

Bounds for Weighted Multivariate Empirical Distribution Functions

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1. Introduction

Let X_1, X_2, \dots be a sequence of independent random vectors, each uniformly distributed over $(0, 1)^d$, $d \in \mathbb{N}$. The first n random vectors determine the empirical d.f. \hat{F}_n in the usual way:

$$\hat{F}_n(t) = n^{-1} \sum_{i=1}^n \prod_{j=1}^d 1_{(0, t_j]}(X_{ij}), \quad t \in (0, 1)^d, \quad (1.1)$$

where X_{ij} is the j -th component of X_i and t_j the j -th component of t . Writing $|t| = \prod_{j=1}^d t_j$ we define

$$\|V_{n, \nu}\| = \sup_{0 < |t| < 1} \frac{n^\nu |\hat{F}_n(t) - |t||}{(|t|(1-|t|))^{1-\nu}}, \quad 0 \leq \nu \leq \frac{1}{2}. \quad (1.2)$$

In the one-dimensional case, much attention has been paid to criteria which determine the almost sure behaviour of $\limsup_{n \rightarrow \infty} a_n \|V_{n, \nu}\|$, where $(a_n)_{n \in \mathbb{N}}$ is a sequence of positive norming constants. Csáki (1974, 1975, 1982) investigated the important value $\nu = \frac{1}{2}$ (i.e. in each point the process is divided by its standard deviation), while Shorack and Wellner (1978) considered the other extreme value $\nu = 0$. Mason (1981) connected these two results and derived criteria for every $\nu \in [0, \frac{1}{2}]$. For arbitrary dimension d the case $\nu = 0$ has been also considered by Mason (1982). In this paper we generalize Csáki's theorem (1974, 1975, 1982) and even Mason's generalization (1981) to arbitrary dimension d . An interesting corollary of this result is a law of the iterated logarithm for $\log \|V_{n, \nu}\|$.

2. Main Results

In this section we present our theorem and its corollaries. The proofs of these are deferred to the next section. Observe that we use sequences of positive norming constants $(a_n)_{n \in \mathbb{N}}$ which differ from those in the introduction.

Theorem. Let $F(t)=|t|$, $t \in (0, 1)^d$, $d \in \mathbb{N}$ and $0 \leq v \leq \frac{1}{2}$.

(i) If $\sum_{n=1}^{\infty} a_n \left(\log \frac{1}{a_n}\right)^{d-1} = \infty$, then

$$\limsup_{n \rightarrow \infty} (na_n)^{1-v} \|V_{n,v}\| = \infty \quad \text{a.s.} \tag{2.1}$$

(ii) If $\sum_{n=1}^{\infty} a_n \left(\log \frac{1}{a_n}\right)^{d-1} < \infty$ and $na_n \downarrow$, then

$$\lim_{n \rightarrow \infty} (na_n)^{1-v} \|V_{n,v}\| = 0 \quad \text{a.s.} \tag{2.2}$$

Corollary 1. There exists no sequence of positive real numbers $(a_n)_{n \in \mathbb{N}}$ such that $na_n \downarrow$ and

$$\limsup_{n \rightarrow \infty} (na_n)^{1-v} \|V_{n,v}\| = 1 \quad \text{a.s.} \tag{2.3}$$

Corollary 2.

$$\limsup_{n \rightarrow \infty} \frac{\log \|V_{n,v}\|}{\log \log n} = (1-v)d \quad \text{a.s.} \tag{2.4}$$

Corollary 3. For all $\alpha > 0$

$$\lim_{n \rightarrow \infty} n^{-\alpha} \|V_{n,v}\| = 0 \quad \text{a.s.} \tag{2.5}$$

Corollary 3 is the multidimensional version of Corollary 1 of Mason (1981).

In order to formulate Corollary 4 we have to introduce some notation. The (open) rectangles $(s_1, t_1) \times \dots \times (s_d, t_d)$ in $(0, 1)^d$ will be written as $R = R(s, t)$. Given an arbitrary rectangle $R \subset (0, 1)^d$ we define $\hat{F}_n\{R\} = n^{-1} \sum_{i=1}^n 1_{R}(X_i)$ and we write $|R|$ for the Lebesgue measure of R .

