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BALANCEDNESS OF THE CLASS OF INFINITE PERMUTATION GAMES AND RELATED CLASSES OF GAMES

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Recently it is proved that all infinite assignment games have a non-empty core. Using this fact, and a technique suggested by L. S. Shapley for finite permutation games, we prove similar results for infinite permutation games. Infinite transportation games can be interpreted as a generalization of infinite assignment games. We show that infinite transportation games are balanced via a related assignment game. By using certain core elements of infinite transportation games it can be shown that infinite pooling games have a non-empty core.

Keywords: Cooperative games; infinite programs; core.

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1. Introduction

Pooling and permutation situations can be seen as special types of one-sided market models. They are related to transportation and assignment problems, respectively, which can be interpreted as two-sided market models.

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An assignment situation is a problem in which agents of two types have to be matched to each other. Shapley and Shubik (1972) introduced cooperative assignment games when both sets of agents are finite and they proved that these games have a non-empty core. Llorca *et al.* (2004) consider cooperative games related to infinite assignment situations, where there are a countably infinite number of agents of each type, and show that there exist core elements.

In a permutation situation with n agents, each player has a job and owns one machine. They can process their jobs on their own machines, or they can cooperate rearranging the jobs and machines to reduce the total costs. The second situation can be described by a cooperative game with transferable utility ($TU\,game$). The class of finite permutation games, introduced by Tijs *et al.* (1984) as cost games, has been shown in several ways to be balanced. In Tijs *et al.* (1984) the result is obtained through a theorem due to Birkhoff (1946) and von Neumann (1953) on double stochastic matrices. The proof given by Curiel and Tijs (1985) is based on a result of Gale (1984) about the existence of equilibrium in discrete exchange economies with money. In Klijn *et al.* (2000) an alternative proof of balancedness of finite permutation games is provided relating the core conditions with the properties of envy-freeness and Pareto efficiency. As Tijs *et al.* (1984) point out, it is also possible to show balancedness exploiting a connection, suggested in 1984 by Shapley in a personal communication, between permutation and assignment games. Quint (1996) shows that all core allocations of a finite permutation game can be obtained in this way.

A transportation problem describes a situation in which demands at several locations for a certain good have to be covered by supplies from other points. The transport of one unit of the good from a supply point to a demand point generates a nonnegative profit. The goal of the suppliers and demanders is to maximize the total profit from transport. The corresponding transportation games can be seen as an extension of assignment games. In Sánchez-Soriano *et al.* (2001b) it is shown that finite transportation games are balanced.

A pooling situation appears when a set of agents which own property rights of several interchangeable commodities decide to cooperate by pooling their property rights in order to achieve as much profit as possible. Potters and Tijs (1987) introduce two types of pooling games arising from pooling situations and prove that both games have non-empty core. It can be shown that the second class of pooling games and transportation games, as defined in Sánchez-Soriano *et al.* (2001b), coincide.

In the next section the relations between assignment and permutation situations are analyzed, and related infinite permutation games are shown to be balanced. In Sec. 3 we study infinite transportation situations and find core elements for the corresponding cooperative TU games. Section 4 is devoted to pooling situations when the number of agents is countable infinite, and it is proved that the core is non-empty. Until this point we have only considered games with a finite value, in order to take into account the possibility of infinite value some comments are given in Sec. 5.

2. Balancedness of Infinite Permutation Games

We start by shortly recalling the finite case. Consider a situation where there are $N = \{1, \ldots, n\}$ agents, each agent $i \in N$ has one job to be processed and one machine to process a job. The machines cannot process more than one job and all jobs have to be done. The value of player i to process his job on the machine owned by agent $\pi(i)$ is the nonnegative reward $a_{i\pi(i)}$, where $\pi : N \to N$ is a permutation in the set of all $N-$permutations, Π_N. Thus a permutation situation can be described by the pair $(N, \mathbf{A})$ where $\mathbf{A}$ is the nonnegative reward matrix, $\mathbf{A} = [a_{i\pi(i)}]_{i \in N}$. From this situation Tijs *et al.* (1984) introduce a cooperative TU game with player set $N = \{1, \ldots, n\}$ and for each non-empty coalition $S \subset N$ the worth, $v^P(S)$, is defined by

$$v^P(S) = \max_{\pi \in \Pi_S} \sum_{i \in S} a_{i\pi(i)}$$

where Π_S is the class of all $N-$permutations with $\pi(i) = i$ for all i outside S. The game (N, v^P) is a so-called finite permutation game.

