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Approximations and Two-Sample Tests Based on *P–P* and *Q–Q* Plots of the Kaplan–Meier Estimators of Lifetime Distributions

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Let F_n and G_n denote the Kaplan–Meier product–limit estimators of lifetime distributions based on two independent samples, and let F_n^{inv} and G_n^{inv} denote their quantile functions. We consider the corresponding *P–P* plot $F_n(G_n^{\text{inv}})$ and *Q–Q* plot $F_n^{\text{inv}}(G_n)$, and establish strong approximations of empirical processes based on these *P–P* and *Q–Q* plots by appropriate sequences of Gaussian processes. It is shown that the rates of approximation we obtain are the best which can be achieved by this method. We apply these results to obtain the limiting distributions of test statistics which are functionals of $F_n(G_n^{\text{inv}}(s))-s$, $G_n(F_n^{\text{inv}}(s))-s$, and $F_n(G_n^{\text{inv}}(s))+G_n(F_n^{\text{inv}}(s))-2s$, and propose solutions to the problem of testing the assumption that the underlying lifetime distributions F and G are equal, in the case where the censoring distributions are arbitrary and unknown.

1. INTRODUCTION

Let F_m and G_n denote Kaplan–Meier estimators of lifetime (or failure) distribution functions based on independent samples of sizes m and n, respectively. In this paper we consider statistical comparison procedures of

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the unknown distributions F and G based on F_m and G_n. We start by some notation and the statement of relevant results in the literature (see, e.g., Csörgő [8, Chap. 8]).

Let $\{X_i, i\geqslant 1\}$, $\{Y_i, i\geqslant 1\}$, $\{U_i, i\geqslant 1\}$, and $\{V_i, i\geqslant 1\}$ be independent sequences of i.i.d. positive random variables with distribution functions $F(x)=P(X_i\leqslant x)$, $G(x)=P(Y_i\leqslant x)$, $H(x)=P(U_i\leqslant x)$, and $K(x)=P(V_i\leqslant x)$ for $i\geqslant 1$ and $-\infty<x\leqslant\infty$. For each $i\geqslant 1$, X_i (resp. Y_i) denotes the uncensored lifetime of the ith individual of the first (resp. second) sample. U_i (resp. V_i) denotes the censoring time at which the ith individual of the first (resp. second) sample is withdrawn. Set $Z_i'=\min(X_i, U_i)$, $Z_i''=\min(Y_i, V_i)$, $\delta_i'=1_{\{X_i\leqslant U_i\}}$, and $\delta_i''=1_{\{Y_i\leqslant V_i\}}$ for $i\geqslant 1$. In the model with random censorship from the right, one observes $\{(Z_i', \delta_i'), 1\leqslant i\leqslant m\}$ and $\{(Z_i'', \delta_i''), 1\leqslant i\leqslant n\}$. The product–limit (PL) estimators introduced by Kaplan and Meier [13] are then given by

$$F_m(x)=1-\prod_{Z'_{(i)}\leqslant x}(1-\delta'_{(i)}/(m-i+1)) \qquad \text{for} \quad -\infty<x\leqslant\infty \tag{1.1}$$

and

$$G_n(x)=1-\prod_{Z''_{(i)}\leqslant x}(1-\delta''_{(i)}/(n-i+1)) \qquad \text{for} \quad -\infty<x\leqslant\infty, \tag{1.2}$$

where $0<Z'_{(1)}\leqslant\cdots\leqslant Z'_{(m)}$ (resp. $0<Z''_{(1)}\leqslant\cdots\leqslant Z''_{(n)}$) are the order statistics of $\{Z_i', 1\leqslant i\leqslant m\}$ (resp. $\{Z_i'', 1\leqslant i\leqslant n\}$), $\{\delta'_{(i)}, 1\leqslant i\leqslant m\}$ (resp. $\{\delta''_{(i)}, 1\leqslant i\leqslant n\}$) being the corresponding values of the δ_i' (resp. of the δ_i'').

We assume throughout that F (resp. G) is differentiable on $(0, \infty)$ with continuous and positive derivative f (resp. g), and hence continuous on $[0, \infty)$ with $F(0)=0$ (resp. $G(0)=0$). Moreover, we assume that H and K are continuous on $(-\infty, \infty)$ and allow $H_-(\infty):=\lim_{x\to\infty}H(x)$ and $K_-(\infty):=\lim_{x\to\infty}K(x)$ to be less than one. $H_-(\infty)=K_-(\infty)=0$ corresponds to the uncensored case, U_i and V_i being then infinite with probability one. Let $F^{\text{inv}}(s)=\inf\{x\geqslant 0: F(x)\geqslant s\}$ and $G^{\text{inv}}(s)=\inf\{x\geqslant 0: G(x)\geqslant s\}$ for $0\leqslant s\leqslant 1$ the quantile functions pertaining to F and G. Set $T_H=\sup\{x: H(x)<1\}$ and $T_K=\sup\{x: K(x)<1\}$, and introduce

$$\begin{aligned} h(s)&=\int_0^s(1-u)^{-2}(1-H(F^{\text{inv}}(u)))^{-1}\,du \qquad \text{for} \quad 0\leqslant s<F(T_H) \\ h(s)&=s \qquad \text{for} \quad s\leqslant 0, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} k(s)&=\int_0^s(1-u)^{-2}(1-K(G^{\text{inv}}(u)))^{-1}\,du \qquad \text{for} \quad 0\leqslant s<G(T_K), \\ k(s)&=s \qquad \text{for} \quad s\leqslant 0. \end{aligned} \tag{1.4}$$

Consider the empirical processes $\alpha'_m(x) = m^{1/2}(F_m(x) - F(x))$ and $\alpha''_n(x) = n^{1/2}(G_n(x) - G(x))$ for $-\infty < x \leqslant \infty$, and the reduced empirical processes

$$a'_m(s) = \alpha'_m(F^{\text{inv}}(s)) = m^{1/2}(F_m(F^{\text{inv}}(s)) - s) \tag{1.5}$$

and

$$a''_n(s) = \alpha''_n(G^{\text{inv}}(s)) = n^{1/2}(G_n(G^{\text{inv}}(s)) - s), \qquad \text{for} \quad 0 \leqslant s \leqslant 1. \tag{1.6}$$

Burke, Csörgő, and Horváth [7] and Major and Rejtő [15] established the following strong approximations of a'_m and a''_n. Assuming that the original probability space is sufficiently rich, it is possible to define two independent standard two-parameter Wiener processes W' and W'' such that, for any fixed $\vartheta_F \in (0, F(T_H))$ and $\vartheta_G \in (0, G(T_K))$, we have almost surely as $m \to \infty$ and $n \to \infty$

$$\| a'_m - m^{-1/2}(1-I)\, W'(h, m) \|_0^{\vartheta_F} =: \| a'_m - A'_m \|_0^{\vartheta_F} = O(m^{-1/2} \log^2 m) \tag{1.7}$$

and

$$\| a''_n - n^{-1/2}(1-I)\, W''(k, n) \|_0^{\vartheta_G} =: \| a''_n - A''_n \|_0^{\vartheta_G} = O(n^{-1/2} \log^2 n), \tag{1.8}$$

where I denotes the identity function, and $\| \varphi \|_c^d := \sup_{c \leqslant x \leqslant d} |\varphi(x)|$.

