The Effect of Information Streams on Capital Budgeting Decisions*

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Abstract

In this paper a new decision rule for capital budgeting is considered. A firm has the opportunity to invest in a project of uncertain profitability. Over time, the firm receives additional information in the form of signals indicating the profitability of the project. The belief that the firm needs to have in a profitable project for investment to be optimal is calculated and analyzed. It is shown that the probability of investing in a project with low profitability is larger when the firm uses a conventional rule like the net present value rule. As a counterintuitive result it is obtained that it can be optimal to undertake the investment at a later point in time in case the expected number of signals per time unit is higher. Also an error measure is discussed that indicates the accuracy of capital budgeting rules in this stochastic environment.

Keywords: Investment analysis, Innovation adoption, Firm behaviour, Bayesian updating.

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1 Introduction

In this paper a firm is considered that faces the decision whether or not to invest in a project. The project's profitability is not known beforehand. However, imperfect signals arrive over time indicating the project either to be good or bad. These signals cause the firm to update its valuation of the project. The aim is to determine the timing of investment as well as the effects of the quantity and the quality of the signals on the investment decision.

The problem can for example be the adoption of a technological innovation whose effectiveness in unknown. One can also think of a firm having the opportunity to enter a new market which involves sunk investment costs. The uncertainty can then for instance be caused by unknown consumer interest, e.g. demand can be favourable or not. Consider for instance the telecommunication sector where there is one company that can supply a new service to its customers. However, the company is uncertain about the possible success of the new service. Occasionally, the firm receives signals from its environment from which it can deduce information concerning the profitability of the new service. Here we can think of market performance of related products and also of more general economic indicators that may influence the market performance of the new service. Another example is given by a pharmaceutical firm that is developing a new drug. Test results are coming in indicating whether the drug is effective or not.

This situation is modelled by considering a project that can be either good or bad. If the project is bad, the optimal strategy is to refrain from investment. Since the firm incurs sunk costs when investing in the project, a loss is suffered in case the project is bad and the firm invests. At irregular intervals, however, the firm receives a signal about the quality of the project. The signals indicate whether the project is good or bad, but it is known to the firm that the signal is imperfect. The points in time at which signals arrive are unknown beforehand. Every time the firm receives a signal it updates its belief that the project is good in a Bayesian way. Therefore, by delaying investment and waiting for more signals to arrive, the firm can predict with higher accuracy whether the market is good or bad. This induces an option value of waiting. The question is how many good signals relative to bad signals the firm needs to observe, to justify investment in the project. We show that this is equivalent to finding a critical level for the belief that the project is good, given the available signals. This belief turns out to depend critically on the quality of the signal, i.e. the probability with which the signals reflect the true state of the world, as well as the frequency at which signals occur over time.

The signals are modelled as two correlated binomially distributed random variables. The first one models the arrival of signals while the latter models its type, i.e. indicating that the project is good or bad. As soon as the firm has invested, the true state of the world is revealed.

This paper is related to several strands of literature. First of all, our model has strong similarities with the standard real options model as developed by McDonald and Siegel (1986) and for which Dixit and Pindyck (1994) develop the basic framework. It is important to note that the way we deal with uncertainty in our model differs crucially from this literature. Within our framework more information becomes available over time, whereas in the standard real-options literature uncertainty is constant over time caused by, for instance, price uncertainty in an existing market. In other words, whereas our model is a decision problem with incomplete information where nature determines the state of the world only at the beginning with information arriving to resolve uncertainty, the framework typically used in the literature is a decision problem with complete information, where nature determines the state of the world at each consecutive point in time. More formally, the stochastic processes in these models have stationary increments that are independent of the past. Examples of processes that are often used are Brownian motion, Poisson process, and Lévy processes. In contrast, the increments of the stochastic process that we consider are not stationary and path-dependent. Typically, the variance of the stochastic process decreases over time. This implies that the standard tools (cf. Oksendal (2000)) cannot be used in our framework.

A second branch of literature to which our paper is related is the R&D literature. In her seminal paper, Reinganum (1981) develops a model of dynamic R&D competition. In this model technological innovations arrive via a Poisson process and the influence of patents is analysed. Again, the stochastic process driving the innovation process has stationary increments that are independent of the past. The paper by Malueg and Tsutsui (1997) introduces learning into the Reinganum framework and is therefore more closely related to this paper. In the endogenous growth literature, Aghion and Howitt (1992) use a similar framework as Reinganum to model Schumpeterian growth.

The papers mentioned above all consider a stream of technological innovations where there is uncertainty about when these innovations become available. Moscarini and Smith (2001) consider a situation where a single decision maker faces a project whose future stream of cash flow is uncertain. The decision maker however receives a signal indicating the quality of the project. This signal is assumed to follow a geometric Brownian motion with (unknown) drift. The task of the decision maker is to infer the drift term. He can reduce the variance of the signal by investing more in R&D at a higher cost. So the decision maker faces an optimal stopping problem, i.e. when to invest (if at all), as well as an optimal control problem, i.e. how much to invest in R&D. Again, the main difference with our approach is the way uncertainty is modelled. Since, in contrast to Moscarini and Smith (2001), in our paper the stochastic increments are non-stationary and path-dependent. For the sake of analytical tractability, we will assume that signals are costless.

