71 Optimal control theory in environmental economics

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1. Introduction

Optimal control theory originated as a mathematical tool to solve problems of dynamic optimization. Applying it to economic problems allows the explicit consideration of time.[2] This makes it suitable for analysing the intertemporal trade-off between current consumption and future pollution or exhaustion of natural resources that is inherent in many environmental problems. Optimal control models consist of an intertemporal objective that must be optimized subject to a set of dynamic equations which specify how some instruments, or control variables, influence the development of the state variables. Examples of control variables in environmental economics are the level of fossil fuel use, the investment in abatement technology and the energy tax rate. State variables are, for instance, the atmospheric concentration of greenhouse gases and the stock of capital. Control theory provides methods of finding optimal levels of the instruments or control variables at each instant of time. Standard references for optimal control theory include Feichtinger and Hartl (1986), Kamien and Schwartz (1991) and Seierstad and Sydsæter (1987). Typically, with the help of Pontryagin's maximum principle,[3] optimality conditions can be formulated in terms of the so-called shadow values of the state variables.

In Section 2 we present a basic model of intertemporal trade-offs and discuss the conclusions that can be drawn from it. Section 3 examines the assumptions that are needed to arrive at these conclusions, such as constant and positive rate of discount and perfect foresight. Section 4 mentions the limitations of optimal control theory in environmental economics. Finally Section 5 reviews some literature in relation to the basic model.

2. A basic model

As regards the application of optimal control theory in environmental economics, it is worthwhile to note that the strands of literature on non-renewable resources (Hotelling, 1931), renewable resources (Gordon, 1954) and environmental resources[4] (Keeler et al., 1972) using optimal control techniques have developed quite separately from each other. Non-renewable and renewable resources are treated elsewhere in this book. The lack of integration in the literature is the more surprising given the analytic similarities between the three, as stressed by Smith (1977) and Dasgupta (1982), among others. These similarities are due to the equivalence of the main problem: to find the optimal trade-off between current and future use of a resource.

Take, for instance, the following optimal control model, where a social planner is supposed to solve the optimization problem:

$$\underset{C}{\text{Max}} \int_0^\infty e^{-rt} U(C(t),S(t))dt \tag{71.1}$$

$$\dot{S}(t) = kG(S(t)) - f(R(t)) \tag{71.2}$$

$$C(t) \leq q(R(t)) \tag{71.3}$$

For notational simplicity, the time dependence of the variables is suppressed hereafter. In this model, the control C is the rate of consumption at t, while the state S is the stock of a natural resource. The variable R denotes the flow of this natural resource and is linked to C by (71.3). The functions $U(\cdot)$, $G(\cdot)$, $q(\cdot)$ and $f(\cdot)$ denote social welfare, the natural regeneration of the resource, production and the transformation of natural resources, respectively. The basic features of any optimal control model are present here: an intertemporal objective function (71.1), to be maximized by the correct choice of a control and a differential equation that describes the development of the state variable (71.2). Additionally the control variable is constrained to be economically meaningful by (71.3). According to whether one wants to discuss a non-renewable, renewable or natural resource, the interpretation and specific characteristics of the functions above change, but the basic model remains the same. Welfare is determined by utility from consumption and costs of extraction, harvest or use. These costs may depend on the resource stock.[5] Moreover, for instance, in the case of environmental resources, the stock itself may influence welfare. In the case of non-renewable resources, the available stock does not grow, so k is zero and $f(R) = R$ is the flow of extraction of the natural resource used for production of $q(R)$ units of output. In the case of renewable resources, k equals unity and $f(R) = R$. The function $q(R)$ denotes the production of goods and services based on resources, for example, fish or timber. In the case of environmental resources, the stock S stands for 'environmental quality'. Accumulated pollution, that is denoted below by P, is another state variable that describes the environmental situation. The basic model is then often further simplified by specifying equation (71.2) as

$$\dot{P} = -\delta P + R \tag{71.2'}$$

This assumes an exponential decay of pollution by natural assimilation. Such a simplification is mathematically convenient. However, it need not be an appropriate description of biological reality.