In the uncensored case, where $H_-(\infty) = K_-(\infty) = 0$, we have $h = k = I/(1-I)$, so that the processes $m^{-1/2}(1-I)\, W'(h, m)$ and $n^{-1/2}(1-I)\, W''(k, n)$ in (1.7)–(1.8) are Brownian bridges. In this case, (1.7) and (1.8) are valid for $\vartheta_F = \vartheta_G = 1$, and coincide with the Kiefer process approximations due to Komlós, Major, and Tusnády [14], F_m and G_n being then the empirical distribution functions based on $X_1, ..., X_m$ and $Y_1, ..., Y_n$. To test the null hypothesis that $F = G$ it is tempting to use Kolmogorov–Smirnov-type statistics such as

$$D^{\nu}_{FG;mn} = \left(\frac{mn}{m+n} \right)^{1/2} \| F_m - G_n \|_0^{\nu}. \tag{1.9}$$

Unfortunately, even when $F = G$ and $H = K$, the limiting distribution of $D^{\nu}_{FG;mn}$ depends on the unknown values of F and H. Moreover, the plot of $F_m - G_n$ has the inconvenience of having a poor visual interpretation. To overcome this latter drawback, one may use quantile–quantile (Q–Q) and probability–probability (P–P) plots, as follows.

Denote by $F^{\text{inv}}_m(s) = \inf\{x \geqslant 0 : F_m(x) \geqslant s\}$ and $G^{\text{inv}}_n(s) = \inf\{x \geqslant 0 : G_n(x) \geqslant s\}$, for $0 \leqslant s \leqslant 1$, the empirical quantile functions pertaining to F_m and G_n. The PL Q–Q plot of F against G is then defined by

$$\Delta_{FG;mn}(x) = F^{\text{inv}}_m(G_n(x)) \qquad \text{for} \quad 0 \leqslant x \leqslant \infty, \tag{1.10}$$

while the PL P–P plot of F against G is defined by

$$\tilde{\Delta}_{FG;mn}(s) = F_m(G_n^{\text{inv}}(s)) \quad \text{for} \quad 0 \leqslant s \leqslant 1. \tag{1.11}$$

Statistics such as in (1.10) and (1.11) have a great appeal since they converge to the identity function (on appropriate intervals) if $F = G$. In the uncensored case such Q–Q and P–P plots have received considerable attention in the recent literature. We refer to Fisher [12], Aly [1, 2], Aly, Csörgő, and Horváth [3], and Beirlant and Deheuvels [4].

The aim of this paper is threefold. In Section 2, we consider approximations of empirical processes based on P–P and Q–Q plots by Gaussian processes. Our theorems yield the best possible rates of approximation given the construction we use, and correspond to the results obtained in the uncensored case by Beirlant and Deheuvels [4]. In Section 3, we establish the asymptotic distribution of the censored version of a statistic due to Deheuvels and Mason [11], which is a two-sample version of Bahadur–Kiefer statistics considered by Deheuvels and Mason [10] and Beirlant and Einmahl [5]. Finally, in Section 4, we present two-sample tests of the hypothesis that $F = G$.

2. Approximations of the PL P–P and Q–Q Plot Processes

Throughout, we consider the case where $m = n$. Extensions of our results to unequal sample sizes can be achieved through additional arguments of minor interest which we omit. It is convenient to introduce the following notation. Let

$$\Gamma_n'(s) = F_n(F^{\text{inv}}(s)), \qquad \Gamma_n''(s) = G_n(G^{\text{inv}}(s)) \quad \text{for} \quad 0 \leqslant s \leqslant 1, \tag{2.1}$$

$$\Gamma_n'^{\text{inv}}(s) = F(F_n^{\text{inv}}(s)), \qquad \Gamma_n''^{\text{inv}}(s) = G(G_n^{\text{inv}}(s)) \quad \text{for} \quad 0 \leqslant s \leqslant 1. \tag{2.2}$$

Define the reduced PL P–P plot process of F against G by

$$\tilde{A}_n(s) = n^{1/2}(\Gamma_n'(\Gamma_n''^{\text{inv}}(s)) - s) \quad \text{for} \quad 0 \leqslant s \leqslant 1, \tag{2.3}$$

and the reduced PL Q–Q plot process of F against G by

$$A_n(s) = n^{1/2}(\Gamma_n'^{\text{inv}}(\Gamma_n''(s)) - s) \quad \text{for} \quad 0 \leqslant s \leqslant 1. \tag{2.4}$$

Whenever $F = G$, $\tilde{A}_n(s)$ has a simple expression in terms of $\tilde{\Delta}_{FF;nn}$, namely

$$\tilde{A}_n(s) = n^{1/2}(F_n(G_n^{\text{inv}}(s)) - s) = n^{1/2}(\tilde{\Delta}_{FF;nn}(s) - s) \quad \text{for} \quad 0 \leqslant s \leqslant 1. \tag{2.5}$$

Such a simple relation does not exist between the reduced PL Q–Q plot process and the PL Q–Q plot $\Delta_{FF;nn}$. However, we establish later on that A_n can be closely approximated when $F=G$ by

$$\hat{A}_n(s)=n^{1/2}f(F^{\mathrm{inv}}(s))(F_n^{\mathrm{inv}}(G_n(F^{\mathrm{inv}}(s)))-F^{\mathrm{inv}}(s))$$
$$=n^{1/2}f(F^{\mathrm{inv}}(s))(\Delta_{FF;nn}(F^{\mathrm{inv}}(s))-F^{\mathrm{inv}}(s)) \qquad \text{for} \quad 0<s<1; \qquad (2.6)$$
$$\hat{A}_n(0)=0.$$

The following process turns out to be a natural approximant of $\tilde{A}_n$, $-A_n$, and $-\hat{A}_n$. Let $\Theta=\min(F(T_H), G(T_K))$, and set

$$M_n(s)=n^{-1/2}(1-s)(W'(h(s),n)-W''(k(s),n))=:A_n'(s)-A_n''(s)$$
$$\text{for} \quad 0\leqslant s<\Theta. \qquad (2.7)$$

The result below gives the rates of approximation of $\tilde{A}_n$, $-A_n$, and $-\hat{A}_n$ by M_n.