The way we model uncertainty is most related to Jensen (1982). The main difference is that in Jensen's model, signals only give information on the probability of the project being good. The probability of a good project is considered to be an unknown parameter. In each period one receives a signal about the true value of the unknown parameter. This signal is used to update the beliefs, just as in our model, i.e. the belief is a conditional probability based on past information. In short, one forms a belief on the belief in a good project. However, in Jensen's model, a good signal not only increases the belief in a good project, but it also increases the firm's probabilistic belief in receiving a good signal in the next period. In other words, the firm not only updates its belief but also the odds of the coin nature flips to determine the project's profitability. In our model it holds that the quality of the signal is independent of past realizations, i.e. the investor exactly knows the odds of the coin that nature flips. Due to this simplification the analysis of our framework provides an explicit expression for the critical value of the belief in a good project at which investing is optimal, contrary to Jensen (1982) who could only show existence. This is the main contribution of the paper. Furthermore, it allows us to simulate the investment problem and the effects of the model parameters on the investment timing. We show that given constant prior odds of a good project, the probability of investment within a certain time interval need not increase in quantity and quality of signals. Another counterintuitive result we obtain is that, given that the project is good, the expected time before investment need not be monotonous in the parameter governing the Poisson arrivals of signals. In other words, it is possible that investment is expected to take place later when the expected number of signals per time unit is higher.

The paper is organised as follows. In Section 2 the formal model is described. After that, the optimal investment decision will be derived in Section 3. In Section 4 an error measure for analysing the performance of capital budgeting rules in this model of investment under uncertainty is introduced. In Section 5 the decision rule from Section 3 will be interpreted using some numerical examples. In the final section some conclusions are drawn and directions for future research are discussed.

2 The Model

Consider a firm that faces the choice of investing in a certain project. The project can be either good, leading to high revenues, U^H, or bad, leading to low revenues U^L.[1] Without loss of generality we assume that $U^L = 0$. The sunk costs involved in investing in the project are given by $I > 0$. Furthermore, it is assumed that there is a constant discount rate, r.

It is assumed that when the firm receives the option to invest, it has a prior belief about the investment project being good or bad. The *ex ante* probability of high revenues is given by

$$\mathbb{P}(H) = p_0.$$

Occasionally, the firm receives a signal indicating the project to be good (denoted by h) or a signal indicating the project to be bad (denoted by l). The probabilities with which these signals occur depend on the true state of

[1] The revenues represent an infinite cash flow discounted at rate r.

project/signal	h	l
H	λ	$1-\lambda$
L	$1-\lambda$	λ

Table 1: Probability of a signal indicating a good or bad market, given the true state of the project. The first row (column) lists the probabilities in case of a good project (good signal) and the second row (column) in case of a bad project (bad signal). A good (bad) project is denoted by H (L), while a good (bad) signal is denoted by h (l).

the project. A correct signal occurs with probability $\lambda > \frac{1}{2}$, see Table 1. As soon as the firm invests in the project, the state of the market is revealed. In reality this may take some time, but we abstract from that. The signals' arrivals are modelled via a Poisson process with parameter $\mu > 0$. The Poisson assumption is made to make the model analytically tractable when using dynamic programming techniques. Hence, denoting the number of signals by n, this boils down to

$$dn(t) = \begin{cases} 1 & \text{with probability } \mu dt, \\ 0 & \text{with probability } 1 - \mu dt, \end{cases}$$

with

$$n(0) = 0.$$

Denoting the number of h-signals by g, the dynamics of g is then given by

$$dg(t) = udn(t),$$

with

$$u = \begin{cases} 1 & \text{with probability } \lambda \text{ if } H \text{ and } 1 - \lambda \text{ if } L, \\ 0 & \text{with probability } 1 - \lambda \text{ if } H \text{ and } \lambda \text{ if } L, \end{cases}$$

and

$$g(0) = 0.$$

For notational convenience the time indices will be suppressed in the remainder of the paper. The belief that revenues are high, i.e. that the project is good, given the number of signals n and the number of h-signals $g \leq n$ is denoted by $p(n, g)$. Now, the conditional expected payoff of the firm can be written as,

$$\mathbb{E}(U|n, g) = p(n, g)(U^H - I) - (1 - p(n, g))I.$$

The structure of the model is such that with respect to the signals there are two main aspects. The first one is the parameter which governs the arrival of the signals, μ. This parameter is a measure for the quantity of the signals, since $1/\mu$ denotes the average time between two signals. The other component is the probability of the correctness of the signal, λ. This parameter is a measure for the quality of the signals. For the model to make sense, it is assumed that $\lambda > \frac{1}{2}$.[2] In this paper learning – or belief updating – takes place by using the Bayesian approach. This, together with the condition $\lambda > \frac{1}{2}$, implies that the belief in high revenues converges to one or to zero if the market is good or bad, respectively, in the long-run. As will be shown in Section 3, quantity and quality together determine the threshold belief in a good project the firm needs to have in order for investment to be optimal.

3 The Optimal Investment Decision

In determining the optimal output level, the firm chooses the output that maximizes its expected profit flow. Since the firm is risk-neutral, it is only interested in the expected values of investing in the project and waiting for more information.

The uncertainty about the true state of the project and the irreversibility of investment induce an option value of waiting for more signals. In this section we will show how to find the critical level for $p(n, g)$ at which the firm is indifferent between investing and waiting, while taking into account

[2]This assumption is not as strong as it seems, for if $\lambda < \frac{1}{2}$ the firm can perform the same analysis replacing λ with $1 - \lambda$. If $\lambda = \frac{1}{2}$ the signals are not informative at all and the firm would do best by making a now-or-never decision, using its *ex ante* belief $p(0, 0) = p_0$.

the option value of waiting. After having determined the critical level we know that it is optimal to invest as soon as $p(n,g)$ exceeds this level.