Corollary 4. Let $F(t)=|t|$, $t \in (0, 1)^d$, $d \in \mathbb{N}$ and $\mu \in (-\infty, 1)$. Then we have

$$\lim_{n \rightarrow \infty} \sup_{|R| \geq \frac{(\log n)^\mu}{n}} \frac{\log \log n}{(\log n)^{1-\frac{1}{2}\mu}} \frac{n^{\frac{1}{2}} |\hat{F}_n\{R\} - |R||}{(|R|(1-|R|))^{\frac{1}{2}}} = \frac{1}{1-\mu} \quad \text{a.s.} \tag{2.6}$$

however,

$$\limsup_{n \rightarrow \infty} \sup_{|R| \geq \frac{1}{n}} \frac{1}{(\log n)^{\frac{1}{2}}} \frac{n^{\frac{1}{2}} |\hat{F}_n\{R\} - |R||}{(|R|(1-|R|))^{\frac{1}{2}}} = \infty \quad \text{a.s.} \tag{2.7}$$

For the proof of this corollary we will require a result of Alexander (1984).

It is easy to see that the distinction between open, half-open and closed rectangles is inessential for the type of results just stated. Also, all of our results remain true if we replace $F(t)=|t|$ by F , whenever F has a density with respect to Lebesgue measure that is bounded away from 0 and ∞ .

3. Proofs

Before beginning the actual proofs we first present two inequalities. The first one can be found in Ruymgaart and Wellner (1982, Corollary 2.4); see also Ruymgaart and Wellner (1984) for related results.

Inequality 1. *There exists $c_1, c_2, c_3 \in (0, \infty)$, only depending on d , such that for any $\theta \in (0, 1)$*

$$P\left(\sup_{|t| \geq \theta} \frac{n^{\frac{1}{2}} |\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}} \geq \lambda\right) \leq c_1 \left(\log \frac{2}{\theta}\right)^d \exp(-c_2 \lambda^2 \psi(c_3 \lambda (n\theta)^{-\frac{1}{2}})), \quad \lambda \geq 0, \tag{3.1}$$

where $\psi: [0, \infty) \rightarrow (0, \infty)$ satisfies $\psi(\sigma) \sim 2\sigma^{-1} \log \sigma$ as $\sigma \rightarrow \infty$.

Inequality 2. *For each $d \in \mathbb{N}$, $0 \leq \nu \leq \frac{1}{2}$, $a \in [1, \infty)$ and $n \geq 3$ we have*

$$P\left(\sup_{0 < |t| \leq (na^{1-\nu})^{-1}} \frac{n^\nu \hat{F}_n(t)}{|t|^{1-\nu}} > 0\right) \leq d(\log(na^{1-\nu}))^{d-1} a^{\frac{-1}{1-\nu}} \tag{3.2}$$

Proof of Inequality 2. Let $|X|_{1:n} = \min_{1 \leq i \leq n} \{|X_i|\}$. Notice that the probability on the left side of (3.2) is equal to

$$P(|X|_{1:n} \leq (na^{1-\nu})^{-1}) = P(\max_{1 \leq i \leq n} (-\log |X_i|) \geq \log(na^{1-\nu})) \leq nP(-\log |X_1| \geq \log(na^{1-\nu})).$$

Observe that $-\log |X_1|$ is a gamma random variable with density $f_d(x) = ((d-1)!)^{-1} x^{d-1} e^{-x} 1_{(0, \infty)}(x)$. Thus

$$\begin{aligned} P(-\log |X_1| \geq \log(na^{1-\nu})) &= \int_{\log(na^{1-\nu})}^{\infty} \frac{x^{d-1} e^{-x}}{(d-1)!} dx \\ &= \sum_{k=0}^{d-1} \{(\log(na^{1-\nu}))^k \cdot (na^{1-\nu} k!)^{-1}\} \\ &\leq d(\log(na^{1-\nu}))^{d-1} (na^{1-\nu})^{-1}. \end{aligned} \tag{3.3}$$

This completes the proof of Inequality 2.

Proof of the Theorem. (i) It is a consequence of a result on the almost sure behaviour of the first order statistic in Geffroy (1958/1959) or Kiefer (1972) that $\sum a_n \left(\log \frac{1}{a_n}\right)^{d-1} = \infty$ implies $P(|X|_{1:n} < \varepsilon a_n \text{ i.o.}) = 1$ for any $\varepsilon > 0$. It can be easily seen that

$$(na_n)^{1-\nu} \|V_{n,\nu}\| \geq \frac{(na_n)^{1-\nu} n^\nu n^{-1}}{2(|X|_{1:n}(1-|X|_{1:n}))^{1-\nu}}. \tag{3.4}$$

Hence we have

$$\limsup_{n \rightarrow \infty} (na_n)^{1-\nu} \|V_{n,\nu}\| \geq \frac{1}{2e^{1-\nu}} \text{ a.s.} \tag{3.5}$$

Letting $\varepsilon \downarrow 0$ proves the first part of our theorem.