THEOREM 2.1. *Assume that $F=G$. Then, for any $\vartheta\in(0,\Theta)$, we have, as $n\to\infty$,*

$$n^{1/4}(\log n)^{-1/2}\,\|\tilde{A}_n-M_n\|_0^{\vartheta}$$
$$\Bigg/\left(\left\|A_n''\left(\frac{1}{1-H(F^{\mathrm{inv}})}+\frac{1}{1-K(F^{\mathrm{inv}})}\right)\right\|_0^{\vartheta}\right)^{1/2}\xrightarrow[P]{}1, \qquad (2.8)$$
$$n^{1/4}(\log n)^{-1/2}\,\|A_n+M_n\|_0^{\vartheta}$$
$$\Bigg/\left(\left\|(A_n'-A_n'')\left(\frac{1}{1-H(F^{\mathrm{inv}})}\right)\right\|_0^{\vartheta}\right)^{1/2}\xrightarrow[P]{}1. \qquad (2.9)$$

Moreover, if $\lim_{u\downarrow 0}(u\log(1/u))^{-1/2}\sup_{0<s\leqslant F^{\mathrm{inv}}(\vartheta);\,|s-t|\leqslant u}|f(t)-f(s)|=0$ *and if f is right continuous and positive at* 0, *then* (2.9) *holds with A_n replaced by $\hat{A}_n$.*

Let W be a standard Wiener process and for $-\infty<s<\Theta$, write $l(s)=h(s)+k(s)$.

COROLLARY 2.1. *As $n\to\infty$*

$$n^{1/4}(\log n)^{-1/2}\,\|\tilde{A}_n-M_n\|_0^{\vartheta}$$
$$\xrightarrow[d]{}(\|(1-I)\,W(k)((1-H(F^{\mathrm{inv}}))^{-1}+(1-K(F^{\mathrm{inv}}))^{-1})\|_0^{\vartheta})^{1/2},$$
$$n^{1/4}(\log n)^{-1/2}\,\|A_n+M_n\|_0^{\vartheta}\xrightarrow[d]{}(\|(1-I)\,W(l)(1-H(F^{\mathrm{inv}}))^{-1}\|_0^{\vartheta})^{1/2}.$$

Remark 2.1. In the uncensored case, $H_-(\infty)=K_-(\infty)=0$, $\Theta=1$, A_n' and A_n'' are Brownian bridges, and (2.8) and (2.9) hold for $\vartheta=1$. While

(2.8) is then similar to Theorem 2.1 of Beirlant and Deheuvels [4], (2.9) is new and may be restated as

$$n^{1/4}(\log n)^{-1/2} \| A_n + M_n \|_0^{\vartheta} / (\| M_n \|_0^{\vartheta})^{1/2} \xrightarrow[P]{} 1 \qquad \text{as} \quad n \to \infty. \quad (2.10)$$

Remark 2.2. Corollary 2.1 implies that the rate $O_P(n^{-1/4}(\log n)^{1/2})$ is the best possible for an approximation of $\tilde{A}_n$, $-A_n$, or $-\hat{A}_n$ by M_n (see Aly [2] for related results).

Remark 2.3. It easily follows that (2.8) (resp. (2.9)) is valid with A_n'' (resp. $A_n' - A_n''$) replaced by a_n'' (resp. A_n).

We now prove Theorem 2.1, making use of the decomposition, with $F = G$,

$$\begin{aligned}
\tilde{A}_n(s) &= n^{1/2}(\Gamma_n'(\Gamma_n''^{\text{inv}}(s)) - \Gamma_n''^{\text{inv}}(s)) + n^{1/2}(\Gamma_n''^{\text{inv}}(s) - s) \\
&=: a_n'(\Gamma_n''^{\text{inv}}(s)) + b_n''(s) =: a_n'(\Gamma_n''^{\text{inv}}(s)) - a_n''(\Gamma_n''^{\text{inv}}(s)) + R_{1,n}(s) \\
&=: A_n'(\Gamma_n''^{\text{inv}}(s)) - A_n''(\Gamma_n''^{\text{inv}}(s)) + \sum_{j=1}^{2} R_{j,n}(s) \\
&= M_n(s + n^{-1/2} b_n''(s)) + \sum_{j=1}^{2} R_{j,n}(s) \\
&=: M_n(s - n^{-1/2} A_n''(s)) + \sum_{j=1}^{3} R_{j,n}(s) =: M_n(s) + \sum_{j=1}^{4} R_{j,n}(s), \qquad (2.11)
\end{aligned}$$

where we have used (1.5) and (2.1)–(2.7). In the following lemmas we evaluate $\| R_{j,n} \|_0^{\vartheta}$ for $j = 1, 2, 3$, and 4.

LEMMA 2.1 (Sander [16]). *We have for all $\vartheta \in (0, \Theta)$ almost surely*

$$\| R_{1,n} \|_0^{\vartheta} = O(n^{-1/2}) \qquad \textit{as} \quad n \to \infty. \qquad (2.12)$$

LEMMA 2.2. *We have for all $\vartheta \in (0, \Theta)$ almost surely*

$$\| R_{2,n} \|_0^{\vartheta} = O(n^{-1/2} \log^2 n) \qquad \textit{as} \quad n \to \infty. \qquad (2.13)$$

Proof. From the obvious inequality

$$\| R_{2,n} \|_0^{\vartheta} \leqslant \| a_n' - A_n' \|_0^{\Gamma_n''^{\text{inv}}(\vartheta)} + \| a_n'' - A_n'' \|_0^{\Gamma_n''^{\text{inv}}(\vartheta)},$$

we use Glivenko–Cantelli for PL estimators, (1.7) and (1.8), to obtain (2.13). ∎

Having established the easy parts of our proof, we turn to the treatment of $R_{3,n}$ and $R_{4,n}$. First, we assume without loss of generality that the Wiener processes W' and W'' in (1.7) and (1.8) are defined on $(-\infty, \infty) \times (0, \infty)$. Using the corresponding definition of M_n as given in (2.7), we see that $M_n(s - n^{-1/2}A''_n(s))$ as given in (2.11) is properly defined.

LEMMA 2.3. *We have for all* $\vartheta \in (0, \Theta)$

$$\| R_{3,n} \|_0^{\vartheta} = O_P(n^{-3/8}(\log n)^{3/4}) \qquad as \quad n \to \infty. \tag{2.14}$$

Proof. Define a sequence of standard two-sided Wiener processes on $(-\infty, l^{\text{inv}}(\Theta))$ by

$$(1 - I)\, W_n(l) = M_n. \tag{2.15}$$

From (2.7) and (2.11) it follows that

$$\begin{aligned} \| R_{3,n} \|_0^{\vartheta} \leqslant n^{-1/2} \, & \| b''_n W_n(l(I + n^{-1/2} b''_n)) \|_0^{\vartheta} \\ & + n^{-1/2} \, \| A''_n W_n(l(I - n^{-1/2} A''_n)) \|_0^{\vartheta} \\ & + \| W_n(l(I + n^{-1/2} b''_n)) - W(l(I - n^{-1/2} A''_n)) \|_0^{\vartheta}. \end{aligned} \tag{2.16}$$