First, we explicitly calculate $p(n,g)$. To simplify matters considerably, define $k := 2g - n$, the number of good signals in excess of bad signals, and $\zeta := \frac{1-p_0}{p_0}$, the unconditional odds of the project being bad. By using Bayes' rule we now obtain:

$$\begin{aligned} p(n,g) &= \frac{\mathbb{P}(n,g|H)\mathbb{P}(H)}{\mathbb{P}(n,g|H)\mathbb{P}(H) + \mathbb{P}(n,g|L)\mathbb{P}(L)} \\ &= \frac{\lambda^g(1-\lambda)^{n-g}p_0}{\lambda^g(1-\lambda)^{n-g}p_0 + (1-\lambda)^g\lambda^{n-g}(1-p_0)} \\ &= \frac{\lambda^k}{\lambda^k + \zeta(1-\lambda)^k} \equiv p(k). \end{aligned} \tag{1}$$

The critical level of k where the firm is indifferent between investing and not investing in the project is denoted by k^*. Note that at any arrival of an h-signal k increases and at any arrival of an l-signal k decreases. Hence, enough h-signals must arrive to reach the critical level. The critical level of the conditional belief in high revenues is denoted by $p^* = p(k^*)$.

Suppose that the state of the process at a particular point in time is given by k. Then there are three possibilities. First, k might be such that $k \geq k^*$ and $p(k) \geq p^*$. Then it is optimal for the firm to directly invest in the project. In this case the value of the project for the firm, denoted by Ω, is given by

$$\Omega(k) = U^H p(k) - I. \tag{2}$$

A second possibility is that, even after a new h-signal arriving, it is still not optimal to invest, i.e. $k < k^* - 1$. We assume that pricing with respect to the objective probability measure implies risk-neutrality concerning the information gathering process. Then the value of the opportunity to invest for the firm, denoted by V_1, must satisfy the following Bellman equation:

$$rV_1(k) = \tfrac{1}{dt}\mathbb{E}(dV_1(k)). \tag{3}$$

Departing from this equation the following second order linear difference equation can be constructed:

$$\begin{aligned} rV_1(k) &= \mu\big[p(k)(\lambda V_1(k+1) + (1-\lambda)V_1(k-1))+ \\ &\quad + (1-p(k))(\lambda V_1(k-1) + (1-\lambda)V_1(k+1)) - V_1(k)\big] \\ &\Leftrightarrow (r+\mu)V_1(k) = \mu\big[(2p(k)\lambda + 1 - \lambda - p(k))V_1(k+1)+ \\ &\quad + (p(k) + \lambda - 2p(k)\lambda)V_1(k-1)\big]. \end{aligned} \tag{4}$$

Eq. (4) states that the value of the option at state k must equal the discounted expected value an infinitesimal amount of time later. Using eq. (1) it holds that

$$2p(k)\lambda + 1 - \lambda - p(k) = \frac{\lambda^{k+1} + \zeta(1-\lambda)^{k+1}}{\lambda^k + \zeta(1-\lambda)^k} \tag{5}$$

and

$$p(k) + \lambda - 2p(k)\lambda = \frac{\lambda(1-\lambda)(\lambda^{k-1} + \zeta(1-\lambda)^{k-1})}{\lambda^k + \zeta(1-\lambda)^k}. \tag{6}$$

Substituting eqs. (5) and (6) in (4), and defining $F(k) := (\lambda^k + \zeta(1-\lambda)^k)V_1(k)$, yields

$$(r+\mu)F(k) = \mu F(k+1) + \mu\lambda(1-\lambda)F(k-1). \tag{7}$$

Eq. (7) is a second order linear homogeneous difference equation which has as general solution

$$F(k) = A\beta^k,$$

where A is a constant and β is a solution of the homogeneous equation

$$\mathcal{Q}(\beta) \equiv \beta^2 - \frac{r+\mu}{\mu}\beta + \lambda(1-\lambda) = 0. \tag{8}$$

Eq. (8) has two real roots,[3] namely

$$\beta_{1,2} = \frac{r+\mu}{2\mu} \pm \tfrac{1}{2}\sqrt{(\tfrac{r}{\mu}+1)^2 - 4\lambda(1-\lambda)}. \tag{9}$$

Note that $\mathcal{Q}(0) = \lambda(1-\lambda) > 0$ and $\mathcal{Q}(1-\lambda) = -\frac{r}{\mu}(1-\lambda) \leq 0$. Since the graph of $\mathcal{Q}$ is an upward pointing parabola we must have $\beta_1 \geq 1-\lambda$ and $0 < \beta_2 < 1-\lambda$ (see Figure 1). The value function $V_1(\cdot)$ is then given by

$$V_1(k) = \frac{F(k)}{\lambda^k + \zeta(1-\lambda)^k} = \frac{A_1\beta_1^k + A_2\beta_2^k}{\lambda^k + \zeta(1-\lambda)^k}. \tag{10}$$

[3]It should be noted that for all λ it holds that $4\lambda(1-\lambda) \leq 1$. Since equality holds iff $\lambda = 1/2$, the homogeneous equation indeed has two real roots for any $\lambda \in (1/2, 1]$.

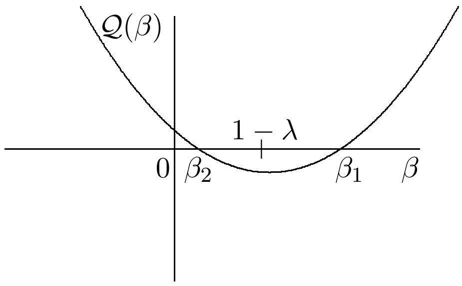

Figure 1: Graph of $\mathcal{Q}$.

