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THE OPEN-LOOP LINEAR QUADRATIC DIFFERENTIAL GAME REVISITED

By Jacob Engwerda

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Jacob Engwerda
Tilburg University
Dept. of Econometrics and O.R.
P.O. Box: 90153, 5000 LE Tilburg, The Netherlands
e-mail: engwerda@uvt.nl

Abstract: In this note we reconsider the indefinite open-loop Nash linear quadratic differential game with an infinite planning horizon. In particular we derive both necessary and sufficient conditions under which the game will have a unique equilibrium.

Keywords: linear-quadratic games, open-loop Nash equilibrium, solvability conditions, Riccati equations.
Jel-codes: C61, C72, C73.

1 Introduction

In the last decades, there is an increased interest in studying diverse problems in economics and optimal control theory using dynamic games. In particular in environmental economics and macroeconomic policy coordination, dynamic games are a natural framework to model policy coordination problems (see e.g. the books and references in Dockner et al. [5] and Engwerda [9]). In these problems, the open-loop Nash strategy is often used as one of the benchmarks to evaluate outcomes of the game. In optimal control theory it is well-known that, e.g., the issue to obtain robust control strategies can be approached as a dynamic game problem (see e.g. [2]).

In this note we consider the open-loop linear quadratic differential game. This problem has been considered by many authors and dates back to the seminal work of Starr and Ho in [17] (see, e.g., [14], [15], [6], [11], [10], [1], [18], [7], [8], [3] and [12]). More specifically, we study in this paper the (regular indefinite) infinite-planning horizon case. The corresponding regular definite (that is the case that the state weighting matrices Q_i (see below) are semi-positive definite) problem has been studied, e.g., extensively in [7] and [8]. Whereas [12] studied the regular indefinite case using a functional analysis approach, under the assumption that the uncontrolled system is stable. In particular, these papers show that, in general, the infinite-planning horizon problem does not have a unique equilibrium. Moreover [12] shows that whenever the game has more than one equilibrium, there will exist an infinite number of equilibria. Furthermore the existence of a unique solution is related to the existence of a so-called strongly stabilizing solution of the set of coupled algebraic Riccati equations (6,7), below.

In this paper we will generalize these results for stabilizable systems using a state-space approach. The outline of this note is as follows. Section two introduces the problem and contains some preliminary results. The main results of this paper are stated in Section three, whereas Section four contains some concluding remarks. The proofs of the main theorems are included in the Appendix.

2 Preliminaries

In this paper we assume that the performance criterion player $i = 1, 2$ likes to minimize is:

$$J_i = \int_0^\infty \{x^T(t)Q_i x(t) + u_i^T(t)R_i u_i(t)\}dt, \tag{1}$$

subject to the linear dynamic state equation

$$\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t),\ x(0) = x_0, \tag{2}$$

Here, the matrices Q_i and R_i are symmetric and R_i are, moreover, assumed to be positive definite, $i = 1, 2$. Notice that we do not make any definiteness assumptions w.r.t. matrix Q_i.

We assume that the matrix pairs (A, B_i), $i = 1, 2$, are stabilizable. So, in principle, each player is capable to stabilize the system on his own.

The open-loop information structure of the game means that we assume that both players only know the initial state of the system and that the set of admissible control actions are functions of time, where time runs from zero to infinity. We assume that the players choose control functions belonging to the set

$$\mathcal{U}_s = \left\{ u \in L_{2,loc} \mid J_i(x_0, u) \text{ exists in } I\!R \cup \{-\infty, \infty\}, \lim_{t \to \infty} x(t) = 0 \right\},$$

where $L_{2,loc}$ is the set of *locally square-integrable* functions, i.e.,

$$L_{2,loc} = \{u[0, \infty) \mid \forall T > 0, \int_0^T u^T(s)u(s)ds < \infty\}.$$

Another set of functions we consider is the class of locally square integrable functions which exponentially converge to zero when $t \to \infty$, $L^e_{2,loc}$. That is, for every $c(.) \in L^e_{2,loc}$ there exist strictly positive constants M and α such that

$$|c(t)| \leq Me^{-\alpha t}.$$

We start our analysis with a result on the regular linear quadratic optimal control problem. Since we were unable to trace this theorem in the literature, an outline of the proof is included.