The first two terms on the RHS of (2.16) are $O_P(n^{-1/2})$ as $n \to \infty$, so we focus on the third one. By Theorem 3 and Corollary 1 in Beirlant and Einmahl [5], as $n \to \infty$

$$\| b''_n + a''_n \|_0^{\vartheta} = O_P(n^{-1/4}(\log n)^{1/2}). \tag{2.17}$$

This, in conjunction with (1.8), yields as $n \to \infty$

$$\| (I + n^{-1/2} b''_n) - (I - n^{-1/2} A''_n) \|_0^{\vartheta} = O_P(n^{-3/4}(\log n)^{1/2}). \tag{2.18}$$

Note also that for any fixed $0 < \varepsilon < \Theta - \vartheta$, as $n \to \infty$,

$$\begin{aligned} & P(s + n^{-1/2} b''_n(s) \in [0, \vartheta + \varepsilon] \text{ and } s - n^{-1/2} A''_n(s) \in [-\varepsilon, \vartheta + \varepsilon], \\ & \qquad \text{for all} \quad 0 \leqslant s \leqslant \vartheta) \to 1. \end{aligned} \tag{2.19}$$

Moreover, on $[0, \vartheta + \varepsilon]$, we have $2 \leqslant l' \leqslant C$, for some $C \in (0, \infty)$, where l' denotes the derivative of l. Note that $l'(s) = 2$ for $s < 0$.

We conclude by showing that, for arbitrary $C' \in (0, \infty)$, as $n \to \infty$,

$$\sup_{\substack{-2\varepsilon \leqslant s,\, t \leqslant C \\ |s - t| \leqslant C' n^{-3/4} (\log n)^{1/2}}} | W_n(s) - W_n(t) | = O_P(n^{-3/8}(\log n)^{3/4}), \tag{2.20}$$

which follows from (2.25) in the sequel. ∎

We now turn to $R_{4,n}$. Recall by (2.11) that

$$R_{4,n}(s) = M_n(s - n^{-1/2}A''_n(s)) - M_n(s) \qquad \text{for} \quad 0 \leqslant s < \Theta. \tag{2.21}$$

Denote by $\{W(t), -\infty < t < \infty\}$ a standard Wiener process extended to the real line. In view of (2.21) and (2.15), the following proposition, in the spirit of Proposition 4 of Deheuvels and Mason [10], gives the proper evaluation of $\| R_{4,n} \|_0^\theta$.

PROPOSITION 2.1. *Let* $-\infty < a < c < d < b < \infty$ *be fixed, and let* $\{\Phi(t), a \leqslant t \leqslant b\}$ *and* $\{\Psi(t), a \leqslant t \leqslant b\}$ *be functions such that*

(i) *Φ is positive and has a derivative φ which is continuous on $[a, b]$;*

(ii) *Ψ has a derivative ψ which is positive and continuous on $[a, b]$.*

Then, there exists an event E of probability one such that, on E, we have for all continuous functions η on $[a, b]$

$$\lim_{u \downarrow 0} \sup_{c \leqslant t \leqslant d} (2u \log(1/u))^{-1/2} \left| \frac{W(\Psi(t + u\eta(t)))}{\Phi(t + u\eta(t))} - \frac{W(\Psi(t))}{\Phi(t)} \right| = \| |\psi\eta|^{1/2}/\Phi \|_c^d. \tag{2.22}$$

Proof. Assume that $0 < C_1 \leqslant \Phi$ and $|\varphi| \leqslant C_2$ on $[a, b]$. On the event of probability one that W is continuous, we have, as $u \downarrow 0$,

$$\sup_{c \leqslant t \leqslant d} (2u \log(1/u))^{-1/2} \left| W(\Psi(t + u\eta(t))) \left(\frac{1}{\Phi(t + u\eta(t))} - \frac{1}{\Phi(t)} \right) \right| \leqslant (C_2/C_1^2)(u/(2\log(1/u)))^{1/2} \| \eta \|_c^d \, \| W(\Psi) \|_a^b \to 0. \tag{2.23}$$

Thus, it suffices to show that

$$\lim_{u \downarrow 0} \sup_{c \leqslant t \leqslant d} (2u \log(1/u))^{-1/2} \left| \frac{W(\Psi(t + u\eta(t)) - W(\Psi(t))}{\Phi(t)} \right| = \sup_{c \leqslant t \leqslant d} \frac{|\psi(t)\, \eta(t)|^{1/2}}{|\Phi(t)|}. \tag{2.24}$$

By changing u into λu, $\lambda > 0$, if (2.24) holds, then it also holds with η replaced by $\lambda\eta$. Thus, excluding the case where $\| |\psi\eta|^{1/2}/\Phi \|_c^d = 0$, we may limit ourselves to proving that (2.24) holds almost surely for all continuous functions η on $[a, b]$ such that $\| |\psi\eta|^{1/2}/\Phi \|_c^d = 1$. We now make use of the following result (see, e.g., Csörgő and Révész [9, pp. 26–29]). There exists an event E' of probability one such that, on E', for all $-\infty < C < D < \infty$,

$$\lim_{u \downarrow 0} \sup_{\substack{C \leqslant t \leqslant D \\ |s-t| \leqslant u}} \frac{|W(s)-W(t)|}{(2u \log(1/u))^{1/2}} = \lim_{u \downarrow 0} \sup_{C \leqslant t \leqslant D} \frac{|W(t \pm u)-W(t)|}{(2u \log(1/u))^{1/2}} = 1. \tag{2.25}$$

In the sequel, we show that (2.24) is true on E' for all continuous functions η on $[a, b]$ satisfying $\| |\psi\eta|^{1/2}/\Phi \|_c^d = 1$. Let η be such a function. By continuity of $\Psi > 0$ and η, for any $0 < \varepsilon < 1$, there exists a $u_1 > 0$ such that for all $0 < u \leqslant u_1$,

$$\begin{aligned} &|(\Psi(t+u\eta(t)) - \Psi(t)) - u\psi(t)\,\eta(t)| \\ &\quad \leqslant \varepsilon u \min(\tfrac{1}{2}, \psi(t)|\eta(t)|) \text{ for all } c \leqslant t \leqslant d. \end{aligned} \tag{2.26}$$