Now we take into account a situation in which there are two types of agents, for example, the owners of the machines (the suppliers, since they can provide a service) and those who have to process jobs on these machines (the demanders, as they have needs to be covered). Denote by M and W these two finite and disjoint sets. When agent $i \in M$ is matched to agent $j \in W$ this gives a nonnegative profit of a_{ij}. This assignment situation can be represented by $(M, W, \mathbf{A})$. In 1972, Shapley and Shubik (1972) introduced cooperative assignment games associated to the matching in which each agent $i \in M$ is coupled to at most one agent $j \in W$ and vice versa. The corresponding assignment game $(M \cup W, v^A)$ is a TU game with player set $M \cup W$. The worth, $v^A(S)$, for a coalition $S \subset N$ is the maximal value that it can obtain by matching its members if $S \cap M \neq \emptyset$ and $S \cap W \neq \emptyset$, otherwise $v^A(S) = 0$. The maximal total value of paired agents in S, can be determined by the following integer program

$$
\begin{aligned}
\max \quad & \sum_{i \in M \cap S} \sum_{j \in W \cap S} a_{ij} x_{ij} \\
\text{s.t.} : \quad & \sum_{i \in M \cap S} x_{ij} \leq 1, \quad \text{for all } j \in W \cap S \\
& \sum_{j \in W \cap S} x_{ij} \leq 1, \quad \text{for all } i \in M \cap S \\
& x_{ij} \in \{0, 1\}, \quad \text{for all } i \in M \cap S, \ j \in W \cap S.
\end{aligned}
\tag{1}
$$

with value $v^A(S)$.

In cooperative game theory we are interested to know how to share the joint profit among the cooperating agents. The core of an assignment game, $C(v^A)$, is the set of distributions of $v^A(N)$ upon which each coalition S will receive at least as much it can obtain on its own. In assignment games core elements are easy to

find because $C(v^A)$ equals the set of optimal solutions of the dual of problem (1), relaxing the integer conditions by nonnegativity.

Curiel (1997) explores the relationship between assignment and permutation games, showing that every assignment game is a permutation game. It is also possible, as Shapley suggested, to find an assignment game related to a given permutation game. The procedure indicated by Shapley goes as follows. Let (N, v^P) be a permutation game with reward matrix $\mathbf{A} = [a_{ij}]_{i \in N, j \in N}$. Consider the assignment game $(M \cup W, v^{AP})$ obtained by duplicating the agents in two sets: the owners of the machines, M, and the owners of the jobs, W. For all group of players S, it is easy to check that its worth in the permutation game can be obtained from this assignment game $v^P(S) = v^{AP}(M_S \cup W_S)$, where $M_S = M \cap S$ and $W_S = W \cap S$.

Now we introduce infinite permutation situations $(N, \mathbf{A})$ where there is a countable infinite number of agents and, therefore, an infinite number of jobs and machines. The corresponding infinite permutation game $(\mathbb{N}, v^P)$ is a cooperative TU game with a countably infinite set of players. As in the finite case, given an infinite permutation situation and its corresponding game we can introduce a related infinite assignment game $(M \cup W, v^{AP})$, with $M = W = \mathbb{N}$ and $v^P(S) = v^{AP}(M_S \cup W_S)$.