Theorem 2.1 *Let $c(.) \in L^e_{2,loc}$, (A, B) stabilizable, Q symmetric and $R > 0$. Consider the minimization of*

$$\int_0^\infty \{x^T(t)Qx(t) + u^T(t)Ru(t)\}dt \tag{3}$$

subject to the state dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) + c(t),\ x(0) = x_0, \tag{4}$$

and $u \in U_s$. Then, the linear quadratic problem (3,4) has a solution for all $x_0 \in \mathbb{R}^n$ if and only if the algebraic Riccati equation

$$Q + A^T K + KA - KBR^{-1}B^T K = 0 \tag{5}$$

has a symmetric stabilizing solution $K(.)$, i.e K is such that $A - BR^{-1}B^T K$ is stable.
Moreover, if the linear quadratic control problem has a solution, then the optimal control in feedback form is

$$u^*(t) = -R^{-1}B^T(Kx^*(t) + m(t)), \text{ where } m(t) = \int_t^{\infty} e^{-(A-SK)^T(t-s)}Kc(s)ds,$$

and $x^*(t)$ is the through this optimal control implied solution of the differential equation

$$\dot{x}^*(t) = (A - SK)x^*(t) - Sm(t) + c(t), \; x^*(0) = x_0.$$

Proof. (Outline) First consider the case $c(.) = 0$. Under the assumption that (A, B) is controllable it follows from e.g. [4] (see also [19]) that the theorem holds. In case (A, B) is stabilizable, one implication follows from a standard completion of squares argument. The reverse implication is obtained by considering the controllability canonical form of the system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t), \; x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix},$$

with (A_{11}, B_1) controllable and A_{22} stable.
Since the optimization problem has a solution for every initial state, it follows that it has in particular a solution for $x_0 = [x_{10}^T, \; 0]^T$. From the above quoted result it follows then that the algebraic Riccati equation $A_{11}^T K_1 + K_1 A_{11} + Q_{11} - K_1 B_1 R^{-1} B_1^T K_1 = 0$ has a stabilizing solution (here $Q_{11} = [I \; 0]Q[I \; 0]^T$). From this it is readily verified, by elementary spelling out (5), that (5) has a stabilizing solution too.
Finally, the fact that the above equivalence continues to hold even when $c(.)$ differs from zero, follows from the general argument that linear terms do not play a role to decide whether a quadratic functional has a minimum or not (see e.g. [13, Section 1.4.2]). □

Let $S_i := B_i R_i^{-1} B_i$. Tightly connected with finding the open-loop Nash equilibria of the game (1,2) is the set of coupled algebraic Riccati equations (ARE) given by

$$0 = Q_1 + A^T P_1 + P_1 A - P_1 S_1 P_1 - P_1 S_2 P_2, \tag{6}$$
$$0 = Q_2 + A^T P_2 + P_2 A - P_2 S_2 P_2 - P_2 S_1 P_1; \tag{7}$$

and the two algebraic Riccati equations,

$$Q_i + A^T K_i + K_i A - K_i S_i K_i = 0, \; i = 1, 2. \tag{8}$$

Similar to, e.g., [7], it can be shown that every solution to the set of equations (6,7) can be obtained as a graph subspace of matrix

$$M := \begin{bmatrix} A & -S_1 & -S_2 \\ -Q_1 & -A^T & 0 \\ -Q_2 & 0 & -A^T \end{bmatrix}, \tag{9}$$

and vice versa. To be more precise

Theorem 2.2 *Let $V \subset \mathbb{R}^{3n}$ be an n-dimensional invariant subspace of M, and let $X_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, 2$, be three real matrices such that*

$$V = \operatorname{Im}\left[X_0^T,\ X_1^T,\ X_2^T\right]^T.$$

If X_0 is invertible, then $P_i := X_i X_0^{-1}$, $i = 1, 2$, is a solution to the set of coupled Riccati equations (6,7) and $\sigma(A - S_1P_1 - S_2P_2) = \sigma(M|_V)$ [1]. Furthermore, the solution (P_1, P_2) is independent of the specific choice of basis of V. $\square$