Fix any $c \leqslant \gamma < \delta \leqslant d$, and set $C = \Psi(\gamma) < D = \Psi(\delta)$ and $M = (1+\varepsilon)\|\psi\eta\|_\gamma^\delta$. We have by (2.26) for all $u > 0$ sufficiently small

$$\begin{aligned} &\sup_{\gamma \leqslant t \leqslant \delta} (2u \log(1/u))^{-1/2} \left| \frac{W(\Psi(t+u\eta(t))) - W(\Psi(t))}{\Phi(t)} \right| \\ &\quad \leqslant \|1/\Phi\|_\gamma^\delta \sup_{\substack{C \leqslant t \leqslant D \\ |s-t| \leqslant Mu}} \frac{|W(s)-W(t)|}{(2u \log(1/u))^{1/2}}, \end{aligned} \tag{2.27}$$

which in turn tends to $(1+\varepsilon)^{1/2} (\|\psi\eta\|_\gamma^\delta)^{1/2} \|1/\Phi\|_\gamma^\delta$ as $u \downarrow 0$. Since there exists an $N_1 \geqslant 1$ such that for all $c \leqslant \gamma < \delta \leqslant d$ with $|\gamma - \delta| \leqslant 1/N_1$, we have

$$(1+\varepsilon)^{1/2} (\|\psi\eta\|_\gamma^\delta)^{1/2} \|1/\Phi\|_\gamma^\delta \leqslant (1+\varepsilon)\| |\psi\eta|^{1/2}/\Phi \|_\gamma^\delta \leqslant 1+\varepsilon. \tag{2.28}$$

By a finite covering of $[c, d]$ with $[\gamma, \delta]$, (2.27) and (2.28) yield

$$\limsup_{u \downarrow 0} \sup_{c \leqslant t \leqslant d} (2u \log(1/u))^{-1/2} \left| \frac{W(\Psi(t+u\eta(t))) - W(\Psi(t))}{\Phi(t)} \right| \leqslant 1+\varepsilon. \tag{2.29}$$

Let $c \leqslant \tau \leqslant d$ be such that $|\psi(\tau)\,\eta(\tau)|^{1/2}/|\Phi(\tau)| = 1$, and set $m = \psi(\tau)\,\eta(\tau)$; let $0 < \varepsilon < |m|/2$. There exists a subinterval $[\gamma, \delta] \subset [c, d]$ with $\gamma \leqslant \tau \leqslant \delta$, such that

$$|\psi(t)\,\eta(t) - m| \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad 1/\Phi(t) \geqslant (1-\varepsilon)/\Phi(\tau) \quad \text{for all} \quad \gamma \leqslant t \leqslant \delta. \tag{2.30}$$

Thus, combining (2.26) and (2.30), we obtain that for all $u>0$ sufficiently small,

$$\sup_{c\leqslant t\leqslant d}(2u\log(1/u))^{-1/2}\left|\frac{W(\Psi(t+u\eta(t)))-W(\Psi(t))}{\Phi(t)}\right|$$
$$\geqslant \sup_{\gamma\leqslant t\leqslant \delta}(2u\log(1/u))^{-1/2}\left|\frac{W(\Psi(t+u\eta(t)))-W(\Psi(t))}{\Phi(t)}\right|$$
$$\geqslant \frac{1-\varepsilon}{\Phi(\tau)}\left(\sup_{C\leqslant t\leqslant D}\frac{|W(t+mu)-W(t)|}{(2u\log(1/u))^{1/2}}\right.$$
$$\left.-\sup_{\substack{C-\varepsilon\leqslant t\leqslant D+\varepsilon\\ |s-t|\leqslant \varepsilon u}}\frac{|W(s)-W(t)|}{(2u\log(1/u))^{1/2}}\right), \tag{2.31}$$

which, by (2.25), tends to

$$(1-\varepsilon)\left(\frac{|\psi(\tau)\,\eta(\tau)|^{1/2}}{\Phi(\tau)}-\frac{\varepsilon^{1/2}}{\Phi(\tau)}\right)=(1-\varepsilon)\left(1-\frac{\varepsilon^{1/2}}{\Phi(\tau)}\right)\qquad \text{as } u\downarrow 0. \tag{2.32}$$

We conclude from (2.31) and (2.32) that

$$\liminf_{u\downarrow 0}\sup_{c\leqslant t\leqslant d}(2u\log(1/u))^{-1/2}\left|\frac{W(\Psi(t+u\eta(t)))-W(\Psi(t))}{\Phi(t)}\right|$$
$$\geqslant (1-\varepsilon)\left(1-\frac{\varepsilon^{1/2}}{\Phi(\tau)}\right). \tag{2.33}$$

Since $\varepsilon<\min(1, |m|/2)$ in (2.29) and (2.33) is arbitrary, we have (2.24) as sought. ▌

We now return to the proof of Theorem 2.1. By choosing $\Phi=1/(1-I)$ and $\Psi=l=h+k$, we see from (1.3) and (1.4) that Φ and Ψ satisfy the assumptions of Proposition 2.1 for $-1<a<c=0<d=\vartheta<b<\Theta$. Consider two independent standard Wiener processes W' and W'' extended to $(-\infty,\infty)$, and define a third Wiener process W through

$$M:=(1-I)\,W(l)=(1-l)(W'(h)+W''(k))=:\Lambda'+\Lambda''. \tag{2.34}$$

It is obvious from (2.7) and (2.34) that, for every $n\geqslant 1$,

$$\{M_n, \Lambda'_n, \Lambda''_n\}=_d\{M, \Lambda', \Lambda''\}. \tag{2.35}$$

Since (2.22), when applied for $\eta=-\Lambda''$, implies that almost surely

$$\lim_{u\downarrow 0}(2u\log(1/u))^{-1/2}\,\|M(I-u\Lambda'')-M\|_0^{\vartheta}/\|\,|l'\Lambda''|^{1/2}\,(1-I)\|_0^{\vartheta}=1,$$

it follows from (2.21) and (2.35) that as $n \to \infty$,

$$n^{1/4}(\log n)^{-1/2} \| R_{4,n} \|_0^\vartheta / \| \, |l' A_n''|^{1/2} (1-I) \|_0^\vartheta \xrightarrow[P]{} 1. \qquad (2.36)$$

Recall, by (1.3) and (1.4), that

$$l' = h' + k' = (1-I)^{-2} ((1 - H(F^{\text{inv}}))^{-1} + (1 - K(F^{\text{inv}}))^{-1}). \qquad (2.37)$$

Thus, by (2.11), (2.36), (2.37) and Lemmas 2.1, 2.2, and 2.3, we obtain (2.8). For the second statement of Theorem 2.1 we use the following decomposition, in the spirit of (2.11):

$$\begin{aligned} A_n(s) &= n^{1/2}(\Gamma_n'^{\text{inv}}(\Gamma_n''(s)) - \Gamma_n''(s)) + n^{1/2}(\Gamma_n''(s) - s) \\ &=: b_n'(\Gamma_n''(s)) + a_n'(s) =: -a_n'(\Gamma_n'^{\text{inv}}(\Gamma_n''(s))) + a_n''(s) + R_{1,n}'(s) \\ &=: -A_n'(\Gamma_n'^{\text{inv}}(\Gamma_n''(s))) + A_n''(s) + \sum_{j=1}^{2} R_{j,n}'(s) \\ &=: -A_n'(s - n^{-1/2} M_n(s)) + A_n''(s) + \sum_{j=1}^{3} R_{j,n}'(s) \\ &=: -M_n(s) + \sum_{j=1}^{4} R_{j,n}'(s). \end{aligned} \qquad (2.38)$$