3. Optimal control theory in economics

We now turn to some remarks on optimal control models in general. Optimal control models applied in economics mostly take the following general form:

$$\underset{u \in U}{\text{Max}} \int_0^\infty e^{-rt} F(x,u)dt \qquad (71.4)$$

subject to

$$\dot{x} = f(x,u) \qquad x(0) = x_0 \qquad (71.5)$$

In the discussion below both x and u are scalars. Thus we restrict the discussion in this section to problems with only one state variable. The discounted stream of some instantaneous objective function $F(x,u)$ is maximized, where the development of the state variable x as steered by the control variable u is described by a differential equation $f(x,u)$. A concept used to derive optimality conditions is the current-value Hamiltonian. For the formulation above, it is given by:

$$H(x,u,\lambda) = F(x,u) + \lambda f(x,u) \qquad (71.6)$$

Here λ is a shadow value or co-state variable and is a function of time. It denotes the increase of the objective function due to a marginal increase of the state variable. At any point in time the decision maker can use the control variable to generate direct contributions to the objective function (represented by the term $F(x,u)$ in the Hamiltonian), or it can use the control variable to change the value of the state variable in order to generate contributions to the objective function in the future. These indirect contributions are measured by the term $\lambda f(x,u)$ in the Hamiltonian. The optimality conditions given by Pontryagin's maximum principle can now loosely be stated as:[6]

If (x^*, u^*) is an optimal solution, then there exists a λ such that

$$u^* = \underset{u \in U}{\text{argmax}}\, H(x^*, u, \lambda) \qquad (71.7)$$

$$\dot{\lambda} = r\lambda - \frac{\partial H(x^*, u^*, \lambda)}{\partial x} \qquad (71.8)$$

Thus the control must maximize the Hamiltonian and one must be able to find a shadow value that satisfies a certain differential equation. The basic model in the previous section has an additional equation that is a constraint on the control variable. Constraints on control or state variables imply that corner solutions should be taken into account. This adds Kuhn–Tucker type of conditions to the optimality conditions.[7] When the optimal value of the control given by (71.7) is inserted in (71.5) and (71.8), we have a system of two differential equations. This is called the modified Hamiltonian system. To be able to solve this system, one needs additional information, namely boundary values for x and λ. For the state variable, its initial value, x_0, is given (in (71.5)). For the shadow value no such initial value exists. In economic applications it frequently holds that the problem has a so-called saddle-point path.[8] This means that there is an optimal path that converges to a steady state. This is a point (x, λ) where the modified Hamiltonian system is stationary, that is, $(\dot{x}, \dot{\lambda}) = (0,0)$. Here the control variable is set such that the value of the state variable does not change, which explains the name steady state. It depends on the characteristics of the specific objective and control functions whether such a point exists, is unique[9] and what are its stability[10] characteristics.[11] If it exists, it gives a boundary value for λ[12] and hence enables us to find one or more candidate optimal solutions.

Some characteristics of the specification of the optimal control problem are worth mentioning. The time horizon is infinite. For trade-off problems, as in the example above, that is probably the most appropriate assumption. In this case, it is unclear where a possibly finite horizon should be located. For other problems, such as those describing a specific project, a finite time horizon may be more suitable.[13] Then the objective function is only evaluated over a finite time period T. It is important to consider the value of the variables at this final time T. The state variables may have some scrap value, since they can either be sold or put to some alternative use. End-point conditions, that describe the properties of an optimal solution at the final time T, should be added to the optimality conditions. Thus some state variables could be required to exactly satisfy a certain value at T, while others should be above some minimum value. To each type of end-point conditions, a different necessary condition on the value of the adjoint shadow value is linked (see, for example, Feichtinger and Hartl, 1986). These are called transversality conditions. They usually provide the additional information that is needed to sort out only a few candidate optimal solutions with the help of the first-order conditions ((71.5), (71.7) and (71.8)).

The instantaneous objective function is discounted at constant and positive rate r. This ensures that the integral converges, provided that $F(x,u)$ is bounded. Especially for environmental economic applications, the assumption of a positive discount rate is disputable. We come back to this in Section 4. When the assumption is dropped, one needs to be careful about the optimality concept used (see Seierstad and Sydsæter, 1987, for alternative optimality concepts).