Here it is important to note that, when the number of l-signals relative to h-signals tends to infinity, then the value of the firm should converge to zero, i.e. $\lim_{k\to-\infty} V(k) = 0$. This implies that we only need to consider the larger root β_1, so that $A_2 = 0$.[4]

In the final case, the value of k is such that it is not optimal to invest in the project right away. However, if the following signal is an h-signal, it will be optimal to invest, i.e $k^* - 1 \leq k < k^*$. In this region the value function $V_2(\cdot)$ for the firm must satisfy eq. (3) with $V_1(\cdot)$ replaced by $V_2(\cdot)$, i.e.

$$\begin{aligned} rV_2(k) = \mu\big[&p(k)(\lambda\Omega(k+1) + (1-\lambda)V_1(k-1)) + (1-p(k)) \\ &(\lambda V_1(k-1) + (1-\lambda)\Omega(k+1)) - V_2(k)\big] \\ \Leftrightarrow (r+\mu)V_2(k) &= \mu\big[(2p(k)\lambda + 1 - \lambda - p(k))\Omega(k+1) + \\ &+ (p(k) + \lambda - 2p(k)\lambda)V_1(k-1)\big]. \end{aligned} \tag{11}$$

Substituting eqs. (2), (5), (6) and (10) into eq. (11) yields

$$\begin{aligned} V_2(k) =& \frac{\mu}{r+\mu}\Big(\lambda U^H p(k) - (\lambda p(k) + (1-\lambda)(1-p(k)))I \\ &+ \lambda(1-\lambda)\tfrac{A_1\beta_1^{k-1}}{\lambda^k + \zeta(1-\lambda)^k}\Big). \end{aligned} \tag{12}$$

If an h-signal arrives, the process jumps to the region where $k \geq k^*$ and if an l-signal arrives the process jumps to the region where $k < k^*$. Therefore the value V_2 is completely determined by $V_1(k-1)$ and $\Omega(k+1)$. The value

[4]This stems from the fact that $\beta_2 < 1-\lambda$, so in $V_1(k)$ and $V_2(k)$ the term β_2^k dominates $(1-\lambda)^k$ if $k \to -\infty$. Hence, if $A_2 \neq 0$, then $V(k) \to \pm\infty$ if $k \to -\infty$.

function V is then given by

$$V(k) = \begin{cases} V_1(k) & \text{if } k < k^* - 1 \\ V_2(k) & \text{if } k^* - 1 \leq k < k^* \\ U^H p(k) - I & \text{if } k \geq k^*, \end{cases} \tag{13}$$

where $V_1(k)$ and $V_2(k)$ are given by (10) and (12), respectively.

To determine A_1 and k^* we solve the continuity condition $V_1(k^* - 1) = V_2(k^* - 1)$ and the value-matching condition $V_2(k^*) = \Omega(k^*)$.[5] The latter equation yields

$$\begin{aligned} A_1 = & \frac{1}{\beta_1^{k^*-1} \mu\lambda(1-\lambda)} [U^H \lambda^{k^*}(r + \mu(1-\lambda)) \\ & - rI(\lambda^{k^*} + \zeta(1-\lambda)^{k^*}) - \mu I(\lambda\zeta(1-\lambda)^{k^*} + (1-\lambda)\lambda^{k^*})]. \end{aligned}$$

Substituting A_1 in the former equation leads to an expression for $p^* \equiv p(k^*)$:

$$p^* = \frac{1}{\Psi(U^H/I - 1) + 1}, \tag{14}$$

where

$$\Psi = \frac{\beta_1(r+\mu)(r+\mu(1-\lambda)) - \mu\lambda(1-\lambda)(r+\mu(1+\beta_1-\lambda))}{\beta_1(r+\mu)(r+\mu\lambda) - \mu\lambda(1-\lambda)(r+\mu(\beta_1+\lambda))}. \tag{15}$$

The threshold number of h-signals relative to l-signals is then given by

$$k^* = \frac{\log(\frac{p^*}{1-p^*}) + \log(\zeta)}{\log(\frac{\lambda}{1-\lambda})}. \tag{16}$$

From eq. (16) it is obtained that k^* decreases with p_0. Hence, less additional information is needed when the initial belief in high revenues is already high.

Next, we check whether the optimal belief p^* is a well-defined probability. The following proposition establishes this result, which is proved in the appendix. It furthermore shows the link between this approach and the traditional net present value rule (NPV). Note that the critical belief under the latter approach is obtained by solving $\mathbb{E}(U|k) = 0$. This yields $p_{NPV} = \frac{I}{U^H}$.

[5] Note that, despite the fact that k is an integer variable, the continuity and the value matching conditions should hold because the critical level k^* can be any real number. Since the realisations of k are discrete, the firm invests as soon as $k = \lceil k^* \rceil$.

Proposition 1 *For $U^H \geq I$ it holds that $p^* \leq 1$. Furthermore, $p^* > p_{NPV}$.*

So, the result that is obtained in the standard real option model, namely that the criterion for investment to be undertaken is less tight under NPV than under the optimal approach, carries over to this model. The reason is the existence of a value of waiting for more information to arrive that reduces uncertainty.

Using eq. (14), one can obtain comparative static results. These are stated in the following proposition, the proof of which is given in the appendix.

Proposition 2 *The threshold belief in a good project, p^*, increases with I, r and λ and decreases with U^H.*

The fact that p^* increases with r is caused by the so-called net present value effect. If r increases, future income is valued less so that the net present value decreases. Therefore, the firm is willing to wait longer with investment until it has more information about the actual state of the project. An increase in λ leads to an increase in p^*, which can be explained by the fact that λ is a measure for the informativeness of the signal. Therefore, it is less costly in terms of waiting time to require a higher level of p^*. This does not necessarily imply that one should wait for more signals to arrive, a point which we elaborate upon in Section 5. It is impossible to get a knife-edged result on the comparative statics with respect to μ, although simulations suggest that in most cases p^* increases with μ, which confirms intuition. The partial derivative of p^* with respect to μ is negative if $r \approx \mu\sqrt{2\lambda - 1}$.