In assignment situations we are interested in how to match, e.g., a set of supply points to a set of demand points such that we obtain the maximal total profit when each supply point is assigned to at most one demand point and vice versa. Consider a problem with an infinite number of production techniques that can be programmed to produce a certain product which can be chosen from an infinite number of possible designs. The marketing policy leads to produce unique pieces. This is an assignment situation in which there is an infinite number of production techniques (suppliers) and an infinite number of patterns (demanders). The goal is to achieve the maximal total reward from matching the techniques with the patterns. Infinite assignment situations and related games are introduced in Llorca *et al.* (2004). They prove that, in case the value for the grand coalition N is finite and in case it is infinite, these games are balanced. Using the result for assignment games, we will show the non-emptiness of the core of infinite permutation games. We describe the procedure through an example based on Llorca *et al.* (2004).

Example 1. Let $(N, \mathbf{A})$ be the infinite permutation situation with a countable infinite number of agents, each of whom can provide a service and at the same time needs to be served. The nonnegative reward matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \dots \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

contains the profits a_{ij} when agent i is served by j. Let $(\mathbb{N}, v^P)$ be the corresponding infinite permutation game. We can consider each agent in $\mathbb{N}$ as a supplier and as well as a demander. Thus we can separate all players in two parts and construct a related infinite assignment game $(M \cup W, v^{AP})$ where $M = W = \mathbb{N}$. The element $(\mathbf{u}; \mathbf{w}) = (1, \frac{1}{2}, \frac{1}{4}, \ldots; 1, \frac{1}{2}, \frac{1}{4}, \ldots)$, belongs to

$$
C(v^{AP}) = \left\{ (\mathbf{u}; \mathbf{w}) \in \mathbb{R}^M \times \mathbb{R}^W \left|
\begin{array}{l}
\sum_{i \in M} u_i + \sum_{i \in W} w_i = v^{AP}(M \cup W) \text{ and} \\
\sum_{i \in M_S} u_i + \sum_{i \in W_S} w_i \geq v^{AP}(S) \\
\text{for all } S \subset M \cup W, \ S \neq \emptyset
\end{array}
\right. \right\}
$$

i.e. the core of this infinite assignment game. Since we have split into two parts each agent in the original permutation situation, if we define $x_i = u_i + w_i$, for all $i \in N$, we obtain $\mathbf{x} = (2, 1, \frac{1}{2}, \ldots)$ an element of the core in the infinite permutation game, as the next theorem states.

Theorem 2. *Let $(N, \mathbf{A})$ be an infinite permutation situation, with (N, v^P) and $(\mathbb{N} \cup \mathbb{N}, v^{AP})$ the corresponding infinite permutation game and the related assignment game, respectively. Then $C(v^P) \neq \emptyset$.*

Proof. According to Theorem 4.1 in Llorca *et al.* (2004) there exists $(\mathbf{u}; \mathbf{w}) \in C(v^{AP})$. Define $x_i = u_i + w_i$, for all $i \in N$, then

$$
\sum_{i \in N} x_i = \sum_{i \in N} u_i + \sum_{i \in N} w_i = v^{AP}(\mathbb{N} \cup \mathbb{N}) = v^P(N),
$$

and

$$
\sum_{i \in S} x_i = \sum_{i \in S} (u_i + w_i) = \sum_{i \in S} u_i + \sum_{i \in S} w_i \geq v^{AP}(M_S \cup W_S) = v^P(S),
$$

for all $S \subset N$ where the inequality holds because $(\mathbf{u}; \mathbf{w})$ is in the core of the infinite assignment game. ☐

3. Owen Vectors for Infinite Transportation Games

A transportation situation describes a problem in which demands at different locations for a certain commodity want to be covered by supplies from other points. The transport of one item of the indivisible good from a supply location i to a demand point j generates a nonnegative profit of t_{ij}. The goal of the suppliers, P, and demanders, Q, is to obtain the maximal profit (if it exists), transporting from origins to destinations as much as possible, with the constraints on supplies and demands. A transportation situation can be represented by $(P, Q, \mathbf{T}, \mathbf{s}, \mathbf{d})$ where $\mathbf{s} = (s_i)_{i \in P}$ and $\mathbf{d} = (d_j)_{j \in Q}$ are the supply and demand positive integer vectors, respectively. Infinite transportation situations arise when the number of the two types of agents (demanders and suppliers) is countably infinite. Infinite assignment situations can be seen as a special case of infinite transportation situations when all supplies and demands are 1.

