiting time is minimised.

Example 5.10.1 Reconsider the graph of Figure 5.8.1 to illustrate a travelling repairman problem. The optimal ordering for the grand coalition is $(1, 2, 3)$, which leads to a total cost of $1 + 3 + 6 = 10$. $\triangleleft$

In this section we discuss the cost allocation problem that is associated with a travelling repairman problem. For this we assume that the edges in a network are public, which means that the repairman can use the edge between any two players.

In order to find a suitable TU-game we start by considering cost games based upon analogous cost games for other cooperative situations discussed in the literature. We also apply the order problem formulation. This framework suggests the marginal cost game. We show that this game has several interesting properties. Furthermore, we also introduce two single-valued solution concepts, among which is the associated generalised Bird solution, and discuss some of their properties.

The structure of this section is as follows. In Subsection 5.10.2 we formally introduce the class of travelling repairman problems and fix notation. In Subsection 5.10.3 we discuss several ways of modelling a *trp* as a TU-game. We represent a *trp* by a corresponding order problem, show that this order problem is a *neop* and discuss properties of the associated marginal cost game. Finally, in Subsection 5.10.4 we discuss the properties of the generalised Bird solution and a context-specific single-valued solution concept.

5.10.2 Travelling repairman problems

Formally, a *travelling repairman problem*, or *trp*, is given by a triple $T = (N, 0, \gamma)$, where N is the finite set of players (nodes) that has to be visited by the repairman. He starts at node 0, called *home*. The function $\gamma : E_{N^0} \to \mathbb{R}_+$ is a non-negative cost function, which can be viewed as the travel time of the repairman from one node to another. An edge $e \in E_{N^0}$ is alternatively denoted by $(i, j) \in E_{N^0}$, with $i, j \in N^0$.

Let $i, j \in N$. If $\gamma(i,k) \geq \gamma(j,k)$ for all $k \in N^0 \backslash \{i,j\}$ we say that player j is *more desirable* than player i.

In a *trp* the single repairman has to visit all players. We consider publicly available networks, which means that the repairman is able to use the edge between any two nodes of the network. As a result, we can impose without loss of generality that the function γ satisfies the triangle inequality. Consequently, the optimal way to visit all players for the repairman is to make a tour[11] in which he visits all the players according to some ordering $\pi \in \Pi$. The individual cost of a player i for a particular ordering π equals the total waiting time of this player, and is given by $k^i(\pi) = \gamma(0, \pi(1)) + \sum_{t=2}^{\pi^{-1}(i)} \gamma(\pi(t-1), \pi(t))$. The operations research problem is to find an ordering $\pi_N^* \in \Pi$ that minimises the total waiting time of the players, *i.e.*, $\sum_{i \in N} k^i(\pi_N^*) = \min_{\pi \in \Pi} \sum_{i \in N} k^i(\pi)$. Note that an optimal ordering need not be unique. Further, Afrati et al. (1986) show that the problem of finding an optimal tour is NP-hard.

5.10.3 The marginal cost game

In this subsection the aim is to find an appropriate TU-game to model *trp*s. We start with some straightforward attempts, based on cost games defined in the literature for other cooperative situations. In the first one we take as a reference point for determining the cost of a subcoalition $S \subseteq N$ the situation in which the players outside S are not present. Note that analogous games are used to model, *e.g.*, both *mcst* situations (Bird (1976)) and *tsp*s (Potters et al. (1992)). We call this the *standard cost game* (N, c_{stan}^T) and, since the waiting time of a player is not influenced by his successors, it can be defined by

$$c_{stan}^T(S) = \min_{\pi \in \Pi_S} \sum_{i \in S} k^i(\pi)$$

for all $S \subseteq N$. Consider the *trp* of Figure 5.8.1. Recall that the optimal ordering for the grand coalition, $\pi_N^* = (1, 2, 3)$, leads to a total cost of 10. The coalitional costs of the standard cost game associated with this *trp* are given in the table below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c_{stan}^T(S)$	1	2	5	4	6	7	10

[11] In the context of *trp*s we use the term *tour* for a path from node 0 to a node $i \in N$ such that all players are visited exactly once.

The costs for coalition $\{1,3\}$ are the result of ordering $\pi = (1,3,2)$ with $k^1(\pi) = 1$ and $k^3(\pi) = 5$. This game is far too optimistic, as, *e.g.*, coalition $\{2\}$ cannot guarantee itself a cost of 2. Only a tour in which player 2 is visited first results in this cost, and there is no reason to assume that players 1 and 3 agree with such a tour as it increases their costs. Furthermore, this game has an empty core.

The cost which the players in S can guarantee themselves is the cost of the worst-case scenario. This is a tour in which the players in $N\backslash S$ are all visited first. Since the players in S cooperate they can choose the last part of the tour optimally, but without cooperation with $N\backslash S$ we can assume that the players in S can *not* influence the order in which the players in $N\backslash S$ are visited. Therefore, the most pessimistic viewpoint is to assume that S is faced with the worst possible ordering for S on the players of the set $N\backslash S$. We call the game modelled in this way the *worst-case game* (N, c^T_{wc}). Note that an analogous game (the tail game) is used for modelling sequencing situations without initial allocation by Klijn and Sánchez (2006).

Let $\pi_{(t,s)}$ be the restriction of ordering $\pi \in \Pi$ from position t to s, hence if $\pi = (\pi(1), \ldots, \pi(t), \ldots, \pi(s), \ldots, \pi(n))$ then $\pi_{(t,s)} = (\pi(t), \ldots, \pi(s))$. The worst-case game is defined by

$$c^T_{wc}(S) = \max_{\pi_{(1,|N\backslash S|)} \in \Pi_{N\backslash S}} \; \min_{\pi_{(|N\backslash S|+1,n)} \in \Pi_{N\backslash S}} \sum_{i \in S} k^i(\pi)$$

for all $S \subseteq N$. The coalitional costs for the *trp* of Figure 5.8.1 according to this game are given in the next table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c^T_{wc}(S)$	10	11	8	18	12	9	10

The costs for coalition $\{2\}$ are the result of ordering $\pi = (3,1,2)$, the worst players 1 and 3 can decide on for player 2. Note that several coalitional costs exceed the cost of the grand coalition. As a result, this game is not monotonic.

More important is that the coalitional costs of this game are based upon unrealistic tours. Consider, *e.g.*, the cost of coalition $\{2\}$. In this tour the total waiting time for players 1 and 3 together is 14. However, this group of players obtains a lower total cost in any other possible tour. Therefore, we argue that it is unreasonable to assume that these players decide on this tour, which leads to the cost of 11 for coalition $\{2\}$.

The most important drawback of this game is, however, that it does not correctly reflect the power of the (coalitions of) players in the *trp* itself. In any reasonable cost allocation based upon this cost game player 3 should contribute less than any other player to the total cost of the grand coalition. However, both players 1 and 2 are more desirable than player 3. Therefore, based on his position in the *trp* player 3 should contribute more than the other two players. Hence, the *trp* is not appropriately described by the associated TU-game (N, c^T_{wc}).

Next we consider a setup in which the players in $N \backslash S$ try to minimise their own waiting time. We call the game set up in this way the *pessimistic game* (N, c^T_{pess}). We define $\Pi^*_R \subseteq \Pi$ as the set of orderings of N which are optimal for $R \subseteq N$. Then (N, c^T_{pess}) can formally be defined by

$$c^T_{pess}(S) \quad = \quad \min_{\pi \in \Pi^*_{N \backslash S}} \sum_{i \in S} k^i(\pi)$$

for all $S \subseteq N$. For the *trp* of Figure 5.8.1 this approach leads to the coalitional costs in the next table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c^T_{pess}(S)$	9	8	6	18	12	9	10

The costs for coalition $\{2\}$ are the result of ordering $\pi = (1, 3, 2)$, the best players 1 and 3 can decide on for themselves. Although $N \backslash S$ minimises its own cost, several coalitional costs still exceed the cost of the grand coalition in this game, which implies that also this game is not monotonic. More importantly, player 3's weak position in the *trp* is not reflected by the coalitional costs of this game either. As a consequence, also this pessimistic game provides in our opinion no appropriate way to model *trp*s.

Hence, the conventional methods to define an appropriate TU-game for the class of *trp*s seem to fail. Next we apply the order problem framework to this class of cooperative situations. The representation of a *trp* by a corresponding order problem is straightforward for two reasons. First of all, orderings of the player set are part of a *trp*. Secondly, a *trp* already includes an individualised cost function based upon these orderings. Let $T = (N, 0, \gamma)$ be a *trp*. We represent this *trp* by the order problem given by $\Omega^T = (N, \Pi, k)$, with N the finite player set, Π the set of orderings of N and $k : \Pi \to \mathbb{R}^N$ given by $k^i(\pi) = \gamma(0, \pi(1)) + \sum_{t=2}^{\pi^{-1}(i)} \gamma(\pi(t-1), \pi(t))$ for all

$i \in N$ and $\pi \in \Pi$. The corresponding direct (N, c_d^T) and marginal (N, c_m^T) cost game are given by (5.1) and (5.2), respectively. Note that the direct cost game coincides with the standard cost game (N, c_{stan}^T). Since any additional predecessor increases the costs for all followers, $\Omega^T = (N, \Pi, k)$ is a *neop*.

Proposition 5.10.2 *Order problem* $\Omega^T = (N, \Pi, k)$ *is a* neop.

Since Ω^T is a *neop* we suggest to model *trps* by the marginal cost game (N, c_m^T). In the remainder of this subsection we discuss some properties of the marginal cost game associated with order problem $\Omega^T = (N, \Pi, k)$ corresponding to *trp* $T = (N, 0, \gamma)$.

Proposition 5.10.3 *Let* $T = (N, 0, \gamma)$ *be a* trp *with corresponding order problem* $\Omega^T = (N, \Pi, k)$ *with associated marginal cost game* (N, c_m^T)*. Then* (N, c_m^T) *is monotonic.*

Proof: Let (N, c_d^T) be the associated direct cost game. Since $\Omega^T = (N, \Pi, k)$ is a *neop* (Proposition 5.10.2), $c_d^T(S) \leq c_d^T(R)$ for all $S \subseteq R \subseteq N$, which implies that (N, c_d^T) is monotonic. Since (N, c_m^T) is the dual of (N, c_d^T), (N, c_m^T) is monotonic as well. □

A TU-game (N, c^T) associated with a *trp* $T = (N, 0, \gamma)$ satisfies *trp desirability* if whenever player j is more desirable than player i, then $c^T(S \cup \{i\}) \geq c^T(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

Proposition 5.10.4 *Let* $T = (N, 0, \gamma)$ *be a* trp *with corresponding order problem* $\Omega^T = (N, \Pi, k)$ *with associated marginal cost game* (N, c_m^T)*. Then* (N, c_m^T) *satisfies* trp *desirability.*

Proof: Let (N, c_d^T) be the associated direct cost game. Let player j be more desirable than player i and let $S \subseteq N \backslash \{i, j\}$. Then

$$\begin{aligned} c_m^T(S \cup \{i\}) &= c_d^T(N) - c_d^T(N \backslash (S \cup \{i\})) \\ &\geq c_d^T(N) - c_d^T(N \backslash (S \cup \{j\})) \\ &= c_m^T(S \cup \{j\}). \end{aligned}$$

□

Proposition 5.10.5 *Let $T = (N, 0, \gamma)$ be a* trp *with associated cost games (N, c^T_{wc}) and (N, c^T_{pess}). The corresponding order problem is given by $\Omega^T = (N, \Pi, k)$ with associated marginal cost game (N, c^T_m). Then $C(N, c^T_m) \subseteq C(N, c^T_{pess}) \subseteq C(N, c^T_{wc})$.*

Proof: Clearly, $c^T_m(N) = c^T_{wc}(N) = c^T_{pess}(N)$. Let $S \subseteq N$. Let $\pi \in \Pi^*_{N \setminus S}$ be such that $c^T_{pess}(S) = \sum_{i \in S} k^i(\pi)$. Then

$$\begin{aligned} c^T_m(S) &= \sum_{i \in N} k^i(\pi^*_N) - \sum_{i \in N \setminus S} k^i(\pi^*_{N \setminus S}) \\ &\leq \sum_{i \in N} k^i(\pi) - \sum_{i \in N \setminus S} k^i(\pi^*_{N \setminus S}) \\ &= \sum_{i \in N \setminus S} k^i(\pi) + \sum_{i \in S} k^i(\pi) - \sum_{i \in N \setminus S} k^i(\pi^*_{N \setminus S}) \\ &= \sum_{i \in N \setminus S} k^i(\pi^*_{N \setminus S}) + c^T_{pess}(S) - \sum_{i \in N \setminus S} k^i(\pi^*_{N \setminus S}) \\ &= c^T_{pess}(S) \end{aligned}$$

for all $S \subseteq N$, which proves that $C(N, c^T_m) \subseteq C(N, c^T_{pess})$. Since it follows immediately that $C(N, c^T_{pess}) \subseteq C(N, c^T_{wc})$ this completes the proof. □

5.10.4 Cost allocation

In this subsection we discuss two single-valued solution concepts for *trp*s. A single-valued *solution* σ on the set of all *trp*s associates with each *trp* $T = (N, 0, \gamma)$ an efficient[12] vector $\sigma(T) \in \mathbb{R}^N$. For a *trp* $T = (N, 0, \gamma)$ and corresponding order problem $\Omega^T = (N, \Pi, k)$ the generalised Bird solution β is given by (5.7).

Proposition 5.10.6 *Let $\Omega^T = (N, \Pi, k)$ be the order problem corresponding to* trp *$T = (N, 0, \gamma)$ and let (N, c^T_m) be the corresponding marginal cost game. Then $\beta \in C(N, c^T_m)$.*

Proof: Since $\Omega^T = (N, \Pi, k)$ is a *neop* (Proposition 5.10.2) the result follows by Corollary 5.2.3. □

[12] A vector $\sigma(T)$ is called efficient if $\sum_{i \in N} \sigma^i(T)$ equals the cost of an optimal tour for *trp* $T = (N, 0, \gamma)$.

Let $T = (N, 0, \gamma^T)$ and $T' = (N, 0, \gamma^{T'})$ be a two *trps* and let σ be a solution. Let T have a unique optimal tour and let edge (i, j) not be part of it. Further, let $\gamma^{T'}(i, j) > \gamma^T(i, j)$ and $\gamma^{T'}(e) = \gamma^T(e)$ for all $e \in E_{N^0} \backslash \{(i, j)\}$. Then σ satisfies *edge monotonicity* if $\sigma^k(T') \leq \sigma^k(T')$ for all $k \in N \backslash \{i, j\}$. If an allocation satisfies edge monotonicity players do not suffer from increasing costs of edges to which they are not adjacent.

Proposition 5.10.7 *The generalised Bird solution β satisfies edge monotonicity.*

Proof: Since β only depends on the optimal tour it gives the same allocation for any two *trps* with the same optimal tour. Consequently, β satisfies edge monotonicity. $\square$

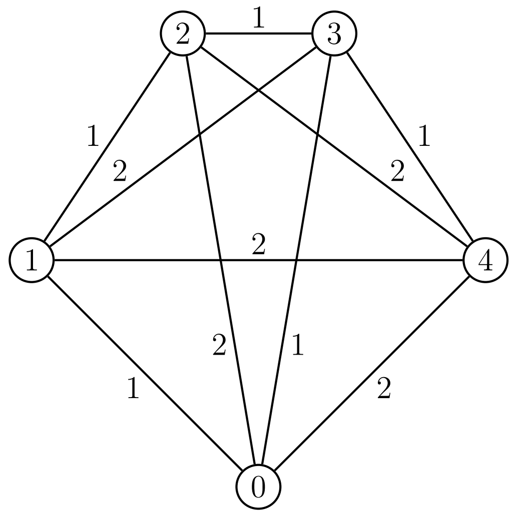
The following property provides a lower-bound for the contribution of a player. Let $T = (N, 0, \gamma)$ be a *trp* and let σ be a solution. Then σ satisfies the *individual cost property* if $\sigma^i(T) \geq \gamma(0, i)$ for all $i \in N$.

Proposition 5.10.8 *The generalised Bird solution β satisfies the individual cost property.*

Proof: It follows from the definition that $\beta^i \geq \min_{\pi \in \Pi} k^i(\pi) = \gamma(0, i)$ for all $i \in N$, which implies that β satisfies the individual cost property. $\square$

The final property we consider requires that more desirable players contribute less. Let $T = (N, 0, \gamma)$ be a *trp* and let σ be a solution. If player j is more desirable than player i, then σ satisfies the *trp desirability property* if $\sigma^i(T) \geq \sigma^j(T)$. The next example illustrates that β does not satisfy the *trp* desirability property.

Example 5.10.9 Consider the four-player *trp* of Figure 5.10.1. The optimal tour is given by $\pi^*_N = (1, 2, 3, 4)$, with a cost of 10. Consequently, the generalised Bird solution is given by $\beta = (1, 2, 3, 4)$. However, player 3 is more desirable than player 1, which implies that β does not satisfy the *trp* desirability property. $\triangleleft$



home
Figure 5.10.1: A travelling repairman problem

Next we introduce a kind of compromise solution based on reasonable upper and lower bounds. Given a *trp* the vectors $\underline{\chi}, \bar{\chi} \in \mathbb{R}^N$ are defined such that

$$\begin{aligned}\underline{\chi}^i &= c_d^T(\{i\}),\\ \bar{\chi}^i &= c_m^T(\{i\}),\end{aligned}$$

for all $i \in N$. Then allocation $\chi \in \mathbb{R}^N$ is defined by

$$\alpha\underline{\chi} + (1-\alpha)\bar{\chi},$$

with $\alpha \in [0,1]$ such that $\sum_{i \in N} \chi^i = c_d(N)$. Since cost function k satisfies condition (5.3), $c_d^T(\{i\}) \leq c_m^T(\{i\})$ for all $i \in N$ and $\sum_{i\in N} c_d^T(\{i\}) \leq c_d^T(N) \leq \sum_{i\in N} c_m^T(\{i\})$, which implies that this solution concept exists for all *trp*s. Furthermore, it is also single-valued.

The minimal rights vector of a cost game (N,c) is given by m_c and the utopia vector by M_c. Game (N,c) is called compromise admissible if $m_c \geq M_c$ and $\sum_{i\in N} m_c^i \geq c(N) \geq \sum_{i\in N} M_c^i$. The *compromise value* τ, defined in Tijs (1981) for revenue instead of cost games, of a compromise admissible cost game is defined by $\tau(N,c) = \alpha M_c + (1-\alpha)m_c$, with α the unique element in $[0,1]$ such that $\sum_{i\in N} \tau^i(N,c) = c(N)$.

Theorem 5.10.10 Let $T = (N, 0, \gamma)$ be a *trp* with corresponding order problem $\Omega^T = (N, \Pi, k)$ and associated marginal cost game (N, c_m^T). Then $\chi = \tau(N, c_m^T)$.

Proof: It suffices to show that $\underline{\chi}$ is equal to the utopia vector $M_{c_m^T}$ and that $\bar{\chi}$ is equal to the minimal rights vector $m_{c_m^T}$. Let $i \in N$. First of all,

$$\begin{aligned}
\underline{\chi}^i &= c_d^T(\{i\}) \\
&= c_m^T(N) - c_m^T(N\backslash\{i\}) \\
&= M_{c_m^T}^i.
\end{aligned}$$

It remains to show that $\bar{\chi}^i = m_{c_m^T}^i$. Hence, by the definition of the minimal rights vector it suffices to show that $\bar{\chi}^i \leq c_m^T(S \cup \{i\}) - \sum_{j \in S} M_{c_m^T}^j$ for all $S \subseteq N\backslash\{i\}$. Let $S \subseteq N\backslash\{i\}$. Then

$$\begin{aligned}
\bar{\chi}^i &= c_m^T(\{i\}) \\
&= c_d^T(N) - c_d(N\backslash\{i\}) \\
&\leq c_d^T(N) - c_d^T(N\backslash(S \cup \{i\})) - \sum_{j \in S} c_d^T(\{j\}) \\
&= c_m^T(S \cup \{i\}) - \sum_{j \in S} c_d^T(\{j\}) \\
&= c_m^T(S \cup \{i\}) - \sum_{j \in S} M_{c_m^T}^j,
\end{aligned}$$

where the inequality follows from the superadditivity of (N, c_d^T). □

Solution χ also satisfies several attractive properties with respect to the underlying *trp*.

Proposition 5.10.11 χ *satisfies edge monotonicity.*

Proof: Let $T = (N, 0, \gamma^T)$ and $T' = (N, 0, \gamma^{T'})$ be a two *trp*s. Let edge (i, j) not be part of the optimal tour for T and let $\gamma^{T'}(i, j) > \gamma^T(i, j)$ and $\gamma^{T'}(e) = \gamma^T(e)$ for all $e \in E_{N^0}\backslash\{(i, j)\}$. Note that $c_d^T(N) = c_d^{T'}(N)$, and $c_d^{T'}(\{\ell\}) \geq c_d^T(\{\ell\})$ and $c_d^{T'}(N\backslash\{\ell\}) = c_d^T(N\backslash\{\ell\})$ for all $\ell \in \{i, j\}$. Therefore, it suffices to show that $c_d^{T'}(\{k\}) = c_d^T(\{k\})$ and $c_m^{T'}(\{k\}) \leq c_m^T(\{k\})$ for all $k \in N\backslash\{i, j\}$.

Let $k \in N\backslash\{i, j\}$. It immediately follows that $c_d^{T'}(\{k\}) = c_d^T(\{k\})$. Furthermore, since $c_d^{T'}(N) = c_d^T(N)$ and $\gamma^{T'}(i,j) \geq \gamma^T(i,j)$ we obtain $c_d^{T'}(N\backslash\{k\}) \geq c_d^T(N\backslash\{k\})$. Consequently,

$$\begin{aligned} c_m^{T'}(\{k\}) &= c_d^{T'}(N) - c_d^{T'}(N\backslash\{k\}) \\ &\leq c_d^T(N) - c_d^T(N\backslash\{k\}) \\ &= c_m^T(\{k\}). \end{aligned}$$

□

Proposition 5.10.12 *χ satisfies the individual cost property.*

Proof: Since $\chi^i \geq c_d^T(\{i\}) = \gamma(i,0)$ for all $i \in N$, χ satisfies the individual cost property. □

Proposition 5.10.13 *χ satisfies the* trp *desirability property.*

Proof: Let $T = (N, 0, \gamma)$ be a *trp* and let $i, j \in N$ be such that j is more desirable than i. Then $\gamma(i,0) \geq \gamma(j,0)$, which implies that $\underline{\chi}^i \geq \underline{\chi}^j$. Furthermore, it follows that $c_d^T(N\backslash\{i\}) \geq c_d^T(N\backslash\{j\})$, which implies that $\bar{\chi}^i \geq \bar{\chi}^j$. □

A drawback of χ is that it is not always an element of the core of the marginal cost game.

Example 5.10.14 Reconsider the *trp* of Figure 5.10.1. Since $\underline{\chi} = (1,2,1,2)$ and $\bar{\chi} = (3,3,3,4)$, the solution χ is given by $\chi = (2\frac{1}{7}, 2\frac{4}{7}, 2\frac{1}{7}, 3\frac{1}{7})$. Further, $c_d^T(\{2,4\}) = 6$, so $c_m^T(\{1,3\}) = 4$, and as $\chi^1 + \chi^3 = 4\frac{2}{7}$ this implies that $\chi \notin C(N, c_m^T)$. ◁

Chapter 6

Transfers, contracts and strategic games

It's a zero sum game, somebody wins,
somebody loses. Money itself isn't lost or
made, it's simply transferred from one
perception to another.

Gordon Gekko, *Wall Street* (1987)

6.1 Introduction

This chapter, which is based on Kleppe et al. (2007), investigates the role of allowing certain aspects of commitment and cooperation within the framework of strategic form games. More in particular, it focuses on the explicit strategic option of costless contracting on monetary transfer schemes with respect to particular outcomes.

Closely related to this chapter are the papers of Jackson and Wilkie (2005) and Yamada (2003). Basically both of these papers allow for a rather broad type of contracts within a setting of mixed extensions of finite strategic form games. In this setting contracts on transfer payments are contingent on the actual choice of specific strategies. Jackson and Wilkie (2005) illustrate that this type of costless contracting does not necessarily lead to efficiency, *i.e.*, maximal total payoff. Yamada (2003) explicitly models the described format of contracting as a strategic option in a two-stage extensive form game and derives a kind a Folk theorem: the payoff configurations supported by *subgame perfect Nash equilibria* of this two-stage *contract game* are characterised.

The objective of this chapter is to analyse contracts on transfer payments of only the simplest form. The contracting will be contingent on the actual occurrence of outcomes, so only on the realisation of strategy combinations and not on the actual choice of individual strategies. This lowers the degree of sophistication required in the cooperative commitments. In particular, it avoids intrinsic problems regarding the non-perceptibility of mixed strategies. Moreover, the two-stage contract game in this chapter allows only for a unanimity type of contracting on sets of outcomes. By restricting to this type of basic contracting and combining this with the more appropriate concept of *virtual* subgame perfection as introduced by García-Jurado and González-Díaz (2006), Yamada's Folk theorem is recovered.

Although the concepts and results in this chapter can be readily extended to games with more players, we restrict our attention to two-player games for expositional purposes.

The first part of the chapter deals with the possibility of making a specific strategy combination individually stable by having a simple monetary transfer scheme contingent on the actual realisation of the corresponding outcome. Such a strategy combination is called a transfer equilibrium. Under standard regularity conditions however (*e.g.*, satisfied for any mixed extension of a finite game) it turns out that the set of transfer equilibria coincides with the set of Nash equilibria. For finite games without the possibility of randomisation, the set of Nash equilibria can be a strict subset of the set of transfer equilibria. This particular subclass is analysed in some detail.

The second and larger part of this chapter models contracting on monetary transfers as an explicit strategic option within a two-stage extensive form setting. The first stage consists of the contracting stage where both players can propose transfer schemes as before but now possibly on multiple outcomes simultaneously. Only if both players fully agree on all transfer proposals ("give or take"), the payoffs of the original game are modified accordingly and the modified game is played in the second stage. We have chosen for this approach, because the preference of a player for a set of contracts does not automatically imply that he is also interested in any subset of these contracts. As a result, both the type of contract proposals and the subsequent implementation mechanism of the proposals are as simple as possible. It is important to note that in this setting implemented contracts on transfer schemes with respect to certain outcomes may lead to the rise of equilibria at outcomes that are not specified in the contracts.

The first main result is a full characterisation of all equilibrium payoff vectors in the same spirit as the well-known Folk theorems in the context of repeated games. It turns out that exactly those payoff vectors that are bounded from below by the individual minimax payoffs and for which the total sum of the payoffs is bounded from above by the maximum of the total payoffs over all outcomes, correspond to Nash equilibria of the two-stage contract game. After arguing that the set of subgame perfect equilibria (Selten (1965)) of the contract game is empty because of the non-existence of equilibria in seemingly irrelevant subgames, we focus attention on the notion of virtual subgame perfect equilibrium. This notion seems especially relevant and suitable in our framework. Roughly speaking, virtual subgame perfection requires players to play best responses only in subgames close to the equilibrium path. The second main result states that exactly those payoff vectors that are individually bounded from below by some equilibrium payoff, and for which there is a similar upper bound as in the case of Nash equilibria, correspond to virtual subgame perfect equilibria of the contract game.

The outline of this chapter is as follows. Section 6.2 analyses the possibility of contracting on a monetary transfer with respect to one particular outcome and investigates the corresponding notion of transfer equilibrium. In Section 6.3 the two-stage contract game that allows for strategic contracting on sets of outcomes is formally introduced and explained. Furthermore, it states and proves the Folk-like theorems with respect to Nash equilibria and virtual subgame perfect equilibria of the contract game.

6.2 Transfer equilibrium

A two-player strategic game is given by $G = (\{1,2\}, \{X^i\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$, with $\{1,2\}$ the player set, X^i the strategy set of player $i \in \{1,2\}$ and $\pi^i : X \to \mathbb{R}$ his payoff function, assigning to each strategy profile $x = (x^1, x^2) \in X$ (with $X = X^1 \times X^2$) a payoff $\pi^i(x)$. In this framework we allow for certain transfers of payoff from one player to the other, so we assume the payoffs to be monetary.

A strategy profile $\hat{x}$ is a *Nash equilibrium* (Nash (1951)) of G, denoted by $\hat{x} \in NE(G)$, if $\pi^i(\hat{x}) \geq \pi^i(x^i, \hat{x}^{-i})$ for all $x^i \in X^i$ and all $i \in \{1,2\}$. Here x^{-i}

is the frequently used shorthand notation for $(x^j)_{j\neq i}$. A Nash equilibrium is usually predicted as the outcome of a game when players are not able to make binding agreements on their strategy choices, but are allowed to communicate before play starts.

In this chapter, we allow the players to cooperate in a limited way. We assume that they have a mechanism which allows them to make an enforceable commitment before play starts on a transfer of money after both players have chosen their prespecified strategy. So, players can agree to commit themselves to any reallocation of $\pi^1(x)+\pi^2(x)$, conditional on the outcome $x \in X$.

Both players also have the option not to cooperate in this way. (Note that if there are more players, we should also allow for partial cooperation, which naturally leads to a partition of the player set into cooperating components.) So, we have to make a distinction between the two possible partitions of the player set. This collection of partitions is denoted by $\mathcal{P} = \{\{\{1\},\{2\}\},\{\{1,2\}\}\}$.

Definition Let $G = (\{1,2\},\{X^i\}_{i\in\{1,2\}},\{\pi^i\}_{i\in\{1,2\}})$ be a two-player strategic game. Then $\hat{x} \in X$ is a *transfer equilibrium* of G if there exist a $(q,P) \in \mathbb{R}^2 \times \mathcal{P}$ such that

$$(i) \quad \sum_{i\in S} q^i = \sum_{i\in S} \pi^i(\hat{x}) \qquad \text{for all } S \in P, \tag{6.1}$$

$$(ii) \quad q^i \geq \pi^i(x^i,\hat{x}^{-i}) \qquad \text{for all } x^i \in X^i \setminus \{\hat{x}^i\} \text{ and all } i \in \{1,2\}. \tag{6.2}$$

The concept of transfer equilibrium is a generalisation of the concept of Nash equilibrium, as is stated in the following lemma.

Lemma 6.2.1 *Each Nash equilibrium is a transfer equilibrium.*

Obviously, any transfer equilibrium x that is supported by $(q,P) = (\pi(x),\{\{1\},\{2\}\})$ is a Nash equilibrium as well. In general, however, a transfer equilibrium need not be a Nash equilibrium, as is illustrated by several examples later on.

The following proposition shows that if G satisfies some regularity conditions, all transfer equilibria correspond to Nash equilibria.