(ii) It is easy to see that we may restrict ourselves without loss of generality to sequences $(a_n)_{n \in \mathbb{N}}$ with $\frac{1}{n^2} \leq a_n \leq \frac{1}{n}$. We first consider the case $\nu = \frac{1}{2}$. Using

$a_n \leq \frac{1}{n}$ we see that $\sum_{n=1}^{\infty} a_n (\log n)^{d-1} < \infty$. We now define

$$U_n = \sup_{0 < |t| \leq b_n} \frac{\hat{F}_n(t) - |t|}{|t|^{\frac{1}{2}}}, \tag{3.6}$$

with $b_n = (n(\log n)^{d-2})^{-1}$, and prove that

$$\lim_{n \rightarrow \infty} n a_n^{\frac{1}{2}} U_n = 0 \quad \text{a.s.} \tag{3.7}$$

It suffices to prove that $\limsup_{n \rightarrow \infty} n a_n^{\frac{1}{2}} U_n \leq 1$ a.s.

Define the following events:

$$A_n = \left\{ U_n \geq \frac{1}{n a_n^{\frac{1}{2}}} \right\}; \quad C_n = A_n A_{n-1}^c. \tag{3.8}$$

According to the Borel-Cantelli lemma we need to prove that $\sum P C_n < \infty$ and $P A_n \rightarrow 0$ as $n \rightarrow \infty$. Define

$$B_{n,k} = \{ \forall_{t: x_{k-1} < |t| \leq x_k \wedge b_n} (n-1) \hat{F}_{n-1}(t) \leq k-1; \exists_{t: x_{k-1} < |t| \leq x_k \wedge b_n} n \hat{F}_n(t) = k \},$$

where $x_k (= x_{n,k})$ is the solution of the equation

$$n x + a_n^{-\frac{1}{2}} x^{\frac{1}{2}} = k, \tag{3.9}$$

i.e.

$$x_k = ((1 + 4nka_n)^{\frac{1}{2}} - 1)^2 / (4n^2 a_n) = \frac{k}{n} \{ 1 - 2(1 + (1 + 4nka_n)^{\frac{1}{2}})^{-1} \}, \tag{3.10}$$

and

$$B'_{n,k} = \{ \sup_{|t| \leq x_k} (n-1) \hat{F}_{n-1}(t) \geq k-1; |X_n| \leq x_k \}.$$

We see that $B_{n,k} \subset B'_{n,k}$.

For any $\tau > 0$ we have the following inclusions (for large n):

$$\begin{aligned} C_n &\subset \left\{ \forall_{t: 0 < |t| \leq b_n} \hat{F}_{n-1}(t) < |t| + \frac{1}{n-1} \left(\frac{|t|}{a_{n-1}} \right)^{\frac{1}{2}}; \right. \\ &\quad \left. \exists_{t: 0 < |t| \leq b_n} \hat{F}_n(t) \geq |t| + \frac{1}{n} \left(\frac{|t|}{a_n} \right)^{\frac{1}{2}} \right\} \\ &\subset \bigcup_{k=1}^{k_0} B_{n,k} \subset \bigcup_{k=1}^{k_0} B'_{n,k}, \end{aligned} \tag{3.11}$$

where $k_0 = \left\lceil \frac{\tau}{n a_n (\log n)^{d-1}} \right\rceil$. (We will choose τ later on.) For the verification of the second inclusion we have to show that $x_{k_0} \geq b_n$ for large n , which follows from an elementary computation using the fact that $\sum_{n=1}^{\infty} a_n (\log n)^{d-1} < \infty$ and