Theorem 2.3 *Let $P_i \in \mathbb{R}^{n\times n}$, $i = 1, 2$, be a solution to the set of coupled Riccati equations (1,2). Then there exist matrices $X_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, 2$, with X_0 invertible, such that $P_i = X_i X_0^{-1}$. Furthermore, the columns of $\left[X_0^T,\ X_1^T,\ X_2^T\right]^T$ form a basis of an n-dimensional invariant subspace of M.* $\square$

The set of (strongly) stabilizing solutions of (6,7) play an important role in the subsequent analysis. Definition a, below, introduces the concept of a stabilizing solution. This notion generalizes the one-player case definition. Definition b, item ii., states that a strongly stabilizing solution has the additional property that the spectrum of the controlled dual system should be in the closed left-half of the complex plane.

Definition 2.4 A solution (P_1, P_2) of the set of algebraic Riccati equations (6,7) is called

a. *stabilizing*, if $\sigma(A - S_1P_1 - S_2P_2) \subset \mathbb{C}^-$;

b. *strongly stabilizing* if

i. it is a stabilizing solution, and

ii.

$$\sigma\left(\begin{bmatrix} -A^T + P_1S_1 & P_1S_2 \\ P_2S_1 & -A^T + P_2S_2 \end{bmatrix}\right) \subset \mathbb{C}_0^+.$$ $\square$

From the above Theorems 2.2, 2.3 it follows immediately that

[1] $\sigma(H)$ denotes the spectrum of matrix H; $\mathbb{C}^- = \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) < 0\}$; $\mathbb{C}_0^+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) > 0\}$.

Corollary 2.5 *(6,7) has a set of stabilizing solutions* (P_1, P_2) *if and only if* M *has an* n*-dimensional stable-invariant graph subspace.* □

Furthermore, the next two important properties of a strongly stabilizing solution are easily obtained.

Theorem 2.6

1. *The set of algebraic Riccati equations (6,7) has a strongly stabilizing solution* (P_1, P_2) *if and only if matrix* M *has an* n*-dimensional stable graph subspace and* M *has* $2n$ *eigenvalues (counting algebraic multiplicities) in* $\mathbb{C}_0^+$.

2. *If the set of algebraic Riccati equations (6,7) has a strongly stabilizing solution, then it is unique.*

Proof.
1. Assume that (6,7) has a strongly stabilizing solution (P_1, P_2). Then (see also Kremer [12]), with

$$T := \begin{bmatrix} I & 0 & 0 \\ -P_1 & I & 0 \\ -P_2 & 0 & I \end{bmatrix},\ TMT^{-1} = \begin{bmatrix} A - S_1P_1 - S_2P_2 & S_1 & S_2 \\ 0 & P_1S_1 - A^T & P_1S_2 \\ 0 & P_2S_1 & P_2S_2 - A^T \end{bmatrix}.$$

Since (P_1, P_2) is a strongly stabilizing solution, by Definition 2.4, matrix M has exact n stable eigenvalues and $2n$ eigenvalues (counted with algebraic multiplicities) in $\mathbb{C}_0^+$. Furthermore, obviously, the stable subspace is a graph subspace.

The converse statement is obtained similarly using the result of Theorem 2.2.
2. Using the result from item 1, Corollary 2.5 shows that there exists exactly one stabilizing solution. So, our solution (P_1, P_2) must be unique. □

Next we state two technical lemmas that are used in the proofs of our main theorems. A proof of them can be found, e.g., in [9]. Lemma 2.7 deals with the stable subspace, E^s, of a linear system.

Lemma 2.7 *Let* $x_0 \in \mathbb{R}^p$, $y_0 \in \mathbb{R}^{n-p}$ *and* $Y \in \mathbb{R}^{(n-p)\times p}$. *Consider the differential equation*

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix},\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

If $\lim_{t\to\infty} x(t) = 0$, *for all* $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \text{Span}\begin{bmatrix} I \\ Y \end{bmatrix}$, *then*

1. $\dim E^s \geq p$, *and*

2. *there exists a matrix* $\bar{Y} \in \mathbb{R}^{(n-p)\times p}$ *such that* $\text{Span}\begin{bmatrix} I \\ \bar{Y} \end{bmatrix} \subset E^s$. □

Lemma 2.8 *Assume there exists an initial state* $x_0 \neq 0$ *such that*

$$x(t) = e^{-A^Tt}x_0 \to 0 \text{ if } t \to \infty \text{ and } B^Tx(t) = 0.$$

Then (A, B) *is not stabilizable.* □

3 Main results

Using the previous results, in the Appendix the following theorem is proved.