If the Hamiltonian is concave in (x,u), Pontryagin's maximum principle provides sufficient conditions for an optimum.[14] This holds, for example, for the basic model in Section 2. It avoids one having to check formally whether (x^*,u^*) really is an optimum. When the Hamiltonian is strictly concave, the solution is, moreover, unique. If the Hamiltonian is linear in u, U must be bounded since, otherwise, one cannot solve (71.7). For a one-dimensional problem, U then is a closed interval and u^* can take one of three forms. It equals one of the interval boundaries, or a specific value in between. The latter is called the singular path. Along an optimal path u may jump between these values at certain moments. Such a solution is called a bang–bang solution. It is, for example, relevant for the optimal harvest or renewable resources. For further discussion see, for instance, Feichtinger and Hartl (1986, ch. 3). An application with a linear Hamiltonian in the field of environmental economics is Tahvonen (1997).

4. Optimal control theory and the environment

Descriptive models used by biologists to model environmental problems (for example, global warming, pollution diffusion, acidification) are often stated as a (large) system of differential equations. Biologists rarely optimize these systems. Economists who include ecological systems in an optimal control model usually make severe simplifying assumptions to keep the model tractable. The question always remains whether essential characteristics of the system are lost in this manner.

This section mentions a couple of usual simplifications. First, to apply optimal control theory, one assumes rationality and forward-looking behaviour. Second, optimal control assumes a deterministic setting. Decisions are made under perfect foresight. But it is the uncertainty in many environmental resource problems, for example global warming, that causes problems for policy making. Introducing uncertainty drastically complicates the analysis.[15] Third, to apply Pontryagin's maximum principle requires that the objective function and the functions that describe the dynamics are piecewise continuous.[16] But further concavity assumptions are often made, to ensure sufficiency. For a great many environmental problems these convexity requirements are problematic. Often the development over time of environmental stocks is characterized by so-called threshold values,[17] so that the dynamics of a certain environmental stock are better described by non-convex S-shaped curves. Dasgupta (1982) discusses this issue. Another aspect of environmental problems that is hard to put into an optimal control framework is the delay between cause and effect. This introduces time explicitly in the optimal control framework. Most basic theorems still apply, but their practical application becomes quite complex. As a final point, in the objective function, future and current welfare are compared by discounting at a constant[18] annual rate $(1 + r)$. Two questions arise: Is it fair to discount over generations?[19] If so, what is the appropriate discount rate? There is no definite answer to the first question. The second question is relevant, since relatively small changes in the discount rate may lead to considerable variation in the key policy variables.

These remarks are meant to give the reader a sound scepticism with respect to policy implications. The important contribution of optimal control theory models is that they provide insight into the key mechanisms.

5. Extensions of the basic model

The example in Section 2 is a basic model type to analyse intertemporal trade-offs in resource use which has led to adaptions and extensions. In the basic model, sustainability in the sense of a stable resource base is ensured when the long-term flow of pollution, $f(R^\infty)$, is (less than or) equal to the natural regenerative capacity of the environment, $kG(S^\infty)$. This is a rather simplistic way to model the complex concept of sustainability.[20] A discussion of more sophisticated ways to include the concept of sustainability in optimal control models is given in Pezzey (1989). The modelling of assimilative capacity of the environment is also too simplistic, especially if the linear specification (71.2′) is chosen. How the assimilative capacity of the environment should be modelled has to depend on the described ecosystem and is in principle an empirical matter. Unfortunately, according to biologists, very little is known about the true process of assimilation. Many authors, having stressed that biologists could not supply them with nice quantitative assimilation functions, take the linear specification (71.2′) as a computationally convenient proxy. Exceptions are, among others, Forster (1975), Dasgupta (1982), Barbier and Markandya (1989) and Pethig (1990). For an analysis of the sensitivity of results to various specifications of the assimilation function see Cesar and de Zeeuw (1995).

Another possible extension of the basic model is to include abatement and other forms of emission reduction. Abatement as an additional control variable is included by, among others, Plourde (1972), Barbier and Markandya (1989) and Van de Ploeg and Withagen (1991). Smith (1972) models recycling. Keeler et al. (1972) include process-integrated changes. The standard result of these types of models with one state variable is a unique saddle-point equilibrium with positive levels of pollution and abatement (and/or other forms of emission reduction) brought about by monotonic[21] environmental taxes. The result that positive levels of pollution are optimal hinges on the assumption that marginal damage costs are negligible for very low levels of pollution (formally: $dU/dP(C,0) = 0$ in the utility function $u(C,P)$). This is analysed in a critical note by Forster (1972) regarding the conclusion of Keeler et al. that non-zero levels of pollution are optimal.