4 Error Analysis

An important question the firm faces is how likely it is that it makes a wrong decision, in the sense that it invests while the project is bad. This question can be answered quantitatively by calculating the probability that k^* is reached while the project is bad. In order to do so, define

$$P^{(k^*)}(k) := \mathbb{P}(\exists_{t \geq 0} : k_t \geq k^* | k_0 = k, L) \tag{17}$$

Of course, for $k \geq k^*$ it holds that $P^{(k^*)}(k) = 1$. A second order linear difference equation can be obtained governing $P^{(k^*)}(k)$. Notice that from

k the process reaches either $k-1$ or $k+1$ with probabilities λ and $1-\lambda$, respectively, given that the project is bad. Therefore, one obtains

$$P^{(k^*)}(k) = (1-\lambda)P^{(k^*)}(k+1) + \lambda P^{(k^*)}(k-1). \tag{18}$$

Using the boundary conditions $P^{(k^*)}(k^*) = 1$ and $\lim_{k\to-\infty} P^{(k^*)}(k) = 0$, one can solve eq. (18), yielding

$$P^{(k^*)}(k) = \left(\frac{\lambda}{1-\lambda}\right)^{k-k^*}. \tag{19}$$

Hence, the probability of a wrong decision decreases when the quality of the signals increases. The *ex ante* probability of a wrong decision is given by $P^{(k^*)}(k_0)$.

The error measure $P^{(k^*)}(\cdot)$ gives a worst-case scenario: the probability that a firm engages in an investment that has low profitability. Another error measure would be given by the probability that the firm forgoes an investment that would have generated a high profit stream, i.e. the probability that k^* is not reached within a certain time T given that the project is good. Note however that since $\lambda > \frac{1}{2}$ this probability equals zero for $T = \infty$. For any finite time T it is possible to calculate the probability that the firm has not invested before T given that the project is good. In order to calculate this probability, denote for all k the pdf of the distribution of the first passage time through k by $f_k(\cdot)$. From Feller (1971, Section 14.6) one obtains that

$$f_k(t) = \left(\frac{1-\lambda}{\lambda}\right)^{-\frac{k}{2}} \frac{k}{t} I_k\left(2\mu\sqrt{\lambda(1-\lambda)}t\right)e^{-\mu t}, \tag{20}$$

where $I_k(\cdot)$ denotes the modified Bessel function with parameter k. This is the *unconditional* density of first passage times. Given the first passage time distribution it holds for all $0 < T < \infty$ and $k < k^*$ that

$$\begin{aligned}\tilde{P}_k^{(k^*)}(T) &:= \mathbb{P}(\neg\exists_{t\in[0,T]} : k_t \geq k^*|H, k_0 = k)\\ &= \mathbb{P}(\forall_{t\in[0,T]} : k_t < k^*|H, k_0 = k)\\ &= 1 - \int_0^T f_{k^*}(t)dt.\end{aligned}$$

Since there is a positive probability mass on the project being bad, the expectation of the time of investment does not exist. However, conditional on the project being good, one can calculate the expected time of investment using the *conditional* density of first passage times, which is obtained in a similar way as eq. (20) and given by

$$\tilde{f}_k(t) = \tfrac{\lambda^k + \zeta(1-\lambda)^k}{1+\zeta} \big(\lambda(1-\lambda)\big)^{-k/2} \frac{k}{t} I_k(2\mu\sqrt{\lambda(1-\lambda)}t)e^{-\mu t}. \tag{21}$$

5 Economic Interpretation

As an example to see how U^H and U^L arise, consider a market where inverse demand is assumed to be given by the following function,

$$P(q) = \begin{cases} Y - q & \text{if } q \leq Y \text{ and } H \\ 0 & \text{otherwise,} \end{cases}$$

where q is the quantity supplied. There is only one supplier so that the firm is a monopolist. The costs of producing q units are given by the cost function

$$C(q) = cq, \qquad c \geq 0.$$

The profit of producing q units is then given by

$$\pi(q) = P(q)q - C(q).$$

Suppose for a moment that the project is good, i.e. that demand is high. Then the maximal revenue to the firm is given by,

$$\begin{aligned} R_g &= \max_q \left\{ \int_0^\infty e^{-rt} \pi(q) dt \right\} \\ &= \max_q \{ \pi(q) \tfrac{1}{r} \}. \end{aligned}$$

Solving for q using the first order condition yields the optimal output level $q^* = \frac{Y-c}{2}$, leading to the maximal profit stream

$$U^H = \tfrac{1}{r}[P(q^*)q^* - C(q^*)]. \tag{22}$$

If the project is bad it is optimal not to produce at all. Hence, the revenue if demand is zero, U^L, is given by,

$$U^L = 0. \tag{23}$$

$Y = 8$	$r = 0.1$
$c = 5$	$\mu = 4$
$I = 12$	$\lambda = 0.8$
$p_0 = \frac{1}{2}$	

Table 2: Parameter values

In Proposition 2 an analytical result for comparative statics is given. To get some feeling for the magnitude of several effects we consider a numerical example. Consider a market structure as described above with parameter values as given in Table 2. So, the discount rate r is set at 10%. The probability of a correct signal is 0.8 and on average four signals arrive every period.