As in (2.11)–(2.37), we evaluate $\| R_{j,n}' \|_0^\vartheta$ for $j = 1, 2, 3, 4$. First, we apply Lemma 2.1 with the replacements of a_n'' and b_n'' by a_n' and b_n', so that for all $\lambda \in (0, \Theta)$,

$$\| b_n' + a_n'(\Gamma_n'^{\text{inv}}) \|_0^\lambda = O(n^{-1/2}) \qquad \text{a.s. as } n \to \infty. \qquad (2.39)$$

By Glivenko–Cantelli for PL estimators, (2.38), and (2.39), for any $\vartheta \in (0, \Theta)$,

$$\| R_{1,n}' \|_0^\vartheta = O(n^{-1/2}) \qquad \text{a.s.} \quad \text{as } n \to \infty. \qquad (2.40)$$

Next, by (1.7) and (1.8), for any $\vartheta \in (0, \Theta)$,

$$\| R_{2,n}' \|_0^\vartheta = O(n^{-1/2} \log^2 n) \qquad \text{a.s.} \quad \text{as } n \to \infty. \qquad (2.41)$$

For $R_{3,n}'$ more work is needed. We first obtain a rough bound, for

$$\Gamma_n'^{\text{inv}}(\Gamma_n''(s)) - s + n^{-1/2} M_n(s) = n^{-1/2}(A_n(s) + M_n(s)). \qquad (2.42)$$

Note that

$$
\begin{aligned}
A_n(s)+M_n(s) &= b'_n(\Gamma''_n(s))+a''_n(s)+M_n(s)\\
&=: -A'_n(\Gamma''_n(s))+A''_n(s)+M_n(s)+T_{1,n}(s)\\
&=: T_{1,n}(s)+T_{2,n}(s). \qquad (2.43)
\end{aligned}
$$

Similar to (2.18), and using (1.8), we have as $n\to\infty$

$$\| T_{1,n}\|_0^\vartheta = O_P(n^{-1/4}(\log n)^{1/2}). \qquad (2.44)$$

Proceeding as in the proof of Lemma 2.3, we see that as $n\to\infty$

$$\| T_{2,n}\|_0^\vartheta = O_P(n^{-1/4}(\log n)^{1/2}). \qquad (2.45)$$

Thus the LHS of (2.42) is $O_P(n^{-3/4}(\log n)^{1/2})$ as $n\to\infty$, uniformly in $0\leqslant s\leqslant \vartheta$.

Proceeding again as in the proof of Lemma 2.3 we see that, as $n\to\infty$,

$$\| R'_{3,n}\|_0^\vartheta = O_P(n^{-3/8}(\log n)^{3/4}). \qquad (2.46)$$

Finally, we have, with Λ' and M as in (2.34),

$$R'_{4,n}(s)=\Lambda'_n(s)-\Lambda'_n(s-n^{-1/2}M_n(s)) =_d \Lambda'(s)-\Lambda'(s-n^{-1/2}M(s)). \qquad (2.47)$$

By Proposition 2.1 with $\Psi=h$, $\Phi=1/(1-I)$, and $\eta=-M$, we have almost surely

$$n^{1/4}(\log n)^{-1/2}\,\|\Lambda'-\Lambda'(I-n^{-1/2}M)\|_0^\vartheta/\|\,|h'M|^{1/2}\,(1-I)\|_0^\vartheta \to 1 \quad \text{as} \quad n\to\infty. \qquad (2.48)$$

Recalling that $h'=(1-I)^{-2}\,(1-H(F^{\text{inv}}))^{-1}$, by (2.47) and (2.48), we obtain

$$n^{1/4}(\log n)^{-1/2}\,\| R'_{4,n}\|_0^\vartheta/(\|M_n/(1-H(F^{\text{inv}}))\|_0^\vartheta)^{1/2} \xrightarrow[P]{} 1 \qquad \text{as} \quad n\to\infty. \qquad (2.49)$$

By combining (2.38) with (2.40), (2.41), (2.46), and (2.49), we readily obtain (2.9).

To complete the proof of Theorem 2.1 it suffices to prove that

$$n^{1/4}(\log n)^{-1/2}\,\|\hat{\Lambda}_n-\Lambda_n\|_0^\vartheta \xrightarrow[P]{} 0 \qquad \text{as} \quad n\to\infty. \qquad (2.50)$$

This can be shown along the lines of the proofs of Lemmas 1 and 2 in Beirlant and Einmahl [5]; the mean ingredient of the proof is the mean value theorem. The straightforward details are left to the reader.

3. A Two-Sample Bahadur–Kiefer Type Test of Fit for P–P Plots

In this section, we consider the statistic defined by

$$\delta_n^\vartheta = n^{1/2} \| F_n(G_n^{\text{inv}}) + G_n(F_n^{\text{inv}}) - 2I \|_0^\vartheta \qquad \text{for} \quad 0 < \vartheta < \Theta. \tag{3.1}$$

Whenever $F = G$, by (2.1), (2.2), and (3.1), we have

$$\delta_n^\vartheta = n^{1/2} \| \Gamma_n'(\Gamma_n''^{\text{inv}}) + \Gamma_n''(\Gamma_n'^{\text{inv}}) - 2I \|_0^\vartheta. \tag{3.2}$$

Recall, by (2.3), that $\tilde{A}_n(s) = n^{1/2}(\Gamma_n'(\Gamma_n''^{\text{inv}}(s)) - s)$ for $0 \leqslant s \leqslant 1$. Define likewise $\tilde{A}_n^*(s) = n^{1/2}(\Gamma_n''(\Gamma_n'^{\text{inv}}(s)) - s)$ for $0 \leqslant s \leqslant 1$. By Theorem 2.1, $\| \tilde{A}_n - M_n \|_0^\vartheta = O_P(n^{-1/4}(\log n)^{1/2})$ as $n \to \infty$. Likewise, by reversing the first and second sample, $\| \tilde{A}_n^* + M_n \|_0^\vartheta = O_P(n^{-1/4}(\log n)^{1/2})$ as $n \to \infty$. Bearing in mind that (3.2) may be rewritten as $\delta_n^\vartheta = \| \tilde{A}_n + \tilde{A}_n^* \|_0^\vartheta$, it follows that, under the null hypothesis that $F = G$,

$$\delta_n^\vartheta = O_P(n^{-1/4}(\log n)^{1/2}) \underset{P}{\to} 0 \qquad \text{as} \quad n \to \infty. \tag{3.3}$$

By (3.3), we see that $n^{1/4}(\log n)^{-1/2}\, \delta_n^\vartheta = O_P(1)$ when $F = G$, whereas the same expression in general tends to infinity with a rate of $n^{3/4}\,(\log n)^{-1/2}$ when $F \neq G$. This motivates the study of the limiting behavior of $n^{1/4}(\log n)^{-1/2}\, \delta_n^\vartheta$ when $F = G$. This problem has been solved in the uncensored case by Deheuvels and Mason [11]. The following theorem extends their results to censored models.