Furthermore, the basic model can be extended to include capital accumulation. Capital can be both a complement for and a substitute to environmental resource use. The possibility of endogenous technological progress through investment in human capital can be taken into account too. Human knowledge can be an environmentally friendly substitute for polluting inputs in the production process. This leads to endogenous growth models that take the environment into consideration.[22] Models that are concerned with pollution related to energy use might include the stock of fossil fuels as a state variable. Inclusion of either of these in the analysis implies a two-state variable model. Due to space limitations we refrain from treating such models here. A general characterization of the dynamics in a two-state variable problem is, for example, Feichtinger et al. (1994). Examples of two- (or even three-) state variable models can be found in Tahvonen (1997), Tahvonen and Kuuluvainen (1993) and Van der Ploeg and Withagen (1991). Nordhaus has written a range of contributions concerning global warming. He uses a model where two-state variables summarize the dynamics of the global warming process, including temperature as a state. His book (1994) on this topic includes an extensive discussion of the model.

Finally, the international dimension was ignored as well as other aspects that give rise to strategic behaviour, such as interactions between the private sector and the government. When strategic interaction between various decision makers is introduced the theory of optimal control is no longer appropriate. Instead (dynamic) game theory must be applied (see Chapter 70).

Other applications in the field of environmental economics analyse the issue of biodiversity (Swanson, 1994) and the optimal reaction of a firm to environmental policy (Kort et al., 1991; Xepapadeas, 1992; Hartl and Kort, 1996). Recent applications of more complex models, either with more than one state variable, or with non-linear dynamics in the modified Hamiltonian system, often use numeric simulation of specified cases to exemplify and extend the analytics (one example is Tahvonen's work).

Notes

1. The research of Talitha Feenstra is sponsored by the Netherlands Organization for Scientific Research (NWO). This chapter is an extended and updated version of the introductory chapter of Cesar (1994).
2. Arrow and Kurz (1970) is a seminal work on the application of dynamic optimization methods in economics.
3. To be explained below.
4. Examples of non-renewable or extractable resources are coal and other minerals; of renewable resources forests and fish; and of environmental resources clean air and clean water.
5. For instance, in the case of fish, it is cheaper to obtain the same harvest from richer fishing grounds than from poorer ones.
6. See, for instance, Seierstad and Sydsæter (1987) for a complete specification.
7. The appendices in Van Hilten, Kort and Van Loon (1993) extensively discuss constrained optimal control problems and their solution.
8. See, for instance, Feichtinger and Hartl (1986), p. 105.
9. If more than one steady state exists, the dynamics can be quite complicated (Skiba, 1978).
10. Kamien and Schwartz (1991, ch. 9, part II) give a complete overview of possible types of steady states that can occur for a one-state variable problem.

11. See also Feichtinger and Hartl (1986), ch. 5.
12. Another possibility is that it must satisfy a so-called transversality condition (see below). However, the validity of such a condition in models with infinite time hori depends on the structure of the problem. See, for example, Seierstad (1977).
13. For instance, in Caputo and Wilen (1995) an application to clean up waste is prese with fixed final time and a scrap value attached to residual waste at T. In Ready Ready (1995), T is the moment a landfill is filled. In their model this T is itself a vai to be chosen optimally.
14. See Seierstad and Sydsæter (1987) for a detailed exposition of the sufficiency thec that are available for optimal control problems.
15. For the theory of stochastic optimal control, see Feichtinger and Hartl (1986, app A8) and the references there.
16. See Seierstad and Sydsæter (1987) for a more precise statement of this.
17. Clarke and Reed (1994), for instance, present a model that includes the possibility irreversible catastrophe when a threshold is crossed.
18. Weitzman (1994) argues that the appropriate discount rate is decreasing over time.
19. See Toman (1994).
20. A well-known definition of sustainable development is 'development that mee needs of the present without compromising the ability of future generations to mee own needs' (World Commission on Environment and Development, 1987).
21. Ulph and Ulph (1994) show that when an exhaustible resource is included in the r the optimal time-path of taxes is no longer monotonic.
22. For example, Bovenberg and Smulders (1995) and Hofkes (1996). Endogenous g models are discussed in the chapter by Withagen and Smulders on this topic (Ch 42).

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