Based on these parameter values the value function is calculated as function of k and depicted in Figure 2.[6] From this figure one can see that the

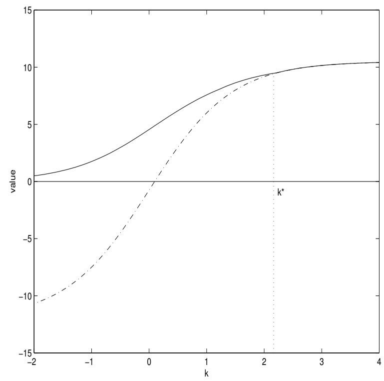

Figure 2: Value function. The dashed line denotes the NPV.

NPV rule prescribes not to invest at the moment the option becomes available ($k = 0$). In fact, in order to invest, the NPV rule demands that the NPV must be positive so that the belief of the firm in high market demand should at least be approximately 0.53 ($k_{NPV} \approx 0.10$). However, our approach specifies that the firm's belief should exceed $p^* \approx 0.96$. This may seem an extremely high threshold, but it implies that the firm invests as soon as $k = 3$, since $k^* \approx 2.23$. The NPV rule prescribes that, in absence

[6]In interpreting Figure 2, notice that realizations of k are discrete, although k^* can be any real number (see Footnote 5).

of l-signals, only one h-signal is needed, while under our approach the firm invests after three h-signals (net from l-signals). From eq. (19) it is obtained that the probability of investing in a bad project while using the optimal approach equals $P^{(k^*)}(0) = 0.00156$. Application of the NPV rule gives $P^{(k_{NPV})}(0) = 0.25$. Hence, the probability of making a wrong decision using the optimal approach is negligible, while it is reasonably large when the NPV rule is used. The other error measure, $\tilde{P}_k^{k^*}(\cdot)$, is depicted in Figure 3 for different values of T. One observes that the error of the second type

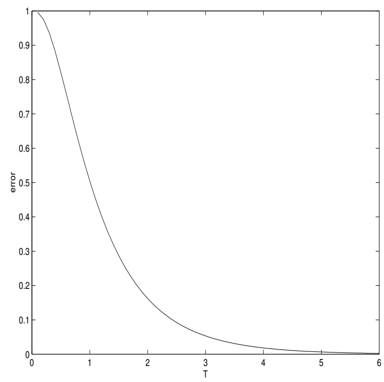

Figure 3: Probability that investment has not taken place before time T given that the project is good.

converges to zero fast. The probability of not having invested by period 6 given that the project is good is already negligible.

Using the same parameters we can see how the critical value k^* changes with λ. From Proposition 2 we can conclude that the critical level for the believe in a good project increases with the quality of the signal λ, as one can also see in the left-hand panel of Figure 4. If λ is higher, then the informativeness of a signal is higher. So, it is more attractive for the firm to demand a higher certainty about the goodness of the market. This belief however, is reached after fewer signals as can be seen from the right-hand panel of Figure 4.

If one takes $Y = 50$, $c = 10$, $I = 500$, $\lambda = 0.8$, $r = 0.1$ and $\mu = 7$, one obtains $p_{NPV} = 0.125$. Since $p_0 = 1/2$ this implies $k_{NPV} < 0$. Hence, the firm invests immediately at time 0 if it applies the NPV rule. So, if the project is bad, the firm invests in the bad project with probability 1. Applying our decision rule gives $p^* = 0.842$, implying that the firm invests

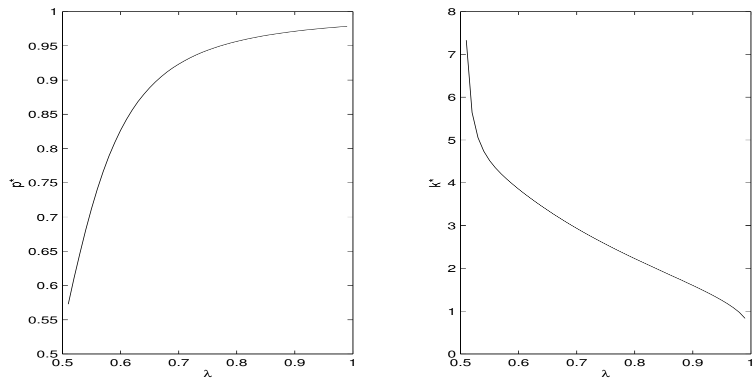

Figure 4: Comparative statics for λ.

if $k = 2$. The probability of a wrong decision then becomes $P^2(0) = 0.06$. Again, our approach greatly reduces this probability compared to the NPV rule.

Consider an example where $U^H = 50$, $I = 30$, $r = 0.1$ and $p_0 = 0.5$. First, we consider the situation where the project is good. Using the conditional first passage time density in eq. (21) one can calculate the expected time until investment takes place as a function of μ and λ, cf. Figures 5 and 6. One can see that both functions are not continuous and the expected

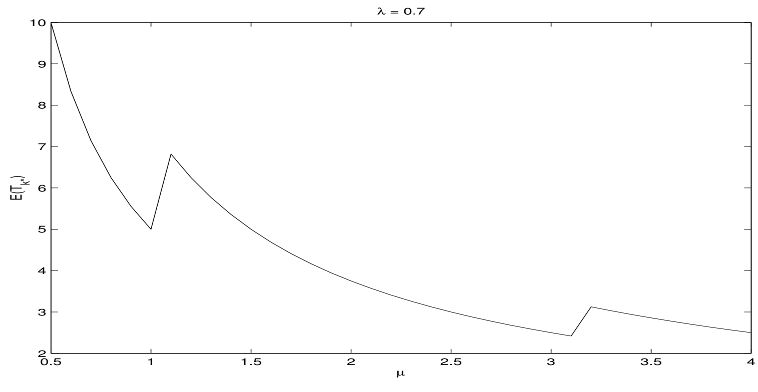

Figure 5: Comparative statics of expected time of investment given a good project for μ with $\lambda = 0.7$ fixed.