Proposition 6.2.2 *Let $G = (\{1,2\}, \{X^i\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a two-player strategic game, with X^i a convex subset of a finite-dimensional Euclidean space and π^i continuous for all $i \in \{1,2\}$. Then $\hat{x}$ is a Nash equilibrium of G if and only if $\hat{x}$ is a transfer equilibrium of G.*

Proof: In view of Lemma 6.2.1, we only have to show the "if" part. Assume that $\hat{x}$ is a transfer equilibrium and assume that both X^1 and X^2 have more than one element (otherwise the proof is straightforward). Let $i \in \{1,2\}$ and let $\varepsilon > 0$. Let $x^i \in X^i \setminus \{\hat{x}^i\}$ be such that $|\pi^i(\hat{x}) - \pi^i(x^i, \hat{x}^{-i})| < \varepsilon$. Note that such an x^i always exists because X^i is a convex subset of an Euclidean space and π^i is continuous. Then (6.2) implies that $q^i \geq \pi^i(\hat{x}) - \varepsilon$ for all $i \in \{1,2\}$, and hence by (6.1),

$$-\varepsilon \leq q^2 - \pi^2(\hat{x}) = \pi^1(\hat{x}) - q^1 \leq \varepsilon.$$

Since this holds for every $\varepsilon > 0$ and π^i is continuous, we obtain $q^i = \pi^i(\hat{x})$ for all $i \in \{1,2\}$. Thus (6.2) implies that $\hat{x}$ is a Nash equilibrium of G. □

Given Proposition 6.2.2, we restrict our attention to games that do not satisfy the regularity conditions mentioned there. In particular, we consider games with a finite number of strategies. A finite two-player game is given by $G = (\{1,2\}, \{M^i\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$, where $M^i = \{1,\ldots,m^i\}$ is the strategy set of player $i \in \{1,2\}$. A typical element of M^i is alternatively denoted by x^i. The set of all strategy profiles is given by $M = M^1 \times M^2$, a typical element of M by x.

The following examples illustrate the concept of transfer equilibrium for finite two-player games. The first example is a prisoners' dilemma and shows that in such a game, the set of transfer equilibria may contain elements that are not Nash equilibria.

Example 6.2.3 Consider the next finite two-player game.

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 3,3 & 0,5 \\ 5,0 & 1,1 \end{bmatrix} \end{array}$$

Both players have two pure strategies: T and B for player 1 and L and R for player 2. If player 1 plays T and player 2 plays R, the payoff equals 0 to the former and 5 to the latter.

Because $x = (B, R)$ is a Nash equilibrium, it is immediately clear using Lemma 6.2.1 that it is a transfer equilibrium as well (with $q = \pi(x)$ and $P = \{\{1\},\{2\}\}$). Furthermore, (B, R) is also supported as a transfer equilibrium by $\{((\delta, 2-\delta), \{\{1,2\}\}) \mid \delta \in [0,2]\}$. The other transfer equilibria are (B, L), supported by $\{((\delta, 5-\delta), \{\{1,2\}\}) \mid \delta \in [3,4]\}$, and the mirror image (T, R), supported by $\{((\delta, 5-\delta), \{\{1,2\}\}) \mid \delta \in [1,2]\}$.

Note that the concept of transfer equilibrium is different from full cooperation, as in that case the players would play (T, L) and divide a total amount of 6 between them. This is, however, impossible as for any transfer of money in that cell, at least one player has an incentive to deviate. $\triangleleft$

In the approach of Jackson and Wilkie (2005), the game in Example 6.2.3 does not have an equilibrium (not even (B, R)). Because of the way they set up their transfer proposals, mixing has to be allowed to sustain any equilibrium in this particular game.

The next example, known as matching pennies, demonstrates that the set of transfer equilibria can be non-empty even when there are no Nash equilibria.

Example 6.2.4 Consider the next finite two-player game.

$$\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 2,0 & 0,2 \\ 0,2 & 2,0 \end{bmatrix} \end{array}$$

It is obvious that the set of Nash equilibria of this game is empty. However, (T, L) supported by $((0,2), \{\{1,2\}\})$ is a transfer equilibrium. In fact any combination of strategies gives rise to a transfer equilibrium in an analogous way. $\triangleleft$

Although the set of transfer equilibria is an extension of the set of Nash equilibria, not all games have transfer equilibria.

Example 6.2.5 Consider the next finite two-player game.

$$\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ M \\ B \end{array} & \begin{bmatrix} 6,6 & 1,9 \\ 4,7 & 6,6 \\ 4,5 & 6,4 \end{bmatrix} \end{array}$$

For none of the strategy profiles it is possible to transfer money in such a way that both players have no incentive to deviate. Therefore, this game has no transfer equilibria. $\triangleleft$

Non-existence of a transfer equilibrium in Example 6.2.5 follows from the following proposition, which provides a necessary and sufficient condition for a transfer equilibrium in a finite two-player game to exist.

Proposition 6.2.6 *Let $G = (\{1,2\}, \{M^i\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a finite two-player game. Strategy profile $\hat{x} \in M$ is a transfer equilibrium of G if and only if*

$$\pi^1(\hat{x}) + \pi^2(\hat{x}) \geq \bar{\pi}^1(\hat{x}) + \bar{\pi}^2(\hat{x}), \tag{6.3}$$

where $\bar{\pi}^i(\hat{x}) = \max_{x^i \in M^i \setminus \{\hat{x}^i\}} \pi^i(x^i, \hat{x}^{-i})$.

Proof: We first prove the "if" part. Let $\hat{x} \in M$ be such that (6.3) holds. If $\pi^i(\hat{x}) \geq \bar{\pi}^i(\hat{x})$ for all $i \in \{1,2\}$, then $\hat{x} \in NE(G)$ and the result follows from Lemma 6.2.1. Otherwise, assume without loss of generality that $\pi^1(\hat{x}) \geq \bar{\pi}^1(\hat{x})$ and $\pi^2(\hat{x}) < \bar{\pi}^2(\hat{x})$. We show that $\hat{x}$ is a transfer equilibrium supported by (q,P), with $q^1 = \bar{\pi}^1(\hat{x})$, $q^2 = \pi^2(\hat{x}) + \pi^1(\hat{x}) - \bar{\pi}^1(\hat{x})$ and $P = \{\{1,2\}\}$. Clearly $q^1 + q^2 = \pi^1(\hat{x}) + \pi^2(\hat{x})$, so (6.1) holds. For (6.2),

$$\begin{aligned} q^1 &= \bar{\pi}^1(\hat{x}) \\ &\geq \pi^1(x^1, \hat{x}^2) \end{aligned}$$

for all $x^1 \in M^1 \setminus \{\hat{x}^1\}$. And,

$$\begin{aligned} q^2 &= \pi^2(\hat{x}) + \pi^1(\hat{x}) - \bar{\pi}^1(\hat{x}) \\ &\geq \bar{\pi}^2(\hat{x}) + \bar{\pi}^1(\hat{x}) - \bar{\pi}^1(\hat{x}) \\ &= \bar{\pi}^2(\hat{x}) \\ &\geq \pi^2(\hat{x}^1, x^2) \end{aligned}$$

for all $x^2 \in M^2 \setminus \{\hat{x}^2\}$.

Secondly, we prove the "only if" statement. Let $\hat{x} \in M$, supported by $(q,P) \in \mathbb{R}^2 \times \mathcal{P}$ be a transfer equilibrium of G. If $\hat{x} \in NE(G)$ then $\pi^i(\hat{x}) \geq \bar{\pi}^i(\hat{x})$ for all

$i \in \{1, 2\}$, from which the assertion follows immediately. If $\hat{x} \notin NE(G)$, then $P = \{\{1, 2\}\}$, since otherwise (6.2) fails for at least one of the players. Then (6.1) and (6.2) give $q^1 + q^2 = \pi^1(\hat{x}) + \pi^2(\hat{x})$ and $q^i \geq \bar{\pi}^i(\hat{x})$ for all $i \in \{1, 2\}$. Hence, $\pi^1(\hat{x}) + \pi^2(\hat{x}) \geq \bar{\pi}^1(\hat{x}) + \bar{\pi}^2(\hat{x})$. □

One consequence of Proposition 6.2.6 is that for any game with $m^i \leq 2$ for all $i \in \{1, 2\}$, the set of transfer equilibria is non-empty. Proposition 6.2.6 also implies that if there exists a transfer equilibrium $\hat{x}$ of G, then $\pi^1(\hat{x})$ is the maximum payoff to player 1 against $\hat{x}^2$ or $\pi^2(\hat{x})$ is the maximum payoff to player 2 against $\hat{x}^1$. So, when looking for a transfer equilibrium, only the cells containing those maxima should be considered, which means that only $m^1 + m^2$ checks are needed.

6.3 Strategic transfer contracts

In the setup of transfer equilibrium as discussed in the previous section, the players have a mechanism to enforce certain commitments between them. This mechanism can be seen as a type of contract in order to transfer money between the players that is executed in case a particular strategy profile is played. By looking at the mechanism from that perspective one could however argue that the contracting possibilities of the players are quite limited. First of all, players are only allowed to sign a single contract and secondly, it is required that the combination of the contract itself and the strategy profile on which it is enforced, constitutes an equilibrium.

In order to overcome these limitations we introduce for the class of finite two-player games a different and more sophisticated contracting model in this section. We assume that before playing the game, the players know which particular allocations of earnings are available. Then each player proposes a *set* of contracts. A single contract describes for one particular strategy combination a reallocation of the corresponding payoffs. We specifically allow the players to propose contracts that discard money. Only in case both players agree on the *entire* contract proposal, the game is modified according to the contract conditions.

Definition Let $G = (\{1, 2\}, \{M^i\}_{i \in \{1,2\}}, \{\pi^i\}_{i \in \{1,2\}})$ be a finite two-player game. A *transfer contract* is a pair $(\hat{x}, q) \in M \times \mathbb{R}^2$ such that $q^1 + q^2 \leq \pi^1(\hat{x}) + \pi^2(\hat{x})$.

Using these transfer contracts, the players play the two-stage *contract game* described as follows.

First stage. Each player $i \in \{1,2\}$ chooses a collection of transfer contracts $\alpha^i \subseteq M \times \mathbb{R}^2$, at most one for each strategy profile. The choices are made simultaneously and independently. After both players have made their choice, the proposed contracts are publicly announced.

Second stage. If both players have chosen the same set of transfer contracts in the first stage, this set is adopted, the payoffs of the game are modified accordingly and the players play this modified game. If the proposed contracts in the first stage do not match, the original game G is played.

We want to point out once more that contracts only come into effect in case both sets of proposed contracts coincide completely. It is therefore not possible that only part of the proposed contracts are enforced. This seems quite natural, as the preference of a player for a set of contracts does not automatically imply that he is also interested in any subset of these contracts.

The main difference between this model and the setup in Section 6.2 is that here contracts are a *strategic* option, as they can be signed on every cell and are not necessarily located at an equilibrium of the ensuing second stage. In particular, it is possible that a contract on one cell results in an equilibrium at another cell.

The *contract game* as described above can be represented by an extensive form game (as modelled in Kreps and Wilson (1982)), denoted by $\Gamma_c(G)$. Its strategic representation is given by $\mathcal{G}_c(G) = (\{1,2\}, \{X^i\}_{i\in\{1,2\}}, \{\pi_c^i\}_{i\in\{1,2\}})$, where for each player i a strategy is a pair $(\alpha^i, f^i) \in X_c^i$ with α^i a collection of transfer contracts and f^i a map which assigns an action $f^i(\bar{\alpha}) \in X^i$ to every pair $\bar{\alpha} = (\bar{\alpha}^1, \bar{\alpha}^2)$ of contract proposals. The payoff function for player $i \in \{1,2\}$ is given by

$$\pi_c^i(\alpha, f) = \begin{cases} q^i & \text{if } \alpha^1 = \alpha^2 \text{ and } (f(\alpha), q) \in \alpha^i, \\ \pi^i(f(\alpha)) & \text{otherwise.} \end{cases}$$

Let us first consider the three examples discussed in Section 6.2. After that we provide necessary and sufficient conditions for a payoff vector to be the result of some Nash equilibrium of this two-stage game. Moreover, we give a similar result for an equilibrium refinement that appears natural in this context, called virtual subgame perfect equilibrium.

For the prisoners' dilemma of Example 6.2.3, the combination of strategies in which both players propose a set of contracts such that the cells (T,R) and (B,L) are replaced by $(0,0)$ constitutes, in combination with player 1 (2) playing T (L) if both players choose these contracts and B (R) otherwise, a Nash equilibrium of the game $\Gamma_c(G)$. Formally, this equilibrium strategy profile $(\hat{\alpha},\hat{f})$ is given by

$$\hat{\alpha}^1 = \hat{\alpha}^2 = \{((T,R),(0,0)),((B,L),(0,0))\},$$
$$\hat{f}^1(\alpha) = \begin{cases} T & \text{if } \alpha = \hat{\alpha}, \\ B & \text{otherwise,} \end{cases}$$
$$\hat{f}^2(\alpha) = \begin{cases} L & \text{if } \alpha = \hat{\alpha}, \\ R & \text{otherwise.} \end{cases}$$

It is clear that unilaterally deviating will not lead to a higher payoff: player i's best response is to play according to $\hat{f}^i$ in the second stage in case both players have played according to $\hat{\alpha}$ in the first stage, and in case player i has deviated in the first stage. In the latter situation the second stage consists of the original prisoners' dilemma game.

For the matching pennies in Example 6.2.4, we establish that, *e.g.*, (T,L), supported by $((0,2),\{1,2\})$, is a transfer equilibrium. However, $\Gamma_c(G)$ does not have a Nash equilibrium with associated payoff vector $(0,2)$, as this outcome is only reachable if both players agree on a set of contracts. In that case, however, player 1 will deviate from $(\hat{\alpha},\hat{f})$ by choosing $\alpha^1=\emptyset$ in combination with a best response to $f^2(\alpha^1,\hat{\alpha}^2)$, leading to a payoff equal to 2.

As a matter of fact, with a similar reasoning one can show that for matching pennies the contract game has no Nash equilibria at all. This is not a consequence of the game being constant-sum (as can be seen by replacing the (T,L) payoffs by (3,0), in which case the same arguments hold), but relates to the minimax payoffs of both players. In the formal analysis of transfer contracts in this section we elaborate on this point.

The game of Example 6.2.5 does not have any transfer equilibria. The corresponding contract game $\Gamma_c(G)$, however, does possess a Nash equilibrium. Consider, *e.g.*, the strategy profile in which both players propose a set of contracts such that all payoffs except at (T,L) are replaced by $(0,0)$. Furthermore, player 1 chooses T if this set of contracts is executed and B otherwise, and player 2 plays L regardless of the contract choice. Then unilaterally deviating will not lead to a higher payoff and hence, this combination of strategies is a Nash equilibrium in the contract game.

Let us now formally analyse the equilibrium payoffs of a contract game. Given a finite two-player game $G = (\{1,2\}, \{M^i\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$, the *minimax payoff vector* $v \in \mathbb{R}^2$ is defined by

$$v^i \quad = \quad \min_{x^{-i}\in M^{-i}} \max_{x^i \in M^i} \pi^i(x)$$

for all $i \in \{1,2\}$.

Theorem 6.3.1 Let $G = (\{1,2\}, \{M^i\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a finite two-player game with minimax payoff vector v. For every $r \in \mathbb{R}^2$ with $r \geq v$ such that $r^1 + r^2 \leq \pi^1(x) + \pi^2(x)$ for some $x \in M$ there exists a Nash equilibrium of the game $\mathcal{G}_c(G)$ with corresponding payoff vector r.

Proof: Let $r \geq v$ and $\bar{x} \in M$ be such that $r^1 + r^2 \leq \pi^1(\bar{x}) + \pi^2(\bar{x})$. Let $\tilde{x} \in M$ be such that

$$\max_{x^i \in M^i} \pi^i(x^i, \tilde{x}^{-i}) \quad = \quad v^i$$

for all $i \in \{1,2\}$. We construct a Nash equilibrium $(\hat{\alpha}, \hat{f})$ of the game $\mathcal{G}_c(G)$ as follows. For each player $i \in \{1,2\}$, the set of transfer contracts is given by

$$\hat{\alpha}^i = \left\{(\bar{x}, r)\right\} \cup \left\{(x,p) \,\middle|\, x \in M \setminus \{\bar{x}\},\, p < v,\, p^1 + p^2 \leq \pi^1(x) + \pi^2(x)\, \forall x \in M \setminus \{\bar{x}\}\right\}.$$

The strategies in the second stage are given for all α by

$$\hat{f}^i(\alpha) = \begin{cases} \bar{x}^i & \text{if } \alpha = \hat{\alpha}, \\ \tilde{x}^i & \text{if } \alpha \neq \hat{\alpha}, \end{cases}$$

for all $i \in \{1,2\}$. Clearly, $\pi(\hat{\alpha}, \hat{f}) = r$. If player 1 chooses (α^1, f^1), then his payoff equals

$$\pi^1((\alpha^1, \hat{\alpha}^2), (f^1, \hat{f}^2)) = \begin{cases} r^1 & \text{if } \alpha^1 = \hat{\alpha}^1,\ f^1(\alpha^1, \hat{\alpha}^2) = \bar{x}^1, \\ p^1 & \text{if } \alpha^1 = \hat{\alpha}^1,\ f^1(\alpha^1, \hat{\alpha}^2) \neq \bar{x}^1, \\ \pi^1(f^1(\alpha^1, \hat{\alpha}^2), \tilde{x}^2) & \text{if } \alpha^1 \neq \hat{\alpha}^1. \end{cases}$$

Given the choice of p^1 and $\tilde{x}^2$,

$$\begin{aligned} \pi^1((\alpha^1, \hat{\alpha}^2), (f^1, \hat{f}^2)) &\leq r^1 \\ &= \pi^1(\hat{\alpha}, \hat{f}). \end{aligned}$$

Similarly, player 2 has no incentive to deviate and $(\hat{\alpha}, \hat{f})$ is a Nash equilibrium of $\mathcal{G}_c(G)$. $\square$

Note that the condition on the payoff vector r in Theorem 6.3.1 is not only sufficient, but also necessary. If $r^i < v^i$ for some player $i \in \{1, 2\}$, then r is not an equilibrium payoff in the contract game, since player i will deviate by proposing no contract and playing his minimax strategy in the second stage.

Theorem 6.3.1 states that every feasible payoff vector larger than the minimax payoff vector of G is supported as the payoff of some Nash equilibrium of the contract game $\Gamma_c(G)$. However, the equilibrium strategy profile constructed in the proof may prescribe unreasonable strategy choices in subgames off the equilibrium path.

Consider, *e.g.*, the game of Example 6.2.5, and the Nash equilibrium $(\hat{\alpha}, \hat{f})$ of the corresponding contract game $\Gamma_c(G)$ presented earlier in this section. In this game, $v = (6, 5)$ and $(\hat{\alpha}, \hat{f})$ is one of the Nash equilibria constructed in the proof of Theorem 6.3.1 with $\bar{x} = (T, L)$, $r = (6, 6)$, $p = (0, 0)$, $\tilde{x}^1 = B$ and $\tilde{x}^2 = L$. The problem with this Nash equilibrium is that after any unilateral deviation from the proposed contract set the players end up in a subgame in which the original game G is played in the second stage, and in that game B is not a best response to L.

In order to deal with this shortcoming, one might consider *subgame perfect equilibria* of $\Gamma_c(G)$ (Selten (1965)). A Nash equilibrium is called subgame perfect if it prescribes a Nash equilibrium in every subgame.

However, the set of subgame perfect equilibria in the contract game is always empty (if $m^i \geq 2$ for all $i \in \{1, 2\}$). Consider the subgame starting at the node where both players have proposed the same collection of contracts in such a way that the modified game in the second stage possesses no Nash equilibrium (notice that this can easily be done). Clearly, any proposed strategy in $\Gamma_c(G)$ does not prescribe a Nash equilibrium in this subgame and hence, no subgame perfect equilibrium exists.

The problem with the concept of subgame perfection in this particular model is that the game has too many subgames, some of which seem not particularly relevant. To tackle this problem, García-Jurado and González-Díaz (2006) introduce the concept of *virtually subgame perfect equilibrium*. For a strategy profile σ in an extensive form game Γ to be a virtually subgame perfect equilibrium, it must prescribe a Nash equilibrium in the σ-relevant subgames of Γ. A subgame of Γ is called σ-relevant if it is Γ itself or if it starts at a node that can be reached from a σ-relevant subgame by at most one unilateral deviation from σ.

Let us once more consider the contract game corresponding to the game of Example 6.2.5. In the Nash equilibrium presented before, both players propose a set of contracts in which all payoffs except at (T, L) are replaced by $(0, 0)$. Then given these contract choices, all σ-relevant subgames correspond to the second stage play of either the game in which all these contracts are executed, or the original game G. This is due to the fact that if only one player deviates from his contract proposal the sets of proposed contracts do not match, in which case the original game G is played in the second stage. In order to end up in a different game in the second stage, both players have to deviate from the equilibrium strategy profile, which means that such a subgame is not σ-relevant.

Hence, a particular strategy profile can only be a virtually subgame perfect equilibrium if it results in a Nash equilibrium in the original game G for each subgame in which the players are called to play this game. Such a strategy profile obviously does not exist in the game of Example 6.2.5 as the subgames in which G is played in the second stage do not possess a Nash equilibrium.

Next, consider the prisoners' dilemma in Example 6.2.3 and the equilibrium strategy profile proposed in this section for the corresponding contract game. Then we see that this strategy profile leads to a Nash equilibrium in all subgames in which G is played. Furthermore, it also constitutes a Nash equilibrium in the subgame in which the proposed contract set comes into effect. Therefore, this strategy profile is a virtually subgame perfect equilibrium.

These two examples indicate that there is a strong relation between the existence of Nash equilibria in the game G on the one hand and the existence of virtually subgame perfect equilibria in the contract game $\Gamma_c(G)$ on the other. The next theorem formalises this result.

Theorem 6.3.2 Let $G = (\{1, 2\}, \{M^i\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a finite two-player game. For every $r \in \mathbb{R}^2$ such that for every $i \in \{1, 2\}$ there exists a Nash equilibrium $\hat{x}(i)$ of G with $r^i \geq \pi^i(\hat{x}(i))$ and such that $r^1 + r^2 \leq \pi^1(x) + \pi^2(x)$ for some $x \in M$, there exists a virtually subgame perfect equilibrium of $\Gamma_c(G)$ with corresponding payoff vector r.

Proof: Let r, $\hat{x}(1)$ and $\hat{x}(2)$ be as stated in the theorem and let $\bar{x} \in M$ be such that $r^1 + r^2 \leq \pi^1(\bar{x}) + \pi^2(\bar{x})$. Define the strategy profile $(\hat{\alpha}, \hat{f})$ as follows.

For each $i \in \{1, 2\}$,

$$\alpha^i = \{(\bar{x}, r)\} \cup \left\{ (x, p) \,\middle|\, x \in M \setminus \{\bar{x}\}, \, p < v, \, p^1 + p^2 \leq \pi^1(x) + \pi^2(x) \, \forall x \in M \setminus \{\bar{x}\} \right\}.$$

The strategies in the second stage are given for all α by

$$\hat{f}(\alpha) = \begin{cases} \bar{x} & \text{if } \alpha = \hat{\alpha}, \\ \hat{x}(1) & \text{if } \alpha^1 \neq \hat{\alpha}^1, \alpha^2 = \hat{\alpha}^2, \\ \hat{x}(2) & \text{if } \alpha^1 = \hat{\alpha}^1, \alpha^2 \neq \hat{\alpha}^2, \\ x^* & \text{otherwise,} \end{cases}$$

where x^* is an arbitrary strategy profile of G.

Obviously, $\pi(\hat{\alpha}, \hat{f}) = r$. We check that $(\hat{\alpha}, \hat{f})$ is a virtually subgame perfect equilibrium of $\Gamma_c(G)$. First, in a similar way as in the proof of Theorem 6.3.1, one can show that $(\hat{\alpha}, \hat{f})$ is a Nash equilibrium of $\mathcal{G}_c(G)$.

The nodes which define a subgame (apart from the root) are the nodes corresponding to each profile of transfer contract collections (α^1, α^2). Of these, only the profiles reachable from unilateral deviations in the first stage, $(\hat{\alpha}^1, \alpha^2)$ and $(\alpha^1, \hat{\alpha}^2)$, give rise to $(\hat{\alpha}, \hat{f})$-relevant subgames. Consider the subgame in which player 1 has chosen $\alpha^1 \neq \hat{\alpha}^1$ in the first stage. In this subgame, $\hat{f}$ prescribes $\hat{x}(1)$, which is a Nash equilibrium in this subgame, because no contract is enforced. Similarly, $\hat{f}$ prescribes a Nash equilibrium in every $(\hat{\alpha}, \hat{f})$-relevant subgame in which player 2 has deviated. Hence, $(\hat{\alpha}, \hat{f})$ is a virtually subgame perfect equilibrium. □

Again, the conditions on the payoff vector r are necessary. For a strategy profile of $\Gamma_c(G)$ to be virtually subgame perfect, it has to prescribe a Nash equilibrium in the second stage in which the original game G is played, since this is always a relevant subgame. If, say, $\pi^1(x) > r^1$ for all $x \in NE(G)$ of G, then player 1 has an incentive to deviate and propose no contract.

Chapter 7

Fall back equilibrium

> Life will not bear refinement. You must do as other people do.
>
> Samuel Johnson (1709 - 1784)

7.1 Introduction

The notion of equilibrium for strategic games, introduced by Nash (1951), is the fundamental concept in non-cooperative game theory. The set of Nash equilibria, however, may be very large and can contain counterintuitive outcomes. In order to overcome these drawbacks Selten (1975) developed the concept of perfectness as a refinement of the Nash equilibrium concept. In the thought experiment underlying perfectness all players make mistakes in such a way that each action is played with positive probability. The notions of properness (Myerson (1978)), robustness (Okada (1983)), strict perfectness (Okada (1984)) and many others originated from Selten's work. Although these refinements differ in their exact concept, the common underlying idea is that an equilibrium should be stable against perturbations in the strategies due to mistakes made by the players of the game. This line of research culminated into the concept of stable sets (Kohlberg and Mertens (1986) and Mertens (1989 and 1991)). A partial overview of this literature can be found in Van Damme (1991).

In this chapter, which is based on Kleppe et al. (2008), we introduce a new equilibrium concept in which the strategy perturbations are based on another type of thought experiment. The idea is that each player faces a small but positive probability that, after all players decided on their action, the action chosen by him is blocked. Therefore, each player has to choose beforehand a back-up action, which he plays in case his first choice action, called *primary action*, is blocked.

The probability with which a player is unable to play his primary action and has to rely on his back-up is assumed to be independent of the particular choice he makes. This probability may, however, vary between the players. It is important to notice that, contrary to the perfectness concept in which players randomly play all other actions by mistake, in our setting players choose their back-up action strategically.

Example 7.1.1 Consider the next 2×4 bimatrix game G, where both players are allowed to randomise between their actions.

$$\begin{array}{cc} & \begin{array}{cccc} e_1^2 & e_2^2 & e_3^2 & e_4^2 \end{array} \\ \begin{array}{c} e_1^1 \\ e_2^1 \end{array} & \begin{bmatrix} 1,7 & 0,0 & 1,5 & 1,6 \\ 1,7 & 1,6 & 1,5 & 0,2 \end{bmatrix} \end{array}$$

Following Borm (1992) we analyse this game graphically in Figure 7.1.1. In this

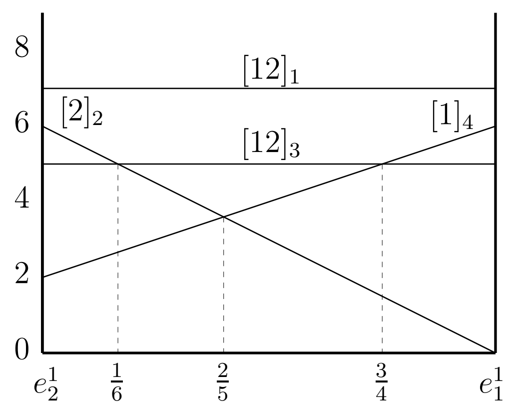


Figure 7.1.1: Graphical representation of G

figure the horizontal axis represents the strategy space of player 1, and each line describes player 2's payoff function corresponding to a particular action (indicated by the subindex). Each label displays player 1's set of pure best replies (either action 1, 2, or both) against the corresponding action of player 2.

In addition to the two proper equilibria on the boundaries of player 1's strategy space, there is a third proper equilibrium $(\frac{2}{5}e_1^1 + \frac{3}{5}e_2^1, e_1^2)$. Here player 1's strategy is the unique strategy for which player 2 is indifferent between his second and fourth action. Hence, although these actions are dominated by both e_1^2 and e_3^2 if player 1 plays (a strategy in the neighbourhood of) $\frac{2}{5}e_1^1 + \frac{3}{5}e_2^1$, the coordination point of these two actions determines a proper equilibrium. The reason is that the concept of proper equilibrium, like many other concepts, assumes full rationality of all players, which in this particular example implies that player 1 has to anticipate the possibility that player 2 makes the mistake of playing $\frac{1}{2}e_2^2 + \frac{1}{2}e_4^2$. Clearly, such an analysis requires a high level of rationality by the players on less relevant payoff levels in the game.

We instead assume that players are boundedly rational in the sense that they only take into account the possibility of a single back-up for each player. This is modelled by only allowing the primary action of a player to be blocked, and not also his back-up action. The set of fall back equilibria of this game is given by $\{(e_1^1, e_1^2)\} \cup \{(e_2^1, e_1^2)\} \cup \mathrm{conv}(\{\frac{1}{6}e_1^1 + \frac{5}{6}e_2^1, \frac{3}{4}e_1^1 + \frac{1}{4}e_2^1\}) \times \{e_2^2\}$. $\triangleleft$

The idea behind fall back equilibrium can also be applied to extensive form games. In that framework one has to make a distinction between the setup with mixed strategies and the setup with behavioural strategies. In the first one players decide on all their (possible) actions beforehand, whereas in the second setup players determine each choice at the moment they actually have to make it. In games with perfect recall, this distinction has no effect on the set of Nash equilibria (Kuhn (1953)). However, for the notion of fall back equilibrium it does matter whether a player faces the possibility of blocked actions once at the beginning of the game, or at each choice moment separately. For a brief discussion on this topic we refer to the end of Section 7.2.

In this chapter, however, we have chosen to focus on mixed extensions of finite non-cooperative games in strategic form. In the thought experiment players act by choosing both a primary and a back-up strategy. These strategies together define a strategy in the *fall back game*. Given that a player can choose between m actions in the original game, the fall back game has $m(m-1)$ actions to choose from, as players are not allowed to choose the same action both as primary and as back-up. The payoffs in the fall back game are the expected payoffs in the original game given the blocking probabilities. In the fall back game players are also allowed to use mixed strategies.

Example 7.1.2 Consider the 3×3 bimatrix game G, which is due to Myerson (1978), depicted below.

$$\begin{array}{cc} & \begin{array}{ccc} e_1^2 & e_2^2 & e_3^2 \end{array} \\ \begin{array}{c} e_1^1 \\ e_2^1 \\ e_3^1 \end{array} & \left[\begin{array}{ccc} 1,1 & 0,0 & -9,-9 \\ 0,0 & 0,0 & -7,-7 \\ -9,-9 & -7,-7 & -7,-7 \end{array}\right] \end{array}$$

Out of the three Nash equilibria of this game, (e_1^1, e_1^2), (e_2^1, e_2^2) and (e_3^1, e_3^2), only the last one is not a perfect equilibrium. The focal point of this game is (e_1^1, e_1^2), and this strategy profile is also the unique proper equilibrium, as the probability that players play the third row/column by mistake is significantly smaller than the probability of making any other mistake.

The strategy profile (e_1^1, e_1^2) is also the unique fall back equilibrium of G. In our framework, however, this is due to the fact that back-up actions are chosen strategically. Therefore, the strategy profile in which both players choose their first action as primary strategy and their second action as back-up strategy forms the unique equilibrium in the fall back game, which supports (e_1^1, e_1^2) as the unique fall back equilibrium. $\triangleleft$

In the thought experiment we introduce for the concept of fall back equilibrium we explicitly allow for strategic choices, as each player himself determines which alternative(s) to play in case the strategy he intended to play is unavailable. This is contrary to the thought experiment underlying proper equilibrium in which players cannot make these decisions as alternatives are ordered (exogenously) based upon the corresponding payoffs (given the opponents' actions). As it turns out, our line of thought culminates for bimatrix games in an alternative and *strategic* characterisation of proper equilibrium, as opposed to the *non-strategic* characterisation of properness by Blume et al. (1991) based on lexicographic belief systems.