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Related to such an infinite transportation situation of an indivisible good we define an infinite transportation game (N, v^T) with player set $N = P \cup Q$ where $P = Q = \mathbb{N}$. As in assignment games, the worth of coalition S equals zero if $S \cap P = \emptyset$ or $S \cap Q = \emptyset$, and the maximal total value of transport among agents in S can be determined by the program

$$
\begin{aligned}
\sup \quad & \sum_{i \in P \cap S} \sum_{j \in Q \cap S} t_{ij} x_{ij} \\
\text{s.t.} : \quad & \sum_{j \in Q \cap S} x_{ij} \leq s_i, \quad \text{for all } i \in P \cap S \\
& \sum_{i \in P \cap S} x_{ij} \leq d_j, \quad \text{for all } j \in Q \cap S \\
& x_{ij} \in \mathbb{Z}_+, \quad \text{for all } i \in P \cap S, \ j \in Q \cap S.
\end{aligned}
\tag{2}
$$

with value $v^T(S)$, otherwise.

Given an infinite transportation situation, $(P, Q, \mathbf{T}, \mathbf{s}, \mathbf{d})$, we can construct a related infinite assignment situation splitting each supply agent $i \in P$ into s_i supply points (with 1 unit of supply), each demand point $j \in Q$ is divided into d_j different players (with demand 1 item), and the per unit profit for all these agents is t_{ij}. Formally, the related assignment situation $(M, W, \mathbf{A})$ has a countably infinite set of supply points $M = \{ir \,|\, i \in P, r \in \{1, \ldots, s_i\}\}$, a countably infinite set of demanders $W = \{jc \,|\, j \in Q, c \in \{1, \ldots, d_j\}\}$, and $a_{ir, \, jc} = t_{ij}$ for all $ir \in M$, $jc \in W$.

The following result states that each solution for the transportation situation determines a solution for the assignment one (and vice versa) by splitting (merging) the corresponding agents and both solutions have the same value. This lemma is given without proof because it resembles that in Sánchez-Soriano *et al.* (2001*a*), Lemma 4.1.

Lemma 3. *Let* $(P, Q, \mathbf{T}, \mathbf{s}, \mathbf{d})$ *be an infinite transportation situation, with* $(\mathbb{N} \cup \mathbb{N}, v^T)$ *and* $(\mathbb{N} \cup \mathbb{N}, v^{AT})$ *the corresponding infinite transportation game and the related assignment game, respectively. Then, for all S,*

$$
v^T(S) = v^{AT}(S^{AT})
$$

where $S^{AT} = \{ir | i \in P \cap S, r \in \{1, \ldots, s_i\}\} \cup \{jc | j \in Q \cap S, c \in \{1, \ldots, d_j\}\}$.

This result also allows us to replace the integer condition by the nonnegativity condition in (2) because it holds for assignment problems. Thus the program

$$
\begin{aligned}
\inf \quad & \sum_{i \in P} s_i u_i + \sum_{j \in Q} d_j w_j \\
\text{s.t.} \quad & u_i + w_j \geq t_{ij} \\
& u_i, w_j \geq 0, \quad \text{for all } i \in P, \ j \in Q
\end{aligned}
\tag{3}
$$

with value $v_d^T(N)$, is the corresponding dual problem for the grand coalition $N = \mathbb{N} \cup \mathbb{N}$, and $O_d^T(N)$ denotes the set of its optimal solutions. An infinite number

of variables and constrains are present in programs (2) and (3) so a duality gap between their values may arise. The next proposition establishes that in infinite transportation games there is no duality gap and there exist optimal dual solutions. Although it is only shown for the grand coalition this holds for all $S \subset N$.