THEOREM 3.1. *Assume that $F = G$. Then, for any $\vartheta \in (0, \Theta)$, we have, as $n \to \infty$,*

$$n^{1/4}(\log n)^{-1/2}\, \delta_n^\vartheta \Big/ \left(\left\| (A_n' - A_n'') \left(\frac{1}{1 - H(F^{\text{inv}})} + \frac{1}{1 - K(F^{\text{inv}})} \right) \right\|_0^\vartheta \right)^{1/2} \underset{P}{\to} 1. \tag{3.4}$$

COROLLARY 3.1. *As $n \to \infty$,*

$$\delta_n^\vartheta \Big/ \left(\log n \left\| \left(F_n(F^{\text{inv}}) - G_n(F^{\text{inv}}) \right) \left(\frac{1}{1 - H(F^{\text{inv}})} + \frac{1}{1 - K(F^{\text{inv}})} \right) \right\|_0^\vartheta \right)^{1/2} \underset{P}{\to} 1. \tag{3.5}$$

In the remainder of this section, we prove Theorem 3.1. Throughout the sequel, we assume that $F=G$, so that (3.2) holds. We make use of the decomposition

$$\begin{aligned} R_n(s) &:= n^{1/2}(\Gamma'_n(\Gamma''^{\text{inv}}_n(s)) + \Gamma''_n(\Gamma'^{\text{inv}}_n(s)) - 2s) = \tilde{A}_n(s) + \tilde{A}^*_n(s) \\ &=: M_n(s - n^{-1/2}A''_n(s)) - M_n(s - n^{-1/2}A'_n(s)) + R''_n(s). \end{aligned} \tag{3.6}$$

LEMMA 3.1. *We have for all* $\vartheta \in (0, \Theta)$

$$\| R''_n \|_0^\vartheta = O_P(n^{-3/8}(\log n)^{3/4}) \qquad as \quad n \to \infty. \tag{3.7}$$

Proof. By (2.11) and Lemmas 2.1, 2.2, and 2.3, we have, as $n \to \infty$,

$$\begin{aligned} &\| M_n(I - n^{-1/2}A'') - \tilde{A}_n \|_0^\vartheta \\ &\quad \leqslant \| R_{1,n} \|_0^\vartheta + \| R_{2,n} \|_0^\vartheta + \| R_{3,n} \|_0^\vartheta = O_P(n^{-3/8}(\log n)^{3/4}). \end{aligned} \tag{3.8}$$

We repeat the same argument by interchanging the two samples; i.e., by changing A'_n (resp. A''_n) into A''_n (resp. A'_n), $M_n = A'_n - A''_n$ into $-M_n = A''_n - A'_n$, and $\tilde{A}_n$ into $\tilde{A}^*_n$ in (3.8). Combining both versions of (3.8) yields (3.7). ▮

By (2.35), we have

$$\begin{aligned} &M_n(I - n^{-1/2}A''_n) - M_n(I - n^{-1/2}A'_n) \\ &\quad =_d M(I - n^{-1/2}A'') - M(I - n^{-1/2}A'). \end{aligned} \tag{3.9}$$

Thus, by (3.6)–(3.9), (2.34), and (2.35), the proof of (3.4) boils down to showing that

$$\begin{aligned} &(2u \log(1/u))^{-1/2} \| M(I - uA') - M(I - uA'') \|_0^\vartheta \\ &\quad \Big/ \left(\left\| (A' - A'') \left(\frac{1}{1 - H(F^{\text{inv}})} + \frac{1}{1 - K(F^{\text{inv}})} \right) \right\|_0^\vartheta \right)^{1/2} \xrightarrow[P]{} 1 \qquad (u \downarrow 0). \end{aligned} \tag{3.10}$$

This follows from the following refinement of Proposition 2.1 (see, e.g., Proposition 2.1 in Deheuvels and Mason [11]).

PROPOSITION 3.1. *Under the assumptions of Proposition* 2.1, *there exists an event* E'' *of probability one such that, on* E'', *we have for all pairs* η' *and* η'' *of continuous functions on* $[a, b]$

$$\begin{aligned} &\lim_{u \downarrow 0} \sup_{c \leqslant t \leqslant d} (2u \log(1/u))^{-1/2} \left| \frac{W(\Psi(t + u\eta'(t)))}{\Phi(t + u\eta'(t))} - \frac{W(\Psi(t + u\eta''(t)))}{\Phi(t + u\eta''(t))} \right| \\ &\quad = \| \, |\psi(\eta' - \eta'')|^{1/2} / \Phi \|_c^d. \end{aligned} \tag{3.11}$$

Proof. The proof is very similar to that of Proposition 2.1. Therefore, we omit details. ▮

By Proposition 3.1 with $\eta' = -\Lambda'$, $\eta'' = -\Lambda''$, $\Psi = l = h + k$, $\Phi = 1/(1-I)$, $c = 0$, $d = \vartheta$, a and b being chosen such that $-1 < a < 0 < \vartheta < b < \Theta$, recalling, by (2.34), that $M = (1-I)\, W(l)$, we obtain (3.10) and (3.4) from (3.11).

Remark 3.1. By a simple modification of the proof of Theorem 3.1 we obtain that, whenever $\{\rho(t), 0 \leqslant t < \Theta\}$ is continuous and positive on $[0, \vartheta]$, we have, for $F = G$,

$$\begin{aligned} &n^{3/4}(\log n)^{-1/2} \,\|(F_n(G_n^{\text{inv}}) + G_n(F_n^{\text{inv}}) - 2I)\, \rho\|_0^{\vartheta} \\ &\quad \xrightarrow[d]{} (\|\rho^2(1-I)\, W(l)((1 - H(F^{\text{inv}}))^{-1} \\ &\quad + (1 - K(F^{\text{inv}}))^{-1})\|_0^{\vartheta})^{1/2} \qquad \text{as} \quad n \to \infty. \end{aligned} \tag{3.12}$$

4. Statistical Applications

We now return to the problem of testing the null hypothesis that $F = G$ given $\{(Z_i', \delta_i'), 1 \leqslant i \leqslant n\}$ and $\{(Z_i'', \delta_i''), 1 \leqslant i \leqslant n\}$. We assume that $F = G$, H, and K are unknown. By Theorems 2.1 and 3.1, and Remark 3.1, we are led to the statistics

$$S_{1,n}'(\rho, \vartheta) = \|(F_n(G_n^{\text{inv}}) - I)\, \rho\|_0^{\vartheta}, \; S_{1,n}''(\rho, \vartheta) = \|(G_n(F_n^{\text{inv}}) - I)\, \rho\|_0^{\vartheta}, \tag{4.1}$$

and

$$S_{2,n}(\rho, \vartheta) = \|(F_n(G_n^{\text{inv}}) + G_n(F_n^{\text{inv}}) - 2I)\, \rho\|_0^{\vartheta}, \tag{4.2}$$

where $\{\rho(t), 0 \leqslant t < \Theta\}$ is a suitable function.