The first result we obtain in this chapter is that the set of fall back equilibria is a non-empty and closed subset of the set of Nash equilibria. We also analyse the relations between fall back equilibrium on one hand and the equilibrium concepts of perfect, proper, strictly perfect and robust on the other. We prove that each robust equilibrium is a fall back equilibrium. Furthermore, for bimatrix games also each proper equilibrium is a fall back equilibrium, and consequently the intersection between the sets of fall back and perfect equilibria is non-empty. For games with more players this relation between proper and fall back equilibrium does not hold. The relation between the sets of fall back and strictly perfect equilibria is restricted to 2×2 bimatrix games. For these games the two sets coincide, otherwise the intersection can be empty, even if the set of strictly perfect equilibria itself is not.

Similar to the way Okada (1984) refines perfectness in strict perfectness we define the concept of strictly fall back equilibrium. It turns out that the sets of fall back and strictly fall back equilibria coincide for bimatrix games. However, for games with more than two players the set of strictly fall back equilibria can be empty.

For bimatrix games also the structure of the set of fall back equilibria is analysed. The main result is that the set of fall back equilibria is the union of finitely many polytopes.

In the thought experiment underlying fall back equilibrium we assume that only one action of each player can be blocked. We also consider two modifications of this concept. We first of all analyse the equilibrium concept that emerges when we allow multiple actions of each player to be blocked. The first main result provided for this concept, called complete fall back equilibrium, is that the set of complete fall back equilibria is a non-empty and closed subset of the set of proper equilibria. Secondly, for bimatrix games the sets of complete fall back and proper equilibria coincide, which means that the concept of complete fall back equilibrium is a strategic characterisation of proper equilibrium.

In the second modification we consider there can only be one blocked action in total. As a result the events of two players being blocked are no longer independent and therefore this equilibrium concept is called dependent fall back equilibrium. We show that for 2×2 bimatrix games the sets of dependent fall back and perfect equilibria coincide, but for bimatrix games in general the intersection between the two sets can be empty.

In the final section of this chapter we discuss bimatrix games in which at least one of the players only has two pure strategies, and for these type of games we characterise the sets of fall back, complete fall back and dependent fall back equilibria.

This chapter is organised as follows. In Section 7.2 we set up notation, formally introduce and characterise the concept of fall back equilibrium for strategic games, and present some basic results. In Section 7.3 we discuss the concept of strictly fall back equilibrium, while in Section 7.4 we consider the relations between fall back equilibrium and other equilibrium concepts. In Section 7.5 we discuss the structure of the set of fall back equilibria for bimatrix games. Section 7.6 covers the analysis of the concept of complete fall back equilibrium, and in Section 7.7 we consider dependent fall back equilibrium. At the end of the latter section we also provide an overview of the relations between all equilibrium concepts discussed in this chapter, both for n-player strategic games and for bimatrix games. Section 7.8 is devoted to the analysis of fall back equilibrium and its related concepts in bimatrix games in which at least one of the players has two pure strategies. We conclude this section with an overview of the relations between all discussed equilibrium concepts, for this type of games.

7.2 Fall back equilibrium

A non-cooperative game in strategic form is given by $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$, with $N = \{1, \ldots, n\}$ the player set, Δ_{M^i} the mixed strategy space of player $i \in N$, with $M^i = \{1, \ldots, m^i\}$ the set of pure strategies, and $\pi^i : \prod_{j\in N} \Delta_{M^j} \to \mathbb{R}$ the Von Neumann Morgenstern expected payoff function of player i. A pure strategy $k \in M^i$ of player i is alternatively denoted by e^i_k, a typical element of Δ_{M^i} by x^i. We denote the probability which x^i assigns to pure strategy (action) k by x^i_k. The set of all strategy profiles is given by $\Delta = \prod_{i\in N} \Delta_{M^i}$, a typical element of Δ by x.

A strategy profile $\hat{x}$ is a *Nash equilibrium* (Nash (1951)) of G, denoted by $\hat{x} \in NE(G)$, if $\pi^i(\hat{x}) \geq \pi^i(x^i, \hat{x}^{-i})$ for all $x^i \in \Delta_{M^i}$ and all $i \in N$. Here $(x^i, \hat{x}^{-i})$ is the frequently used shorthand notation for the strategy profile $(\hat{x}^1, \ldots, \hat{x}^{i-1}, x^i, \hat{x}^{i+1}, \ldots, \hat{x}^n)$.

The *carrier* of a strategy x^i is given by $C(x^i) = \{k \in M^i \,|\, x^i_k > 0\}$, the *pure best reply correspondence* of player i by $PB^i(x^{-i}) = \{k \in M^i \,|\, \pi^i(e^i_k, x^{-i}) \geq \pi^i(e^i_\ell, x^{-i})$ for all $\ell \in M^i\}$. Clearly, $\hat{x} \in NE(G)$ if and only if $C(\hat{x}^i) \subseteq PB^i(\hat{x}^{-i})$ for all $i \in N$.

Next we formalise the thought experiment as described in the introduction of this chapter. The action set for player i in the associated fall back game (only defined if $m^j \geq 2$ for all $j \in N$) is given by $\tilde{M}^i = \{(k, \ell) \in M^i \times M^i \,|\, k \neq \ell\}$. Hence, the total number of actions in the fall back game for player i is $\tilde{m}^i = m^i(m^i - 1)$. An action $(k, \ell) \in \tilde{M}^i$ consists of a *primary action* k and a *back-up action* ℓ. Let

$\varepsilon = (\varepsilon^1, \ldots, \varepsilon^n)$ be an n-tuple of (small) non-negative probabilities. The interpretation of player i's action (k, ℓ) in the fall back game is that he plays in the original game with probability $1-\varepsilon^i$ primary action k and with probability ε^i back-up action ℓ.

The *fall back game* $\tilde{G}(\varepsilon)$ is given by $\tilde{G}(\varepsilon) = (N, \{\Delta_{\tilde{M}^i}\}_{i\in N}, \{\pi^i_\varepsilon\}_{i\in N})$, with $\pi^i_\varepsilon : \prod_{j\in N} \Delta_{\tilde{M}^j} \to \mathbb{R}$ the extended expected payoff function of player i. The pure strategy $(k, \ell) \in \tilde{M}^i$ is alternatively denoted by $e^i_{k\ell}$. The payoff function π^i_ε is for pure strategy combinations formally defined by

$$\pi^i_\varepsilon((e^j_{k^j\ell^j})_{j\in N}) = \sum_{S\subseteq N} (\prod_{j\in S}(1-\varepsilon^j) \prod_{j\in N\setminus S} \varepsilon^j)\pi^i((e^j_{k^j})_{j\in S}, (e^j_{\ell^j})_{j\in N\setminus S}).$$

A typical element of $\Delta_{\tilde{M}^i}$ is denoted by ρ^i, where $\rho^i_{k\ell}$ is the probability which ρ^i assigns to pure strategy (k, ℓ). Note that ρ^i assigns probabilities to pure strategies (k, ℓ) of the fall back game, not to primary and back-up actions separately. The set of all strategy profiles is given by $\tilde{\Delta} = \prod_{i\in N} \Delta_{\tilde{M}^i}$, an element of $\tilde{\Delta}$ will be denoted by ρ.

Definition Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game. A strategy profile $x \in \Delta$ is a *fall back equilibrium* of G if there exists a sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of n-tuples of positive real numbers converging to zero, and a sequence $\{\rho_t\}_{t\in\mathbb{N}}$ such that $\rho_t \in NE(\tilde{G}(\varepsilon_t))$ for all $t \in \mathbb{N}$, converging to $\rho \in \tilde{\Delta}$, with $x^i_k = \sum_{\ell\in M^i\setminus\{k\}} \rho^i_{k\ell}$ for all $k \in M^i$ and all $i \in N$. The set of fall back equilibria of G is denoted by $FBE(G)$.

In the thought experiment underlying the concept of fall back equilibrium each player faces the small but positive probability that, after all players decided on their action, the action chosen by him is blocked. In that case the player plays the back-up action he chose beforehand. This is modelled by letting players play the fall back game in which each action consists of a primary action, played with a probability close to one, and a back-up action, played with the remaining probability. A fall back equilibrium of the original game is then deduced from the limit point of a sequence of Nash equilibria of the corresponding fall back games when the blocking probabilities converge to zero.

Theorem 7.2.1 Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game. Then the set of fall back equilibria of G is a non-empty and closed subset of the set of Nash equilibria of G.

Proof: We first show non-emptiness. Let $\{\varepsilon_t\}_{t\in\mathbb{N}}$ be a sequence of n-tuples of positive real numbers converging to zero. Take a sequence $\{\rho_t\}_{t\in\mathbb{N}}$ such that $\rho_t \in NE(\tilde{G}(\varepsilon_t))$ for all $t \in \mathbb{N}$. Because the strategy spaces are compact there exists a subsequence of $\{\rho_t\}_{t\in\mathbb{N}}$ converging to, say, $\rho \in \tilde{\Delta}$. Define $x \in \Delta$ by $x^i_k = \sum_{\ell\in M^i\backslash\{k\}} \rho^i_{k\ell}$ for all $k \in M^i$ and all $i \in N$. By definition $x \in FBE(G)$.

Next we prove that each fall back equilibrium is a Nash equilibrium. Take $x \in FBE(G)$. We prove that $x \in NE(G)$ by showing that $C(x^i) \subseteq PB^i(x^{-i})$ for all $i \in N$. Take a sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of n-tuples of positive real numbers converging to zero and a sequence $\{\rho_t\}_{t\in\mathbb{N}}$ such that $\rho_t \in NE(\tilde{G}(\varepsilon_t))$ for all $t \in \mathbb{N}$, converging to $\rho \in \tilde{\Delta}$, with $x^i_k = \sum_{\ell\in M^i\backslash\{k\}} \rho^i_{k\ell}$ for all $k \in M^i$ and all $i \in N$. Let $i \in N$ and $k \in C(x^i)$. Then for sufficiently large $t \in \mathbb{N}$ we have that $(k,\ell) \in C(\rho^i_t)$ for some $\ell \in M^i\backslash\{k\}$. Hence,

$$\pi^i_{\varepsilon_t}(e^i_{k\ell}, \rho^{-i}_t) \;\;\geq\;\; \pi^i_{\varepsilon_t}(e^i_{rs}, \rho^{-i}_t)$$

for every $(r,s) \in \tilde{M}^i$. Taking t to infinity, we find by continuity of π^i_ε,

$$\pi^i_0(e^i_{k\ell}, \rho^{-i}) \;\;\geq\;\; \pi^i_0(e^i_{rs}, \rho^{-i})$$

for every $(r,s) \in \tilde{M}^i$. Since $\pi^i_0(e^i_{k\ell}, \rho^{-i}) = \pi^i(e^i_k, x^{-i})$ and similarly $\pi^i_0(e^i_{rs}, \rho^{-i}) = \pi^i(e^i_r, x^{-i})$, it follows that

$$\pi^i(e^i_k, x^{-i}) \;\;\geq\;\; \pi^i(e^i_r, x^{-i})$$

for every $r \in M^i$ and hence $k \in PB^i(x^{-i})$.

Finally we show that $FBE(G)$ is closed. Take a converging sequence $\{x_t\}_{t\in\mathbb{N}}$ with $x_t \in FBE(G)$ for all $t \in \mathbb{N}$, with limit $x \in \Delta$. For all $t \in \mathbb{N}$ there exists a sequence $\{\varepsilon_{tr}\}_{r\in\mathbb{N}}$ of n-tuples of positive real numbers converging to zero and a sequence $\{\rho_{tr}\}_{r\in\mathbb{N}}$ converging to $\rho_t \in \tilde{\Delta}$, with $x^i_{t,k} = \sum_{\ell\in M^i\backslash\{k\}} \rho^i_{t,k\ell}$ for all $k \in M^i$ and all $i \in N$, such that

$$x_{tr} \in NE(\tilde{G}(\varepsilon_{tr}))$$

for all $r \in \mathbb{N}$. Considering the sequences $\{\varepsilon_{tt}\}_{t\in\mathbb{N}}$ and $\{x_{tt}\}_{t\in\mathbb{N}}$ one readily establishes that $x \in FBE(G)$. $\square$

Although the definition of fall back equilibrium is natural in its interpretation, the fact that the size of the payoff matrices is larger in the fall back game than in the original game makes further analysis complicated. Therefore, we now provide an alternative characterisation of fall back equilibrium.

For a (sufficiently small) blocking vector $\delta \in \mathbb{R}^N_+$, the *blocking game* $G(\delta) = (N, \{\Delta_{M^i}(\delta^i)\}_{i\in N}, \{\pi^i\}_{i\in N})$ is defined to be the game which only differs from $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ in the sense that the strategy spaces are restricted to

$$\Delta_{M^i}(\delta^i) = \{x^i \in \Delta_{M^i} \mid x^i_k \leq 1 - \delta^i \text{ for all } k \in M^i\}$$

for all $i \in N$ and the domains of the payoff functions are restricted accordingly. Define the set of all strategy profiles of the blocking game by $\Delta(\delta) = \prod_{j\in N} \Delta_{M^j}(\delta^j)$.

Note that the strategy spaces of the blocking game, with $\delta > 0$, restrict each player to play at least two of his original actions with positive probability, but also allow him to play some actions with zero probability.

Lemma 7.2.2 *Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game. Let $\{\varepsilon_t\}_{t\in\mathbb{N}}$ and $\{\delta_t\}_{t\in\mathbb{N}}$ be sequences of n-tuples of positive real numbers converging to zero such that $\varepsilon_t = \delta_t$ for all $t \in \mathbb{N}$, with corresponding fall back and blocking games $\tilde{G}(\varepsilon_t) = (N, \{\Delta_{\tilde{M}^i}\}_{i\in N}, \{\pi^i_{\varepsilon_t}\}_{i\in N})$ and $G(\delta_t) = (N, \{\Delta_{M^i}(\delta^i_t)\}_{i\in N}, \{\pi^i\}_{i\in N})$ respectively.*

Then for each sequence $\{\rho_t\}_{t\in\mathbb{N}}$ converging to ρ, with $\rho_t \in \tilde{\Delta}$ for all $t \in \mathbb{N}$, there exists a sequence $\{x_t\}_{t\in\mathbb{N}}$ converging to x, with $x_t \in \Delta(\delta_t)$ for all $t \in \mathbb{N}$, such that $x^i_k = \sum_{\ell\in M^i\setminus\{k\}} \rho^i_{k\ell}$ for all $k \in M^i$ and all $i \in N$, and $\pi^i(x_t) = \pi^i_{\varepsilon_t}(\rho_t)$ for all $i \in N$ and all $t \in \mathbb{N}$.

Conversely, for each sequence $\{x_t\}_{t\in\mathbb{N}}$ converging to x, with $x_t \in \Delta(\delta_t)$ for all $t \in \mathbb{N}$, there exists a sequence $\{\rho_t\}_{t\in\mathbb{N}}$ converging to ρ, with $\rho_t \in \tilde{\Delta}$ for all $t \in \mathbb{N}$, such that $x^i_k = \sum_{\ell\in M^i\setminus\{k\}} \rho^i_{k\ell}$ for all $k \in M^i$ and all $i \in N$, and $\pi^i_{\varepsilon_t}(\rho_t) = \pi^i(x_t)$ for all $i \in N$ and all $t \in \mathbb{N}$.

Proof: Let $\{\rho_t\}_{t\in\mathbb{N}}$ be a sequence converging to $\rho \in \tilde{\Delta}$, with $\rho_t \in \tilde{\Delta}$ for all $t \in \mathbb{N}$. We define the sequence $\{x_t\}_{t\in\mathbb{N}}$ such that for all $t \in \mathbb{N}$

$$x^i_{t,k} = (1 - \delta^i_t) \sum_{\ell\in M^i\setminus\{k\}} \rho^i_{t,k\ell} + \delta^i_t \sum_{\ell\in M^i\setminus\{k\}} \rho^i_{t,\ell k} \tag{7.1}$$

for all $k \in M^i$ and all $i \in N$. Then $x_t \in \Delta(\delta_t)$ for all $t \in \mathbb{N}$ and $\{x_t\}_{t\in\mathbb{N}}$ converges to x, with $x^i_k = \sum_{\ell \in M^i \setminus \{k\}} \rho^i_{k\ell}$ for all $k \in M^i$ and all $i \in N$. Furthermore, because $\varepsilon_t = \delta_t$ for all $t \in \mathbb{N}$ the strategy profile x_t puts the same probabilities on the actions of the game G as ρ_t for all $t \in \mathbb{N}$. Therefore, $\pi^i(x_t) = \pi^i_{\varepsilon_t}(\rho_t)$ for all $i \in N$ and all $t \in \mathbb{N}$.

The reverse statement is shown similarly, with the sequence $\{\rho_t\}_{t\in\mathbb{N}}$ defined in such a way that equation (7.1) is satisfied. Note that since $x_t \in \Delta(\delta_t)$ and $\varepsilon_t = \delta_t$ for all $t \in \mathbb{N}$, and therefore at least two actions of each player i of game G are played with a probability of at least ε^i_t, it is always possible to construct such a sequence $\{\rho_t\}_{t\in\mathbb{N}}$. □

Note that the restrictions with respect to strategies of the original game are the same in the blocking game as in the fall back game (if $\delta = \varepsilon$). As a consequence of Lemma 7.2.2, a fall back equilibrium can also be defined in terms of Nash equilibria of blocking games.

Theorem 7.2.3 Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game. Then a strategy profile $x \in \Delta$ is a fall back equilibrium of G if and only if there exists a sequence $\{\delta_t\}_{t\in\mathbb{N}}$ of blocking vectors of positive real numbers converging to zero and a sequence $\{x_t\}_{t\in\mathbb{N}}$ converging to x such that $x_t \in NE(G(\delta_t))$ for all $t \in \mathbb{N}$.

Proof: We just prove the "only if" part, the reverse statement can be shown analogously. Assume $\hat{x} \in FBE(G)$. Then by definition there exists a sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of n-tuples of positive real numbers converging to zero, and a sequence $\{\hat{\rho}_t\}_{t\in\mathbb{N}}$ converging to $\hat{\rho} \in \tilde{\Delta}$, with $\hat{x}^i_k = \sum_{\ell \in M^i \setminus \{k\}} \hat{\rho}^i_{k\ell}$ for every $k \in M^i$ and all $i \in N$, such that $\hat{\rho}_t \in NE(\tilde{G}(\varepsilon_t))$ for all $t \in \mathbb{N}$. By Lemma 7.2.2 there exists a sequence $\{\hat{x}_t\}_{t\in\mathbb{N}}$ converging to $\hat{x} \in \Delta$, with $\hat{x}_t \in \Delta(\delta_t)$ for all $t \in \mathbb{N}$, such that $\pi^i(\hat{x}_t) = \pi^i_{\varepsilon_t}(\hat{\rho}_t)$ for all $i \in N$ and all $t \in \mathbb{N}$.

Let $i \in N$. We show that $\pi^i(\hat{x}_t) \geq \pi^i(x^i_t, \hat{x}^{-i}_t)$ for all $x^i_t \in \Delta_{M^i}(\delta^i_t)$ and all $t \in \mathbb{N}$, which proves that $\hat{x}_t \in NE(G(\delta_t))$ for all $t \in \mathbb{N}$ and therefore completes the proof. Let $t \in \mathbb{N}$ and let $(x^i_t, \hat{x}^{-i}_t) \in \Delta(\delta_t)$. Then by Lemma 7.2.2 we can take a strategy $(\rho^i_t, \hat{\rho}^{-i}_t) \in \tilde{\Delta}$ such that $\pi^i_{\varepsilon_t}(\rho^i_t, \hat{\rho}^{-i}_t) = \pi^i(x^i_t, \hat{x}^{-i}_t)$.

Since $\hat{\rho}_t \in NE(\tilde{G}(\varepsilon_t))$ we obtain

$$\begin{aligned}\pi^i(x_t^i, \hat{x}_t^{-i}) &= \pi^i_{\varepsilon_t}(\rho_t^i, \hat{\rho}_t^{-i}) \\ &\leq \pi^i_{\varepsilon_t}(\hat{\rho}_t) \\ &= \pi^i(\hat{x}_t).\end{aligned}$$

Consequently, $\pi^i(\hat{x}_t) \geq \pi^i(x_t^i, \hat{x}_t^{-i})$ for all $x_t^i \in \Delta_{M^i}(\delta_t^i)$ and all $t \in \mathbb{N}$. □

Since a blocking game with $\delta > 0$ only excludes the possibility for any player to play an original action with probability one, we obtain the following proposition.

Proposition 7.2.4 *Let $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$ be an n-player strategic game and let $x \in \Delta$ be such that $|C(x^i)| > 1$ for all $i \in N$. Then x is a fall back equilibrium of G if and only if x is a Nash equilibrium of G.*

In the thought experiment underlying perfectness (Selten (1975)) players also play a perturbed game. Let $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$ be an n-player strategic game. A pertubation vector for player $i \in N$ is given by $\varepsilon^i \in \mathbb{R}^{M^i}$, with $\varepsilon_k^i > 0$ for all $k \in M^i$ and $\sum_{k \in M^i} \varepsilon_k^i \leq 1$. Then the ε-perturbed game $H(\varepsilon) = (N, \{\Delta_{M^i}(\varepsilon^i)\}_{i \in N}, \{\pi^i\}_{i \in N})$ is defined to be the game which only differs from G in the sense that the strategy spaces are restricted to

$$\Delta_{M^i}(\varepsilon^i) = \{x^i \in \Delta_{M^i} \,|\, x_k^i \geq \varepsilon_k^i \text{ for all } k \in M^i\}$$

for all $i \in N$ and the domains of the payoff functions are restricted accordingly. Define the set of all strategy profiles of the ε-perturbed game by $\Delta(\varepsilon) = \prod_{j \in N} \Delta_{M^j}(\varepsilon^j)$.

Definition Let $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$ be an n-player strategic game. A strategy profile $x \in \Delta$ is a *perfect equilibrium* of G if there exists a sequence $\{\varepsilon_t\}_{t \in \mathbb{N}}$ of pertubation vectors converging to zero, and a sequence $\{x_t\}_{t \in \mathbb{N}}$ converging to x, such that $x_t \in NE(H(\varepsilon_t))$ for all $t \in \mathbb{N}$. The set of perfect equilibria of G is denoted by $PE(G)$.

Apparently, an important difference between fall back and perfect equilibrium is that in the thought experiment underlying perfectness all actions have to be played with positive probability and for fall back equilibrium only at least two. This observation leads to the following proposition.

Proposition 7.2.5 *Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game such that $m^i = 2$ for all $i \in N$. Then the sets of fall back and perfect equilibria of G coincide.*

We conclude this section with an example in which we show that the idea behind fall back equilibrium can also be applied to extensive form games, but that in the setup with mixed strategies this leads to a different concept as in the setup with behavioural strategies. Moreover, the two concepts may support different leaves of the extensive form game as a fall back equilibrium

Example 7.2.6 Consider the extensive form game Γ depicted in Figure 7.2.1. Note that in the corresponding normal form game player 1 has eight actions. Since only one of player 1's actions can be blocked he is always able to play A followed by C by choosing ACE as primary and ACF as back-up strategy. Consequently, the strategy profile in which player 1 plays ACE and player 2 plays x constitutes a fall back equilibrium, with outcome $(3, 1)$.

In the setup with behavioural strategies players determine their choice at the moment they actually have to make it. Hence, when we apply the notion of fall back equilibrium, given this setup, to the extensive form game of Figure 7.2.1, it follows that when player 1 chooses C (at the node after his play of A and player 2's choice of x) there is a positive probability that C is blocked. Consequently, where in the setup with mixed strategies player 1 was able to play C with probability one, this is not possible in the setup with behavioural strategies. As a result, after playing A, there always is a positive probability that player 1 plays D if player 2 chooses x. Therefore, player 2 will always choose y as primary strategy, which implies that outcome $(3, 1)$ cannot be supported as a fall back equilibrium in this setup. $\triangleleft$

7.3 Strictly fall back equilibrium

Okada (1984) refines the perfectness concept to strict perfectness by requiring that for *every* sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of pertubation vectors converging to zero there exists a sequence $\{x_t\}_{t\in\mathbb{N}}$ of strategy profiles converging to x such that $x_t \in NE(G(\varepsilon_t))$ for all $t \in \mathbb{N}$. In a similar way we introduce the concept of strictly fall back equilibrium.

Definition Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game. A

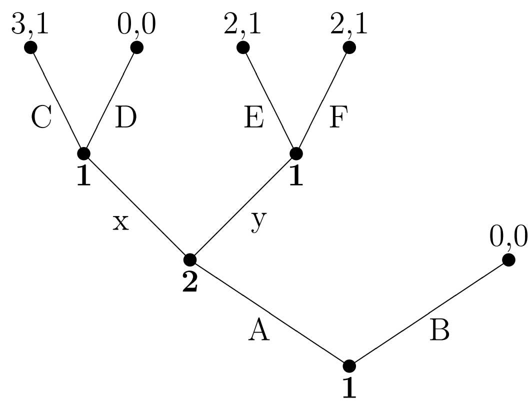


Figure 7.2.1: Extensive form game Γ

strategy profile $x \in \Delta$ is a *strictly fall back equilibrium* of G if for every sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of n-tuples of positive real numbers converging to zero there exists a sequence $\{\rho_t\}_{t\in\mathbb{N}}$ such that $\rho_t \in NE(\tilde{G}(\varepsilon_t))$ for all $t \in \mathbb{N}$, converging to $\rho \in \tilde{\Delta}$, with $x^i_k = \sum_{\ell \in M^i \setminus \{k\}} \rho^i_{k\ell}$ for all $k \in M^i$ and all $i \in N$. The set of strictly fall back equilibria of G is denoted by $SFBE(G)$.

Note that if we impose in Theorem 7.2.3 the requirement *for every* sequence $\{\delta_t\}_{t\in\mathbb{N}}$ of blocking vectors of positive real numbers converging to zero, we get in the same way an equivalent characterisation of strictly fall back equilibrium in terms of blocking vectors.

Since the sets of fall back and perfect equilibria are refined to a strict concept in a similar way, we obtain the following corollary.

Corollary 7.3.1 *Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game, with $m^i = 2$ for all $i \in N$. Then the sets of strictly fall back and strictly perfect equilibria of G coincide.*

Since the set of strictly perfect equilibria can be empty for three-player games with action sets of size two for all players, also the set of strictly fall back equilibria can be empty if the number of players is three.

However, for any strategic game with only two players, *i.e.* bimatrix games, the set of strictly fall back equilibria is non-empty since in that case it coincides with the non-empty set of fall back equilibria. Before we can prove this result we first have to provide a second characterisation of fall back equilibrium, which can only be applied to bimatrix games. This characterisation is convenient as it does not make use of perturbed games or converging sequences.

Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game and let $x \in \Delta$. Then, we first of all define the *pure second best reply correspondence* of player i by

$$PSB^i(x^{-i}) = \Big\{k \in M^i \,\Big|\, \exists\, \ell \in M^i\backslash\{k\} : \pi^i(e^i_\ell, x^{-i}) \geq \pi^i(e^i_k, x^{-i}), \\ \pi^i(e^i_k, x^{-i}) \geq \pi^i(e^i_r, x^{-i})\,\forall r \in M^i\backslash\{\ell\}\Big\}.$$

Note that if $|PB^i(x^{-i})| > 1$, then $PB^i(x^{-i}) = PSB^i(x^{-i})$. Also note that the correspondences PB^i and PSB^i are upper-semi-continuous.

In the blocking game the strategy of each player is composed of primary and back-up strategies. The preferences of player i over possible actions are independent of δ^i, as they only depend on the strategies of the other players. Furthermore, if the probability on the back-up strategies, $\bar{\delta}^j$, of player $j \neq i$ in a blocking game corresponding to a bimatrix game is sufficiently close to zero, the set of best replies for player i in the fall back game is, by upper-semi-continuity of PB^i and PSB^i, the same for all $\delta^j \in (0, \bar{\delta}^j]$. Consequently, the best replies of both players in a blocking game corresponding to a bimatrix game are independent of the blocking vector $\delta \in \mathbb{R}^2_{++}$ when δ is sufficiently close to zero.

Proposition 7.3.2 *Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a bimatrix game. Then a strategy profile $x = (x^1, x^2) \in \Delta$ is a fall back equilibrium of G if and only if one of the following three statements is satisfied.*

1. *$|C(x^1)| > 1$, $|C(x^2)| > 1$ and $x \in NE(G)$.*
2. *For $i, j \in \{1,2\}, i \neq j$: $|C(x^i)| > 1$, $|C(x^j)| = 1$ and there exists a strategy $\bar{x}^j \in \Delta_{M^j}$ such that $C(\bar{x}^j) \cap C(x^j) = \emptyset$ and a blocking probability $\bar{\delta}^j > 0$, such that for all $\delta^j \in (0, \bar{\delta}^j]$ the strategy profile $\hat{x} = (x^i, \hat{x}^j)$, with $\hat{x}^j = (1-\delta^j)x^j + \delta^j \bar{x}^j$, satisfies*

$$\begin{aligned} C(x^i) &\subseteq PB^i(\hat{x}^j), \\ C(x^j) &\subseteq PB^j(x^i), \\ C(\bar{x}^j) &\subseteq PSB^j(x^i). \end{aligned}$$

3. $|C(x^1)| = |C(x^2)| = 1$ *and there exists for all* $i \in \{1,2\}$ *a strategy* $\bar{x}^i \in \Delta_{M^i}$ *such that* $C(\bar{x}^i) \cap C(x^i) = \emptyset$ *and a blocking probability* $\bar{\delta}^i > 0$, *such that for all* $\delta \in \mathbb{R}^2_{++}$, *with* $\delta^i \in (0, \bar{\delta}^i]$ *for all* $i \in \{1,2\}$, *the strategy profile* $(\hat{x}^1, \hat{x}^2)$, *with* $\hat{x}^i = (1-\delta^i)x^i + \delta^i \bar{x}^i$ *for all* $i \in \{1,2\}$, *satisfies*

$$\begin{aligned} C(x^1) &\subseteq PB^1(\hat{x}^2), \\ C(x^2) &\subseteq PB^2(\hat{x}^1), \\ C(\bar{x}^1) &\subseteq PSB^1(\hat{x}^2), \\ C(\bar{x}^2) &\subseteq PSB^2(\hat{x}^1). \end{aligned}$$

Proof: We first prove the "if" part. We do this by distinguishing between the three cases. If the first statement is satisfied, the result follows immediately from Proposition 7.2.4.

Next assume that the second statement is satisfied. Let $\{\delta_t\}_{t\in\mathbb{N}}$ be a sequence of pairs of positive real numbers converging to zero. Define the sequence $\{\hat{x}_t\}_{t\in\mathbb{N}}$ such that

$$\begin{aligned} \hat{x}^i_t &= x^i, \\ \hat{x}^j_t &= (1-\delta^j_t)x^j + \delta^j_t \bar{x}^j \end{aligned}$$

for all $t \in \mathbb{N}$. Then the sequence $\{\hat{x}_t\}_{t\in\mathbb{N}}$ converges to x and $\hat{x}_t \in \Delta(\delta_t)$ for all $t \in \mathbb{N}$. Let $t \in \mathbb{N}$ be such that $\delta^j_t \leq \bar{\delta}^j$. We prove that $\hat{x}_t \in NE(G(\delta_t))$ by showing that both players play best replies in $G(\delta_t)$. Since $C(x^i) \subseteq PB^i(\hat{x}^j)$ player i plays by playing $\hat{x}^i_t$ a best reply against $\hat{x}^j_t$ in $G(\delta_t)$. Furthermore, since $C(x^j) \subseteq PB^j(x^i)$ player j puts by playing $\hat{x}^j_t$ maximal probability on best reply actions against $\hat{x}^i_t$, and as $C(\bar{x}^j) \subseteq PB^j(\hat{x}^i)$ he puts the remaining probability on second best reply actions against $\hat{x}^i_t$.