time of investment is not monotonic with respect to μ. This stems from the fact that the realisations of k are discrete. Hence, for certain combinations of μ and λ, the threshold jumps from $\lceil k^* \rceil$ to $\lceil k^* \rceil + 1$. If p^* increases in μ (as

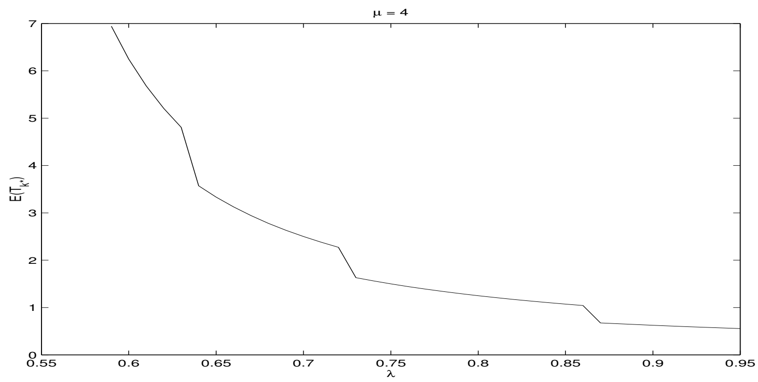

Figure 6: Comparative statics of expected time of investment given a good project for λ with $\mu = 4$ fixed.

it usually does), k^* is also increasing in μ. If, as a result, $\lceil k^* \rceil$ increases with unity, one additional good signal (in excess of bad signals) is needed before it is optimal to undertake the project. This implies that the expected time before investment jumps upwards. Immediately after a jump, the expected time decreases continuously with μ, as intuition suggests, until the threshold jumps again.

Concerning the comparative statics with respect to λ we already observed that an increase in p^* can lead to a decrease in k^*. This implies that for certain values of λ the threshold $\lceil k^* \rceil$ decreases with unity. As soon as this happens, there is a downward jump in the expected time of investment. So, for λ the discreteness of k works in the same direction as the increase of the quality of the signals.

We also analyse the comparative statics of the probability of investment before time $T = 20$ with respect to the parameters μ and λ using the unconditional first passage time density in eq. (20), cf. Figure 7. One can see that this probability is not monotonically increasing in μ and λ. Particularly, one can see from Figure 8 that, taking $\lambda = 0.7$, the comparative statics for μ are both non-continuous and non-monotonic. The explanation for this behaviour is the same as for the comparative statics of the expected time of investment given a good project. Note, however, that th increase in the probability of investment after each jump increases less fast. This is due to the fact that a higher threshold needs to be reached.

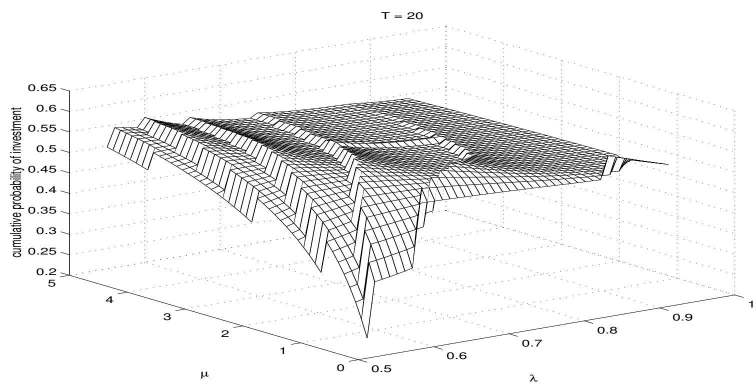

Figure 7: Comparative statics of the probability of investment before $T = 20$ for λ and μ.

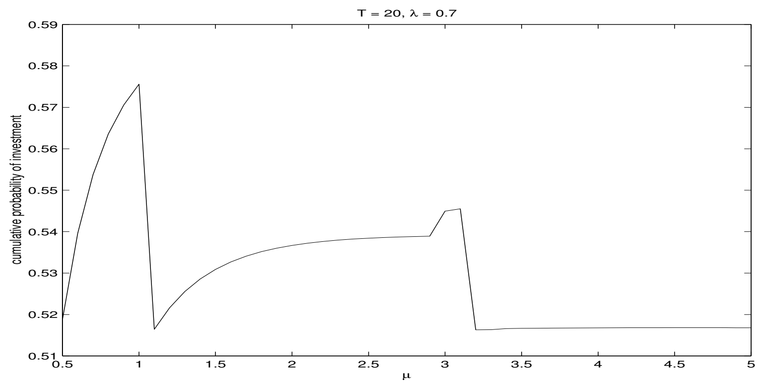

Figure 8: Comparative statics of the probability of investment before $T = 20$ with $\lambda = 0.7$ for μ.

6 Conclusions

In this paper a situation was analysed where a firm has the opportunity to invest in a project. Initially, the profitability of the project is unknown, but as time passes the firm receives signals about the profitability of the investment. There are two types of signals: one type indicating the project to be profitable and the other type indicating it to be unprofitable. The present paper differs from the standard literature on investment under uncertainty (see Dixit and Pindyck (1994)) in that uncertainty diminishes in the course of time. The firm has a – subjective – a priori belief about the profitability of the project. A posterior belief about the profitability is obtained in a

Bayesian way each time a signal arrives. It turns out that it is optimal for the firm to invest as soon as its belief in a profitable project exceeds a certain critical level. An analytical expression for this critical level is provided and it is seen that this level depends crucially on the reliability and the quantity of the signals and the firm's discount rate. Given the initial belief in a good project the critical level can be translated in a number of signals indicating a good project net from signals indicating a bad project. In other words, from the critical belief it can be derived how many "good" signals in excess of "bad" signals are needed before it is optimal for the firm to invest.