Proposition 4. *Let* $(P, Q, \mathbf{T}, \mathbf{s}, \mathbf{d})$ *be an infinite transportation situation, and* $(\mathbb{N} \cup \mathbb{N}, v^T)$ *the corresponding infinite transportation game. Then*

$$v^T(N) = v_d^T(N) \ and \ O_d(N) \neq \emptyset.$$

Proof. Theorem 4.1 in Llorca *et al.* (2004) states that the core of infinite assignment games is always non-empty. So, let $(\mathbf{u}; \mathbf{w}) \in C(v^{AT})$. Then, $u_{ir} + w_{jc} \geq t_{ij}$ for all $ir \in M$, $jc \in W$. Therefore,

$$\sum_{r=1}^{s_i} \sum_{c=1}^{d_j} (u_{ir} + w_{jc}) = d_j \sum_{r=1}^{s_i} u_{ir} + s_i \sum_{c=1}^{d_j} w_{jc} \geq \sum_{r=1}^{s_i} \sum_{c=1}^{d_j} t_{ij} = s_i d_j t_{ij},$$

for all $i \in P$, $j \in Q$. Then dividing by $s_i d_j$ one obtains that

$$\sum_{r=1}^{s_i} u_{ir}/s_i + \sum_{c=1}^{d_j} w_{jc}/d_j \geq t_{ij}.$$

Define $\bar{u}_i := \sum_{r=1}^{s_i} u_{ir}/s_i$ and $\overline{w}_j := \sum_{c=1}^{d_j} w_{jc}/d_j$, which can be interpreted as average prices. Then $\bar{u}_i \geq 0$, $\overline{w}_j \geq 0$, and $\bar{u}_i + \overline{w}_j \geq t_{ij}$ for all $i \in P$, $j \in Q$. Hence,

$$v^T(N) = v^{AT}(N^{AT}) = v_d^{AT}(N^{AT}) = \sum_{i \in P} s_i \bar{u}_i + \sum_{j \in Q} d_j \overline{w}_j \geq v_d^T(N)$$

where the first equality follows from Lemma 3, the second one from the absence of duality gap in infinite assignment problems, the third one because $(\mathbf{u}; \mathbf{w})$ is an optimal solution of the dual assignment problem for the grand coalition, and the inequality follows because $(\overline{\mathbf{u}}; \overline{\mathbf{w}})$ is a feasible dual solution in the transportation situation. According to Theorem 3.1 in Anderson and Nash (1987) we know that weak duality holds and therefore

$$v^T(N) = \sum_{i \in P} s_i \bar{u}_i + \sum_{j \in Q} d_j \overline{w}_j = v_d^T(N).$$

So we can conclude that there is no duality gap and $(\overline{\mathbf{u}}, \overline{\mathbf{w}})$ is a dual optimal solution in the transportation situation for the grand coalition. □

An element of the so-called Owen set,

$$\text{Owen}(P, Q, \mathbf{T}, \mathbf{s}, \mathbf{d}) = \left\{ \mathbf{z} \in \mathbb{R}^N \ \middle| \ \begin{array}{l} \exists (\mathbf{u}; \mathbf{w}) \in O_d^T(N) \text{ such that } z_k = s_k u_k \text{ if } k \in P \\ \text{and } z_k = d_k w_k \text{ if } k \in Q \end{array} \right\},$$

turns out to be an element of the core of the infinite transportation game, as the next theorem states.