Theorem 3.1 *If the linear quadratic differential game (1,2) has an open-loop Nash equilibrium for every initial state, then*

1. *M has at least n stable eigenvalues (counted with algebraic multiplicities). More in particular, there exists a p-dimensional stable M-invariant subspace S, with $p \geq n$, such that*

$$\mathrm{Im}\begin{bmatrix} I \\ V_1 \\ V_2 \end{bmatrix} \subset S,$$

for some $V_i \in I\!R^{n\times n}$.

2. *the two algebraic Riccati equations (8) have a symmetric stabilizing solution $K_i(.)$, $i = 1, 2$.*

Conversely, if $v^T(t) =: [x^T(t), \psi_1^T(t), \psi_2^T(t)]$ is an asymptotically stable solution of $\dot{v}(t) = Mv(t),\ x(0) = x_0$, and the two algebraic Riccati equations (8) have a stabilizing solution then,

$$u_i^* := -R_i^{-1}B_i^T\psi_i(t),\ \ i = 1, 2,$$

provides an open-loop Nash equilibrium for the linear quadratic differential game (1,2). □

Remark 3.2 From this theorem one can draw a number of conlusions concerning the existence of open-loop Nash equilibria. A general conclusion is that this number depends critically on the eigenstructure of matrix M. We will distinguish some cases. To that end, let s denote the number (counting algebraic multiplicities) of stable eigenvalues of M.
1. If $s < n$, still for some initial state there may exist an open-loop Nash equilibrium. Consider, e.g., the case that $s = 1$. Then, for every $x_0 \in \text{Span }[I,\ 0,\ 0]v$, where v is an eigenvector corresponding with the stable eigenvalue, the game has a Nash equilibrium.
2. In case $s \geq 2$, the situation might arise that for some initial states there exists an infinite number of equilibria. A situation in which there are an infinite number of Nash equilibrium actions occurs if, e.g., v_1 and v_2 are two independent eigenvectors in the stable subspace of M for which $[I,\ 0,\ 0]v_1 = \mu[I,\ 0,\ 0]v_2$, for some scalar μ. In such a situation,

$$x_0 = \lambda[I,\ 0,\ 0]v_1 + (1-\lambda)\mu[I,\ 0,\ 0]v_2,$$

for an arbitrary scalar $\lambda \in I\!R$. The resulting equilibrium control actions, however, differ for each λ (apart from some exceptional cases). □

Similar to [3, Theorem 6.22] it can be shown that

Theorem 3.3 *Assume that*

1. *the set of coupled algebraic Riccati equations (6,7) has a stabilizing solution; and*
2. *the two algebraic Riccati equations (8) have a symmetric stabilizing solution $K_i(.)$, $i = 1, 2$.*

Then the linear quadratic differential game (1,2) has an open-loop Nash equilibrium for every initial state. Moreover, one set of equilibrium actions is given by:

$$u_i^*(t) = -R_i^{-1} B_i^T P_i \Phi(t,0) x_0, \; i = 1, 2. \tag{10}$$

Here $\Phi(t,0)$ satisfies the transition equation $\dot{\Phi}(t,0) = (A - S_1 P_1 - S_2 P_2)\Phi(t,0)$; $\Phi(t,t) = I$. □

The above reflections raise the question whether it is possible to find conditions under which the game has a unique equilibrium for every initial state. The next Theorem 3.4 gives such conditions. Moreover, it shows that in that case the unique equilibrium actions can be synthesized as a state feedback. The proof of this theorem is provided in the Appendix.