$na_n \downarrow$ imply

$$\lim_{n \rightarrow \infty} na_n(\log n)^d = 0. \tag{3.12}$$

We are now going to derive an upper bound for PC_n . The inclusions in (3.11) yield

$$PC_n \subseteq \sum_{k=1}^{k_0} PB'_{n,k}. \tag{3.13}$$

Using (3.3), it is easy to see that

$$P\left(\sup_{|t| \leq x_k} (n-1)\hat{F}_{n-1}(t) \geq k-1\right) \leq \binom{n-1}{k-1} x_k^{k-1} \left(\sum_{i=0}^{d-1} \left(\log \frac{1}{x_k}\right)^i\right)^{k-1}. \tag{3.14}$$

Hence we have

$$PB'_{n,k} \leq \binom{n-1}{k-1} x_k^k \left(\sum_{i=0}^{d-1} \left(\log \frac{1}{x_k}\right)^i\right)^k \leq \frac{k(n x_k)^k}{k!} \left(\sum_{i=0}^{d-1} \left(\log \frac{1}{x_k}\right)^i\right)^k. \tag{3.15}$$

It can be seen from (3.9) that $x_k \leq k^2 a_n$. This yields for large n

$$\begin{aligned} PB'_{n,k} &\leq \frac{k}{n} (cnk^2 a_n)^k \frac{1}{k!} (\log n)^{k(d-1)} \\ &= ck^3 a_n (cnk^2 a_n)^{k-1} \frac{1}{k!} (\log n)^{k(d-1)}, \end{aligned} \tag{3.16}$$

where $c \in (0, \infty)$ is a constant depending on d . Using $k \leq k_0$ for a sufficiently small τ we see that

$$PB'_{n,k} \leq ck^3 (ck\tau)^{k-1} \frac{1}{k!} a_n (\log n)^{d-1} \leq c\left(\frac{1}{2}\right)^{k-1} a_n (\log n)^{d-1}, \tag{3.17}$$

which entails $PC_n \leq 2ca_n(\log n)^{d-1}$, hence $\sum PC_n < \infty$.

For the proof of $PA_n \rightarrow 0$ as $n \rightarrow \infty$ we need

$$A_{n,k} = \left\{ \sup_{|t| \leq x_k} n\hat{F}_n(t) \geq k \right\}. \tag{3.18}$$

Using $x_{k_0} \geq b_n$ we see that

$$A_n \subset \bigcup_{k=1}^{k_0} A_{n,k}. \tag{3.19}$$

We have

$$PA_{n,k} \leq \binom{n}{k} x_k^k \left(\sum_{i=0}^{d-1} \left(\log \frac{1}{x_k}\right)^i\right)^k \leq \frac{(nx_k)^k}{k!} \left(\sum_{i=0}^{d-1} \left(\log \frac{1}{x_k}\right)^i\right)^k. \tag{3.20}$$

Using (3.15)–(3.17) we see that

$$PA_{n,k} \leq c \frac{n}{k} \left(\frac{1}{2}\right)^{k-1} a_n (\log n)^{d-1} \quad \text{for } k \leq k_0; \tag{3.21}$$

hence

$$PA_n \leq c \sum_{k=1}^{k_0} (\frac{1}{2})^{k-1} na_n (\log n)^{d-1} \leq 2cna_n (\log n)^{d-1} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{3.22}$$

Our next step is investigating the a.s. behaviour of

$$H_n = \sup_{0 < |t| \leq b_n} \frac{|t| - \hat{F}_n(t)}{|t|^{\frac{1}{2}}}. \tag{3.23}$$

We immediately see that

$$H_n \leq \sup_{0 < |t| \leq b_n} |t|^{\frac{1}{2}} = \frac{1}{n^{\frac{1}{2}} (\log n)^{\frac{d-2}{2}}}, \tag{3.24}$$

yielding that

$$\lim_{n \rightarrow \infty} na_n^{\frac{1}{2}} H_n = 0 \text{ a.s.} \tag{3.25}$$

Now consider $\sup_{b_n < |t| < 1} \frac{|\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}}$; using Inequality 1 we find for large n , with $p_n = \frac{1}{na_n (\log n)^d}$,

$$\begin{aligned} P \left(\sup_{b_n < |t| < 1} \frac{|\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}} \geq \frac{1}{na_n^{\frac{1}{2}}} \right) &\leq c_4 (\log n)^d \exp(-c_2 (\log n)^d p_n \psi(c_3 (\log n)^{d-1} p_n^{\frac{1}{2}})) \\ &\leq c_4 (\log n)^d \exp(-c_5 p_n^{\frac{1}{2}} \log n) \\ &\leq c_4 (\log n)^d n^{-c_5 p_n^{\frac{1}{2}}} \leq \frac{1}{n^2}, \end{aligned} \tag{3.26}$$

where $c_4, c_5 \in (0, \infty)$ are constants depending on d . Applying the Borel-Cantelli lemma shows that

$$\lim_{n \rightarrow \infty} na_n^{\frac{1}{2}} \sup_{b_n < |t| < 1} \frac{|\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}} = 0 \text{ a.s.} \tag{3.27}$$