Theorem 5. *Let $(P, Q, \mathbf{T}, \mathbf{s}, \mathbf{d})$ be an infinite transportation situation and $(\mathbb{N} \cup \mathbb{N}, v^T)$ be the corresponding infinite transportation game. Then*

$$C(v^T) \supset \mathrm{Owen}(P, Q, \mathbf{T}, \mathbf{s}, \mathbf{d}) \neq \emptyset.$$

Proof. Let $\mathbf{z} \in \mathrm{Owen}(P, Q, \mathbf{T}, \mathbf{s}, \mathbf{d})$, which is always non-empty because $O_d^T(N) \neq \emptyset$, and let $(\mathbf{u}; \mathbf{w}) \in O_d^T(N)$ be such that $z_k = s_k u_k$ if $k \in P$ and $z_k = d_k w_k$ if $k \in Q$. Then

$$\sum_{k \in N} z_k = \sum_{i \in P} s_i u_i + \sum_{j \in Q} d_j w_j = v_d^T(N) = v^T(N),$$

where the last equality follows from the no gap result in Proposition 4.

Take a coalition $S \subset N$, $S \neq \emptyset$. If $S \subset P$ or $S \subset Q$ then $\sum_{k \in S} z_k \geq 0 = v^T(S)$ because $z_k \geq 0$ for all $k \in N$. When $S \cap P \neq \emptyset$ and $S \cap Q \neq \emptyset$, we have that $u_i + w_j \geq t_{ij}$ for all $i \in P$, $j \in Q$, and this is in particular true for all $i \in S \cap P$, $j \in S \cap Q$. Therefore stability

$$\sum_{k \in S} z_k = \sum_{i \in S \cap P} s_i u_i + \sum_{j \in S \cap Q} d_j w_j \geq v^T(S)$$

holds, for all $S \subset N$. Thus we obtain that $\mathbf{z}$ belongs to $C(v^T)$. $\qquad\square$

4. Core Elements of Infinite Pooling Games

Again we start with the finite case. A pooling situation arises when a set of agents, who own property rights of several interchangeable commodities, decide to cooperate sharing these property rights. Their goal is to maximize the total profit. Thus, a finite pooling situation can be described by a $6-$tuple $(P, M, \mathbf{d}, \mathbf{s}, \mathbf{O}, \mathbf{E})$, where P is the finite set of agents, M the finite set of commodities, $\mathbf{s} \in \mathbb{Z}_{++}^M$ and $\mathbf{d} \in \mathbb{Z}_{++}^P$, $\mathbf{E} \in \mathbb{R}_+^{P \times M}$ and $\mathbf{O} \in \mathbb{Z}_+^{P \times M}$ with $\sum_{i \in P} O_{ij} = s_j$. A feasible distribution for a pooling situation is a matrix $\mathbf{X} \in \mathbb{Z}_+^{P \times M}$ with integer entries and such that $\sum_{i \in P} x_{ij} \leq s_j, \forall j \in M$, and $\sum_{j \in M} x_{ij} \leq d_i, \forall i \in P$. To find an optimal allocation of the commodities is to obtain a feasible distribution which maximizes the total profit: $\sum_{i \in P, j \in M} e_{ij} x_{ij}$.

Associated with a pooling situation, Potters and Tijs (1987) introduced a finite cooperative TU game (N, v^O) with player set $N = P$ and the value for a coalition $S \subset N$ is

$$
\begin{aligned}
\max \quad & \sum_{i \in P \cap S} \sum_{j \in M \cap S} e_{ij} x_{ij} \\
\text{s.t}: \quad & \sum_{j \in M \cap S} x_{ij} \leq d_i, \quad \text{for all } i \in P \cap S \\
& \sum_{i \in P \cap S} x_{ij} \leq \sum_{i \in P \cap S} O_{ij}, \quad \text{for all } j \in M \cap S \\
& x_{ij} \in \mathbb{Z}_+, \quad \text{for all } i \in P \cap S, \ j \in M \cap S
\end{aligned}
\tag{4}
$$

what the members in S can gain by pooling their rights.

Going to infinite pooling situations and their corresponding games, they occur when the number of players and the set of commodities are considered countably infinite. Just as in the finite case, the value $v^O(S)$ of coalition S can be determined by the linear program (4), replacing the maximum by the supremum since the sets P and M are countable infinite.