We finally assume that the third statement holds. Let $\{\delta_t\}_{t\in\mathbb{N}}$ be a sequence of pairs of positive real numbers converging to zero.

Define the sequence $\{\hat{x}_t\}_{t\in\mathbb{N}}$ such that for all $i \in \{1,2\}$,

$$\hat{x}_t^i \;=\; (1-\delta_t^i)x^i + \delta_t^i \bar{x}^i$$

for all $t \in \mathbb{N}$. Then the sequence $\{\hat{x}_t\}_{t\in\mathbb{N}}$ converges to x and $\hat{x}_t \in \Delta(\delta_t)$ for all $t \in \mathbb{N}$. Let $t \in \mathbb{N}$ be such that $\delta_t^i \leq \bar{\delta}^i$ for all $i \in \{1,2\}$. We prove that $\hat{x}_t \in NE(G(\delta_t))$ by showing that player 1 plays a best reply in $G(\delta_t)$. Showing that player 2 plays a best reply can be done analogously. Since, $C(x^1) \subseteq PB^1(\hat{x}^2)$ player 1 puts by playing $\hat{x}_t^1$ maximal probability on best reply actions against $\hat{x}_t^2$, and as $C(\bar{x}^1) \subseteq PB^1(\hat{x}^2)$ the remaining probability on second best reply actions against $\hat{x}_t^2$. Hence, player 1 plays a best reply in $G(\delta_t)$.

We now show the "only if" part. Let $x \in FBE(G)$. If $|C(x^1)| > 1$ and $|C(x^2)| > 1$ it follows from Proposition 7.2.4 that the first statement is satisfied. Otherwise, by Theorem 7.2.3 there exists a sequence of blocking vectors $\{\delta_t\}_{t\in\mathbb{N}}$ of positive real numbers converging to zero and a sequence of strategy profiles $\{x_t\}_{t\in\mathbb{N}}$ converging to x such that $x_t \in NE(G(\delta_t))$ for all $t \in \mathbb{N}$.

Let $i \in \{1,2\}$. Since $x_t \in \Delta(\delta_t)$ for all $t \in \mathbb{N}$, if $|C(x^i)| = 1$, then for all $t \in \mathbb{N}$ there exists a strategy $\bar{x}_t^i \in \Delta_{M^i}$, with $C(\bar{x}_t^i) \cap C(x^i) = \emptyset$, such that $x_t^i = (1-\xi_t^i)x^i + \xi_t^i \bar{x}_t^i$, with $\xi_t^i \geq \delta_t^i$. Take $\hat{t} \in \mathbb{N}$ sufficiently large, and define

$$\hat{x}^i(\xi^i) = (1-\xi^i)x^i + \xi^i \bar{x}_{\hat{t}}^i.$$

Then, by the upper-semi-continuity of PB^j and PSB^j, we obtain that $PB^j(\hat{x}^i(\xi^i)) = PB^j(\hat{x}^i(\tilde{\xi}^i))$ and $PSB^j(\hat{x}^i(\xi^i)) = PSB^j(\hat{x}^i(\tilde{\xi}^i))$ for all $\xi^i, \tilde{\xi}^i \in (0, \xi_{\hat{t}}^i]$.

Take for all $i \in \{1,2\}$ some $\delta^i \in (0, \xi_{\hat{t}}^i]$ and define $\hat{x}$ such that

$$\hat{x}^i = \begin{cases} x^i & \text{if } |C(x^i)| > 1, \\ \hat{x}^i(\delta^i) & \text{if } |C(x^i)| = 1 \end{cases}$$

for all $i \in \{1,2\}$. Then we show that either the second or the third statement is satisfied.

Let us first assume that $|C(x^i)| > 1$ and $|C(x^j)| = 1$. We show that for strategy $\hat{x} = (x^i, \hat{x}^j(\delta^j))$ it holds that $C(x^i) \subseteq PB^i(\hat{x}^j(\delta^j))$, $C(x^j) \subseteq PB^i(x^i)$ and

$C(\bar{x}^j_{\hat{t}}) \subseteq PSB^j(x^i)$. Note that $x^i_{\hat{t}}$ is a best reply against $x^j_{\hat{t}}$ in $G(\delta_{\hat{t}})$. Since $x^i_{\hat{t}}$ is close to x^i (as $x^i_{\hat{t}}$ converges to x^i and $\hat{t}$ was chosen sufficiently large) we obtain that $C(x^i) \subseteq PB^i(x^j_{\hat{t}})$. Because $PB^i(x^j_{\hat{t}}) = PB^i(\hat{x}^j(\delta^j))$ we obtain $C(x^i) \subseteq PB^i(\hat{x}^j(\delta^j))$. Then, $x \in FBE(G)$ and therefore $x \in NE(G)$, which immediately gives the second result that $C(x^j) \subseteq PB^j(x^i)$. Furthermore, since $x^j_{\hat{t}}$ is a best reply against $x^i_{\hat{t}}$ in $G(\delta_{\hat{t}})$, $C(\bar{x}^j_{\hat{t}}) \subseteq PSB^j(x^i_{\hat{t}})$, and as $PSB^j(x^i_{\hat{t}}) = PSB^j(x^i)$ it follows that $C(\bar{x}^j_{\hat{t}}) \subseteq PSB^j(x^i)$.

We now consider the case $|C(x^1)| = |C(x^2)| = 1$. We show that for $\hat{x} = (\hat{x}^1(\delta^1), \hat{x}^2(\delta^2))$ it holds that $C(x^1) \subseteq PB^1(\hat{x}^2(\delta^2))$ and $C(\bar{x}^1) \subseteq PSB^1(\hat{x}^2(\delta^2))$. Showing that $C(x^2) \subseteq PB^2(\hat{x}^1(\delta^1))$ and $C(\bar{x}^2) \subseteq PSB^2(\hat{x}^1(\delta^1))$ can be done analogously. Since $x^1_{\hat{t}}$ is a best reply against $x^2_{\hat{t}}$ in $G(\delta_{\hat{t}})$ it must hold (since $\hat{t}$ was chosen sufficiently large) that $C(x^1) \subseteq PB^1(x^2_{\hat{t}})$ and $C(\bar{x}^1) \subseteq PSB^1(x^2_{\hat{t}})$. As $PB^1(x^2_{\hat{t}}) = PB^1(\hat{x}^2(\delta^2))$ and $PSB^1(x^2_{\hat{t}}) = PSB^1(\hat{x}^2(\delta^2))$ we obtain both $C(x^1) \subseteq PB^1(\hat{x}^2(\delta^2))$ and $C(\bar{x}^1) \subseteq PSB^1(\hat{x}^2(\delta^2))$. □

By the use of Proposition 7.3.2 we can prove the following theorem.

Theorem 7.3.3 Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a bimatrix game. Then the sets of fall back and strictly fall back equilibria of G coincide.

Proof: Since the set of strictly fall back equilibria refines the set of fall back equilibria we only have to show that $FBE(G) \subseteq SFBE(G)$. Let $x \in FBE(G)$. Then one of the three statements of Proposition 7.3.2 is satisfied. Let $\{\delta_t\}_{t\in\mathbb{N}}$ be a sequence of blocking vectors of positive real numbers converging to zero and let for all $i \in \{1,2\}$, $\bar{x}^i \in \Delta_{M^i}$ be such that it fulfills all conditions of the satisfied statement. We define the sequence $\{\hat{x}_t\}_{t\in\mathbb{N}}$ such that for all $t \in \mathbb{N}$

$$\hat{x}^i_t = \begin{cases} x^i & \text{if } |C(x^i)| > 1, \\ (1-\delta^i_t)x^i + \delta^i_t \bar{x}^i & \text{if } |C(x^i)| = 1 \end{cases}$$

for all $i \in \{1,2\}$. Then the sequence $\{\hat{x}_t\}_{t\in\mathbb{N}}$ converges to x, $\hat{x}_t \in \Delta(\delta_t)$ for all $t \in \mathbb{N}$ and for $t \in \mathbb{N}$ sufficiently large $\hat{x}_t \in NE(G(\delta_t))$. Consequently, $x \in SFBE(G)$. □

7.4 Relations to other refinements

In this section we discuss the relation of fall back equilibrium to the concepts of perfect, proper, strictly perfect and robust equilibrium. We start with the relation between fall back and proper equilibrium (Myerson (1978)).

Definition Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game. A strategy profile $x \in \Delta$ is a *proper equilibrium* of G if there exists a sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of positive real numbers converging to zero, and a sequence $\{x_t\}_{t\in\mathbb{N}}$ of completely mixed strategy profiles converging to x such that x_t is ε_t-proper for all $t \in \mathbb{N}$, *i.e.*,

$$\pi^i(e^i_k, x^{-i}_t) < \pi^i(e^i_\ell, x^{-i}_t) \quad\Rightarrow\quad x^i_{t,k} \le \varepsilon_t x^i_{t,\ell} \tag{7.2}$$

for all $k, \ell \in M^i$ and all $i \in N$. The set of proper equilibria of G is denoted by $PR(G)$.

Note that by replacing $\varepsilon_t x^i_{t,\ell}$ on the right hand side of equation (7.2) by ε_t, one obtains an alternative characterisation of perfect equilibrium.

The relation between fall back and proper equilibrium is such that for any bimatrix game each proper equilibrium is a fall back equilibrium, which is illustrated by Example 7.1.1. Only the sets of pure best and pure second best replies determine whether a strategy profile is a fall back equilibrium, which can also be seen in the bimatrix game characterisation of Proposition 7.3.2. In the concept of proper equilibrium, however, all lower-level sets of best replies may be relevant as well. Hence, for bimatrix games, any strategy profile that satisfies the conditions for proper equilibrium also satisfies the conditions for fall back equilibrium.

Theorem 7.4.1 Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a bimatrix game. Then each proper equilibrium of G is a fall back equilibrium of G.

Proof: Let $x \in \Delta$ be a proper equilibrium. Then by definition there exists a sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of positive real numbers converging to zero and a sequence $\{x_t\}_{t\in\mathbb{N}}$ of completely mixed strategy profiles converging to x such that x_t is ε_t-proper for all $t \in \mathbb{N}$, *i.e.*,

$$\pi^i(e^i_k, x^{-i}_t) < \pi^i(e^i_\ell, x^{-i}_t) \quad\Rightarrow\quad x^i_{t,k} \le \varepsilon_t x^i_{t,\ell}$$

for all $k, \ell \in M^i$ and all $i \in N$.

We show that for this particular x one of the three statements of Proposition 7.3.2

is satisfied and hence that $x \in FBE(G)$. If $|C(x^1)| > 1$ and $|C(x^2)| > 1$, then the fact that each proper equilibrium is a Nash equilibrium gives by Proposition 7.2.4 that statement 1 of Proposition 7.3.2 is fulfilled. Otherwise, take $\hat{t} \in \mathbb{N}$ sufficiently large. Then, by upper-semi-continuity we obtain that $PB^i(x_t^{-i}) \subseteq PB^i(x^{-i})$ and $PSB^i(x_t^{-i}) \subseteq PSB^i(x^{-i})$ for all $i \in N$ and all $t \geq \hat{t}$.

In the remainder of this proof we make use of the following notation. Let $i \in \{1, 2\}$. Then, for a given strategy $x^i \in \Delta_{M^i}$ the vector $x^i(-k)$ is defined such that for all $\ell \in M^i$

$$x^i_\ell(-k) = \begin{cases} 0 & \text{if } \ell = k, \\ x^i_\ell & \text{otherwise.} \end{cases}$$

Note that $x^i(-k)$ is not necessarily a strategy, as the probabilities might not sum up to 1. Moreover, if $|C(x^i)| = 1$ for some $i \in \{1, 2\}$ we assume in this proof without loss of generality that $x^i = e^i_1$ and introduce the set $Q^i(t) = \{\ell \in M^i \backslash \{1\} \,|\, x^i_{t,\ell} > \varepsilon_t x^i_{t,r}$ for all $r \in M^i \backslash \{1\}\}$. Let the strategy $\bar{x}^i$ in that case be defined by

$$\bar{x}^i_\ell = \begin{cases} \frac{x^i_{\hat{t},\ell}}{\sum_{r \in Q^i(\hat{t})} x^i_{\hat{t},r}} & \text{for all } \ell \in Q^i(\hat{t}), \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the case that $|C(x^i)| > 1$ and $|C(x^j)| = 1$. Take $\delta^j > 0$ sufficiently small and define $\hat{x} = (x^i, \hat{x}^j)$, with $\hat{x}^j = (1 - \delta^j)e^j_1 + \delta^j \bar{x}^j$. Then we show that $C(x^i) \subseteq PB^i(\hat{x}^j)$, $C(x^j) \subseteq PB^j(x^i)$ and $C(\bar{x}^j) \subseteq PSB^j(x^i)$ as in statement 2 of Proposition 7.3.2.

We first show that $C(x^i) \subseteq PB^i(\hat{x}^j)$. Without loss of generality let $1 \in C(x^i)$. Since $x \in NE(G)$

$$\pi^i(e^i_1, e^j_1) \;\; \geq \;\; \pi^i(e^i_k, e^j_1) \tag{7.3}$$

for all $k \in M^i$. Furthermore, since the sequence $\{x_t\}_{t \in \mathbb{N}}$ is ε_t-proper and converges to x it holds that $1 \in PB^i(x^j_{\hat{t}})$, which implies that

$$\pi^i(e^i_1, x^j_{\hat{t}}) \;\; \geq \;\; \pi^i(e^i_k, x^j_{\hat{t}})$$

for all $k \in M^i$.

Since $x^j_{\hat{t}}$ is sufficiently close to e^j_1 we obtain that

$$\pi^i(e^i_1, x^j_{\hat{t}}(-1)) \quad \geq \quad \pi^i(e^i_k, x^j_{\hat{t}}(-1))$$

for all $k \in PB^i(e^j_1)$. Using the fact that $x^j_{\hat{t},\ell} > \varepsilon_{\hat{t}} x^j_{\hat{t},r}$ for all $\ell \in Q^j(\hat{t})$, $r \notin Q^j(\hat{t}) \cup \{1\}$ this results in

$$\pi^i(e^i_1, \bar{x}^j) \quad \geq \quad \pi^i(e^i_k, \bar{x}^j) \tag{7.4}$$

for all $k \in PB^i(e^j_1)$. Combining (7.3) and (7.4) we find

$$\pi^i(e^i_1, \hat{x}^j) \quad \geq \quad \pi^i(e^i_k, \hat{x}^j)$$

for all $k \in M^j$. Hence, $C(x^i) \subseteq PB^i(\hat{x}^j)$.

Since $x \in NE(G)$ we immediately obtain that $C(x^j) \subseteq PB^j(x^i)$. It remains to be shown that $C(\bar{x}^j) \subseteq PSB^j(x^i)$. Properness of x implies that $Q^j(\hat{t}) \subseteq PSB^j(x^i_{\hat{t}})$ and hence

$$\begin{aligned} C(\bar{x}^j) &= Q^j(\hat{t}) \\ &\subseteq PSB^j(x^i_{\hat{t}}) \\ &\subseteq PSB^j(x^i). \end{aligned}$$

Finally consider the case that $|C(x^1)| = |C(x^2)| = 1$. Take $\delta^i > 0$, $i \in \{1,2\}$, sufficiently small and define $\hat{x} = (\hat{x}^1, \hat{x}^2)$, with $\hat{x}^i = (1-\delta^i)e^i_1 + \delta^j \bar{x}^i$ for all $i \in \{1,2\}$. We prove that statement 3 of Proposition 7.3.2 is satisfied by showing that $C(x^1) \subseteq PB^1(\hat{x}^2)$ and that $C(\bar{x}^1) \subseteq PSB^1(\hat{x}^2)$. Showing that $C(x^2) \subseteq PB^2(\hat{x}^1)$ and $C(\bar{x}^2) \subseteq PSB^2(\hat{x}^1)$ can be done analogously. The proof that $C(x^1) \subseteq PB^1(\hat{x}^2)$ is similar to the proof that $C(x^i) \subseteq PB^i(\hat{x}^j)$ for the previous case with $|C(x^i)| > 1$ and $|C(x^j)| = 1$. Hence, we only have to show that $C(\bar{x}^1) \subseteq PSB^1(\hat{x}^2)$. Assume without loss of generality that $2 \in C(\bar{x}^1)$. Since $C(\bar{x}^1) = Q^1(\hat{t}) \subseteq PSB^1(x^2_{\hat{t}})$, it holds that $2 \in PSB^1(x^2_{\hat{t}})$, which implies that

$$\pi^1(e^1_2, x^2_{\hat{t}}) \geq \pi^1(e^1_k, x^2_{\hat{t}})$$

for all $k \in M^1 \backslash \{1\}$.

Since $x^2_{\hat{t}}$ is close to e^2_1

$$\pi^1(e^1_2, x^2_{\hat{t}}(-1)) \geq \pi^1(e^1_k, x^2_{\hat{t}}(-1))$$

for all $k \in PSB^1(e^2_1)\backslash\{1\}$. Using the fact that $x^2_{\hat{t},\ell} > \varepsilon_{\hat{t}} x^2_{\hat{t},r}$ for all $\ell \in Q^2(\hat{t})$, $r \notin Q^2(\hat{t}) \cup \{1\}$, this results in

$$\pi^1(e^1_2, \bar{x}^2) \geq \pi^1(e^2_k, \bar{x}^2) \tag{7.5}$$

for all $k \in PSB^1(e^2_1)\backslash\{1\}$. Furthermore, since $PSB^1(x^2_{\hat{t}}) \subseteq PSB^1(e^2_1)$, we know that

$$\pi^1(e^1_2, e^2_1) \geq \pi^1(e^1_k, e^2_1) \tag{7.6}$$

for all $k \in M^1\backslash\{1\}$. As a result of equations (7.5) and (7.6)

$$\pi^1(e^1_2, \hat{x}^2) \geq \pi^1(e^1_k, \hat{x}^2)$$

for all $k \in M^1\backslash\{1\}$, which implies that $C(\bar{x}^1) \subseteq PSB^1(\hat{x}^2)$. $\square$

The following example shows that for games with three players the set of proper equilibria need not be a subset of the set of fall back equilibria.

Example 7.4.2 Consider the following three-player game G in which the third player chooses the left (e^3_1) or the right (e^3_2) matrix.

$$\begin{array}{c} \\ e^1_1 \\ e^1_2 \\ e^1_3 \end{array} \begin{array}{c} \begin{array}{ccc} e^2_1 & e^2_2 & e^2_3 \end{array} \\ \begin{bmatrix} 10,10,1 & 5,5,1 & 0,5,1 \\ 10,0,1 & 0,0,1 & 5,0,1 \\ 0,-1,1 & 5,10,1 & 0,0,1 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} e^2_1 & e^2_2 & e^2_3 \end{array} \\ \begin{bmatrix} 0,0,0 & 0,0,0 & 0,0,0 \\ 0,0,0 & 0,0,0 & 0,10,0 \\ 0,0,0 & 0,0,0 & 0,0,0 \end{bmatrix} \end{array}$$

The strategy profile $x = (e^1_1, e^2_1, e^3_1)$ is a proper equilibrium since for the sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$, with $\varepsilon_t = \frac{2}{t}$ for all $t \in \mathbb{N}$, converging to zero the sequence $\{x_t\}_{t\in\mathbb{N}}$ converging to x is ε_t-proper for all $t \in \mathbb{N}$, with x_t given by $x^1_t = (1 - \frac{1}{t} - \frac{1}{t^2})e^1_1 + \frac{1}{t}e^1_2 + \frac{1}{t^2}e^1_3$, $x^2_t = (1 - \frac{1}{t} - \frac{1}{t^2})e^2_1 + \frac{1}{t}e^2_2 + \frac{1}{t^2}e^2_3$ and $x^3_t = (1 - \frac{1}{10t})e^3_1 + \frac{1}{10t}e^3_2$.

However, x is not a fall back equilibrium, which can be seen by considering a corresponding blocking game $G(\delta_t)$. In such a game player 3 will always play $x^3_t = (1 - \delta^3_t)e^3_1 + \delta^3_t e^3_2$. Player 1, however, plays his third row with zero probability for any strategy combination close to x. Knowing this, player 2 always prefers e_3^2 to e_2^2, due to the payoff of 10 in the second row of the right matrix. As a consequence, player 1 prefers e_2^1 to e_1^1, which implies that for any sequence of blocking vectors $\{\delta_t\}_{t\in\mathbb{N}}$ converging to zero there does not exist a sequence $\{x_t\}_{t\in\mathbb{N}}$ converging to x such that $x_t \in NE(G(\delta_t))$ for all $t \in \mathbb{N}$. Therefore, x is not a fall back equilibrium.

One of the fall back equilibria of this game is $x' = (e_3^1, e_2^2, e_1^3)$, which requires player 1 to play a weakly dominated strategy. So clearly, x' is not a proper (or perfect) equilibrium, as in the corresponding thought experiment all strategies are played with strictly positive probability. In any corresponding blocking game however, player 2 plays e_1^2 with zero probability for any strategy profile close to x', and consequently player 1 can maximise his profit by playing $(1-\delta_t^1)e_3^1 + \delta_t^1 e_1^1$ for all $t \in \mathbb{N}$.

In this game there are also some equilibria that are both proper and fall back, *e.g.*, (e_2^1, e_1^2, e_1^3). The question whether in general the intersection between the sets of fall back and proper equilibria can be empty is still open. $\triangleleft$

Next we consider the relations between the concepts of fall back equilibrium on the one hand and perfect and strictly perfect equilibrium (Okada (1984)) on the other. Note that since the set of proper equilibria refines the set of perfect equilibria Theorem 7.4.1 implies that for all bimatrix games the intersection between the sets of fall back and perfect equilibria is non-empty.

The intersection between the sets of fall back and strictly perfect equilibria, however, can be empty for bimatrix games. We first give the definition of strictly perfect equilibrium and then provide an example in which we show that the intersection between the two sets is empty. This example is due to Vermeulen and Jansen (1996), who use it to show that not each strictly perfect equilibrium is a proper equilibrium.

Reconsider the ε-perturbed game $H(\varepsilon) = (N, \{\Delta_{M^i}(\varepsilon^i)\}_{i\in N}, \{\pi^i\}_{i\in N})$ introduced in Section 7.2.

Definition Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game. A strategy profile $x \in \Delta$ is a *strictly perfect equilibrium* of G if for every sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of pertubation vectors converging to zero, there exists a sequence $\{x_t\}_{t\in\mathbb{N}}$ converging to x, such that $x_t \in NE(H(\varepsilon_t))$ for all $t \in \mathbb{N}$. The set of strictly perfect equilibria of G is denoted by $SPE(G)$.

Example 7.4.3 Consider the next 3×4 bimatrix game G.

$$
\begin{array}{c}
\\ e_1^1 \\ e_2^1 \\ e_3^1
\end{array}
\begin{array}{c}
\begin{array}{cccc} \;\; e_1^2 & \quad e_2^2 & \quad e_3^2 & \quad e_4^2 \end{array} \\
\begin{bmatrix}
1,3 & 0,0 & 0,2 & 0,2 \\
7,0 & 7,0 & -3,0 & 0,0 \\
0,0 & 1,3 & 0,2 & 0,2
\end{bmatrix}
\end{array}
$$

In their paper Vermeulen and Jansen (1996) show that (e_2^1, e_4^2) is a strictly perfect equilibrium. We demonstrate that there is no other strictly perfect equilibrium and show that this strategy profile is not a fall back equilibrium. We start, however, by repeating the argument given by Vermeulen and Jansen (1996) to show that (e_2^1, e_4^2) is indeed a strictly perfect equilibrium of G.

Let $\{\varepsilon_t\}_{t\in\mathbb{N}}$ be a sequence of pertubation vectors converging to zero. Define the sequence $\{x_t\}_{t\in\mathbb{N}}$ such that for all $t \in \mathbb{N}$ with $\varepsilon_{t,1}^1 \geq \varepsilon_{t,3}^1$, x_t is given by $x_t^1 = \varepsilon_{t,1}^1 e_1^1 + (1-3\varepsilon_{t,1}^1)e_2^1 + 2\varepsilon_{t,1}^1 e_3^1$ and $x_t^2 = \varepsilon_{t,1}^2 e_1^2 + (\frac{7}{6}\varepsilon_{t,1}^2 + \varepsilon_{t,2}^2 + \varepsilon_3^2)e_2^2 + (\frac{14}{3}\varepsilon_{t,1}^2 + 2\varepsilon_{t,2}^2 + 2\varepsilon_{t,3}^2)e_3^2 + (1 - \frac{41}{6}\varepsilon_1^2 - 3\varepsilon_2^2 - 3\varepsilon_3^2)e_4^2$, and for all $t \in \mathbb{N}$ with $\varepsilon_{t,1}^1 < \varepsilon_{t,3}^1$, x_t is given by $x_t^1 = 2\varepsilon_{t,3}^1 e_1^1 + (1-3\varepsilon_{t,3}^1)e_2^1 + \varepsilon_{t,3}^1 e_3^1$ and $x_t^2 = (\varepsilon_{t,1}^2 + \frac{7}{6}\varepsilon_{t,2}^2 + \varepsilon_{t,3}^2)e_1^2 + \varepsilon_{t,2}^2 e_2^2 + (2\varepsilon_{t,1}^2 + \frac{14}{3}\varepsilon_{t,2}^2 + 2\varepsilon_{t,3}^2)e_3^2 + (1 - 3\varepsilon_{t,1}^2 - \frac{41}{6}\varepsilon_{t,2}^2 - 3\varepsilon_{t,3}^2)e_4^2$. Then $x_t \in NE(H(\varepsilon_t))$ for all $t \in \mathbb{N}$, and the sequence $\{x_t\}_{t\in\mathbb{N}}$ converges to (e_2^1, e_4^2), which implies that (e_2^1, e_4^2) is a strictly perfect equilibrium.

For player i the carrier of strategy x^i in game $H(\varepsilon)$ is given by $C_\varepsilon(x^i) = \{k \mid x_k^i > \varepsilon_k^i\}$. The set of Nash equilibria of G is given by $\{e_2^1\} \times \text{conv}(\{e_1^2, e_2^2, \frac{1}{3}e_1^2 + \frac{2}{3}e_3^2, \frac{1}{3}e_2^2 + \frac{2}{3}e_3^2, \frac{3}{19}e_1^2 + \frac{3}{19}e_2^2 + \frac{13}{19}e_3^2, e_4^2\}) \cup \text{conv}(\{\frac{2}{3}e_1^1 + \frac{1}{3}e_3^1, \frac{1}{3}e_1^1 + \frac{2}{3}e_3^1\}) \times \text{conv}(\{e_3^2, e_4^2\}) \cup \text{conv}(\{\frac{2}{3}e_1^1 + \frac{1}{3}e_3^1, \frac{1}{3}e_1^1 + \frac{2}{3}e_3^1, e_2^1\}) \times \{e_4^2\}$.

We first show that $x \in \text{conv}(\{\frac{2}{3}e_1^1 + \frac{1}{3}e_3^1, \frac{1}{3}e_1^1 + \frac{2}{3}e_3^1\}) \times \text{conv}(\{e_3^2, e_4^2\}) \cup \text{conv}(\{\frac{2}{3}e_1^1 + \frac{1}{3}e_3^1, \frac{1}{3}e_1^1 + \frac{2}{3}e_3^1, e_2^1\}) \times \{e_4^2\}$ is not a strictly perfect equilibrium. Let $x_1^1 \geq x_3^1$ and take $\varepsilon > 0$ such that $\varepsilon_1^2 > \varepsilon_2^2$, and assume that $x(\varepsilon) \in NE(H(\varepsilon))$. As a result player 2 plays a best reply in $H(\varepsilon)$, implying that $x_2^2(\varepsilon) = \varepsilon_2^2$ and hence, $\pi^1(e_1^1, x^2(\varepsilon)) > \pi^1(e_3^1, x^2(\varepsilon))$. Therefore, $3 \notin PB^1(x^2(\varepsilon))$, which implies that $x(\varepsilon)$ is not close to x. By symmetry we obtain the same result if $x_1^1 \leq x_3^1$. Consequently, there does not exist a sequence of equilibria in the ε-perturbed games converging to x as in the definition of strictly perfect equilibrium, which implies that x is not a strictly perfect equilibrium.

Next $x \in \{e_2^1\} \times \text{conv}(\{e_1^2, e_2^2, \frac{1}{3}e_1^2 + \frac{2}{3}e_3^2, \frac{1}{3}e_2^2 + \frac{2}{3}e_3^2, \frac{3}{19}e_1^2 + \frac{3}{19}e_2^2 + \frac{13}{19}e_3^2, e_4^2\})$. Here x^1 is fixed, but we distinguish between the strategies of player 2.

Consider first of all the case $C(x^2) = \{1, 2, 3\}$. Take $\varepsilon > 0$ and assume $x(\varepsilon) \in NE(H(\varepsilon))$. Since player 2 plays a best reply in $H(\varepsilon)$ it must hold that $3x_1^1(\varepsilon) = 3x_3^1(\varepsilon) = 2x_1^1(\varepsilon) + 2x_3^1(\varepsilon)$, which implies that $x_1^1(\varepsilon) = x_3^1(\varepsilon) = 0$. This means, however, that $x(\varepsilon) \notin \Delta(\varepsilon)$. Consequently, x is not a strictly perfect equilibrium.

We now consider the case in which $C(x^2) \subseteq \{1, 2\}$. If $x_1^2 \geq x_2^2$ we define $\varepsilon > 0$ such that $\varepsilon_3^1 > \varepsilon_1^1$ and let $x(\varepsilon) \in NE(H(\varepsilon))$. Since player 1 plays a best reply in $H(\varepsilon)$ we obtain $x^1(\varepsilon) = \varepsilon_1^1 e_1^1 + (1 - \varepsilon_1^1 - \varepsilon_3^1)e_2^1 + \varepsilon_3^1 e_3^1$. Combining this with $\varepsilon_3^1 > \varepsilon_1^1$ gives $x_1^2(\varepsilon) = \varepsilon_1^2$, and as a result $x(\varepsilon)$ is not close to x. By symmetry, the same result can be obtained if $x_1^2 \leq x_2^2$. Hence, $x \notin SPE(G)$.

Let the carrier of x^2 equal $\{1, 3\}$, $\{1, 4\}$ or $\{1, 3, 4\}$. Take $\varepsilon > 0$ such that $\varepsilon_2^1 > 2\varepsilon_3^1$ and assume $x(\varepsilon) \in NE(H(\varepsilon))$. Since player 1 plays a best reply in $H(\varepsilon)$ we obtain $x_3^1(\varepsilon) = \varepsilon_3^1$. However, since player 2 plays a best reply in $H(\varepsilon)$ we get $x_1^1(\varepsilon) = 2x_3^1(\varepsilon)$, which implies that $x_1^1(\varepsilon)$ must equal $2\varepsilon_3^1$. Since $\varepsilon_1^1 > 2\varepsilon_3^1$ it follows that $x(\varepsilon) \notin \Delta(\varepsilon)$ and hence, that x is no strictly perfect equilibrium. By symmetry we can get the same result if the carrier of x^2 equals $\{2, 3\}$, $\{2, 4\}$ or $\{2, 3, 4\}$. Hence, (e_2^1, e_4^2) is the unique strictly perfect equilibrium of this game.