Theorem 3.4 *The linear quadratic differential game (1,2) has a unique open-loop Nash equilibrium for every initial state if and only if*

1. *The set of coupled algebraic Riccati equations (6,7) has a strongly stabilizing solution, and*
2. *the two algebraic Riccati equations (8) have a stabilizing solution.*

Moreover, the unique equilibrium actions are given by (10). □

Example 3.5
1. Consider the system

$$\dot{x}(t) = -2x(t) + u_1(t) + u_2(t), \; x(0) = x_0;$$

and cost functions

$$J_1 = \int_0^\infty \{x^2(t) + u_1^2(t)\}dt \text{ and } J_2 = \int_0^\infty \{4x^2(t) + u_2^2(t)\}dt.$$

The eigenvalues of M are $\{-3, 2, 3\}$. An eigenvector corresponding with the eigenvalue -3 is $[5,\ 1,\ ,4]^T$.
So, according Theorem 2.6 item 1, the with this game corresponding set of algebraic Riccati equations (6,7) has a strongly stabilizing solution. Furthermore, since $q_i > 0$, $i = 1, 2$, the two algebraic Riccati equations (8) have a stabilizing solution. Consequently, this game has a unique open-loop Nash equilibrium for every initial state x_0.
2. Reconsider the game in item 1, but with the system dynamics replaced by

$$\dot{x}(t) = 2x(t) + u_1(t) + u_2(t), \; x(0) = x_0.$$

Then M has the eigenvalues $\{-3, -2, 3\}$. Since M has two stable eigenvalues, it follows from Theorem 2.6 item 1 that the with this game corresponding set of algebraic Riccati equations (6,7) does not

have a strongly stabilizing solution. So, see Theorem 3.4, the game does not have for every initial state a unique open-loop Nash equilibrium.
On the other hand, since $[1,\ 1,\ 4]^T$ is an eigenvector corresponding with $\lambda = -3$, it follows from the Corollaries 2.5 and Theorem 3.3 that the game does have an open-loop Nash equilibrium for every initial state that permits a feedback synthesis.
In fact for every initial state there are an infinite number of equilibria. For every $\alpha \in I\!R$ the equilibrium actions $u_1^*(t) = -2(e^{-5t}x_0 - \alpha e^{-3t})$, $u_2^*(t) = -2(e^{-5t}x_0 + \alpha e^{-3t})$ yield an open-loop Nash equilibrium. $\square$

4 Concluding Remarks

In this note we considered the regular indefinite infinite-planning horizon linear-quadratic differential game. Both necessary conditions and sufficient conditions were derived for the existence of an open-loop Nash equilibrium. Moreover, conditions were presented that are both necessary and sufficient for the existence of a unique equilibrium.
The above results can be generalized straightforwardly to the N-player case. Furthermore, since Q_i are assumed to be indefinite, the obtained results can be directly used to (re)derive properties for the zero-sum game, which plays, e.g., an important role in robustness analysis. If players discount their future loss, similar to [7], it follows from Theorem 3.4 that if the discount factor is "large enough" the game has generically a unique open-loop Nash equilibrium. Finally we conclude from (17) that the conclusion in [12], that if the game has an open-loop Nash equilibrium for every initial state either there is a unique equilibrium or an infinite number of equilibria, applies in general.

Appendix

Proof of Theorem 3.1.
"$\Rightarrow$ part" Suppose that u_1^*, u_2^* are a Nash solution. That is,

$$J_1(u_1, u_2^*) \geq J_1(u_1^*, u_2^*) \text{ and } J_2(u_1^*, u_2) \geq J_2(u_1^*, u_2^*).$$

From the first inequality we see that for every $x_0 \in I\!R^n$ the (nonhomogeneous) linear quadratic control problem to minimize

$$J_1 = \int_0^\infty \{x^T(t)Q_1x(t) + u_1^T(t)R_1u_1(t)\}dt,$$

subject to the (nonhomogeneous) state equation

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2^*(t),\ x(0) = x_0,$$

has a solution. This implies, see Theorem 2.1, that the algebraic Riccati equation (8) has a stabilizing solution (with $i = 1$). In a similar way it follows that also the second algebraic Riccati equation must have a stabilizing solution. Which completes the proof of point 2.
To prove point 1. we consider Theorem 2.1 in some more detail. According Theorem 2.1 the minimization problem