Summarizing (3.8), (3.25) and (3.27) yields

$$\lim_{n \rightarrow \infty} na_n^{\frac{1}{2}} \sup_{0 < |t| < 1} \frac{|\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}} = 0 \text{ a.s.} \tag{3.28}$$

We are now going to consider arbitrary $v \in [0, \frac{1}{2})$ and we shall prove

$$\lim_{n \rightarrow \infty} (na_n)^{1-v} \sup_{0 < |t| < 1} \frac{n^v |\hat{F}_n(t) - |t||}{|t|^{1-v}} = 0 \text{ a.s.} \tag{3.29}$$

It suffices to prove that

$$P \left(na_n^{1-v} \sup_{0 < |t| < 1} \frac{|\hat{F}_n(t) - |t||}{|t|^{1-v}} \geq 1 \text{ i.o.} \right) = 0. \tag{3.30}$$

Set $c_n = (na_n)^{v-1}$ and define

$$D = \left\{ \sup_{0 < |t| < 1} \frac{n^v |\hat{F}_n(t) - |t||}{|t|^{1-v}} \geq c_n \text{ i.o.} \right\},$$

$$E = \left\{ \sup_{0 < |t| \leq \frac{1}{nc_n^{1-v}}} \frac{n^v |\hat{F}_n(t) - |t||}{|t|^{1-v}} \geq c_n \text{ i.o.} \right\},$$

$$F = \left\{ \sup_{\frac{1}{nc_n^{1-v}} \leq |t| < 1} \frac{n^v |\hat{F}_n(t) - |t||}{|t|^{1-v}} \geq c_n \text{ i.o.} \right\}.$$

We see that $P(D) \leq P(E) + P(F)$.

We will first show that $P(E) = 0$. Note that for all $t \in (0, 1)^d$ such that $|t| \leq (nc_n^{1-v})^{-1}$, we have $n^v |t|^v \leq c_n^{1-v}$. Since necessarily $c_n \rightarrow \infty$, $c_n^{1-v} \leq \frac{1}{2}c_n$ for large enough n . Hence $E \subset E'$, where

$$E' = \left\{ \sup_{0 < |t| \leq \frac{1}{nc_n^{1-v}}} \frac{n^v \hat{F}_n(t)}{|t|^{1-v}} \geq \frac{1}{2}c_n \text{ i.o.} \right\}.$$

Let $n_r = 2^r$ for $r \in \mathbb{N}$. Obviously

$$P(E') \leq P \left(\max_{n_r < n \leq n_{r+1}} \sup_{0 < |t| \leq \frac{1}{nc_n^{1-v}}} \frac{n^v \hat{F}_n(t)}{|t|^{1-v}} \geq \frac{1}{2}c_{n_r} \text{ i.o.} \right), \tag{3.31}$$

which, since $c_n \uparrow$ and $n\hat{F}_n(t) \uparrow$ as a function of n for fixed t , is less than or equal to

$$P \left(\sup_{0 < |t| \leq \frac{1}{n_r(c_{n_r})^{1-v}}} \frac{2n_{r+1} \hat{F}_{n_{r+1}}(t)}{|t|^{1-v}} \geq \frac{1}{2}c_{n_r} \text{ i.o.} \right).$$

Now

$$\begin{aligned} & \sum_{r=1}^{\infty} P \left(\sup_{0 < |t| \leq \frac{1}{n_r(c_{n_r})^{1-v}}} \frac{n_{r+1} \hat{F}_{n_{r+1}}(t)}{|t|^{1-v}} \geq 4^{-1}c_{n_r} \right) \\ & \leq \sum_{r=1}^{\infty} P \left(\sup_{0 < |t| \leq \frac{1}{n_{r+1}(c'_{n_r})^{1-v}}} \frac{n_{r+1} \hat{F}_{n_{r+1}}(t)}{|t|^{1-v}} > 0 \right), \end{aligned}$$

where $c'_{n_r} = 2^{v-1}c_{n_r}$.