We can consider each player formed by two parts: As a demander of a service and as the owner of property rights of several commodities. Therefore, if we take into account the total property rights $\sum_{i \in P} O_{ij}$ for each commodity $j \in M$ and split the player set into $P \cup M$, we will deal with an infinite transportation situation $(P, M, \mathbf{T}, \mathbf{d}, \mathbf{s})$ related to the infinite pooling situation $(P, M, \mathbf{d}, \mathbf{s}, \mathbf{O}, \mathbf{E})$ where $\mathbf{T} = \mathbf{E}$ and $s_j = \sum_{i \in P} O_{ij}$, for all $j \in M$. The corresponding infinite transportation game $(\mathbb{N} \cup \mathbb{N}, v^{TO})$ satisfies

$$v^O(S) = v^{TO}(\{S \cap P\} \cup \{S \cap M\}), \tag{5}$$

where $S \cap P = S$ and $S \cap M := \{j \in M \mid \sum_{i \in P \cap S} O_{ij} = s_j > 0\}$.

Theorem 6. *Let $(\mathbb{N}, \mathbb{N}, \mathbf{d}, \mathbf{s}, \mathbf{O}, \mathbf{E})$ be an infinite pooling situation, with $(\mathbb{N}, v^O)$ and $(\mathbb{N} \cup \mathbb{N}, v^{TO})$ the corresponding infinite pooling game and the related transportation game, respectively. Then $C(v^O) \neq \emptyset$.*

Proof. According to Proposition 4, there exists $(\mathbf{u}; \mathbf{w}) \in \mathbb{R}^P \times \mathbb{R}^M$ a solution of the dual program for the grand coalition. This element can be seen as a shadow price vector for the resources, d_i for agents in P and $s_j = \sum_{i \in P} O_{ij}$ for all $j \in M$, and it is such that

$$\sum_{i \in P} d_i u_i + \sum_{j \in M} s_j w_j = v_d^{TO}(\mathbb{N} \cup \mathbb{N}) = v^{TO}(\mathbb{N} \cup \mathbb{N}) \tag{6}$$

and

$$\sum_{i \in S \cap P} d_i u_i + \sum_{j \in S \cap M} s_j w_j \geq v_d^{TO}(\{S \cap P\} \cup \{S \cap M\})$$

$$= v^{TO}(\{S \cap P\} \cup \{S \cap M\}), \tag{7}$$

for all $S \subset N$. Since each player $i \in P$ is a demander of a service and, at the same time, he has rights over $j \in M$, let us consider $z_i = d_i u_i + \sum_{j \in M} O_{ij} w_j$ for all $i \in P$. We will show that the vector $\mathbf{z} \in \mathbb{R}^{\mathbf{P}}$, where each coordinate can be interpreted as what agent i receives from his demand and ownership parts, belongs to $C(v^O)$.

$$\sum_{i \in P} z_i = \sum_{i \in P} d_i u_i + \sum_{i \in P} \sum_{j \in M} O_{ij} w_j = \sum_{i \in P} d_i u_i + \sum_{j \in M} \left(\sum_{i \in P} O_{ij} \right) w_j$$

$$= \sum_{i \in P} d_i u_i + \sum_{j \in M} s_j w_j = v^{TO}(\mathbb{N} \cup \mathbb{N}) = v^O(N),$$

where the last two equalities hold from (6) and (5). Further, for each $S \subset N$

$$
\begin{aligned}
\sum_{i \in S \cap P} z_i &= \sum_{i \in S \cap P} d_i u_i + \sum_{i \in S \cap P} \sum_{j \in S \cap M} O_{ij} w_j \\
&= \sum_{i \in S \cap P} d_i u_i + \sum_{j \in S \cap M} \left(\sum_{i \in S \cap P} O_{ij} \right) w_j \\
&= \sum_{i \in P} d_i u_i + \sum_{j \in M} s_j w_j \geq v^{TO}(\{S \cap P\} \cup \{S \cap M\}) = v^O(S),
\end{aligned}
$$

where the inequality holds from (7) and the last equality from (5). $\qquad\square$