Next we show by contradiction that $(e_2^1, e_4^2) \notin FBE(G)$. Let us assume that $(e_2^1, e_4^2) \in FBE(G)$. Then by statement 3 of Proposition 7.3.2 there exists an $\hat{x}$ given by $\hat{x} = ((1-\delta^1)e_2^1 + \delta^1\bar{x}^1, (1-\delta^2)e_4^2 + \delta^2\bar{x}^2)$, with $C(\bar{x}^1) \subseteq \{1, 3\}$ and $C(\bar{x}^2) \subseteq \{1, 2, 3\}$, and $\delta > 0$, satisfying four properties. We show that not all four of them can hold simultaneously, and as a consequence that $(e_2^1, e_4^2) \notin FBE(G)$.

Let us first consider the case that $\hat{x}_1^2 > \hat{x}_2^2$. Then, since $C(\bar{x}^1) \subseteq PSB^1(\hat{x}^2)$ we obtain that $\bar{x}^1 = e_1^1$. However, then $C(x^2) \nsubseteq PB^2(\hat{x}^1)$, which is a contradiction. By symmetry we obtain the same result if $\hat{x}_1^2 < \hat{x}_2^2$. Hence, it must hold that $\hat{x}_1^2 = \hat{x}_2^2$.

If $\hat{x}_1^2 = \hat{x}_2^2 = 0$, then $\bar{x} = e_3^2$, but then $C(x^1) \nsubseteq PB^1(\hat{x}^2)$. Consequently, $\hat{x}_1^2 = \hat{x}_2^2 > 0$. In that case it follows from $C(\bar{x}^2) \subseteq PSB^2(\hat{x}^1)$ that $3\hat{x}_1^1 = 3\hat{x}_3^1$. Furthermore, $C(x^2) \subseteq PB^2(\hat{x}^1)$ then implies that $3\hat{x}_1^1 = 3\hat{x}_3^1 \leq 2\hat{x}_1^1 + 2\hat{x}_3^1$. This is only possible if $\hat{x}_1^1 = \hat{x}_3^1 = 0$, which is not allowed, as $C(\bar{x}^1) \subseteq \{1, 3\}$.

Hence, statement 3 of Proposition 7.3.2 can not be satisfied and hence, $(e_2^1, e_4^2) \notin FBE(G)$. $\triangleleft$

We now focus on the relation between fall back and robust equilibrium (Okada (1983)).

Definition Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game. A strategy profile $\hat{x} \in \Delta$ is a *robust equilibrium* of G if for all $j \in N$ there exists an open neighbourhood $U^j(\hat{x}^j)$ of $\hat{x}^j \in \Delta_{M^j}$ such that for all $i \in N$

$$\pi^i(\hat{x}^i, \breve{x}^{-i}) \quad \geq \quad \pi^i(x^i, \breve{x}^{-i})$$

for all $x^i \in \Delta_{M^i}$ and all $\breve{x}^{-i} \in \prod_{r\in N\setminus\{i\}} U^r(\hat{x}^r)$. The set of robust equilibria of G is denoted by $RB(G)$.

Theorem 7.4.4 Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game. Then each robust equilibrium of G is a strictly fall back equilibrium of G.

Proof: Let $\hat{x} \in \Delta$ be a robust equilibrium. Then by definition for all $j \in N$ there exists an open neighbourhood $U^j(\hat{x}^j)$ of $\hat{x}^j \in \Delta_{M^j}$ such that for all $i \in N$

$$\pi^i(\hat{x}^i, \breve{x}^{-i}) \quad \geq \quad \pi^i(x^i, \breve{x}^{-i}) \tag{7.7}$$

for all $x^i \in \Delta_{M^i}$ and all $\breve{x}^{-i} \in \prod_{r\in N\setminus\{i\}} U^r(\hat{x}^r)$.

Let $\{\delta_t\}_{t\in\mathbb{N}}$ be a sequence of blocking vectors of positive real numbers converging to zero, and let for all $t \in \mathbb{N}$ the blocking game be given by $G(\delta_t) = (N, \{\Delta_{M^i}(\delta_t)\}_{i\in N}, \{\pi^i\}_{i\in N})$. Then we construct a sequence $\{\hat{x}_t\}_{t\in\mathbb{N}}$ converging to $\hat{x}$ such that $\hat{x}_t \in NE(G(\delta_t))$ for all $t \in \mathbb{N}$. This shows by Theorem 7.2.3 that $\hat{x}$ is a fall back equilibrium.

Define $N^* = \left\{ i \in N \middle| |C(\hat{x}^i)| > 1 \right\}$ and $N' = N\setminus N^*$. Assume without loss of generality that for each $i \in N'$, $\hat{x}^i = e^i_1$. We introduce for all $t \in \mathbb{N}$ the game $G_t(\hat{x}) = (N, \{\Delta_{\hat{M}^i}\}_{i\in N}, \{\hat{\pi}^i_t\}_{i\in N})$. For all $i \in N^*$, $\hat{M}^i = \{f^i_1\}$ and for all $i \in N'$, $\hat{M}^i = \{f^i_2, \ldots, f^i_{m^i}\}$. For all $i \in N'$ the payoff function $\hat{\pi}^i_t$ is the mixed extension of

$$\hat{\pi}^i_t(f^i_{k^i}, (f^j_{k^j})_{j\in N\setminus\{i\}}) = \pi^i(e^i_{k^i}, (\hat{x}^j)_{j\in N^*}, ((1-\delta^j_t)e^j_1 + \delta^j_t e^j_{k^j})_{j\in N'\setminus\{i\}})$$

for all $(f^j_{k^j})_{j\in N} \in \prod_{j\in N} \hat{M}^j$, $t \in \mathbb{N}$. Since each $i \in N^*$ is a dummy player in $G_t(\hat{x})$, $t \in \mathbb{N}$, we do not need to specify their payoff functions explicitly. Then, let

$\tilde{x}_t \in NE(G_t(\hat{x}))$ for all $t \in \mathbb{N}$. For all $t \in \mathbb{N}$ and all $i \in N'$ we define $\bar{x}^i_t \in \Delta_{M^i}$ to be the extension of $\tilde{x}^i_t \in \Delta_{\hat{M}^i}$ to Δ_{M^i}, in the sense that $\bar{x}^i_{t,k} = \tilde{x}^i_{t,k}$ for all $k \in \hat{M}^i$, $\bar{x}^i_{t,1} = 0$. Further, for all $t \in \mathbb{N}$ and $i \in N^*$, $\bar{x}^i_t = \hat{x}^i$. Next define the sequence $\{\hat{x}_t\}_{t\in\mathbb{N}}$ such that

$$\hat{x}^i_t = (1 - \delta^i_t)\hat{x}^i + \delta^i_t \bar{x}^i_t$$

for all $i \in N$ and all $t \in \mathbb{N}$. Note that since $\bar{x}^i_{t,1} = 0$ for all $i \in N'$ and all $t \in \mathbb{N}$ we obtain that $\hat{x}_t \in \Delta(\delta_t)$ for sufficiently large $t \in \mathbb{N}$. Also note that since $\bar{x}^i_t = \hat{x}^i$ for all $i \in N^*$ and all $t \in \mathbb{N}$, we obtain that $\hat{x}^i_t = \hat{x}^i$ for all $t \in \mathbb{N}$.

Hence, $\hat{x}_t \in \Delta(\delta_t)$ for sufficiently large $t \in \mathbb{N}$, and $\{\hat{x}_t\}_{t\in\mathbb{N}}$ converges to $\hat{x}$. Take $\hat{t} \in \mathbb{N}$ such that for all $i \in N$, $\hat{x}^i_t \in \Delta_{M^i}(\delta^i_t) \cap U^i(\hat{x}^i)$ for all $t \geq \hat{t}$. Then, we complete the proof by showing that $\hat{x}^i_t$ is a best reply against $\hat{x}^{-i}_t$ in $G(\delta_t)$ for all $i \in N$ and for all $t \geq \hat{t}$. Let $i \in N$ and $t \geq \hat{t}$. First of all, from (7.7) it follows that

$$\pi^i(\hat{x}^i, \hat{x}^{-i}_t) \quad \geq \quad \pi^i(x^i, \hat{x}^{-i}_t) \tag{7.8}$$

for all $x^i \in \Delta_{M^i}$. If $i \in N^*$, then $\hat{x}^i_t = \hat{x}^i \in \Delta_{M^i}(\delta_t)$ and $\hat{x}^i_t$ is a best reply against $\hat{x}^{-i}_t$ in $G(\delta_t)$. So, assume $i \in N'$. Then it remains to be shown that $\bar{x}^i_t \in PSB^i(\hat{x}^{-i}_t)$. Since $\tilde{x}_t \in NE(G_t(\hat{x}))$,

$$\hat{\pi}^i_t(\tilde{x}^i_t, \tilde{x}^{-i}_t) \geq \hat{\pi}^i_t(\dot{x}^i_t, \tilde{x}^{-i}_t)$$

for all $\dot{x}^i_t \in \Delta_{\hat{M}^i}$. As a result, we obtain by the definition of $\hat{\pi}^i_t$ that

$$\pi^i(\bar{x}^i_t, \hat{x}^{-i}_t) \geq \pi^i(x^i_t, \hat{x}^{-i}_t) \tag{7.9}$$

for all $x^i_t \in \Delta_{M^i\backslash\{1\}}$. Hence, $\bar{x}^i_t \in PSB^i(\hat{x}^{-i}_t)$. Combining (7.8) and (7.9) results in

$$\pi^i(\hat{x}^i_t, \hat{x}^{-i}_t) \quad \geq \quad \pi^i(x^i_t, \hat{x}^{-i}_t)$$

for all $x^i_t \in \Delta_{M^i}(\delta^i_t)$, which implies that $\hat{x}^i_t$ is a best reply against $\hat{x}^{-i}_t$ in $G(\delta_t)$.

Since the sequence $\{\delta_t\}_{t\in\mathbb{N}}$ was arbitrarily chosen this implies that each robust equilibrium is a strict fall back equilibrium. $\square$

7.5 Structure of the set of fall back equilibria

For bimatrix games the set of Nash equilibria is the union of finitely many polytopes (Jansen (1981)). The main result provided in this section is that this is also true for the set of fall back equilibria. In order to obtain this result we need several preliminary lemmas.

We first introduce some additional notation. For a bimatrix game $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ the strategies $x^1, \tilde{x}^1 \in \Delta_{M^1}$ are *reply-equivalent* if the following two statements hold:

$$\begin{aligned} PB^2(x^1) &= PB^2(\tilde{x}^1), \\ PSB^2(x^1) &= PSB^2(\tilde{x}^1). \end{aligned}$$

By $\mathcal{V}_1, \ldots, \mathcal{V}_{r^1}$ we denote the finitely many reply-equivalence classes in Δ_{M^1}. In a similar way a reply-equivalence relation can be defined for the strategies of player 2. The reply-equivalence classes in Δ_{M^2} are denoted by $\mathcal{W}_1, \ldots, \mathcal{W}_{r^2}$. Note that since the sets of pure best and pure second best replies are determined by linear inequalities, the closure of each reply-equivalence class is a polytope.

By the use of Jansen (1993) we obtain the following two lemmas.

Lemma 7.5.1 *Let H be a face of cl$(\mathcal{V}_s)$, $s \in \{1, \ldots, r^1\}$, or of cl$(\mathcal{W}_t)$, $t \in \{1, \ldots, r^2\}$. Then all the elements in relint(H) are reply-equivalent.*

Lemma 7.5.2 *If the intersection of the closure of two reply-equivalence classes is non-empty, then this intersection is a face of both polytopes.*

Given a bimatrix game $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ and reply-equivalence classes $\mathcal{V}_s$, $s \in \{1, \ldots, r^1\}$, and $\mathcal{W}_t$, $t \in \{1, \ldots, r^2\}$, the (s,t)-*fall back component* is defined by $FB_{st}(G) = \{x \in FBE(G) \,|\, x^1 \in \mathrm{cl}(\mathcal{V}_s), x^2 \in \mathrm{cl}(\mathcal{W}_t)\}$.

Proposition 7.5.3 *For every $s \in \{1, \ldots, r^1\}$, $t \in \{1, \ldots, r^2\}$, $FB_{st}(G)$ is the cartesian product of two polytopes, which are faces of cl$(\mathcal{V}_s)$ and cl$(\mathcal{W}_t)$, respectively.*

Proof: Consider the reply-equivalence class $\mathcal{V}_s$, $s \in \{1, \ldots, r^1\}$, and let H be a face of $\mathrm{cl}(\mathcal{V}_s)$. Take $x^1 \in \mathrm{relint}(H)$ and $x^2 \in \mathrm{cl}(\mathcal{W}_t)$, $t \in \{1, \ldots, r^2\}$. We show that whenever $(x^1, x^2) \in FB_{st}(G)$ it holds that $(\tilde{x}^1, x^2) \in FB_{st}(G)$ for all $\tilde{x}^1 \in \mathrm{relint}(H)$. The fact that the set of fall back equilibria is closed then shows that $(\tilde{x}^1, x^2) \in FB_{st}(G)$ for all $\tilde{x}^1 \in H$, which completes the proof.

Let $(x^1, x^2) \in FB_{st}(G)$. If $|C(x^1)| = 1$, then $|H| = 1$ and the statement follows immediately. So, assume $|C(x^1)| > 1$. We distinguish between two cases. We first assume that $|C(x^2)| > 1$. Then by Proposition 7.2.4 $x \in NE(G)$, which implies that $C(x^1) \subseteq PB^1(x^2)$ and $C(x^2) \subseteq PB^2(x^1)$. Let $\tilde{x}^1 \in \mathrm{relint}(H)$. Then $C(\tilde{x}^1) = C(x^1)$, and hence $C(\tilde{x}^1) \subseteq PB^1(x^2)$. Furthermore, by Lemma 7.5.1 we obtain $PB^2(\tilde{x}^1) = PB^2(x^1)$ and therefore, $C(x^2) \subseteq PB^2(\tilde{x}^1)$. Consequently, $(\tilde{x}^1, x^2) \in FB_{st}(G)$.

Next we assume that $|C(x^2)| = 1$. Let $\bar{x}^2 \in \Delta_{M^2}$ with $C(\bar{x}^2) \cap C(x^2) = \emptyset$ and $\bar{\delta}^2 > 0$ be such that for all $\delta^2 \in (0, \bar{\delta}^2]$ the strategy profile $(x^1, \hat{x}^2) \in \Delta$, with $\hat{x}^2 = (1 - \delta^2)x^2 + \delta^2\bar{x}^2$ satisfies $C(x^1) \subseteq PB^1(\hat{x}^2)$, $C(x^2) \subseteq PB^2(x^1)$ and $C(\bar{x}^2) \subseteq PSB^2(x^1)$. Note that by Proposition 7.3.2 this is possible. We show that these conditions are also satisfied for $(\tilde{x}^1, x^2)$. Since $x^1, \tilde{x}^1 \in \mathrm{relint}(H)$, by Lemma 7.5.1 we conclude $PB^2(x^1) = PB^2(\tilde{x}^1)$ and $PSB^2(x^1) = PSB^2(\tilde{x}^1)$, and furthermore $C(x^1) = C(\tilde{x}^1)$. Consequently, $C(\tilde{x}^1) = C(x^1) \subseteq PB^1(\hat{x}^2)$, $C(x^2) \subseteq PB^2(x^1) = PB^2(\tilde{x}^1)$ and $C(\bar{x}^2) \subseteq PSB^2(x^1) = PSB^2(\tilde{x}^1)$.

Hence, by Proposition 7.3.2 we obtain that $(\tilde{x}^1, x^2) \in FBE(G)$, and as a consequence $(\tilde{x}^1, x^2) \in FB_{st}(G)$. □

Since there are only finitely many combinations of reply-equivalence classes we obtain the following theorem.

Theorem 7.5.4 Let G be a bimatrix game. Then the set of fall back equilibria of G is the union of finitely many polytopes.

A *maximal fall back component* is a fall back component not properly contained in another fall back component. From Example 7.1.1 it follows that a maximal Nash subset may contain more than one maximal fall back component and that a maximal fall back component need not be the face of a maximal Nash subset. The latter result in particular implies that an extreme element of a maximal fall back component need not be an extreme element of a maximal Nash subset. Furthermore, Lemma 7.5.2 and Proposition 7.5.3 imply that the intersection of two maximal fall back components is either empty or a face of both maximal fall back components.

7.6 Complete fall back equilibrium

In this section we discuss a modification of the concept of fall back equilibrium. In the thought experiment underlying fall back equilibrium each player faces the possibility that, after all players decided on their action, the action chosen by him is blocked. In that case the player plays a back-up action, chosen by him beforehand. We assume that a back-up action is never blocked.

In this section we consider the possibility that any number of actions of each player is blocked. Consequently, players have to decide beforehand on a second back-up action in case the first back-up action is blocked and a third back-up action in case the second back-up cannot be played either, *etc.* Hence, each player must decide on a complete ordering of his actions. If all actions of a player turn out to be blocked the game is not played and all players receive zero payoff. This thought experiment is modelled by a corresponding *complete fall back game*. The equilibrium concept that is based on this thought experiment is called complete fall back equilibrium.

Note that not playing the game is not an option a player can choose, but that this can only be the result of a player not being able to play any of his actions. Therefore, the zero payoff to each player if this situation occurs is strategically irrelevant, as any fixed amount would result in the same set of equilibria. In order to avoid the possibility that the game is not played we could also have chosen for a setup in which at most all but one actions of each player are blocked. This results in a similar equilibrium concept, but one that leads to different equilibria. This alternative equilibrium concept will not be discussed in this chapter.

Let us formalise the concept introduced above. The action set in the complete fall back game for player $i \in N$ equals the set of all orderings of the action set M^i, and is given by Ω^i. Hence, the total number of actions in the complete fall back game for player i equals $\tilde{m}^i = m^i!$. A typical element of Ω^i is denoted by σ^i, where the action on position s of σ^i is given by $\sigma^i(s) \in M^i$. By $\Omega^i_k \subseteq \Omega^i$, $k \in M^i$, we denote the set of orderings of M^i such that $\sigma^i(1) = k$ for all $\sigma^i \in \Omega^i_k$. Similar to the concept of fall back equilibrium we assume that each action of player i is blocked with the same probability, denoted by ε^i, but we allow for different probabilities among players. Hence, let $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^n)$ be an n-tuple of (small) non-negative probabilities. If player i plays action $\sigma^i \in \Omega^i$ in the complete fall back game he plays with probability $(1-\varepsilon^i)(\varepsilon^i)^{s-1}$ action $\sigma^i(s)$ of the game G for $s \in \{1, \ldots, m^i\}$. With probability $(\varepsilon^i)^{m^i}$ all actions of player i are blocked and the payoff to all players is defined to be zero.

The *complete fall back game* $\tilde{G}^C(\varepsilon)$ is given by $\tilde{G}^C(\varepsilon) = (N, \{\Delta_{\Omega^i}\}_{i\in N}, \{\pi^{C,i}_\varepsilon\}_{i\in N})$, with $\pi^{C,i}_\varepsilon : \prod_{j\in N} \Delta_{\Omega^j} \to \mathbb{R}$ the extended expected payoff function to player i. A pure strategy $\sigma^i \in \Omega^i$ will be alternatively denoted by e^i_σ. For pure strategy combinations $\pi^{C,i}_\varepsilon$ is formally given by

$$\pi^{C,i}_\varepsilon((e^j_{\sigma^j})_{j\in N}) = \sum_{(k^1,\ldots,k^n)\in\prod_{r\in N} M^r} (\prod_{j\in N}(1-\varepsilon^j)(\varepsilon^j)^{\sigma^{j^{-1}}(k^j)-1})\pi^i((e^j_{k^j})_{j\in N}).$$

A typical element of Δ_{Ω^i} will be denoted by ρ^i, the probability which ρ^i assigns to pure strategy σ^i is given by ρ^i_σ. The set of all strategy profiles is given by $\tilde{\Delta}^C = \prod_{i\in N} \Delta_{\Omega^i}$, an element of $\tilde{\Delta}^C$ by ρ.

Definition Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game. A strategy profile $x \in \Delta$ is a *complete fall back equilibrium* of G if there exists a sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of n-tuples of positive real numbers converging to zero, and a sequence $\{\rho_t\}_{t\in\mathbb{N}}$ such that $\rho_t \in NE(\tilde{G}^C(\varepsilon_t))$ for all $t \in \mathbb{N}$, converging to $\rho \in \tilde{\Delta}^C$, with $x^i_k = \sum_{\sigma^i\in\Omega^i_k} \rho^i_\sigma$ for all $k \in M^i$ and all $i \in N$. The set of complete fall back equilibria of G is denoted by $CFBE(G)$.

Note that from the above definition it immediately follows that each completely mixed Nash equilibrium is a complete fall back equilibrium. Moreover, we obtain the result that each complete fall back equilibrium is a Nash equilibrium.

Theorem 7.6.1 Let G be an n-player strategic game. Then the set of complete fall back equilibria of G is a non-empty and closed subset of the set of Nash equilibria of G.

The proof of this theorem is analogous to the proof of Theorem 7.2.1 for fall back equilibrium.

As the thought experiment underlying the concept of complete fall back equilibrium takes into account all levels of best replies and the thought experiment underlying fall back equilibrium only the first and second, one might expect that the set of complete fall back equilibria refines the set of fall back equilibria. This is however not the case, as can be seen in Example 7.4.2. In this example the strategy profile (e_1^1, e_1^2, e_1^3), which is not a fall back equilibrium, is a complete fall back equilibrium. This is due to the fact that each action in the complete fall back game puts positive probability on all actions of the original game G. Therefore, player 1 is unable to play e_3^1 with zero probability, which is the main reason for (e_1^1, e_1^2, e_1^3) not being a fall back equilibrium.

The setup of complete fall back equilibrium is closely related to that of proper equilibrium. In both concepts the replies of each player are ordered in such a way that a complete set of levels of best replies is obtained. The properness concept then requires that replies of a lower level are played with *some* significant smaller probability than replies from a higher level. The concept of complete fall back equilibrium is however more restrictive, as for each action in the complete fall back game the probability on the actions of the original game are given. Hence, by requiring that players play a best reply in the complete fall back game the probability on each best reply level of the original game is fixed. This is the reason why the set of complete fall back equilibria refines the set of proper equilibria.

Theorem 7.6.2 Let $G = (N, \{\Delta_{M^i}\}_{i\in N}, \{\pi^i\}_{i\in N})$ be an n-player strategic game. Then each complete fall back equilibrium of G is a proper equilibrium of G.

Proof: Let $x \in CFBE(G)$. Then by definition there exists a sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of n-tuples of positive real numbers converging to zero, and a sequence $\{\rho_t\}_{t\in\mathbb{N}}$ such that $\rho_t \in NE(\tilde{G}^C(\varepsilon_t))$ for all $t \in \mathbb{N}$, converging to $\rho \in \tilde{\Delta}^C$, with $x_k^i = \sum_{\sigma^i\in\Omega_k^i} \rho_\sigma^i$ for all $k \in M^i$ and all $i \in N$.

We define the sequence $\{x_t\}_{t\in\mathbb{N}}$ such that for all $t \in \mathbb{N}$

$$x_{t,k}^i = \sum_{\sigma^i\in\Omega^i} (1-\varepsilon_t^i)(\varepsilon_t^i)^{\sigma^{i^{-1}}(k)-1} \rho_{t,\sigma}^i$$

for all $k \in M^i$ and all $i \in N$. Note that x_t^i puts the same probability on the actions of the game G as ρ_t^i for all $i \in N$ and all $t \in \mathbb{N}$, and that the sequence $\{x_t\}_{t\in\mathbb{N}}$

converges to x. However, since for all $t \in \mathbb{N}$ there is a probability of $(\varepsilon^i_t)^{m^i}$ that all actions of player i are blocked x^i_t is not a probability distribution. For that reason we define the sequence $\{\hat{x}_t\}_{t\in\mathbb{N}}$ such that for all $t \in \mathbb{N}$

$$\hat{x}^i_{t,k} = \frac{x^i_{t,k}}{1-(\varepsilon^i_t)^{m^i}}$$

for all $k \in M^i$ and all $i \in N$. Let the sequence $\{\hat{\varepsilon}_t\}_{t\in\mathbb{N}}$ be given by $\hat{\varepsilon}_t = \max_{i\in N} \frac{\varepsilon^i_t}{1-(\varepsilon^i_t)^{m^i}}$ for all $t \in \mathbb{N}$.

Let $i \in N$ and let $\pi^i(e^i_k, \hat{x}^{-i}_{\hat{t}}) < \pi^i(e^i_\ell, \hat{x}^{-i}_{\hat{t}})$ for some $k, \ell \in M^i$ and some $\hat{t} \in \mathbb{N}$. Since $\rho_t \in NE(G^C(\varepsilon_t))$ for all $t \in \mathbb{N}$ it holds that $\hat{x}^i_{\hat{t},k} \leq \frac{\varepsilon^i_{\hat{t}}}{1-(\varepsilon^i_{\hat{t}})^{m^i}} \hat{x}^i_{\hat{t},\ell}$. Hence,

$$\begin{aligned}\hat{x}^i_{\hat{t},k} &\leq \frac{\varepsilon^i_{\hat{t}}}{1-(\varepsilon^i_{\hat{t}})^{m^i}} \hat{x}^i_{\hat{t},\ell} \\ &\leq \hat{\varepsilon}_{\hat{t}} \hat{x}^i_{\hat{t},\ell}.\end{aligned}$$

Consequently, $\{\hat{\varepsilon}_t\}_{t\in\mathbb{N}}$ is a sequence of positive real numbers converging to zero and $\{\hat{x}_t\}_{t\in\mathbb{N}}$ is a sequence of completely mixed strategy profiles converging to x such that for all $t \in \mathbb{N}$

$$\pi^i(e^i_k, \hat{x}^{-i}_t) < \pi^i(e^i_\ell, \hat{x}^{-i}_t) \;\Rightarrow\; \hat{x}^i_{t,k} \leq \hat{\varepsilon}_t \hat{x}^i_{t,\ell}$$

for all $k, \ell \in M^i$ and all $i \in N$. Hence, x is a proper equilibrium. $\square$

The following example shows that the set of complete fall back equilibria can be a strict subset of the set of proper equilibria.

Example 7.6.3 Consider the following three-player game in which the third player chooses the left (e^3_1) or the right (e^3_2) matrix.

$$\begin{array}{c} \\ e^1_1 \\ e^1_2 \\ e^1_3 \end{array}
\begin{array}{c} \begin{array}{ccc} e^2_1 & e^2_2 & e^2_3 \end{array} \\ \begin{bmatrix} 10,10,10 & 0,10,0 & 0,0,1 \\ 10,1,0 & 2,0,0 & 0,0,0 \\ 0,0,0 & 0,0,0 & 0,0,0 \end{bmatrix} \end{array}
\begin{array}{c} \begin{array}{ccc} e^2_1 & e^2_2 & e^2_3 \end{array} \\ \begin{bmatrix} 1,0,10 & 0,1,0 & 0,0,0 \\ 0,0,0 & 0,0,0 & 0,0,0 \\ 0,0,5 & 0,0,0 & 0,0,0 \end{bmatrix} \end{array}$$

As stated before, a difference between proper and complete fall back equilibrium is that for the former concept one has more freedom in choosing the probabilities with which actions are played in the corresponding thought experiment. Therefore, in this example the players can coordinate the probabilities on the lower level actions in such a way that $x = (e_1^1, e_1^2, e_1^3)$ is a proper equilibrium.

Consider the sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$, with $\varepsilon_t = \frac{1}{t}$ for all $t \in \mathbb{N}$, converging to zero and the sequence $\{x_t\}_{t\in\mathbb{N}}$ converging to $x \in \Delta$, with x_t for all $t \in \mathbb{N}$ given by $x_t^1 = (1 - \frac{1}{25t} - \frac{1}{1000t^2})e_1^1 + \frac{1}{25t}e_2^1 + \frac{1}{1000t^2}e_3^1$, $x_t^2 = (1 - \frac{1}{100t} - \frac{1}{100t^2})e_1^2 + \frac{1}{100t}e_2^2 + \frac{1}{100t^2}e_3^2$ and $x_t^3 = (1 - \frac{3}{100t})e_1^3 + \frac{3}{100t}e_2^3$. Then x_t is ε_t-proper for all $t \in \mathbb{N}$ and hence, x is a proper equilibrium.

Let $t \in \mathbb{N}$ and consider x_t. It is clear that although $\pi^1(e_1^1, x_t^{-1}) > \pi^1(e_2^1, x_t^{-1}) > \pi^1(e_3^1, x_t^{-1})$ there is no $\varepsilon_t^1 > 0$ such that $x_{t,1}^1 = 1 - \varepsilon_t^1$, $x_{t,2}^1 = \varepsilon_t^1 - (\varepsilon_t^1)^2$ and $x_{t,3}^1 = (\varepsilon_t^1)^2 - (\varepsilon_t^1)^3$. Consequently, $x_t \notin NE(\tilde{G}^C(\varepsilon_t))$. In fact there does not exist a sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of positive real numbers converging to zero and a sequence $\{x_t\}_{t\in\mathbb{N}}$ of strategy profiles converging to x such that $x_t \in NE(\tilde{G}^C(\varepsilon_t))$ for all $t \in \mathbb{N}$. Consequently, x is not a complete fall back equilibrium. $\triangleleft$

Although a lack of freedom for the notion of complete fall back equilibrium makes that in general not each proper equilibrium is a complete fall back equilibrium, for bimatrix bimatrix the sets of proper and complete fall back equilibria coincide. Consequently, the concept of complete fall back equilibrium is a strategic characterisation of proper equilibrium.

Theorem 7.6.4 Let G be a bimatrix game. Then the sets proper and complete fall back equilibria of G coincide.

By Theorem 7.6.2 we only have to show that $PR(G) \subseteq CFBE(G)$. That proof is an extension of the proof of Theorem 7.4.1 in which it is shown that for bimatrix games the set of fall back equilibria contains the set of proper equilibria.

Note that this theorem also implies that for bimatrix games complete fall back equilibrium is a refinement of fall back equilibrium.

7.7 Dependent fall back equilibrium

In the previous section we modify the notion of fall back equilibrium in such a way that in the underlying thought experiment any number of actions of each player can be blocked. This idea results in the concept of complete fall back equilibrium. In this section we also alter the concept of fall back equilibrium, but here we consider a thought experiment in which at most one single action in total can be blocked. Hence, after all players decided on their action there is a small but positive probability that the action of *one* of the players is blocked. In that case this player plays his back-up action, chosen by him beforehand, while all other players play their primary action. This thought experiment is modelled by letting players play a *dependent fall back game*. The equilibrium concept that is based on this thought experiment is called dependent fall back equilibrium.

Let us formalise this thought experiment. The action set in the dependent fall back game for player $i \in N$ equals the action set of the fall back game, hence $\tilde{M}^i = \{(k,\ell) \in M^i \times M^i \,|\, k \neq \ell\}$. Consequently, his total number of actions is $\tilde{m}^i = m^i(m^i - 1)$. An action $(k,\ell) \in \tilde{M}^i$ consists of a primary action k and a back-up action ℓ. Let $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^n)$ be an n-tuple of (small) non-negative probabilities. The interpretation, given that each player $i \in N$ plays action (k^i, ℓ^i) in the dependent fall back game, is that in the original game with probability $1 - \sum_{j \in N} \varepsilon^j$ each player $i \in N$ plays k^i, and with probability ε^j player j plays ℓ^j, while k^i is played by all players $i \in N \backslash \{j\}$.