Application of Inequality 2 gives that this last series is less than or equal to

$$2d \sum_{r=1}^{\infty} (\log(n_r c_{n_r}^{1-v}))^{d-1} / c_{n_r}^{1-v}. \tag{3.32}$$

Since for large enough x $(\log x)^{d-1}/x \downarrow$, we have for $r_0 \in \mathbb{N}$ large enough:

$$\begin{aligned}
& \sum_{r=r_0}^{\infty} \sum_{n_{r-1} < n \leq n_r} (\log(nc_n^{\frac{1}{1-v}}))^{d-1} / (nc_n^{\frac{1}{1-v}}) \\
& \geq \sum_{r=r_0}^{\infty} (n_r - n_{r-1}) \{ (\log(nc_{n_r}^{\frac{1}{1-v}}))^{d-1} / n_r c_{n_r}^{\frac{1}{1-v}} \} \\
& = \frac{1}{2} \sum_{r=r_0}^{\infty} (\log(nc_{n_r}^{\frac{1}{1-v}}))^{d-1} / c_{n_r}^{\frac{1}{1-v}}. \tag{3.33}
\end{aligned}$$

We see immediately now that finiteness of the series in (ii) implies that the series in (3.32) is finite. Therefore by the Borel-Cantelli lemma $P(E')=0$, which in turn implies that $P(E)=0$.

We will now show that $P(F)=0$. Notice that

$$\begin{aligned}
& \sup_{\frac{1}{nc_n^{1-v}} \leq |t| < 1} \frac{n^v |\hat{F}_n(t) - |t||}{|t|^{1-v}} \\
& \leq \sup_{0 < |t| < 1} \frac{n^v |\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}} \left(\frac{1}{nc_n^{1-v}} \right)^{\frac{1}{2}-v}} \\
& = c_n^{\frac{1}{2}-v} \sup_{0 < |t| < 1} \frac{n^{\frac{1}{2}} |\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}}. \tag{3.34}
\end{aligned}$$

Hence $P(F) \leq P(F')$, where

$$F' = \left\{ \sup_{0 < |t| < 1} \frac{n^{\frac{1}{2}} |\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}} \geq c_n^{2(1-v)} \text{ i.o.} \right\},$$

but we now can use (3.28), i.e. the case $v = \frac{1}{2}$, which gives that $P(F')=0$, which in turn implies $P(F)=0$. Thus we have shown that $P(D)=0$. This completes the proof of (3.29).

Noting that $0 < y \leq \frac{1}{2}$ implies $1 < (1-y)^{v-1} \leq 2$, we see that it remains to prove for $0 \leq v \leq \frac{1}{2}$

$$\lim_{n \rightarrow \infty} (na_n)^{1-v} \sup_{\frac{1}{2} < |t| < 1} \frac{n^v |\hat{F}_n(t) - |t||}{(1-|t|)^{1-v}} = 0 \text{ a.s.} \tag{3.35}$$

With the same approach as in Einmahl et al. (1984, proof of Theorem 2.2) we can prove, using results of Ruymgaart and Wellner (1982, Corollary 2.3) or Alexander (1982, Corollary 6.2) that “large d -dimensional points” behave as “small (or large) 1-dimensional points”, i.e.

$$\begin{aligned}
& \sum_{n=1}^{\infty} a'_n < \infty \text{ and } na'_n \downarrow \text{ imply} \\
& \lim_{n \rightarrow \infty} (na'_n)^{1-v} \sup_{0 < |t| < 1} \frac{n^v |\hat{F}_n(t) - |t||}{(1-|t|)^{1-v}} = 0 \text{ a.s.} \tag{3.36}
\end{aligned}$$

We omit the proof of (3.36), because it is straightforward though tedious. Q.E.D.

Proof of Corollary 2. Applying the theorem for $a_n = (n(\log n)^d)^{-1}$ and for $a_n = (n(\log n)^{d+\varepsilon})^{-1}$, $\varepsilon > 0$, gives the desired result. Q.E.D.

Proof of Corollary 4. "Large d -dimensional rectangles" also have the same behaviour as "small 1-dimensional points". That means that (3.36) holds true with t replaced by R (cf. Einmahl et al. (1984, proof of Theorem 3.2)). Taking $\nu = \frac{1}{2}$ and combining this with Corollary 3.9 of Alexander (1984) proves this corollary. Q.E.D.

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