5. Concluding Remarks

In this paper we have considered infinite assignment games with finite value. When $v^A(\mathbb{N} \cup \mathbb{N}) = +\infty$, there is no duality gap and the utopia payoff can be used to assure balancedness of infinite assignment games. This allocation $(\mathbf{u}^*, \mathbf{w}^*) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ is obtained giving to each agent what he can expect at most, i.e.

$$
u_i^* = \sup_{j \in W} a_{ij} \forall\, i \in \mathbb{N} \text{ and } w_j^* = \sup_{i \in M} a_{ij} \forall\, j \in \mathbb{N},
$$

and it would play the same role as the optimal dual solutions to prove the non-emptiness of the core in infinite permutation and transportation games with unbounded value.

We have shown the non-emptiness of the core in infinite permutation games using the idea suggested by L. S. Shapley for the finite case. As it was pointed out in the introduction there are other techniques to prove balancedness in finite permutation games. So it remains an open question how they could be adjusted to this infinite context, especially what is related to the Birkhoff-von Neumann result on doubly stochastic matrices.

In Sec. 3 we have analyzed the problem of finding core elements when we have to transport indivisible goods. These results can be applied with infinitely divisible goods when the total supply and demand are infinite, assuming that all suppliers and demanders want to provide or to receive a positive amount. This is due to the fact that this kind of transportation situations can be read as infinite transportation situations with indivisible goods.

References

Anderson, E. J. and Nash, P. [1987] *Linear Programming in Infinite Dimensional Spaces* (New York, Wiley).

Birkhoff, G. [1946] "Tres observaciones sobre el Algebra Lineal," *Revista de la Universidad Nacional de Tucumán* **A5**, 147–151.

Curiel, I. and Tijs S. [1985] "Assignment games and permutation games," *Methods of Operations Research* **54**, 323–334.

Curiel, I. [1997] *Cooperative Game Theory and Applications* (Boston, Kluwer Academic Publishers).

Gale, D. [1984] "Equilibrium in a discrete economy with money," *International Journal of Game Theory* **13**, 61–64.

Klijn, F., Tijs, S. and Hamer, H. [2000] "Balancedness of permutation games and envy-free allocations in indivisible good economies," *Economics Letters* **69**, 323–326.

Llorca, N., Sánchez-Soriano, J., Tijs, S. and Timmer, J. [2004] "The core and related solution concepts for infinite assignment games," *TOP* **12**, 331–350.

Potters, J. A. M. and Tijs, S. [1987] "Pooling: Assignment with property rights," *Methods of Operations Research* **57**, 495–508.

Quint, T. [1996] "On one-sided versus two-sided matching games," *Games and Economic Behavior* **16**, 124–134.

Sánchez-Soriano, J., Llorca, N., Tijs, S. and Timmer, J. [2001a] "Semi-infinite assignment and transportation games," in *Semi-Infinite Programming: Recent Advances*, Goberna, M. A. and López, M. A. (eds.) (Kluwer, Dordrecht), pp. 349–363.

Sánchez-Soriano, J., López, M. A. and García-Jurado, I. [2001b] "On the core of transportation games," *Mathematical Social Sciences* **41**, 215–222.

Shapley, L. S. and Shubik, M. [1972] "The assignment game i: The core," *International Journal of Game Theory* **1**, 111–130.

Tijs, S., Parthasarathy, T., Potters, J. A. M. and Rajendra Prasad, V. [1984] "Permutation games: Another class of totally balanced games," *OR Spectrum* **6**, 119–123.

von Neumann, J. [1953] "A certain zero-sum two-person game equivalent to optimal assignment problem," in *Contributions to the Theory of Games II*, (eds.) Kuhn, H. W. and Tucker, A. W. (Princeton University Press, Princeton), pp. 5–12.