$$\min_{u_1} J_1(x_0, u_1, u_2^*) = \int_0^\infty \{x_1^T(t)Q_1x_1(t) + u_1^T(t)R_1u_1(t)\}dt,$$

where

$$\dot{x_1} = Ax_1 + B_1u_1 + B_2u_2^*,\ x_1(0) = x_0,$$

has a unique solution. Its solution is

$$\tilde{u}_1(t) = -R_1^{-1}B_1^T(K_1x_1(t) + m_1(t)) \text{ with } m_1(t) = \int_t^{\infty} e^{-(A-S_1K_1)^T(t-s)}K_1B_2u_2^*(s)ds \tag{11}$$

and K_1 the stabilizing solution of the algebraic Riccati equation

$$Q_1 + A^TX + XA - XS_1X = 0. \tag{12}$$

Notice that, since the optimal control $\tilde{u}_1$ is uniquely determined, and by definition the equilibrium control u_1^* solves the optimization problem, $u_1^*(t) = \tilde{u}_1(t)$. Consequently,

$$\begin{aligned}
\frac{d(x(t) - x_1(t))}{dt} &= Ax(t) + B_1u_1^*(t) + B_2u_2^*(t) - ((A - S_1K_1)x_1(t) - S_1m_1(t) + B_2u_2^*(t)) \\
&= Ax(t) - S_1(K_1x_1(t) + m_1(t)) - Ax_1(t) + S_1K_1x_1(t) + S_1m_1(t) \\
&= A(x(t) - x_1(t)).
\end{aligned}$$

Since $x(0) - x_1(0) = x_0 - x_0 = 0$ it follows that $x_1(t) = x(t)$.
In a similar way we obtain from the minimization of J_2, with u_1^* now entering into the system as an external signal, that

$$u_2^*(t) = -R_2^{-1}B_2^T(K_2x(t) + m_2(t)) \text{ with } m_2(t) = \int_t^{\infty} e^{-(A-S_2K_2)^T(t-s)}K_2B_1u_1^*(s)ds \tag{13}$$

and K_2 the stabilizing solution of the algebraic Riccati equation

$$Q_2 + A^TX + XA - XS_2X = 0.$$

By straightforward differentiation of $m_i(t)$ in (11) and (13), respectively, we obtain

$$\begin{aligned}
\dot{m}_1(t) &= -(A - S_1K_1)^Tm_1(t) - K_1B_2u_2^*(t), \text{ and} \\
\dot{m}_2(t) &= -(A - S_2K_2)^Tm_2(t) - K_2B_1u_1^*(t).
\end{aligned} \tag{14}$$

Next, introduce $\psi_i(t) := K_ix(t) + m_i(t),\ i = 1, 2$. Using (14) and (12) we get

$$\begin{aligned}
\dot{\psi}_1(t) &= K_1\dot{x}(t) + \dot{m}_1(t) \\
&= K_1(A - S_1K_1)x(t) - K_1S_1m_1(t) + K_1B_2u_2^*(t) - K_1B_2u_2^*(t) - (A - S_1K_1)^Tm_1(t) \\
&= (-Q_1 - A^TK_1)x(t) - K_1S_1m_1(t) - (A - S_1K_1)^Tm_1(t) \\
&= -Q_1x(t) - A^T(K_1x(t) + m_1(t)) \\
&= -Q_1x(t) - A^T\psi_1(t).
\end{aligned} \tag{15}$$

Similarly it follows that $\dot{\psi}_2(t) = -Q_2x(t) - A^T\psi_2(t)$. Consequently, $v^T(t) := [x^T(t),\ \psi_1^T(t),\ \psi_2^T(t)]$, satisfies

$$\dot{v}(t) = Mv(t),\ \text{with } v_1(0) = x_0.$$

Since by assumption, for arbitrary x_0, $v_1(t)$ converges to zero it is clear from Lemma 2.7 by choosing consecutively $x_0 = e_i,\ i = 1, \cdots, n$, that matrix M must have at least n stable eigenvalues (counting algebraic multiplicities). Moreover, the other statement follows from the second part of this lemma. Which completes this part of the proof.
"$\Leftarrow$ part" Let u_2^* be as claimed in the theorem, that is

$$u_2^*(t) = -R_2^{-1} B_2^T \psi_2.$$

We next show that then necessarily u_1^* solves the optimization problem

$$\min_{u_1} \int_0^\infty \{\tilde{x}^T(t) Q_1 \tilde{x}(t) + u_1^T R_1 u_1(t)\} dt,$$

subject to

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B_1 u_1(t) + B_2 u^*(t),\ \tilde{x}(0) = x_0.$$