The *dependent fall back game* $\tilde{G}^D(\varepsilon)$ is given by $\tilde{G}^D(\varepsilon) = (N, \{\Delta_{\tilde{M}^i}\}_{i \in N}, \{\pi_\varepsilon^{D,i}\}_{i \in N})$, with $\pi_\varepsilon^{D,i} : \prod_{j \in N} \Delta_{\tilde{M}^j} \to \mathbb{R}$ the extended expected payoff function of player i. Pure strategy $(k,\ell) \in \tilde{M}^i$ is alternatively denoted by $e^i_{k\ell}$. The payoff function $\pi_\varepsilon^{D,i}$ is for pure strategy profiles formally defined by

$$\pi_\varepsilon^{D,i}((e^j_{k^j \ell^j})_{j \in N}) = (1 - \sum_{j \in N} \varepsilon^j) \pi^i((e_{k^j})_{j \in N}) + \sum_{j \in N} \varepsilon^j \pi^i((e_{k^r})_{r \in N \backslash \{j\}}, e_{\ell^j}).$$

A typical element of $\Delta_{\tilde{M}^i}$ is denoted by ρ^i, where $\rho^i_{k\ell}$ is the probability which ρ^i assigns to pure strategy (k,ℓ). Note that ρ^i assigns probabilities to pure strategies (k,ℓ) of the dependent fall back game, not to primary and back-up actions separately. The set of all strategy profiles is given by $\tilde{\Delta} = \prod_{i \in N} \Delta_{\tilde{M}^i}$, an element of $\tilde{\Delta}$ will be denoted by ρ.

Definition Let $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$ be an n-player strategic game. A strategy profile $x \in \Delta$ is a *dependent fall back equilibrium* of G if there exists a sequence $\{\varepsilon_t\}_{t \in \mathbb{N}}$ of n-tuples of positive real numbers converging to zero, and a sequence $\{\rho_t\}_{t \in \mathbb{N}}$ such that $\rho_t \in NE(\tilde{G}^D(\varepsilon_t))$ for all $t \in \mathbb{N}$, converging to $\rho \in \tilde{\Delta}$, with $x^i_k = \sum_{\ell \in M^i \backslash \{k\}} \rho^i_{k\ell}$ for all $k \in M^i$ and all $i \in N$. The set of dependent fall back equilibria of G is denoted by $DFBE(G)$.

The following example shows that not each dependent fall back equilibrium is either a fall back equilibrium or a complete fall back equilibrium. In Example 7.8.10 we show that the reversed results are not valid either.

Example 7.7.1 Consider the 2×3 bimatrix game G depicted below.

$$\begin{array}{c} \\ e_1^1 \\ e_2^1 \end{array} \begin{array}{c} \begin{array}{ccc} e_1^2 & e_2^2 & e_3^2 \end{array} \\ \begin{bmatrix} 1,2 & 1,1 & 0,0 \\ 1,2 & 0,2 & 1,2 \end{bmatrix} \end{array}$$

In this game strategy profile (e_2^1, e_1^2) is not a fall back equilibrium, because player 2's unique best reply in $\tilde{G}(\varepsilon)$ to e_{21}^1 is e_{12}^2, but against e_{12}^2 player 1 prefers e_{12}^1 over e_{21}^1. Analogously, we can show that (e_2^1, e_1^2) is not a complete fall back equilibrium of G.

In the concept of dependent fall back equilibrium, however, it is not possible that the actions of both players are blocked simultaneously. Consequently, player 2's payoff against e_{21}^1 is the same for both e_{12}^2 and e_{13}^2 and against e_{13}^2 player 1's best reply is e_{21}^1. As a result (e_{21}^1, e_{13}^2) is a Nash equilibrium of $\tilde{G}^D(\varepsilon)$ and hence, $(e_2^1, e_1^2) \in DFBE(G)$. ◁

Theorem 7.7.2 Let $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$ be an n-player strategic game. Then the set of dependent fall back equilibria of G is a non-empty and closed subset of the set of Nash equilibria of G.

The proof of this theorem is analogous to the proof of Theorem 7.2.1 for fall back equilibrium.

For the analysis of the concept of fall back equilibrium the alternative characterisation based on blocking games is very useful. Recall that the idea behind the blocking games, which have the same dimension as the original game, is that the strategy space of each player is restricted. However, for the notion of dependent fall back equilibrium the events of two players having a blocked action are not independent. Therefore, we can not restrict the strategy spaces in a similar way. Hence, for further analysis of the concept of dependent fall back equilibrium only the original definition can be used. As a consequence, the results we obtain for this concept are limited only to bimatrix games. For these type of games we study the relation to perfect equilibrium.

We first give the definition of dominance. Let $\Delta_{M^{-i}}$ denote $\prod_{j\in N\setminus\{i\}}\Delta_{M^j}$. We say that a strategy $x^i \in \Delta_{M^i}$ is dominated by a strategy $\bar{x}^i \in \Delta_{M^i}$ whenever $\pi^i(\bar{x}^i, x^{-i}) \geq \pi^i(x^i, x^{-i})$ for all $x^{-i} \in \Delta_{M^{-i}}$, and $\pi^i(\bar{x}^i, x^{-i}) > \pi^i(x^i, x^{-i})$ for some $x^{-i} \in \Delta_{M^{-i}}$. A strategy $\bar{x}^i$ strictly dominates x^i if $\pi^i(\bar{x}^i, x^{-i}) > \pi^i(x^i, x^{-i})$ for all $x^{-i} \in \Delta_{M^{-i}}$. Furthermore, a strategy is called (strictly) dominant if it (strictly) dominates all other strategies. This definition of dominance implies that for a bimatrix game a strategy profile is a perfect equilibrium if and only if it is a Nash equilibrium that consists of two undominated strategies (Van Damme (1991)).

Proposition 7.7.3 *Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a 2×2 bimatrix game. Then the sets of dependent fall back and perfect equilibria of G coincide.*

Proof: We first state that whenever e^i_1, $i \in \{1,2\}$, is a dominant strategy for 2×2 bimatrix game G this automatically implies that e^i_{12} is a strictly dominant strategy in $\tilde{G}^D(\varepsilon)$ for $\varepsilon > 0$ small enough. This is due to the fact that the action against which e^i_1 is a strictly better response than e^i_2 in the original game is played with positive probability for any strategy profile in the dependent fall back game.

Next we show that $DFBE(G) \subseteq PE(G)$. Let $\hat{x} \in DFBE(G)$. We show that $\hat{x} \in PE(G)$ by proving that $\hat{x}^1$ is an undominated strategy. The proof that $\hat{x}^2$ is undominated is analogous. Suppose $\hat{x}^1$ is a dominated strategy. Then, since $m^1 = 2$ we know that $\hat{x}^1$ is dominated by a pure strategy and hence, without loss of generality we can assume that e^1_1 is a dominant strategy and $\hat{x}^1 \neq e^1_1$. As a consequence, for $\varepsilon > 0$ small enough e^1_{12} is a strictly dominant strategy in $\tilde{G}^D(\varepsilon)$, which implies that in every dependent fall back equilibrium player 1 plays e^1_1. Since $\hat{x}^1 \neq e^1_1$, this implies that $\hat{x} \notin DFBE(G)$, which is a contradiction. Therefore, $\hat{x}^1$ must be an undominated strategy.

We now demonstrate that $PE(G) \subseteq DFBE(G)$. Let $\hat{x} \in PE(G)$. We first assume that there exists a dominant strategy for player 1. This strategy must be pure and unique, and without loss of generality we can assume that this strategy is e^1_1 and $\hat{x}^1 = e^1_1$. As a result, for $\varepsilon > 0$ small enough e^1_{12} is a strictly dominant strategy in the dependent fall back game. Then, let $\{\varepsilon_t\}_{t\in\mathbb{N}}$ be a sequence of pairs of positive real numbers converging to zero and construct $\{\hat{\rho}_t\}_{t\in\mathbb{N}}$ such that $\hat{\rho}^1_t = e^1_{12}, \hat{\rho}^2_{t,12} = \hat{x}^2_1$ and $\hat{\rho}^2_{t,21} = \hat{x}^2_2$ for all $t \in \mathbb{N}$. Then for $t \in \mathbb{N}$ large enough,

$$\pi^{D,2}_{\varepsilon}(e^1_{12}, \hat{\rho}^2_t) \quad \geq \quad \pi^{D,2}_{\varepsilon}(e^1_{12}, \rho^2_t)$$

for all $\rho^2_t \in \Delta_{\tilde{M}^2}$, because $\hat{x}^2$ is an undominated strategy in G. Hence, $\hat{x} \in DFBE(G)$. A similar result holds if there exists a dominant strategy for player 2.

Next we assume that both players do not have a dominant strategy. In that case every Nash equilibrium of G is perfect, which means that it suffices to show that every Nash equilibrium is also a dependent fall back equilibrium. Without loss of generality we assume that $\pi^1(e^1_1, e^2_1) > \pi^1(e^1_2, e^2_1)$. Then there exists an $\hat{x}^2$ defined by $\hat{x}^2_1 = \frac{\pi^1(e^1_2,e^2_2)-\pi^1(e^1_1,e^2_2)}{(\pi^1(e^1_1,e^2_1)-\pi^1(e^1_2,e^2_1))+(\pi^1(e^1_2,e^2_2)-\pi^1(e^1_1,e^2_2))}$ such that

$$PB^1(x^2) = \begin{cases} \{1\} & \text{for all } x^2 \text{ such that } x^2_1 \in (\hat{x}^2_1, 1], \\ \{1,2\} & \text{for } x^2 = \hat{x}^2, \\ \{2\} & \text{for all } x^2 \text{ such that } x^2_1 \in [0, \hat{x}^2_1). \end{cases} \tag{7.10}$$

Let us now consider the sequence of dependent fall back games $\tilde{G}^D(\varepsilon_t)$, with $\{\varepsilon_t\}_{t\in\mathbb{N}}$ a sequence of pairs of positive real numbers converging to zero. It follows that for each $t \in \mathbb{N}$ there exists a $\hat{\rho}^2_t \in \Delta_{\tilde{M}^2}$, which results in a different, but similar pure best reply correspondence for player 1 as the one in G given by (7.10). By the continuity of the payoff function the sequence $\{\hat{\rho}^2_t\}_{t\in\mathbb{N}}$ converges to $\hat{\rho}^2$, with $\hat{\rho}^2_{12} = \hat{x}^2_1$. Since a similar argument holds for the best reply correspondences of player 2, this completes the proof. $\square$

Note that this implies by Proposition 7.2.5 that for 2×2 bimatrix games also the sets of fall back and dependent fall back equilibria coincide.

The result that the sets of perfect and dependent fall back equilibria are equivalent is only valid for 2×2 bimatrix games. However, for all $2 \times m^2$ bimatrix games the intersection between the two sets is non-empty.

Proposition 7.7.4 *Let* $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ *be a* $2 \times m^2$ *bimatrix game. Then the intersection between the sets of dependent fall back and perfect equilibria is non-empty.*

Proof: Take a sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of pairs of positive real numbers converging to zero and a sequence $\{\hat{\rho}_t\}_{t\in\mathbb{N}}$ such that $\hat{\rho}_t \in PE(\tilde{G}^D(\varepsilon_t))$ for all $t \in \mathbb{N}$, converging to $\hat{\rho} \in \tilde{\Delta}$, and define $\hat{x} \in \Delta$ such that $\hat{x}^i_k = \sum_{\ell \in M^i\setminus\{k\}} \hat{\rho}^i_{k\ell}$ for all $k \in M^i$ and all $i \in \{1,2\}$.

Since $\tilde{G}^D(\varepsilon_t)$ is a bimatrix game for all $t \in \mathbb{N}$, and the set of perfect equilibria is non-empty for every bimatrix game, we know that such a sequence $\{\hat{\rho}_t\}_{t\in\mathbb{N}}$ exists. Then, by definition $\hat{x} \in DFBE(G)$. We now prove that $\hat{x} \in PE(G)$ by showing that $\hat{x}^i$ is an undominated strategy in G for all $i \in \{1,2\}$.

We first consider $\hat{x}^1$. Since $m^1 = 2$, if player 1 does not have a dominant strategy in the game G, then $\hat{x}^1$ is obviously undominated. Let us therefore assume that player 1 does have a dominant strategy in the game G. Without loss of generality we assume that this strategy is e^1_1. Then for $\varepsilon > 0$ small enough e^1_{12} is a dominant strategy in $\tilde{G}^D(\varepsilon)$. Hence, since $\hat{\rho}_t \in PE(\tilde{G}^D(\varepsilon_t))$ for all $t \in \mathbb{N}$ it follows that for $t \in \mathbb{N}$ large enough $\hat{\rho}^1_t$ equals e^1_{12} and as a result $\hat{x}^1 = e^1_1$. Consequently, $\hat{x}^1$ is an undominated strategy.

We now focus on player 2. Since $\hat{x} \in DFBE(G)$ we know that strategy $\hat{\rho}^2 \in \Delta_{\tilde{M}^2}$ forms a Nash equilibrium with some $\rho^1 \in \Delta_{\tilde{M}^1}$ in the game $\tilde{G}^D(0)$. By contradiction we prove that $\hat{\rho}^2$ is undominated in $\tilde{G}^D(0)$, which implies that $\hat{x}^2$ is undominated in G. So, let us assume that $\hat{\rho}^2$ is dominated by some $\bar{\rho}^2 \in \Delta_{\tilde{M}^2}$ in $\tilde{G}^D(0)$. Without loss of generality

$$\pi^{D,2}_0(e^1_{12}, \bar{\rho}^2) = \pi^{D,2}_0(e^1_{12}, \hat{\rho}^2), \tag{7.11}$$
$$\pi^{D,2}_0(e^1_{21}, \bar{\rho}^2) > \pi^{D,2}_0(e^1_{21}, \hat{\rho}^2). \tag{7.12}$$

Consequently, $\hat{\rho} = (e^1_{12}, \hat{\rho}^2)$. Moreover, all actions in $C^2(\hat{\rho}^2)$ must give the highest payoff to player 2 against e^1_{12}. Furthermore, given that $\bar{\rho}^2$ exists we know that there also exists a pure strategy satisfying both (7.11) and (7.12). Without loss of generality we can assume that $\hat{x}^2 \neq e^2_1$ and that the pure strategy satisfying both (7.11) and (7.12) is e^2_{12}. Now, we define $\rho^{2,*} \in \Delta_{\tilde{M}^2}$ by $e^2_{1\ell}$, with $\ell \in C(\hat{x}^2)\setminus\{1\}$. Note that since there always exists an action $\ell \in C(\hat{x}^2)$ unequal to 1, strategy $\rho^{2,*}$ can be constructed. Then, $\hat{\rho}^2$ is also dominated by $\rho^{2,*}$ in $\tilde{G}^D(0)$.

Hence,

$$\begin{aligned}\pi_0^{D,2}(e_{12}^1,\rho^{2,*}) &= \pi_0^{D,2}(e_{12}^1,\hat{\rho}^2),\\ \pi_0^{D,2}(e_{21}^1,\rho^{2,*}) &> \pi_0^{D,2}(e_{21}^1,\hat{\rho}^2).\end{aligned}$$

Moreover, because of the way the back-up action of $\rho^{2,*}$ was chosen,

$$\begin{aligned}\pi_{\varepsilon_t}^{D,2}(e_{12}^1,\rho^{2,*}) &> \pi_{\varepsilon_t}^{D,2}(e_{12}^1,\hat{\rho}^2),\\ \pi_{\varepsilon_t}^{D,2}(e_{21}^1,\rho^{2,*}) &> \pi_{\varepsilon_t}^{D,2}(e_{21}^1,\hat{\rho}^2)\end{aligned}$$

for all $t \in \mathbb{N}$. Then we define for all $\delta > 0$, $\rho_t^{2,\delta}$ such that $\rho_t^{2,\delta} = \hat{\rho}_t^2 + \delta(\rho^{2,*} - \hat{\rho}^2)$ for all $t \in \mathbb{N}$. Then there exists a $\delta > 0$ such that $\rho_t^{2,\delta}$ is an element of $\Delta_{\tilde{M}^2}$ for all $t \in \mathbb{N}$. Furthermore, strategy $\rho_t^{2,\delta}$ dominates $\hat{\rho}_t^2$ for all $t \in \mathbb{N}$, which contradicts the assumption that $\hat{\rho}_t \in PE(\tilde{G}^D(\varepsilon_t))$ for all $t \in \mathbb{N}$. Hence, $\hat{\rho}^2$ cannot be dominated in the game $\tilde{G}^D(0)$ and consequently $\hat{x}^2$ is an undominated strategy. $\square$

In the above proof we show that for all $2 \times m^2$ bimatrix games a strategy profile $\hat{x} \in \Delta$, defined by $\hat{x}_k^i = \sum_{\ell \in M^i \setminus \{k\}} \hat{\rho}_{k\ell}^i$ for all $k \in M^i$ and all $i \in \{1,2\}$, with $\{\varepsilon_t\}_{t\in\mathbb{N}}$ a sequence of pairs of positive real numbers converging to zero and $\{\hat{\rho}_t\}_{t\in\mathbb{N}}$ a sequence such that $\hat{\rho}_t \in PE(\tilde{G}^D(\varepsilon_t))$ for all $t \in \mathbb{N}$, converging to $\hat{\rho} \in \tilde{\Delta}$, is a perfect equilibrium. Unfortunately, this is not true for bimatrix games in general. Therefore, the intersection between the sets of perfect and dependent fall back equilibria can be empty.

Example 7.7.5 Consider the following 3×4 bimatrix game G.

$$\begin{array}{c}\\ e_1^1\\ e_2^1\\ e_3^1\end{array}\begin{array}{c}\begin{array}{cccc} e_1^2 & e_2^2 & e_3^2 & e_4^2\end{array}\\ \left[\begin{array}{cccc} 10,2 & 3,2 & 1,0 & 10,10\\ 11,1 & 1,2 & 10,0 & 10,1\\ 9,2 & 5,1 & 10,0 & 10,1\end{array}\right]\end{array}$$

In the game $\tilde{G}^D(0)$ player 1's strategy e_{12}^1 forms a Nash equilibrium, *e.g.*, in combination with e_{41}^2. Let us say that $\hat{\rho}^1 = e_{12}^1$. Then this strategy is dominated by $\bar{\rho}^1 = \frac{1}{2}e_{21}^1 + \frac{1}{2}e_{32}^1$, but not by any pure strategy. Therefore, it is impossible to define some $\rho^{1,*} \in \Delta_{\tilde{M}^1}$ similar to the strategy constructed in the proof of Proposition 7.7.4 such that it dominates $\hat{\rho}^1$ in every game $\tilde{G}^D(\varepsilon)$. Hence, a setup similar to the one in the proof of Proposition 7.7.4 can not be obtained.

Moreover, we can show that for this game the intersection between the sets of perfect and dependent fall back equilibria is empty. Player 1's pure strategy e^1_1 is dominated (by $\frac{1}{2}e^1_2 + \frac{1}{2}e^1_3$) and as a consequence the unique perfect equilibrium is given by $x = (\frac{1}{2}e^1_2+\frac{1}{2}e^1_3, \frac{2}{3}e^2_1+\frac{1}{3}e^2_2)$. Note that for determining a perfect equilibrium e^1_1 is unimportant, because in the thought experiment underlying perfectness player 2 makes the mistake of playing e^2_3 with positive probability.

In the dependent fall back framework, however, player 1 cannot discard his first action, as player 2 can choose his back-up action strategically and will never play e^2_3. Indeed, in the dependent fall back game player 1 must always play e^1_{12} or e^1_{13} with positive probability, not converging to zero, in order to sustain a Nash equilibrium. Therefore, $(\frac{1}{2}e^1_2 + \frac{1}{2}e^1_3, \frac{2}{3}e^2_1 + \frac{1}{3}e^2_2) \notin DFBE(G)$ and consequently, the intersection between the sets of perfect and dependent fall back equilibria is empty for this bimatrix game. ◁

In this section we have shown that the intersection between the sets of perfect and dependent fall back equilibria is non-empty for $2 \times m^2$ bimatrix games. However, this result does not hold for bimatrix games in general, as is shown in Example 7.7.5. Note that in that example a fourth column is needed to obtain that each dependent fall back equilibrium makes use of a dominated strategy. Consequently, for 3×3 bimatrix games the intersection between the sets of perfect and dependent fall back equilibria is non-empty. The formal proof is similar to the proof of Proposition 7.7.4, and is therefore omitted.

Proposition 7.7.6 *Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a 3×3 bimatrix game. Then the intersection between the sets of dependent fall back and perfect equilibria of G is non-empty.*

We conclude this section with a graphical overview of the relations between all the equilibrium concepts discussed in this chapter, for both n-player strategic games (Figure 7.7.1) and bimatrix games (Figure 7.7.2). The questions whether the set of robust equilibria is a subset of the sets of complete fall back, proper and dependent fall back equilibria are still open. All other relations not depicted in these two figures are known to be non-existent.

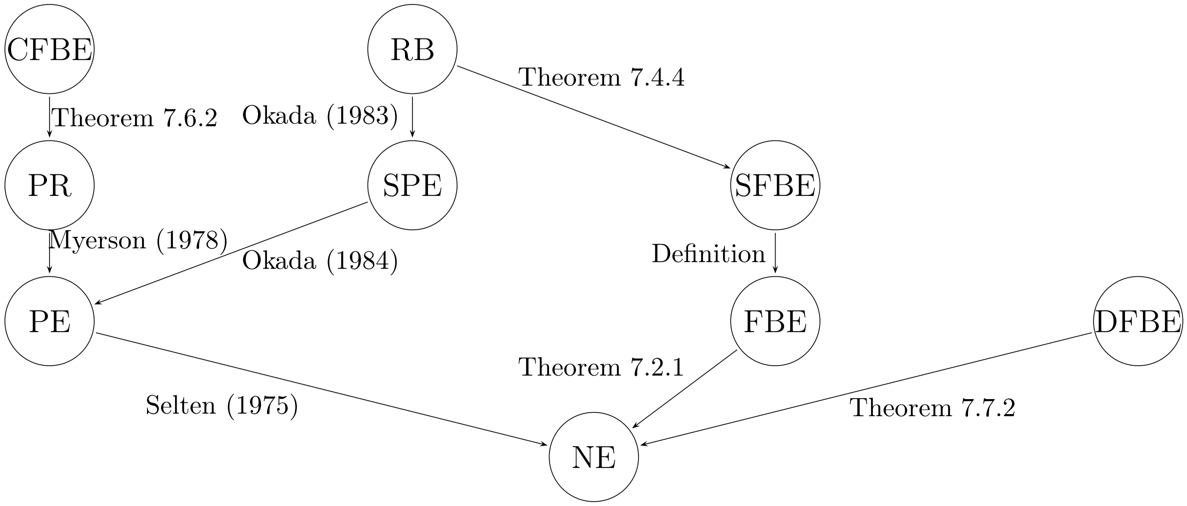


Figure 7.7.1: Relations for n-player strategic games

7.8 $2 \times m^2$ bimatrix games

In this section we study fall back equilibrium, and its variations complete and dependent fall back equilibrium, for $2 \times m^2$ bimatrix games. Note first of all that for a $2 \times m^2$ bimatrix game player 1 also has two actions in the corresponding (complete/dependent) fall back game. Therefore, we can use the method developed in Borm (1992) to find all Nash equilibria of a (complete/dependent) fall back game, which give rise to (complete/dependent) fall back equilibria of the original game. The main goal of this section is, however, to determine the various sets of equilibria directly from the original game.

We start this section by introducing some notation and definitions for $2 \times m^2$ bimatrix games. Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i \in \{1,2\}}, \{\pi^i\}_{i \in \{1,2\}})$ be a $2 \times m^2$ bimatrix game. Consider the function $g : \Delta_{M^1} \to \mathbb{R}$, defined by $g(x^1) = \max_{k \in M^2} \pi^2(x^1, e^2_k)$ for all $x^1 \in \Delta_{M^1}$. Obviously, g is a piecewise linear function, and there exists a minimal number $v+1$ ($\leq m^2+1$) of strategies in Δ_{M^1}, $e^1_2 = x^1(0), x^1(1), \ldots, x^1(v) = e^1_1$, such that g is affine on $[x^1(r-1), x^1(r)]$ for every $r \in \{1, 2, \ldots, v\}$. Note that g, which is called the upper envelope, represents player 2's payoff corresponding to his best reply function.

Further, a pure strategy $k \in M^2$ of player 2 will be provided with label [1] if $PB^1(e^2_k) = \{1\}$, with label [2] if $PB^1(e^2_k) = \{2\}$, and with label [12] if $PB^1(e^2_k) = \{1,2\}$. Let $I([1]) = \{k \in M^2 \,|\, PB^1(e^2_k) = \{1\}\}$ represent the set of player 2's pure

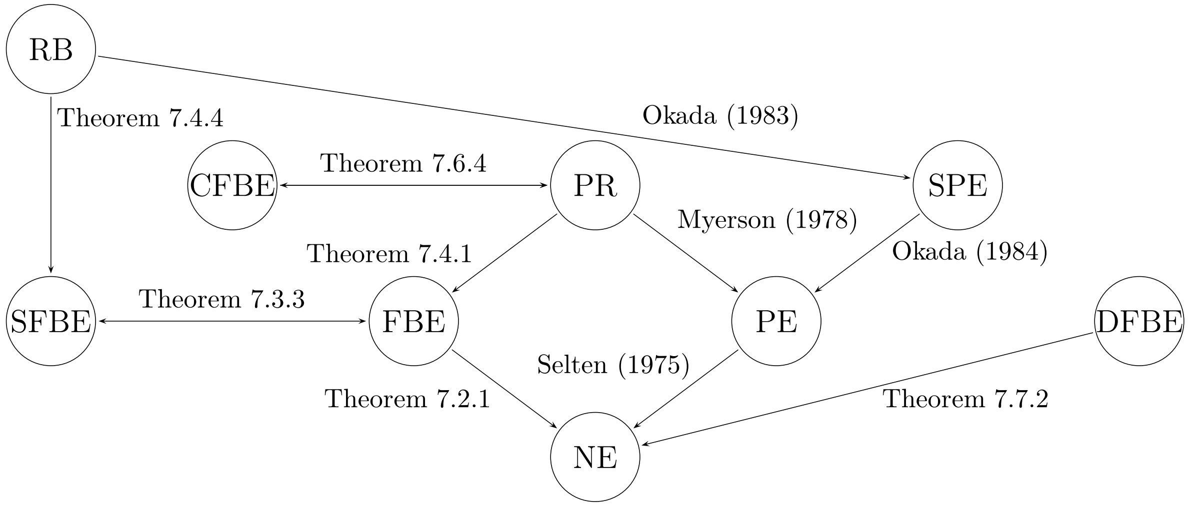


Figure 7.7.2: Relations for bimatrix games

strategies with label [1]. The sets $I([2])$ and $I([12])$ are defined analogously. In addition, for $r \in \{1, 2, \ldots, v\}$ let $I_r = PB^2(\frac{1}{2}x^1(r-1) + \frac{1}{2}x^1(r))$ denote the set of pure best replies to any strategy in the open interval $(x^1(r-1), x^1(r))$.

Then for $x^1 \in \Delta_{M^1}$, let the set $S(x^1)$ of solutions to x^1 be defined by $S(x^1) = \{x^2 \in \Delta_{M^2} \mid x \in NE(G)\}$. The set $S(x^1)$ is the convex hull of finitely many extreme points. These points can be divided into two sets; pure and coordination solutions. Let us define the interior of Δ_{M^1} by $\dot{\Delta}_{M^1} = \Delta_{M^1} \backslash \{e^1_1, e^1_2\}$. Then we define the set $PS(x^1)$ of pure solutions to x^1 by $PS(x^1) = \{e^2_k \in \Delta_{M^2} \mid (x^1, e^2_k) \in NE(G)\}$, which results in

$$PS(x^1) = \begin{cases} \{e^2_k \in \Delta_{M^2} \mid k \in PB^2(x^1, [12])\} & \text{if} \quad x^1 \in \dot{\Delta}_{M^1}, \\ \{e^2_k \in \Delta_{M^2} \mid k \in PB^2(x^1, [12]) \cup PB^2(x^1, [1])\} & \text{if} \quad x^1 = e^1_1, \\ \{e^2_k \in \Delta_{M^2} \mid k \in PB^2(x^1, [12]) \cup PB^2(x^1, [2])\} & \text{if} \quad x^1 = e^1_2. \end{cases}$$

Here $PB^2(x^1, [12]) = PB^2(x^1) \cap I([12])$ denotes the set of pure best replies to x^1 with label [12]. The sets $PB^2(x^1, [1])$ and $PB^2(x^1, [2])$ are defined in a similar way.

As mentioned above, pure solutions may not be the only extreme points of the set of solutions, as there can also be coordination solutions. Let $k \in I([1])$ and $\ell \in I([2])$. Then there exists a unique strategy $x^2(k, \ell) \in \Delta_{M^2}$ such that $C(x^2(k, \ell)) = \{k, \ell\}$ and $\pi^1(e^1_1, x^2(k, \ell)) = \pi^1(e^1_2, x^2(k, \ell))$. In particular, $x^2_k = \frac{\pi^1(e^1_2,e^2_\ell)-\pi^1(e^1_1,e^2_\ell)}{(\pi^1(e^1_1,e^2_k)-\pi^1(e^1_2,e^2_k))+(\pi^1(e^1_2,e^2_\ell)-\pi^1(e^1_1,e^2_\ell))}$. Then we define the set $CS(x^1)$ of coordination solutions to $x^1 \in \Delta_{M^1}$ by $CS(x^1) = \{x^2(k, \ell) \in \Delta_{M^2} \mid k \in PB^2(x^1, [1]),$ $\ell \in PB^2(x^1, [2])\}$. As a consequence, $S(x^1) = \text{conv}(PS(x^1) \cup CS(x^1))$. Hence, if

$x^2 \in S(x^1)$, then $(x^1, x^2) \in NE(G)$.

After these preliminaries we first of all discuss the concept of fall back equilibrium for $2 \times m^2$ bimatrix games. Reconsider the 2×4 bimatrix game of Example 7.1.1. We delete in this example the upper envelope, which in this case corresponds to action e_3^2 for all $x^1 \in \Delta_{M^1}$. Then all of player 1's strategies with a non-empty set of solutions in the game without action e_3^2, $x^1 \in \{e_2^1, [\frac{1}{6}e_1^1 + \frac{5}{6}e_2^1, \frac{3}{4}e_1^1 + \frac{1}{4}e_2^1], e_1^1\}$, correspond to strategies that constitute with some $x^2 \in \Delta_{M^2}$ a fall back equilibrium in the original game. We formalise this statement in Theorem 7.8.1 in which we characterise the set of fall back equilibria for $2 \times m^2$ bimatrix games. Let us first introduce some additional notation.

Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i \in \{1,2\}}, \{\pi^i\}_{i \in \{1,2\}})$ be a $2 \times m^2$ bimatrix game. Then for $x^2 \in \Delta_{M^2}$, with $C(x^2) \neq M^2$ the game $G_{x^2} = (\{1,2\}, \{\Delta_{M^i_{x^2}}\}_{i \in \{1,2\}}, \{\pi^i\}_{i \in \{1,2\}})$ is defined to be the game which only differs from G in the sense that player 2's action set is restricted to $M^2 \backslash C(x^2)$, with the domains of the payoff functions restricted accordingly. Define the set of all strategy profiles of G_{x^2} by $\Delta_{x^2} = \Delta_{M^1_{x^2}} \times \Delta_{M^2_{x^2}}$. The set of solutions according to G_{x^2} is given by $S_{x^2}(x^1)$, by $PB^2_{x^2}(x^1)$ we denote player 2's set of best replies, and $I_{x^2,r}$, with $r \in \{1, \ldots, v_{x^2}\}$, denotes player 2's set of best replies in an open interval between two intersection points on the upper envelope.