An interesting extension of the present model is to look at what happens when the firm is not a monopolist, but if there are rivalling firms to invest in the same project. This requires using game theoretic concepts in the present setting. In the standard real options framework such an analysis has been carried out by e.g. Huisman (2001), Lambrecht and Perraudin (1999) and Boyer et al. (2001).

Another topic for further research is to include costs for receiving the signals. In this way one obtains a model for optimal sampling, closely related to statistical decision theory. For the standard real options model this has been done by Moscarini and Smith (2001). An interpretation of such a model could be that a firm can decide once about the intensity and quality of R&D, leading to a combination of μ and λ. If one assumes a cost function for μ and λ one can solve a two stage decision problem where the first stage consists of determining R&D intensity and quantity, while the second stage consists of the timing of investment. In fact, this paper solves the second stage. With simulations one could solve the first stage, using our analysis as an input. Since the value stream depends on the (rather complicated) first passage density of the threshold, analytical results can probably not be found. One could even try to extend the model to a situation where the firm can continuously adjust its R&D intensity and quality, adding again to the complexity of the problem.

Finally, one could extend the idea of diminishing uncertainty. For instance, to look at a market where two firms are competing, with imperfect information about each each other's cost functions. Gradually, firms receive signals on each other's behaviour from which they infer the opponent's cost function, which then influences their strategies.

Appendix

A Proof of Proposition 1

Denote the denominator of Ψ by $d(\Psi)$. Analogously, we denote the numerator of Ψ by $n(\Psi)$. Using $\beta_1 \geq 1-\lambda$, it is easy to derive that $\Psi < 1$. If $r=0$, it holds that $\beta_1 = \lambda$. Therefore,

$$n(0) = \lambda\mu^2(1-\lambda) - \mu\lambda(1-\lambda)\mu = 0.$$

Furthermore, using that $\beta_1 \geq (1-\lambda)$ and $\frac{\partial\beta_1}{\partial r} > 0$, it can be obtained that $\frac{dn(\Psi)}{dr} > 0$. So, $\Psi > 0$ and p^* is a well-defined probability. Furthermore, since $\Psi < 1$, it holds that

$$\begin{aligned} p^* &= \frac{1}{\Psi(\frac{U^H}{I} - 1) + 1} \\ &> \frac{I}{U^H} = p_{NPV}. \end{aligned}$$

□

B Proof of Proposition 2

Simple calculus gives the result for U^H and I. To prove the proposition for r, μ, and λ, let us first derive the comparative statics of β_1 for these parameters. First, take r. The total differential of $\mathcal{Q}$ with respect to r is given by

$$\frac{\partial\mathcal{Q}}{\partial\beta_1}\frac{\partial\beta_1}{\partial r} + \frac{\partial\mathcal{Q}}{\partial r} = 0.$$

From Figure 1 one can see that $\frac{\partial\mathcal{Q}}{\partial\beta_1} > 0$. Furthermore, $\frac{\partial\mathcal{Q}}{\partial r} = -\frac{\beta_1}{\mu} < 0$. Hence, it must hold that $\frac{\partial\beta_1}{\partial r} > 0$. In a similar way one obtains $\frac{\partial\beta_1}{\partial\mu} < 0$ and $\frac{\partial\beta_1}{\partial\lambda} > 0$.

The numerator and denominator of Ψ can be written in the following form

$$\begin{aligned} n(\Psi) &= \eta(r,\mu,\lambda) - 2\mu(1-\lambda)\zeta(r,\mu,\lambda), \\ d(\Psi) &= \eta(r,\mu,\lambda) - 2\mu(1-\lambda)\nu(r,\mu,\lambda), \end{aligned}$$

where

$$\eta(r,\mu,\lambda) = \beta_1(r+\mu)(r+\mu\lambda) - \mu\lambda(1-\lambda)(r+\mu(\beta_1+\lambda)),$$
$$\zeta(r,\mu,\lambda) = \beta_1(r+\mu) - \mu\lambda(1-\lambda),$$
$$\nu(r,\mu,\lambda) = r(1-\lambda) + \mu(1-\lambda)^2.$$

Since $\Psi > 0$, this implies that to determine the sign of the derivative of Ψ with respect to one of the parameters, one only needs to compare the respective derivatives of $\zeta(\cdot)$ and $\nu(\cdot)$. Note that

$$\begin{aligned}\frac{\partial\zeta(\cdot)}{\partial r} &= \beta_1 + \frac{\partial\beta_1}{\partial r} r > \beta_1 \\ &\geq 1-\lambda = \frac{\partial\nu(\cdot)}{\partial r}.\end{aligned} \tag{24}$$

Hence, $\frac{\partial\Psi}{\partial r} < 0$ and $\frac{\partial p^*}{\partial r} > 0$.

For λ a similar exercise can be done, yielding

$$\begin{aligned}\frac{\partial\zeta(\cdot)}{\partial\lambda} &= \mu(2\lambda-1) + (r+\mu)\frac{\partial\beta_1}{\partial\lambda} > 0 \\ &> -\big(r + 2\mu(1-\lambda)\big) = \frac{\partial\nu(\cdot)}{\partial\lambda}.\end{aligned} \tag{25}$$

Hence, $\frac{\partial p^*}{\partial\lambda} > 0$. □

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