Since, by assumption, the algebraic Riccati equation

$$Q_1 + A^T K_1 + K_1 A - K_1 S_1 K_1 = 0 \tag{16}$$

has a stabilizing solution, according Theorem 2.1, the above minimization problem has a solution. This solution is given by

$$\tilde{u}_1^*(t) = -R^{-1} B_1^T (K_1 \tilde{x} + m_1),\ \text{where } m_1 = \int_t^\infty e^{-(A - S_1 K_1)^T (t-s)} K_1 B_2 u_2^*(s) ds.$$

Next, introduce

$$\tilde{\psi}_1(t) := K_1 \tilde{x}(t) + m_1(t).$$

Then, similar to (15) we obtain

$$\dot{\tilde{\psi}}_1 = -Q_1 \tilde{x} - A^T \tilde{\psi}_1.$$

Consequently, $x_d(t) := x(t) - \tilde{x}(t)$ and $\psi_d(t) := \psi_1(t) - \tilde{\psi}_1(t)$ satisfy

$$\begin{bmatrix} \dot{x}_d(t) \\ \dot{\psi}_d(t) \end{bmatrix} = \begin{bmatrix} A & -S_1 \\ -Q_1 & -A^T \end{bmatrix} \begin{bmatrix} x_d(t) \\ \psi_d(t) \end{bmatrix},\ \begin{bmatrix} x_d(0) \\ \psi_d(0) \end{bmatrix} = \begin{bmatrix} 0 \\ p \end{bmatrix},\ \text{for some } p \in I\!R^n.$$

Notice that matrix $\begin{bmatrix} A & -S_1 \\ -Q_1 & -A^T \end{bmatrix}$ is the Hamiltonian matrix associated with the algebraic Riccati equation (16). Recall that the spectrum of this matrix is symmetric w.r.t. the imaginary axis. Since by assumption the Riccati equation (16) has a stabilizing solution, we know that its stable invariant subspace is given by Span$[I\ K_1]^T$. Therefore, with E^u representing a basis for the unstable subspace, we can write

$$\begin{bmatrix} 0 \\ p \end{bmatrix} = \begin{bmatrix} I \\ K_1 \end{bmatrix} v_1 + E^u v_2,$$

for some vectors $v_i,\ i=1,2$. However, it is easily verified that due to our asymptotic stability assumption both $x_d(t)$ and $\psi_d(t)$ converge to zero if $t\to\infty$. So, v_2 must be zero. From this it follows now directly that $p=0$. Since the solution of the differential equation is uniquely determined, and $[x_d(t)\ \psi_d(t)]=[0\ 0]$ solve it, we conclude that $\tilde{x}(t)=x(t)$ and $\tilde{\psi}_1(t)=\psi_1(t)$. Or stated differently, u_1^* solves the minimization problem.
In a similar way it is shown that for u_1 given by u_1^*, player two his optimal control is given by u_2^*. Which proves the claim. $\square$.

Proof of Theorem 3.4.
"$\Rightarrow$ part" That the Riccati equations (8) must have a stabilizing solution follows directly from Theorem 3.1.
Assume that matrix M has a s-dimensional stable graph subspace S, with $s>n$. Let $\{b_1,\cdots,b_s\}$ be a basis for S. Denote $d_i:=[I,0,0]b_i$ and assume (without loss of generality) that Span $[d_1,\cdots,d_n]=I\!R^n$. Then $d_{n+1}=\mu_1d_1+\cdots+\mu_nd_n$ for some $\mu_i,\ i=1,\cdots,n$. Furthermore, let $x_0=\alpha_1d_1+\cdots+\alpha_nd_n$. Then also for arbitrary $\lambda\in[0,1]$,

$$\begin{aligned} x_0 &= \lambda(\alpha_1d_1+\cdots+\alpha_nd_n)+(1-\lambda)(d_{n+1}-\mu_1d_1-\cdots-\mu_nd_n) \\ &= [I,0,0]\{\lambda(\alpha_1b_1+\cdots+\alpha_nb_n)+(1-\lambda)(b_{n+1}-\mu_1b_1-\cdots-\mu_nb_n)\} \\ &= [I,0,0]\{(\lambda\alpha_1-(1-\lambda)\mu_1)b_1+\cdots+(\lambda\alpha_n-(1-\lambda)\mu_n)b_n+(1-\lambda)b_{n+1}\}. \end{aligned}$$