We illustrate some of these definitions by the use of Example 7.1.1. The upper envelope of G consists of one interval and the set of solutions of G is given by $S(x^1) = \{e_1^2\}$ for all $x^1 \in [e_2^1, e_1^1]$. However, for G_{x^2} the upper envelope consists of three intervals, $[e_2^1, \frac{1}{6}e_1^1 + \frac{5}{6}e_2^1]$, $[\frac{1}{6}e_1^1 + \frac{5}{6}e_2^1, \frac{3}{4}e_1^1 + \frac{1}{4}e_2^1]$ and $[\frac{3}{4}e_1^1 + \frac{1}{4}e_2^1, e_1^1]$, and furthermore, the set of solutions is dependent on x^1, as, *e.g.*, $S_{x^2}(e_2^1) = \{e_2^2\}$ and $S_{x^2}(\frac{1}{2}e_1^1 + \frac{1}{2}e_2^1) = \{e_3^2\}$.

Theorem 7.8.1 Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i \in \{1,2\}}, \{\pi^i\}_{i \in \{1,2\}})$ be a $2 \times m^2$ bimatrix game and $x \in \Delta$. Then x is a fall back equilibrium of G if and only if the following three assertions hold:

(I) $x^2 \in S(x^1)$.

(II) If $x^1 = e_1^1$, then $C(x^2) \subseteq I_v$. If $x^1 = e_2^1$, then $C(x^2) \subseteq I_1$.

(III) If $x^2 = e_k^2$, with $k \in I([12])$, then there exists an $\tilde{x}^2 \in \Delta_{M^2_{x^2}}$ such that

(i) $\tilde{x}^2 \in S_{x^2}(x^1)$.

(ii) If $x^1 = e^1_1$, then $C(\tilde{x}^2) \subseteq I_{x^2,v}$, if $x^1 = e^1_2$, then $C(\tilde{x}^2) \subseteq I_{x^2,1}$.

Proof: Let us first show the "if" part. Assume there exists a strategy profile $x \in \Delta$ satisfying conditions (I) - (III). Then we define for $\delta > 0$ small the strategy profile $\hat{x} \in \Delta$ such that

$$\hat{x}^1 = \begin{cases} x^1 & \text{if } |C(x^1)| > 1, \\ (1-\delta^1)e^1_k + \delta^1 e^1_\ell & \text{otherwise,} \end{cases}$$

with $\{k\} = C(x^1)$, and $\ell \neq k$, and

$$\hat{x}^2 = \begin{cases} x^2 & \text{if } |C(x^2)| > 1, \\ (1-\delta^2)e^2_k + \delta^2 \bar{x}^2 & \text{otherwise,} \end{cases}$$

with $\{k\} = C(x^2)$, and $\bar{x}^2 = \tilde{x}^2$ if $k \in I([12])$, $\bar{x}^2 \in PSB^2(\hat{x}^1)\backslash\{j\}$ otherwise. We show that $\hat{x} \in \Delta$ satisfies one of the conditions of Proposition 7.3.2, proving that $x \in FBE(G)$.

We first assume that $|C(x^i)| > 1$ for all $i \in \{1,2\}$. Then $x \in NE(G)$, because $x^2 \in S(x^1)$, which implies that the first condition of Proposition 7.3.2 is fulfilled.

Secondly, we assume that $|C(x^1)| > 1$ and $|C(x^2)| = 1$. We show that $C(x^1) \subseteq PB^1(\hat{x}^2)$, $C(x^2) \subseteq PB^2(x^1)$ and $C(\bar{x}^2) \subseteq PSB^2(x^1)$. Since $x^1 \in \dot{\Delta}_{M^1}$, $x^2 \in S(x^1)$ and $\tilde{x}^2 \in S_{x^2}(x^1)$ we obtain that $PB^1(x^2) = PB^1(\tilde{x}^2) = \{1,2\}$. Hence, $C(x^1) \subseteq PB^1(\hat{x}^2)$. From the fact that $x^2 \in S(x^1)$ it follows immediately that $C(x^2) \subseteq PB^2(x^1)$. Furthermore, since $\tilde{x}^2 \in S_{x^2}(x^1)$ and $\bar{x}^2 = \tilde{x}^2$ we get $C(\bar{x}^2) \subseteq PSB^2(x^1)$.

Next we assume that $|C(x^1)| = 1$ and $|C(x^2)| > 1$. We show that $C(x^2) \subseteq PB^2(\hat{x}^1)$, $C(x^1) \subseteq PB^1(x^2)$ and $C(\bar{x}^1) \subseteq PSB^1(x^2)$. Without loss of generality $x^1 = e^1_1$. Then, since $C(x^2) \subseteq I_v$ it holds that $C(x^2) \subseteq PB^2(\hat{x}^1)$. The fact that $x^2 \in S(x^1)$ immediately gives the result $C(x^1) \subseteq PB^1(x^2)$. Furthermore, since $|M^1| = 2$ we obtain $C(\bar{x}^1) \subseteq PSB^1(x^2)$.

Finally, we assume that both $|C(x^1)| = 1$ and $|C(x^2)| = 1$. We show that $C(x^1) \subseteq PB^1(\hat{x}^2)$, $C(x^2) \subseteq PB^2(\hat{x}^1)$, $C(\bar{x}^1) \subseteq PSB^1(x^2)$ and $C(\bar{x}^2) \subseteq PSB^2(x^1)$. Without loss of generality $x^1 = e^1_1$. Since $x^2 \in S(x^1)$ we know that $C(x^1) \subseteq PB^1(x^2)$.

If $PB^1(x^2) = \{1\}$ this implies that $C(x^1) \subseteq PB^1(\hat{x}^2)$. Otherwise, $PB^1(x^2) = \{1, 2\}$ and then since $\tilde{x}^2 \in S_{x^2}(x^1)$ it must hold that $PB^1(\tilde{x}^2) \neq \{2\}$, which implies that also in that case $C(x^1) \subseteq PB^1(\hat{x}^2)$. The fact that $C(x^2) \subseteq I_v$ leads to $C(x^2) \subseteq PB^2(\hat{x}^1)$. Furthermore, since $|M^1| = 2$ it holds that $C(\bar{x}^1) \subseteq PSB^1(\hat{x}^2)$. Finally, since $C(\tilde{x}^2) \subseteq I_{x^2,v}$ we obtain that $C(\bar{x}^2) \subseteq PSB^2(\hat{x}^1)$.

Next we prove the "only if" part. So, let us assume $x \in FBE(G)$. Consequently, $x \in NE(G)$, which implies that $x^2 \in S(x^1)$. Therefore, the first condition of Theorem 7.8.1 is always satisfied. Furthermore, since $x \in FBE(G)$ we know that one of the statements of Proposition 7.3.2 is satisfied. We prove that as a result also conditions (II) and (III) of Theorem 7.8.1 hold. We first assume that the first statement of Proposition 7.3.2 is fulfilled. Since the strategies of both players are mixed, the conditions (II) and (III) hold automatically for this case.

Secondly, we assume that the second statement holds, with $i = 1$ and $j = 2$. Statement (II) is automatically satisfied, and since $C(\bar{x}^2) \subseteq PSB^2(x^1)$ we know that there exists a $\tilde{x}^2 \in \Delta_{M_q^2}$ such that $\tilde{x}^2 \in S_{x^2}(x^1)$, which fulfills condition (III.i). Statement (III.ii) holds automatically.

Next we assume that the second statement holds, with $i = 2$ and $j = 1$. Since $C(x^2) \subseteq PB^2(\hat{x}^1)$ we obtain $C(x^2) \subseteq I_v$, satisfying (II). Statement (III) is automatically satisfied.

Since $C(x^2) \subseteq PB^2(\hat{x}^1)$ we obtain $C(x^2) \subseteq I_v$, satisfying (II). Without loss of generality $x^1 = e_1^1$. Then if $x^2 = e_k^2$, with $k \in I([1])$ statement (III) is satisfied automatically. Otherwise, $k \in I([12])$, and then the fact that $C(\bar{x}^2) \subseteq PSB^2(\hat{x}^1)$ implies that $C(\bar{x}^2) \subseteq PSB^2(x^1)$, and as a consequence there exists a strategy $\tilde{x}^2$ such that $\tilde{x}^2 \in S_{x^2}(x^1)$. Moreover, $C(\bar{x}^2) \subseteq PSB^2(\hat{x}^1)$ leads to the fact that $C(\tilde{x}^2) \subseteq I_{x^2,v}$. As a result, (III) is fulfilled. $\square$

Note that both in Theorem 7.8.1 itself and in the corresponding proof the blocking game characterisation of fall back equilibrium, as given in Proposition 7.3.2, is crucial and insightful, as all results follow from the idea that players are not allowed to play a pure strategy with probability one in the thought experiment underlying fall back equilibrium.

Next we discuss the concept of complete fall back equilibrium. As a result of Theorem 7.6.4 we know that the sets of complete fall back and proper equilibria coincide for $2 \times m^2$ bimatrix games. Therefore, we can follow the characterisation by Borm (1992) for proper equilibrium. Let us first give Borm (1992)'s theorem characterising perfect equilibrium.

Theorem 7.8.2 Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a $2 \times m^2$ bimatrix game and $x \in \Delta$. Then x is a perfect equilibrium of G if and only if the following three assertions hold:

(I) If $I([1]) \neq \emptyset$ and $I([2]) = \emptyset$, then $x^1 = e^1_1$. If $I([2]) \neq \emptyset$ and $I([1]) = \emptyset$, then $x^1 = e^1_2$.

(II) $x^2 \in S(x^1)$.

(III) If $x^1 = e^1_1$, then $C(x^2) \subseteq I_v$. If $x^1 = e^1_2$, then $C(x^2) \subseteq I_1$.

Then, let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a $2 \times m^2$ bimatrix game. The $2 \times m^2$ bimatrix game $\bar{G} = (\{1,2\}, \{\Delta_{\bar{M}^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ is obtained from G by deleting from M^2 all actions $k \in I([12])$, and restricting the payoff functions accordingly.

Theorem 7.8.3 Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a $2 \times m^2$ bimatrix game and $x \in \Delta$. If $I([12]) = \{1,\ldots,m^2\}$, then the sets of proper and perfect equilibria of G coincide. Otherwise x is a proper equilibrium of G if and only if the following three assertions hold:

(I) There is an $\bar{x}^2 \in \Delta_{\bar{M}^2}$ such that $(x^1, \bar{x}^2) \in PE(G)$.

(II) $x^2 \in S(x^1)$.

(III) If $x^1 = e^1_1$, then $C(x^2) \subseteq I_v$. If $x^1 = e^1_2$, then $C(x^2) \subseteq I_1$.

The idea behind both complete fall back and proper equilibrium is that all actions are played with positive probability, but that an action with a lower payoff is played with significant smaller probability than an action with a higher payoff. As a result, actions of player 2 against which player 1 is indifferent have no influence on player 1's decision, as long as he is not indifferent to *some* action of player 2. Consequently, the set of complete fall back equilibria for the 2×4 bimatrix game of Example 7.1.1 is given by $CFBE(G) = \{e_1^1, e_1^2\} \cup \{\frac{2}{5}e_1^1 + \frac{3}{5}e_2^1, e_1^2\} \cup \{e_2^1, e_1^2\}$.

In the remainder of this section we discuss the concept of dependent fall back equilibrium for $2 \times m^2$ bimatrix games. Due to the rather complicated structure of the dependent fall back game it is harder to characterise the set of dependent fall back equilibria for $2 \times m^2$ bimatrix games. Therefore, we examine the dependent fall back game in more detail.

Let $\varepsilon = (\varepsilon^1, \varepsilon^2)$ be a vector of small but positive probabilities and consider the $2 \times \tilde{m}^2$ dependent fall back game $\tilde{G}^D(\varepsilon)$. In the remainder of this section we take $\varepsilon > 0$ small enough. A similar notation as for the game $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ can be used to describe $\tilde{G}^D(\varepsilon)$. For this game the corner points of the intervals describing the upper envelope are given by $e_{21}^1 = \rho_\varepsilon^1(0), \rho_\varepsilon^1(1), \ldots, \rho_\varepsilon^1(\tilde{v}) = e_{12}^1$. The exact position of these points depends on ε, but both $\tilde{v}$ and the order of the intersection points are independent of ε. This result follows from Lemma 7.8.5. Furthermore, also the sets indicating the labels of player 2's actions are independent of ε, and given by $\tilde{I}([1])$, $\tilde{I}([2])$ and $\tilde{I}([12])$, where $\tilde{I}([1]) = \{(k,\ell) \in \tilde{M}^2 \mid PB^1(e_{k\ell}^2) = \{(1,2)\}\}$ and the other sets are defined analogously.

Lemma 7.8.4 *Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a $2 \times m^2$ bimatrix game and let $e_{k\ell}^2 \in \Delta_{\tilde{M}^2}$ be a pure strategy of player 2 in the corresponding dependent fall back game $\tilde{G}^D(\varepsilon)$. Then the following three statements hold for all $(k,\ell) \in \tilde{M}^2$:*

1. *$(k,\ell) \in \tilde{I}([1])$ if and only if one of the following two assertions holds:*

   (a) $k \in I([1])$.

   (b) $k \in I([12])$ and $\ell \in I([1])$.

2. *$(k,\ell) \in \tilde{I}([2])$ if and only if one of the following two assertions holds:*

   (a) $k \in I([2])$.

   (b) $k \in I([12])$ and $\ell \in I([2])$.

3. *$(k,\ell) \in \tilde{I}([12])$ if and only if $k \in I([12])$ and $\ell \in I([12])$.*

Lemma 7.8.4 directly relates the labels of the dependent fall back game to those of the original game. Next we provide a lemma that gives a direct relation between the intersection points of the lines describing player 2's payoffs according to the pure strategies of the dependent fall back game and those of the original game.

Lemma 7.8.5 *Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a $2 \times m^2$ bimatrix game and let $\hat{x}^1 \in \dot{\Delta}_{M^1}$ be such that $\pi^2(\hat{x}^1, e^2_k) = \pi^2(\hat{x}^1, e^2_\ell)$ for some $k, \ell \in M^2, k \neq \ell$. Then for $\rho^{1,I}_\varepsilon \in \dot{\Delta}_{\tilde{M}^1}$, with $\rho^{1,I}_{\varepsilon,12} = \hat{x}^1_1$, the following three results hold for all $r \in M^2 \backslash \{k, \ell\}$:*

1. $\pi^2_\varepsilon(\rho^{1,I}_\varepsilon, e^2_{rk}) = \pi^2_\varepsilon(\rho^{1,I}_\varepsilon, e^2_{r\ell})$.
2. $\pi^2_\varepsilon(\rho^{1,M}_\varepsilon, e^2_{kr}) = \pi^2_\varepsilon(\rho^{1,M}_\varepsilon, e^2_{\ell r})$, *with* $\rho^{1,M}_{\varepsilon,12} = \rho^{1,I}_{\varepsilon,12} + \frac{\varepsilon^1}{1-2\varepsilon^1-\varepsilon^2}(2\rho^{1,I}_{\varepsilon,12} - 1)$.
3. $\pi^2_\varepsilon(\rho^{1,O}_\varepsilon, e^2_{k\ell}) = \pi^2_\varepsilon(\rho^{1,O}_\varepsilon, e^2_{\ell k})$, *with* $\rho^{1,O}_{\varepsilon,12} = \rho^{1,I}_{\varepsilon,12} + \frac{\varepsilon^1}{1-2\varepsilon^1-2\varepsilon^2}(2\rho^{1,I}_{\varepsilon,12} - 1)$.

Note that this lemma is valid for all intersection points in $\dot{\Delta}_{M^1}$ and not only for those on the upper envelope. Further, the superscripts I, M and O correspond to the notions of inside, middle and outside, respectively. The reason for this is that $\rho^{1,O}_{\varepsilon,12} < \rho^{1,M}_{\varepsilon,12} < \rho^{1,I}_{\varepsilon,12} < \frac{1}{2}$ whenever $\hat{x}^1_1 < \frac{1}{2}$. For $\hat{x}^1_1 > \frac{1}{2}$ it is the other way around and $\rho^{1,O}_\varepsilon = \rho^{1,M}_\varepsilon = \rho^{1,I}_\varepsilon$ for $\hat{x}^1_1 = \frac{1}{2}$. This means that in the graphical representation of a dependent fall back game each group of strategies $\rho^{1,O}_\varepsilon$, $\rho^{1,M}_\varepsilon$ and $\rho^{1,I}_\varepsilon$ is ordered in the way their superscripts indicate. Note further that $\rho^{1,I}_\varepsilon$ is independent of ε.

Example 7.8.6 Consider the following 2×4 payoff matrix for player 2.

$$\begin{array}{cc} & \begin{array}{cccc} e^2_1 & e^2_2 & e^2_3 & e^2_4 \end{array} \\ \begin{array}{c} e^1_1 \\ e^1_2 \end{array} & \begin{bmatrix} 0 & 3 & 6 & 9 \\ 8 & 7 & 6 & 5 \end{bmatrix} \end{array}$$

As also Figure 7.8.1 indicates, all lines corresponding to the pure strategies of player 2 intersect at $x^1 = \frac{1}{4}e^1_1 + \frac{3}{4}e^1_2$. In the dependent fall back game player 2 has twelve pure strategies. However, the corresponding lines do not all intersect at exactly the same $\rho^1 \in \Delta_{\tilde{M}^1}$. This is illustrated by Figure 7.8.2, which only gives the lines that are part of the upper envelope, around the point $(\frac{1}{4}e^1_{12} + \frac{3}{4}e^1_{21}, 6)$. Note that the fact that not all twelve lines intersect with each other at the same point on the upper envelope is due to the dependence between the players' executed strategies.

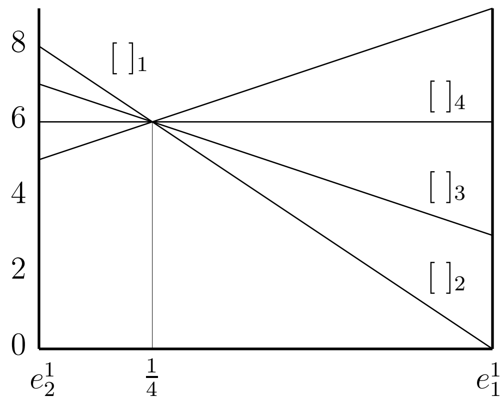


Figure 7.8.1: The original game of Example 7.8.6

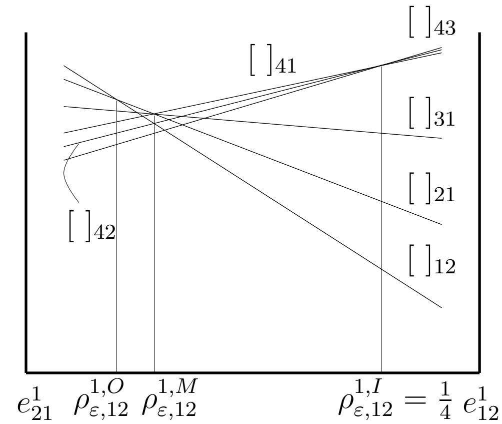


Figure 7.8.2: The dependent fall back game of Example 7.8.6

As a result not all Nash equilibria at such an intersection point of the original game are dependent fall back equilibria. Note, however, that by Lemma 7.8.5 the structure of the upper envelope around such a point is fixed and for any point $x^1 \in \Delta_{M^1}$ with at least three different intersecting lines, the dependent fall back game has exactly three intersection points on the upper envelope, given by the corresponding strategies $\rho_\varepsilon^{1,O}$, $\rho_\varepsilon^{1,M}$ and $\rho_\varepsilon^{1,I}$. We discuss the consequences for the set of dependent fall back equilibria in these situations in more detail later on. $\triangleleft$

For a particular bimatrix game G we are obviously interested in the set $NE(\tilde{G}^D(\varepsilon))$, as $DFBE(G)$ results from that set. Crucial for the determination of the Nash equilibrium set of a dependent fall back game are the intersection points on the upper envelope and the labels of the pure strategies corresponding to these lines. As a result of the above lemmas we do not need to consider the dependent fall back game itself for this, as the intersection points can be found by the use of Lemma 7.8.5 and the labels follow from Lemma 7.8.4.

Example 7.8.7 Consider the 2×3 bimatrix game depicted below.

$$\begin{array}{cc} & \begin{array}{ccc} e_1^2 & e_2^2 & e_3^2 \end{array} \\ \begin{array}{c} e_1^1 \\ e_2^1 \end{array} & \begin{bmatrix} 0,0 & 2,6 & 1,8 \\ 1,8 & 0,6 & 1,2 \end{bmatrix} \end{array}$$

The Nash equilibrium set of this game is given by $NE(G) = \{(e_2^1, e_1^2)\} \cup \{(\frac{1}{4}e_1^1 +$

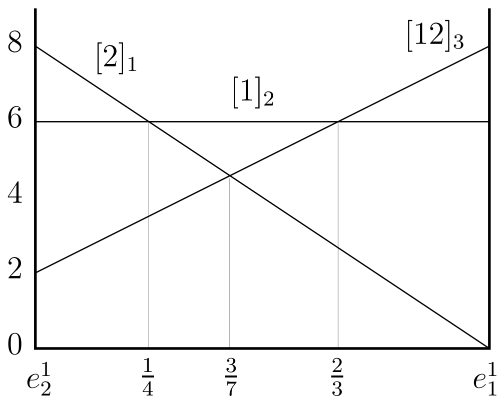


Figure 7.8.3: The game of Example 7.8.7

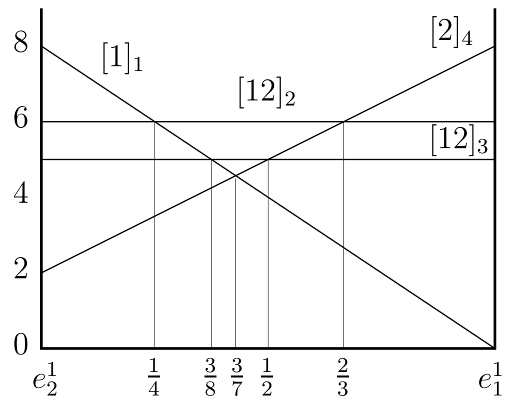


Figure 7.8.4: The game of Example 7.8.8

$\frac{3}{4}e^1_2, \frac{2}{3}e^2_1 + \frac{1}{3}e^2_2)\} \cup \text{conv}(\{\frac{2}{3}e^1_1 + \frac{1}{3}e^1_2, e^1_1\}) \times \{e^2_3\}$. Furthermore, $NE(\tilde{G}^D(\varepsilon)) = \{(e^1_{21}, e^2_{12})\} \cup \{((\frac{1}{4} - \frac{1}{2}\frac{\varepsilon^1}{1-2\varepsilon^1-2\varepsilon^2})e^1_{12} + (\frac{3}{4} + \frac{1}{2}\frac{\varepsilon^1}{1-2\varepsilon^1-2\varepsilon^2})e^1_{21}, (\frac{2}{3} + \frac{\varepsilon^2}{3-6\varepsilon^1-6\varepsilon^2})e^2_{12} + (\frac{1}{3} - \frac{\varepsilon^2}{3-6\varepsilon^1-6\varepsilon^2})e^2_{21})\} \cup \{(e^1_{12}, e^2_{32})\}$. Consequently, the set of dependent fall back equilibria of this game is given by $DFBE(G) = \{(e^1_2, e^2_1)\} \cup \{(\frac{1}{4}e^1_1 + \frac{3}{4}e^1_2, \frac{2}{3}e^2_1 + \frac{1}{3}e^2_2)\} \cup \{(e^1_1, e^2_3)\}$.

In this example many Nash equilibria are dependent fall back equilibria. However, from the set $\text{conv}(\{\frac{2}{3}e^1_1 + \frac{1}{3}e^1_2, e^1_1\}) \times \{e^2_3\}$ only the corner solution $\{(e^1_1, e^2_3)\}$ is a dependent fall back equilibrium. The reason is that player 2's action $(3, 2)$ is an element of $\tilde{I}([1])$ and not of $\tilde{I}([12])$, which is due to the fact that $2 \in I([1])$, where 2 is the unique element of $PSB^2(x^1)$ for all $x^1 \in (\frac{2}{3}e^1_1 + \frac{1}{3}e^1_2, e^1_1]$. Therefore, for all x^1 in this interval player 2 is unable to choose a back-up strategy that results in a Nash equilibrium in the dependent fall back game. ◁

The set of x^1's for which player 2 is able to choose a primary and back-up strategy such that there exists a Nash equilibrium in the dependent fall back game is called the set of *fall back strategies* of player 1 and is defined by $\Delta^{FB}_{M^1} = \{x^1 \in \Delta_{M^1} \mid \text{there exists } x^2 \in \Delta_{M^2} \text{ such that } (x^1, x^2) \in DFBE(G)\}$. This set is characterised later on but note that a particular $x^1 \in \dot{\Delta}_{M^1}$ can only be an element of this set if $S(x^1) \neq \emptyset$ and there exists an $x^2 \in \text{conv}(PSB^2(x^1))$ for which player 1 is indifferent between e^1_1 and e^1_2, because in order to make player 1 indifferent player 2 should play strategies as primary and back-up for which both e^1_1 and e^1_2 are best replies. Hence, whether x^1 is an element of the set of fall back strategies does not only depend on the labels of the strategies in the set $PB^2(x^1)$, but also on those in the set $PSB^2(x^1)$. Obviously, $x^1 \in \Delta^{FB}_{M^1}$ is a necessary condition for $(x^1, x^2) \in DFBE(G)$.

Note that by Theorem 7.8.2 for perfect equilibria the set of second best replies in itself is irrelevant and that as a result for Example 7.8.7 the set of perfect equilibria coincides with the set of Nash equilibria and is therefore unequal to the set of dependent fall back equilibria.

Example 7.8.8 Consider the following 2×4 bimatrix game.

$$\begin{array}{cc} & \begin{array}{cccc} e_1^2 & e_2^2 & e_3^2 & e_4^2 \end{array} \\ \begin{array}{c} e_1^1 \\ e_2^1 \end{array} & \left[\begin{array}{cccc} 1,0 & 1,6 & 1,5 & 0,8 \\ 0,8 & 1,6 & 1,5 & 1,2 \end{array}\right] \end{array}$$

We first of all consider the game without the third column. Then the only change compared to Example 7.8.7 is in the labels indicating player 1's best replies. The set of Nash equilibria of this game is given by $NE(G) = \text{conv}(\{\frac{1}{4}e_1^1 + \frac{3}{4}e_2^1, \frac{2}{3}e_1^1 + \frac{1}{3}e_2^1\}) \times \{e_2^2\}$.

Obviously, also the dependent fall back game only differs in the labels and the set of Nash equilibria of the dependent fall back game is given by $NE(\tilde{G}^D(\varepsilon)) = \{(\frac{3}{7}e_{12}^1 + \frac{4}{7}e_{21}^1, \frac{1}{2}e_{21}^2 + \frac{1}{2}e_{24}^2)\}$. This results in $DFBE(G) = \{(\frac{3}{7}e_1^1 + \frac{4}{7}e_2^1, e_2^2)\}$, which coincides with the set of proper equilibria.

Next we consider the entire game, hence including column 3. Then the set of Nash equilibria is exactly the same as before, because player 2's action 3 is not an element of $PB^2(x^1)$ for any $x^1 \in \Delta_{M^1}$. However, action $3 \in PSB^2(x^1)$ for $x^1 \in [\frac{3}{8}e_1^1 + \frac{5}{8}e_2^1, \frac{1}{2}e_1^1 + \frac{1}{2}e_2^1]$, which means that it may influence the Nash equilibrium set of the dependent fall back game.

This is indeed the case as $NE(\tilde{G}^D(\varepsilon)) = \text{conv}(\{\frac{3}{8}e_{12}^1 + \frac{5}{8}e_{21}^1, \frac{1}{2}e_{12}^1 + \frac{1}{2}e_{21}^1\}) \times \{e_{23}^2\}$. Therefore, $DFBE(G) = \text{conv}(\{\frac{3}{8}e_1^1 + \frac{5}{8}e_2^1, \frac{1}{2}e_1^1 + \frac{1}{2}e_2^1\}) \times \{e_2^2\}$. It follows from Theorem 7.8.3 that the third column does not influence the set of proper equilibria, as it has label [12]. However, for dependent fall back equilibrium we only have to consider the sets of pure best and pure second best replies. In this example, that back-up strategy is e_3^2 for $x^1 \in [\frac{3}{8}e_1^1 + \frac{5}{8}e_2^1, \frac{1}{2}e_1^1 + \frac{1}{2}e_2^1] = \Delta_{M^1}^{FB}$. $\triangleleft$

Next we start with the characterisation of the set $\Delta_{M^1}^{FB}$ for a $2 \times m^2$ bimatrix game $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ starting at the extreme points, hence for $x^1 \in \{e_1^1, e_2^1\}$. Then, $e_1^1 \in \Delta_{M^1}^{FB}$ if there exist strategies $k \in I_v$ and $\ell \in PSB^2(e_1^1)$ such that (i) $k \in I([1])$ or (ii) $k \in I([12])$ and $\ell \notin I([2])$. Clearly, $(e_1^1, e_k^2) \in DFBE(G)$ in both cases, which means that the set of dependent fall back equilibria is non-empty. Note however, that in case several lines coincide on the upper envelope at $x^1 = e_1^1$ this fall back equilibrium need not be unique in this point. Example 7.8.7 demonstrates option (ii), for $k = 3$ and $\ell = 2$.

For $x^1 = e_2^1$ it is the other way around, hence $e_2^1 \in \Delta_{M^1}^{FB}$ if there exist strategies $k \in I_1$ and $\ell \in PSB^2(e_2^1)$ such that (i) $k \in I([2])$ or (ii) $k \in I([12])$ and $\ell \notin I([1])$. Obviously, $(e_2^1, e_k^2) \in DFBE(G)$ if either one of the two options holds. Example 7.8.7 illustrates the first situation, with $k = 1$. In the following theorem we characterise the set of dependent fall back equilibria for $x^1 \in \{e_1^1, e_2^1\}$.

Theorem 7.8.9 Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a $2 \times m^2$ bimatrix game and $x \in \Delta$, with $x^1 \in \{e_1^1, e_2^1\}$. Then x is a dependent fall back equilibrium of G if and only if the following three assertions hold:

(I) $x^1 \in \Delta_{M^1}^{FB}$.

(II) $x^2 \in S(x^1)$.

(III) If $x^1 = e_1^1$, then $C(x^2) \subseteq I_v$. If $x^1 = e_2^1$, then $C(x^2) \subseteq I_1$.

Note that the third statement indicates that x^2 should not only be the best reply against e_1^1 (or e_2^1), but to all x^1 in the neighborhood of e_1^1 (or e_2^1) as well, which is due to the fact that there is a strictly positive probability that player 1 has to deviate to his other action. The exact same assertion should hold for all other refinements discussed in this section.

For $x^1 \in \dot{\Delta}_{M^1}$ it is more difficult to indicate whether x^1 is a fall back strategy of player 1, which is due to the structure around intersection points as shown in Example 7.8.6. Therefore, we have to introduce some additional notation before we first of all give the three conditions that describe which x^1s in the interior of Δ_{M^1} are elements of $\Delta_{M^1}^{FB}$, and secondly and more importantly characterise the dependent fall back equilibria for $2 \times m^2$ bimatrix games.