The question to be answered is to find the appropriate critical values at level $\alpha \in (0, 1)$. Following the usual large sample approximations, we may use approximate critical values by rejecting the assumption that $F = G$ whenever $S_{1,n}'(\rho, \vartheta) \geqslant c_\rho'(\alpha, \vartheta)\, n^{-1/2}$, $S_{1,n}''(\rho, \vartheta) \geqslant c_\rho''(\alpha, \vartheta)\, n^{-1/2}$, or $S_{2,n}(\rho, \vartheta) \geqslant c_\rho(\alpha, \vartheta)\, n^{-3/4}(\log n)^{1/2}$, respectively, where $c_\rho'(\alpha, \vartheta) = c_\rho''(\alpha, \vartheta)$ and $c_\rho(\alpha, \vartheta)$ are given by

$$P(\|\rho W(l)(1-I)\|_0^{\vartheta} \geqslant c_\rho'(\alpha, \vartheta)) = \alpha \tag{4.3}$$

and

$$P\left(\left(\left\|\rho^2 W(l)(1-I)\left(\frac{1}{1-H(F^{\text{inv}})} + \frac{1}{1-K(F^{\text{inv}})}\right)\right\|_0^{\vartheta}\right)^{1/2} \geqslant c_\rho(\alpha, \vartheta)\right) = \alpha. \tag{4.4}$$

The problem is not solved yet, since the expressions in (4.3) and (4.4) depend upon the unknown values of l, H, K, and F. To overcome this difficulty, we introduce estimators of the unknown factors in (4.3) and (4.4) by setting for $-\infty < s < \infty$

$$\begin{aligned} J'_n(s) &= n^{-1}\#\{1 \leqslant i \leqslant n : Z'_i < s\}, \\ J''_n(s) &= n^{-1}\#\{1 \leqslant i \leqslant n : Z''_i < s\}, \\ \tilde{J}'_n(s) &= n^{-1}\#\{1 \leqslant i \leqslant n : Z'_i \leqslant s, \delta'_i = 1\}, \\ \tilde{J}''_n(s) &= n^{-1}\#\{1 \leqslant i \leqslant n : Z''_i \leqslant s, \delta''_i = 1\}, \end{aligned} \tag{4.5}$$

and for $0 \leqslant s < \mu_n := \min(\max\{F_n(Z'_i) : 1 \leqslant i \leqslant n, \delta'_i = 1\}, \max\{G_n(Z''_i) : 1 \leqslant i \leqslant n, \delta''_i = 1\})$

$$\begin{aligned} h_n(s) &= \int_0^{F_n^{\mathrm{inv}}(s)} (1 - J'_n(t))^{-2}\, d\tilde{J}'_n(t), \\ k_n(s) &= \int_0^{G_n^{\mathrm{inv}}(s)} (1 - J''_n(t))^{-2}\, d\tilde{J}''_n(t); \end{aligned} \tag{4.6}$$

h_n and k_n are strongly uniformly consistent estimators of h and k, respectively, on any interval $[0, \vartheta]$, $0 < \vartheta < \Theta$ (see e.g. Lemma 6.2 in Burke, Csörgő, and Horváth [6]). By all this, our first choice of ρ $(= \rho_n$ now) appropriate to (4.3) is given by

$$\rho_n(s) = (1-s)^{-1} (l_n(\vartheta))^{-1/2}, \qquad \text{where } l_n := h_n + k_n, \qquad \text{for } 0 \leqslant s < \mu_n. \tag{4.7}$$

It is now straightforward from the preceding arguments that (for $F = G$)

$$\begin{aligned} &\lim_{n\to\infty} P(n^{1/2} S'_{1,n}(\rho_n, \vartheta) \geqslant c) \\ &\quad = \lim_{n\to\infty} P(n^{1/2} S''_{1,n}(\rho_n, \vartheta) \geqslant c) = P(\| W \|_0^1 \geqslant c) \\ &\quad = 1 - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \exp\left(-\frac{\pi^2 (2j+1)^2}{8c^2}\right) \qquad \text{for } c > 0 \end{aligned} \tag{4.8}$$

(see, e.g., (1.5.2) in Csörgő and Révész [9]). Thus, by (4.8), we may choose $c'_{\rho_n}(\alpha, \vartheta) = c''_{\rho_n}(\alpha, \vartheta)$ to be equal to the value of c which renders the right hand side of (4.8) equal to α. We so obtain a first solution to our two-sample testing problem.

A similar argument can be used for $S_{2,n}$. The appropriate choice of ρ is here given by

$$\rho_n^*(s) = (1-s)^{-1/2} (l_n(\vartheta))^{-1/4} \left(\frac{1}{1 - H_n(F_n^{\text{inv}}(s))} + \frac{1}{1 - K_n(G_n^{\text{inv}}(s))} \right)^{-1/2}, \tag{4.9}$$

where H_n and K_n denote strongly uniformly consistent estimators of H and K. Since J_n' and J_n'' are strongly consistent estimators of $1-(1-F)(1-H)$ and of $1-(1-F)(1-K)$, respectively, one may set

$$\begin{aligned} 1 - H_n(s) &= (1 - J_n'(s))/(1 - F_n(s)), \\ 1 - K_n(s) &= (1 - J_n''(s))/(1 - G_n(s)). \end{aligned} \tag{4.10}$$

In view of (4.9) and (4.10), we may simplify our choice of ρ by the observation that $1 - F_n(F_n^{\text{inv}}(s))$ and $1 - G_n(G_n^{\text{inv}}(s))$ can be replaced by $1-s$. This leads to

$$\rho_n(s) = (1-s)^{-1} (l_n(\vartheta))^{-1/4} \left(\frac{1}{1 - J_n'(F_n^{\text{inv}}(s))} + \frac{1}{1 - J_n''(G_n^{\text{inv}}(s))} \right)^{-1/2}. \tag{4.11}$$

Now we obtain reasily that (for $F = G$)

$$\begin{aligned} \lim_{n \to \infty} & P(n^{3/4}(\log n)^{-1/2} S_{2,n}(\rho_n, \vartheta) \geq c) \\ &= P(\| W \|_0^1 \geq c^2) \\ &= 1 - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \exp\left(-\frac{\pi^2 (2j+1)^2}{8c^4} \right) \qquad \text{for} \quad c > 0. \end{aligned} \tag{4.12}$$

Finally, we choose $c_{\rho_n}(\alpha, \vartheta)$ equal to the value of c which renders the RHS of (4.12) equal to α.

Other choices of ρ may be used likewise. Also, similar applications can be presented for the Q–Q plots. We will not present such results here for the sake of brevity.

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