The set of pure outside best replies for a given $x^1 \in \dot{\Delta}_{M^1}$ is defined by $POB^2(x^1) = PB^2(\dot{x}^1)$, with $\dot{x}^1$ such that $\dot{x}_1^1 = x_1^1 + \xi(2x_1^1 - 1)$, with $\xi > 0$ very small. In Example 7.8.7, *e.g.*, $POB^2(\frac{1}{4}e_1^1 + \frac{3}{4}e_2^1) = \{1\}$ and $POB^2(\frac{2}{3}e_1^1 + \frac{1}{3}e_2^1) = \{3\}$.

Furthermore, we define the set of pure outside second best replies by $POSB^2(x^1) = PSB^2(\dot{x}^1) \cap PB^2(x^1)$. In Example 7.8.7, $POSB^2(\frac{1}{4}e_1^1 + \frac{3}{4}e_2^1) =$

$POSB^2(\frac{2}{3}e_1^1 + \frac{1}{3}e_2^1) = \{2\}$. Note that the set of pure outside second best replies is non-empty if and only if $|PB^2(x^1)| > 1$.

Finally, we also define the set of pure inside best replies, which is given by $PIB^2(x^1) = PB^2(\ddot{x}^1)$, with $\ddot{x}^1$ such that $\ddot{x}_1^1 = x_1^1 - \xi(2x_1^1 - 1)$, with $\xi > 0$ very small. In Example 7.8.7, $PIB^2(\frac{1}{4}e_1^1 + \frac{3}{4}e_2^1) = POSB^2(\frac{1}{4}e_1^1 + \frac{3}{4}e_2^1) = \{2\}$ and $PIB^2(\frac{2}{3}e_1^1 + \frac{1}{3}e_2^1) = POSB^2(\frac{2}{3}e_1^1 + \frac{1}{3}e_2^1) = \{2\}$. Note that $POSB^2(x^1) = PIB^2(x^1)$ for any $x^1 \in \Delta_{M^1}$ with $|PB^2(x^1)| = 2$. However, when $|PB^2(x^1)| \neq 2$ this is in general not the case.

Also note that $POB^2(x^1)$, $POSB^2(x^1)$ and $PIB^2(x^1)$ are all subsets of $PB^2(x^1)$ for all $x^1 \in \dot{\Delta}_{M^1}$. Then, $x^1 \in \dot{\Delta}_{M^1}$ is an element of $\Delta_{M^1}^{FB}$ if and only if one of the following three conditions holds:

1. There exist strategies $k \in PB^2(x^1, [1])$ and $\ell \in PB^2(x^1, [2])$. In that case each one of these two strategies can either be used as back-up for the other resulting in a coordination solution of these strategies as a dependent fall back equilibrium strategy, or they can be combined and used as a back-up strategy for a third pure strategy $r \in PB^2(x^1, [12])$. The first situation arises, *e.g.*, in Example 7.8.7 at $x^1 = \frac{1}{4}e_1^1 + \frac{3}{4}e_2^1$.

2. There exist strategies $k \in PB^2(x^1, [12])$, $\ell \in PSB^2(x^1, [1])$ and $r \in PSB^2(x^1, [2])$. In that case player 2 can choose a combination between the strategies ℓ and r as back-up for his primary strategy k, which results in $(x^1, e_k^2) \in DFBE(G)$. Example 7.8.8 without the third column demonstrates this at $x^1 = \frac{3}{7}e_1^1 + \frac{4}{7}e_2^1$.

3. There exist strategies $k \in PB^2(x^1, [12])$ and $\ell \in PSB^2(x^1, [12])$ and either $k \in POB^2(x^1) \cup PIB^2(x^1)$ or $x^1 = \frac{1}{2}e_1^1 + \frac{1}{2}e_2^1$. When $PB^2(x^1) \neq PSB^2(x^1)$, then $k \in POB^2(x^1) \cup PIB^2(x^1)$ and one can obviously take $e_{k\ell}^2$ as a solution to ρ^1 corresponding to x^1, as, *e.g.*, in the complete version of Example 7.8.8 for $x^1 \in [\frac{3}{8}e_1^1 + \frac{5}{8}e_2^1, \frac{1}{2}e_1^1 + \frac{1}{2}e_2^1]$. This is also possible when $|PB^2(x^1)| = 2$. However for an $x^1 \in \dot{\Delta}_{M^1}$ with at least three different intersecting lines it is slightly more difficult, and the condition that $k \in POB^2(x^1) \cup PIB^2(x^1)$ is needed if $x^1 \neq \frac{1}{2}e_1^1 + \frac{1}{2}e_2^1$.

   This can be explained by the use of Example 7.8.6. There the lines of strategies e_{23}^2 and e_{32}^2 are not part of the upper envelope. Hence, if both e_2^2 and e_3^2 have a label [12] and strategies 1 and 4 both a label [1] (or [2]) the set of dependent fall back equilibria for $x^1 = \frac{1}{4}e^1_1 + \frac{3}{4}e^1_2$ is empty. Hence, it is required that $k \in POB^2(x^1) \cup PIB^2(x^1)$. If $k \in POB^2(x^1)$ there always is a Nash equilibrium at intersection point $\rho^{1,M}_{\varepsilon}$, and depending on the labels of the other lines maybe even at $\rho^{1,O}_{\varepsilon}$ as well, and if $k \in PIB^2(x^1)$ there certainly is a Nash equilibrium at $\rho^{1,I}_{\varepsilon}$, and possibly also at $\rho^{1,M}_{\varepsilon}$. Hence, in those cases $x^1 \in \Delta^{FB}_{M^1}$. For $x^1 = \frac{1}{2}e^1_1 + \frac{1}{2}e^1_2$, $\rho^{1,O}_{\varepsilon} = \rho^{1,M}_{\varepsilon} = \rho^{1,I}_{\varepsilon}$, and therefore, if $k \in PB^2(x^1, [12])$ and $\ell \in PSB^2(x^1, [12])$, then we immediately obtain that $x^1 \in \Delta^{FB}_{M^1}$.

We now know when an $x^1 \in \Delta_{M^1}$ is part of the fall back strategies of player 1. However, not all solutions to such an $x^1 \in \Delta^{FB}_{M^1}$ are also dependent fall back equilibria. In Example 7.8.6, *e.g.*, we can see that when three or more lines intersect with each other on the upper envelope of the original game not all corresponding lines intersect with each other on the upper envelope of the dependent fall back game. Strategies out of the sets $POB^2(x^1)$, $POSB^2(x^1)$, $PIB^2(x^1)$ and $PB^2(x^1)\backslash(POB^2(x^1) \cup POSB^2(x^1) \cup PIB^2(x^1))$ give rise to three intersection points, which lead to different sets of Nash equilibria.

Example 7.8.10 Consider the following 2×4 bimatrix game G.

$$\begin{array}{c c} & \begin{array}{cccc} e^2_1 & e^2_2 & e^2_3 & e^2_4 \end{array} \\ \begin{array}{c} e^1_1 \\ e^1_2 \end{array} & \begin{bmatrix} 1,0 & 1,3 & 0,6 & 0,9 \\ 0,8 & 1,7 & 1,6 & 1,5 \end{bmatrix} \end{array}$$

Note that player 2's payoff matrix is the same as in Example 7.8.6. The set of Nash equilibria of the dependent fall back game is given by $NE(\tilde{G}^D(\varepsilon)) = \{(\frac{1}{4} - \frac{1}{2}\frac{\varepsilon^1}{1-2\varepsilon^1-\varepsilon^2})e^1_{12} + (\frac{3}{4} + \frac{1}{2}\frac{\varepsilon^1}{1-2\varepsilon^1-\varepsilon^2})e^1_{21})\} \times \text{conv}(\{\rho^2(21,31), \rho^2(21,41)\})$, where $\rho^2(k\ell, rs)$ is a coordination solution similar to $x^2(k,\ell)$ as defined for $2 \times m^2$ bimatrix games. This Nash equilibrium set leads to $DFBE(G) = \{(\frac{1}{4}e^1_1 + \frac{3}{4}e^1_2, e^2_2)\}$. Hence, although $x^1 \in \Delta^{FB}_{M^1}$ for $x^1 = \frac{1}{4}e^1_1 + \frac{3}{4}e^1_2$ not all solutions to this x^1 are dependent fall back equilibria; the coordination solutions $x^2(1,3)$ and $x^2(1,4)$ do not sustain, because lines corresponding to a primary strategy out of $POB^2(x^1)$ (pure strategy 1 in this example) do not intersect with lines corresponding to the primary strategies out of $PB^2(x^1)\backslash(POB^2(x^1) \cup POSB^2(x^1))$ (pure strategies 3 and 4) on the upper envelope of the dependent fall back game.

Note that by Theorem 7.8.1, *e.g.*, strategy profile $(\frac{1}{4}e^1_1 + \frac{3}{4}e^1_2, \frac{1}{2}e^2_1 + \frac{1}{2}e^2_3) \in FBE(G)$, which implies that not every fall back equilibrium is a dependent fall back equilibrium. Furthermore, by Theorem 7.8.3 this strategy profile is also a complete fall back equilibrium. Consequently, the set of complete fall back equilibria is not a subset of the set of dependent fall back equilibria. $\triangleleft$

From the above it follows that to characterise the set of dependent fall back equilibria there is a need for additional restrictions. These are given by the requirement that x^2 is an element of $S_{FB}(x^1)$, the set of *fall back solutions* to $x^1 \in \dot{\Delta}_{M^1}$. Let us give all six sets that together define this set $S_{FB}(x^1)$.

For all $x^1 \in \dot{\Delta}_{M^1}$ a strategy $e_k^2 \in PS_{FB}^O(x^1)$, the outside set of pure fall back solutions to x^1, if and only if $|POB^2(x^1)| = 1$ and one of the following two assertions holds:

1. $k \in POB^2(x^1, [12])$ and one of the following two statements is fulfilled:
   (a) There exists a strategy $\ell \in POSB^2(x^1, [12])$.
   (b) There exist strategies $\ell \in POSB^2(x^1, [1])$ and $r \in POSB^2(x^1, [2])$.
2. $k \in POSB^2(x^1, [12])$ and one of the following two statements is fulfilled:
   (a) There exists a strategy $\ell \in POB^2(x^1, [12])$.
   (b) There exist strategies $\ell, r \in POB^2(x^1) \cup POSB^2(x^1)$ such that $\ell \in I([1])$ and $r \in I([2])$.

For all $x^1 \in \dot{\Delta}_{M^1}$ a strategy $x^2(k, \ell) \in CS_{FB}^O(x^1)$, the outside set of coordination fall back solutions to x^1, if and only if $|POB^2(x^1)| = 1$ and the following two assertions hold:

1. $x^2(k, \ell) \in CS(x^1)$.
2. $k, \ell \in POB^2(x^1) \cup POSB^2(x^1)$.

For all $x^1 \in \dot{\Delta}_{M^1}$ a strategy $e_k^2 \in PS_{FB}^M(x^1)$, the middle set of pure fall back solutions to x^1, if and only if one of the following two assertions holds:

1. $|POB^2(x^1)| = 1$, $k \in (PB^2(x^1) \backslash POB^2(x^1)) \cap I([12])$ and one of the following two statements is fulfilled:

   (a) There exists a strategy $\ell \in POB^2(x^1, [12])$.

   (b) There exist strategies $\ell \in PB^2(x^1, [1])$ and $r \in PB^2(x^1, [2])$.

2. $|POB^2(x^1)| > 1$, $k \in PB^2(x^1, [12])$ and one of the following two statements is fulfilled:

   (a) There exists a strategy $\ell \in POB^2(x^1, [12])$, with $\ell \neq k$.

   (b) There exist strategies $\ell \in PB^2(x^1, [1])$ and $r \in PB^2(x^1, [2])$.

For all $x^1 \in \dot{\Delta}_{M^1}$ a strategy $x^2(k, \ell) \in CS^M_{FB}(x^1)$, the middle set of coordination fall back solutions to x^1, if and only if the following two assertions hold:

1. $x^2(k, \ell) \in CS(x^1)$.

2. If $|POB^2(x^1)| = 1$, then $k, \ell \notin POB^2(x^1)$.

For all $x^1 \in \dot{\Delta}_{M^1}$ a strategy $e^2_k \in PS^I_{FB}(x^1)$, the inside set of pure fall back solutions to x^1, if and only if $k \in PIB^2(x^1, [12])$ and one of the following two assertions holds:

1. There exists a strategy $\ell \in PSB^2(x^1, [12])$, with $\ell \neq k$.

2. There exist strategies $\ell \in PSB^2(x^1, [1])$ and $r \in PSB^2(x^1, [2])$.

For all $x^1 \in \dot{\Delta}_{M^1}$ a strategy $x^2(k, \ell) \in CS^I_{FB}(x^2)$, the inside set of coordination fall back solutions to x^1, if and only if the following two assertions hold:

1. $x^2(k, \ell) \in CS(x^1)$.

2. $k, \ell \in PIB^2(x^1)$.

As a result the outside, middle and inside set of fall back solutions to $x^1 \in \dot{\Delta}_{M^1}$ are given by $S^O_{FB}(x^1) = \text{conv}(PS^O_{FB}(x^1) \cup CS^O_{FB}(x^1))$, $S^M_{FB}(x^1) = \text{conv}(PS^M_{FB}(x^1) \cup CS^M_{FB}(x^1))$ and $S^I_{FB}(x^1) = \text{conv}(PS^I_{FB}(x^1) \cup CS^I_{FB}(x^1))$, respectively. The set of fall back solutions to $x^1 \in \dot{\Delta}_{M^1}$ is defined as $S_{FB}(x^1) = S^O_{FB}(x^1) \cup S^M_{FB}(x^1) \cup S^I_{FB}(x^1)$. Note that $S_{FB}(x^1)$ is not equal to the convex hull of its subsets but only to the union. This is due to the fact that the three separate sets of fall back solutions result from different intersection points on the upper envelope; at an intersection point $x^1 \in \dot{\Delta}_{M^1}$ the set $S^O_{FB}(x^1)$ corresponds, *e.g.*, to the intersection point $\rho^{1,O}_{\varepsilon}$ of the dependent fall back game, and this set is empty whenever x^1 does not correspond to an intersection point on the upper envelope of the original game.

Note further that these sets do not only contain equilibria on intersection points of the upper envelope, but also the equilibria on line segments, as these are elements of $S^I_{FB}(x^1)$.

Whenever $x^1 \in \dot{\Delta}_{M^1}$, then $x^2 \in S_{FB}(x^1)$ implies that $x^1 \in \Delta^{FB}_{M^1}$. Therefore the theorem that characterises the dependent fall back equilibria for $x^1 \in \dot{\Delta}_{M^1}$ can be given in the following way.

Theorem 7.8.11 Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a $2 \times m^2$ bimatrix game and $x \in \Delta$. If $x^1 \in \dot{\Delta}_{M^1} \backslash \{\frac{1}{2}e^1_1 + \frac{1}{2}e^1_2\}$, then (x^1, x^2) is a dependent fall back equilibrium of G if and only if $x^2 \in S_{FB}(x^1)$. If $x^1 = \frac{1}{2}e^1_1 + \frac{1}{2}e^1_2$, then (x^1, x^2) is a dependent fall back equilibrium of G if and only if $x^1 \in \Delta^{FB}_{M^1}$ and $x^2 \in S(x^1)$.

To complete the relations for $2 \times m^2$ bimatrix games between all equilibrium concepts discussed in this chapter we show that every robust equilibrium is a dependent fall back equilibrium for $2 \times m^2$ bimatrix games. We first give Borm (1992)'s theorem characterising the set of robust equilibrium for $2 \times m^2$ bimatrix games.

Theorem 7.8.12 Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a $2 \times m^2$ bimatrix game and $x \in \Delta$. Then x is a robust equilibrium of G if and only if x is a perfect equilibrium of G and one of the following three assertions holds:

(I) $x^1 = e^1_1$ and, either $PB^1(x^2) = \{1\}$ or $I([2]) = \emptyset$.

(II) $x^1 = e^1_2$ and, either $PB^1(x^2) = \{2\}$ or $I([1]) = \emptyset$.

(III) $x^1 \in \dot{\Delta}_{M^1} \backslash \{x^1(r)\}_{r\in\{1,\ldots,v-1\}}$ and $I([12]) = \{1, \ldots, n\}$.

Proposition 7.8.13 *Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a $2 \times m^2$ bimatrix game. Then each robust equilibrium of G is a dependent fall back equilibrium of G.*

Proof: Let $x \in RB(G)$. We first assume that $x^1 = e^1_1$. (The proof for $x^1 = e^1_2$ is analogous.) Since $x \in PE(G)$ it follows that (II) and (III) of Theorem 7.8.9 are satisfied. Hence, it suffices to show that $x^1 \in \Delta^{FB}_{M^1}$. Consequently, it must hold that there exists a strategy $k \in I_v$ and a strategy $\ell \in PSB^2(e^2_1)$ such that either (i) $k \in I([1])$ or (ii) $k \in I([12])$ and $\ell \notin I([2])$. By (I) and (III) of Theorem 7.8.12 we easily obtain that indeed either (i) or (ii) is valid.

Next we assume that (III) is satisfied. If $x^1 = \frac{1}{2}e^1_1 + \frac{1}{2}e^1_2$, we must show that $x^1 \in \Delta^{FB}_{M^1}$ and $x^2 \in S(x^1)$. The latter holds by the fact that $x \in PE(G)$. Furthermore, since $I([12]) = \{1, \ldots, n\}$ there exist strategies $k \in PB^2(x^1, [12])$ and $\ell \in PSB^2(x^1, [12])$, which implies that $x^1 \in \Delta^{FB}_{M^1}$.

If $x^1 \neq \frac{1}{2}e^1_1 + \frac{1}{2}e^1_2$ we must show that $x^2 \in S_{FB}(x^1)$. Since $I([12]) = \{1, \ldots, n\}$ it follows that x^2 cannot be a coordination solution. Hence, it suffices to show that x^2 is an element of the convex hull of a set of pure solutions. Let $k \in C(x^2)$. We show that $e^2_k \in PS^I_{FB}$, which completes the proof. Since $x^1 \in \dot{\Delta}_{M^1} \backslash \{x^1(r)\}_{r \in \{1,\ldots,v-1\}}$, $k \in PIB^2(x^1)$ and since $I([12]) = \{1, \ldots, n\}$, $k \in PIB^2(x^1, [12])$. Furthermore, as $I([12]) = \{1, \ldots, n\}$ there exists a strategy $\ell \in PSB^2(x^1, [12])$, with $\ell \neq k$. Hence, $e^2_k \in PS^I_{FB}(x^1)$. $\square$

We conclude this section with a graphical overview of the relations for $2 \times m^2$ bimatrix games between all equilibrium concepts discussed in this chapter. Note that all relations not present in this figure are indeed known to be non-existent.

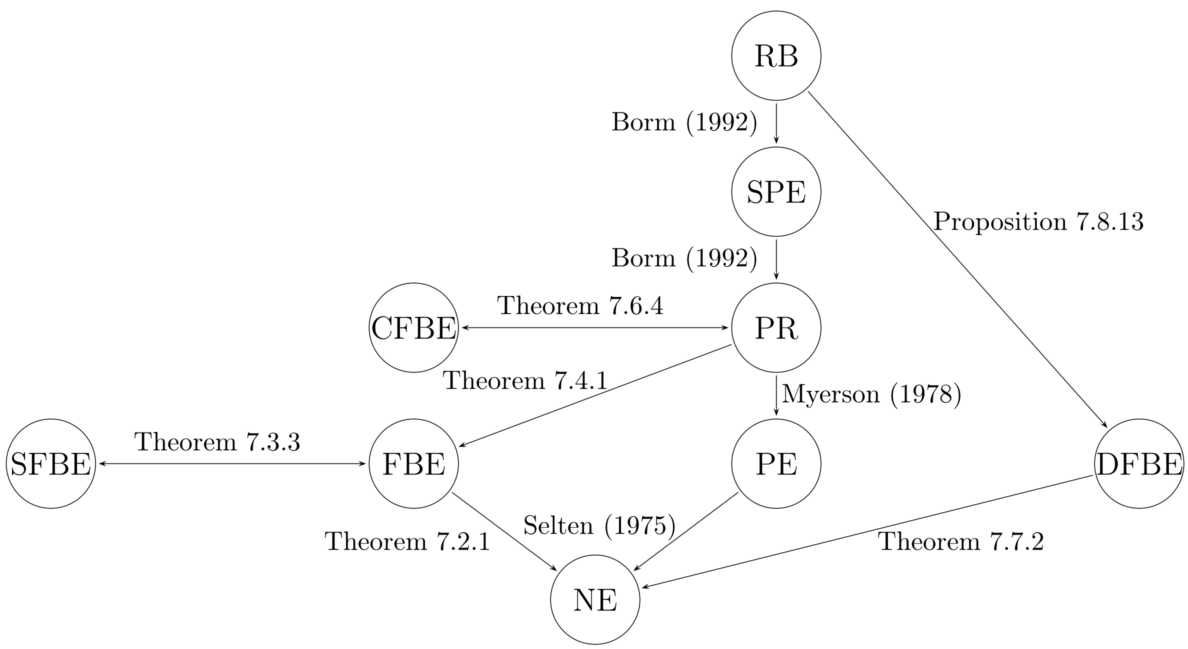


Figure 7.8.5: Relations for $2 \times m^2$ bimatrix games

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Het modelleren van interactief gedrag, en oplossingsconcepten

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Samenvatting

In dit proefschrift behandelen we enkele onderwerpen uit de speltheorie. Deze stroming binnen de economische wetenschappen heeft als doel om situaties met verschillende beslissingsnemers (spelers) wiskundig te modelleren en te analyseren. Speltheorie is onder te verdelen in twee takken: *coöperatieve* speltheorie en *niet-coöperatieve speltheorie*. Niet-coöperatieve speltheorie behandelt situaties waarin spelers, om wat voor reden dan ook, geen bindende afspraken kunnen maken.

Coöperative speltheorie daarentegen bestudeert situaties waarin spelers de mogelijkheid hebben tot samenwerking om op die manier (extra) opbrengsten te genereren of kosten te besparen. Men richt zich binnen deze tak van de speltheorie met name op het bestuderen van eerlijke allocaties van de gezamenlijke opbrengsten. Hiervoor wordt vaak gebruik gemaakt van een *coöperatief spel*. In een coöperatief spel wordt ieder mogelijke coalitie van spelers geassocieerd met een bepaalde waarde, die correspondeert met de opbrengsten die deze coalitie kan behalen zonder medewerking van de spelers buiten de coalitie. Deze *coalitiewaarden* kunnen dan gebruikt worden als referentiekader voor het verdelen van de opbrengsten van de grote coalitie (de coalitie van alle spelers) over de spelers van het spel. Het meest fundamentele

oplossingsconcept voor coöperatieve spelen is de *core*. Een allocatie is een core element als het voldoet aan twee eigenschappen. Allereerst moet een allocatie *efficient* zijn, wat inhoudt dat de opbrengsten van de grote coalitie exact verdeeld moet worden over de spelers. Ten tweede moet een allocatie *stabiel* zijn. Een allocatie is stabiel als er geen coalitie van spelers bestaat met een coalitiewaarde hoger dan de opbrengsten uit de allocatie.

In het eerste gedeelte van dit proefschrift behandelen we verschillende onderwerpen uit de coöperatieve speltheorie. In Hoofdstuk 3 analyseren we oplossingsconcepten voor coöperatieve spelen. Twee van de meest bekende en bestudeerde één-punts-oplossingsconcepten voor coöperatieve spelen zijn de *nucleolus* en de *prenucleolus*. De prenucleolus is de unieke allocatie in de *preimputatieverzameling* (de verzameling van alle efficiente allocaties) waartegen het maximale bezwaar per coalitie minimaal is. Veel minder bekend en bestudeerd zijn de gerelateerde concepten de *per capita nucleolus* en de *per capita prenucleolus*. De per capita prenucleolus is de unieke allocatie in de preimputatieverzameling waartegen het maximale bezwaar *per speler* van een coalitie minimaal is. Kortweg zijn onder de prenucleolus alle *coalities* even belangrijk, terwijl gegeven de per capita prenucleolus alle *spelers* van alle coalities even belangrijk zijn.

In een uitgebreid overzicht laten we zien aan welke eigenschappen de per capita prenucleolus voldoet en we gebruiken enkele van deze eigenschappen om dit oplossingsconcept te karakteriseren. Verder onderzoeken we de relaties tussen de per capita prenucleolus en andere oplossingsconcepten voor coöperatieve spelen. Een soortgelijke analyse passen we toe voor de per capita nucleolus, en ook voor de gerelateerde concepten de *per capita prekernel* en de *per capita kernel*. Bovendien presenteren we ook een nieuwe karakterisering van de core.

In de Hoofdstukken 4 en 5 is ons uitganspunt niet een coöperatief spel, maar een onderliggende *coöperative situatie*. Een coöperative situatie bestaat in het algemeen uit een groep spelers die kunnen kiezen uit een verzameling alternatieven en elk van deze alternatieven resulteert in bepaalde kosten voor (de groep) spelers. Deze verzameling alternatieven en bijbehorende kosten zijn over het algemeen afkomstig van een combinatorisch optimaliseringsprobleem waarin verschillende spelers de controle hebben over delen van dit probleem. Men kan hier bijvoorbeeld denken aan een *minimum opspannende boom probleem*. In een dergelijk probleem dienen een aantal spelers zich rechtstreeks of indirect te verbinden met een bepaalde voorziening. Dit kan bijvoorbeeld een elektriciteitscentrale of watervoorziening zijn. Door samen te werken kunnen de spelers een netwerk vormen met minimale kosten. Een dergelijk netwerk wordt een *minimum opspannende boom (mob)* genoemd. Men kan de verzameling van alle opspannende bomen dan zien als de verzameling alternatieven, met een *mob* als optimaal alternatief. Als een optimaal alternatief gevonden is, dan is het zaak om de bijbehorende totale kosten eerlijk te verdelen over alle spelers zodanig dat iedere speler daadwerkelijk wil samenwerken om dit optimale alternatief te bewerkstelligen. Daarom leidt een coöperatieve situatie in het algemeen tot twee vragen; welk alternatief moet er gekozen worden en hoe moeten de bijbehorende kosten verdeeld worden over de spelers? Om de tweede vraag te beantwoorden kunnen we coöperatieve situaties modelleren als een coöperatief spel.

In Hoofdstuk 4 bestuderen we een specifieke klasse van coöperatieve situaties, *publieke netwerk situaties met congestie*. Deze klasse van coöperatieve situaties generaliseert de klasse van minimum opspannende boom problemen in de zin dat de kosten van het gebruik van een bepaalde verbinding van het netwerk in deze klasse afhangen van het aantal gebruikers van deze verbinding. Voor toepassingen kan men denken aan computer netwerken met een hoofdserver, aan communicatienetwerken met een unieke informatieverstrekker of aan distributiecentra met meerdere leveranciers op een publiek wegennetwerk. We bestuderen met name publieke netwerk situaties met *convexe* congestie kosten, wat betekent dat de gemiddelde kosten voor het gebruik van een verbinding toenemen met het aantal gebruikers van deze verbinding. Voor deze klasse presenteren we een algoritme dat voor iedere coalitie van spelers het optimale netwerk genereert. Verder beargumenteren we dat dit soort situaties passend gemodelleerd worden door het marginale congestie kostenspel. We bewijzen dat dit spel *concaaf* is, wat kortweg betekent dat samenwerken met een grotere groep meer kostenbesparing oplevert dan samenwerken met een kleinere groep. Als een gevolg hiervan wil in een concaaf spel iedereen met iedereen samenwerken. We introduceren ook een één-punts-oplossingsconcept dat gebasseerd is op drie principes van gelijke behandeling van spelers.

In Hoofdstuk 5 pakken we het algemener aan. Het uitgangspunt is een willekeurige coöperatieve situatie en de centrale vraag is hoe we de kosten van het optimale alternatief moeten verdelen. Hiervoor maken we gebruik van coöperatieve spelen. In het algemeen is het echter niet duidelijk welk spel het best past bij welke situatie. Daarom presenteren we een algemeen model dat gebruikt kan worden als een gids voor het vinden van een passend coöperatief spel. Daarnaast kan hetzelfde model ook gebruikt worden voor het vinden van core elementen van de bijbehorende spelen. Dit model passen we toe op verschillende (bekende) klassen van coöperatieve situaties, waaronder *volgorde problemen zonder beginvolgorde*, minimum opspannende boom problemen, *permutatie situaties zonder initiële allocatie* en *handelsreizigersproblemen*. We besteden extra aandacht aan de nieuwe klasse van *reperateursproblemen*, waarin een reperateur een groep spelers moet bezoeken en de kosten van de spelers afhangen van de tijd waarop zij op de reperateur moeten wachten. Voor een toepassing kan men bijvoorbeeld denken aan fabrieken met kapotte machines, die opbrengsten mislopen doordat er niet geproduceerd kan worden. Voor deze klasse van coöperatieve situaties introduceren en analyseren wij een passend coöperatief spel. Bovendien bekijken we twee één-punts-oplossingsconcepten.

Hoofdstuk 6 vormt een brug tussen coöperatieve en niet-coöperatieve speltheorie. In dat hoofdstuk onderzoeken we de rol van het toestaan van bepaalde vormen van samenwerking in niet-coöperatieve spelen in strategische vorm, ook wel *strategische spelen* genoemd. In een strategisch spel kiezen alle spelers tegelijkertijd één strategie uit hun persoonlijke verzameling van mogelijke strategieën. De uitbetaling aan iedere speler is in het algemeen afhankelijk van zijn eigen keuze en van de keuze(s) van de andere speler(s). Normaal gesproken kunnen spelers in een strategisch spel geen bindende afspraken maken. In dit hoofdstuk staan we echter toe dat de spelers een strategische optie hebben om bepaalde contracten af te sluiten. Deze contracten bepalen de uitbetaling aan de spelers nadat een bepaalde combinatie van strategieën gespeeld is. We analyseren dit soort situaties en besteden in het bijzonder aandacht aan evenwichtsgedrag van de spelers.

Hoofdstuk 7 staat volledig in het teken van strategische spelen. Voor dergelijke spelen is het *Nash evenwicht* het fundamentele oplossingsconcept. Een combinatie van strategieën is een Nash evenwicht als geen enkele speler zijn uitbetaling kan verhogen door af te wijken van zijn gekozen strategie. De verzameling van Nash evenwichten kan echter erg groot zijn en bovendien kan het tegen-intuitieve uitkomsten bevatten. Daarom zijn er in de loop der jaren meerdere verfijningen van dit evenwichtsconcept geintroduceerd, zoals *perfecte* en *propere evenwichten*. Wij introduceren en analyseren een nieuw evenwichtsconcept, *fall back evenwichten* geheten, waarin het idee is dat een evenwicht stabiel moet zijn tegen het wegvallen van bepaalde strategieën.

We laten zien dat de verzameling van fall back evenwichten een niet-lege en gesloten deelverzameling is van de verzameling van Nash evenwichten. Verder onderzoeken we de relaties tussen fall back evenwichten aan de ene kant, en verschillende andere evenwichtsconcepten, waaronder perfecte en propere evenwichten, aan de andere kant. Naast fall back evenwichten introduceren en analyseren we bovendien drie gerelateerde evenwichtsconcepten, zijnde *strict fall back*, *complete fall back* en *dependent fall back evenwichten*.

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