Next consider

$$v_\lambda:=(\lambda\alpha_1-(1-\lambda)\mu_1)b_1+\cdots+(\lambda\alpha_n-(1-\lambda)\mu_n)b_n+(1-\lambda)b_{n+1}.$$

Notice that $v_{\lambda_1}\neq v_{\lambda_2}$ whenever $\lambda_1\neq\lambda_2$.
According Theorem 3.1 all solutions $v^T(t)=[x^T,\ \psi_1^T,\ \psi_2^T]$ of $\dot{v}(t)=Mv(t),\ v(0)=v_\lambda$, induce then open-loop Nash equilibrium strategies

$$u_{i,\lambda}:=-R_i^{-1}B_i^T\psi_{i,\lambda}(t),\ i=1,2. \tag{17}$$

Since by assumption for every initial state there is a unique equilibrium strategy it follows on the one hand that the by these equilibrium strategies induced state trajectory $x_\lambda(t)$ coincides for all λ and, on the other hand, that

$$B_i^T\psi_{i,\lambda_1}(t)=B_i^T\psi_{i,\lambda_2}(t),\ \forall\lambda_1,\lambda_2\in[0,1]. \tag{18}$$

Since $\dot{\psi}_{i,\lambda}=-Q_ix_\lambda(t)-A^T\psi_{i,\lambda}$ it follows that

$$\dot{\psi}_{i,\lambda_1}-\dot{\psi}_{i,\lambda_2}=-A^T(\psi_{i,\lambda_1}-\psi_{i,\lambda_2})\text{ and }B_i^T(\psi_{i,\lambda_1}(t)-\psi_{i,\lambda_2}(t))=0. \tag{19}$$

Notice that both $\psi_{i,\lambda_1}(t)$ and $\psi_{i,\lambda_2}(t)$ converge to zero. Furthermore, since $v_{\lambda_1}\neq v_{\lambda_2}$ whenever $\lambda_1\neq\lambda_2$, $\{b_1,\cdots,b_{n+1}\}$ are linearly independent and Span$[d_1,\cdots,d_n]=I\!R^n$, it can be easily verified that at least for one i, $\psi_{i,\lambda_1}(0)\neq\psi_{i,\lambda_2}(0)$, for some λ_1 and λ_2. Therefore, by Lemma 2.8, it follows from (19) that (A,B_i) is not stabilizable. But this violates our basic assumption. So, our assumption that $s>n$ must have been wrong and we conclude that matrix M has an n-dimensional stable graph subspace and that the dimension of the subspace corresponding with non-stable eigenvalues is $2n$. By Theorem 2.6 the set of Riccati equations (6,7) has then a strongly stabilizing solution.
"$\Leftarrow$ part" Since by assumption the stable subspace, E^s, is a graph subspace we know that every

initial state, x_0, can be written uniquely as a combination of the first n entries of the basisvectors in E^s. Consequently, with every x_0 there corresponds a unique ψ_1 and ψ_2 for which the solution of the differential equation $\dot{z}(t) = Mz(t)$, with $z_0^T = [x_0^T,\ \psi_1^T,\ \psi_2^T]$, converges to zero. So, according Theorem 3.1, for every x_0 there is a Nash equilibrium. On the other hand we have from the proof of Theorem 3.1 that all Nash equilibrium actions (u_1^*, u_2^*) satisfy

$$u_i^*(t) = -R_i^{-1}B_i^T\psi_i(t),\ \ i = 1,2,$$

where $\psi_i(t)$ satisfy the differential equation

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = M \begin{bmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{bmatrix},\ \text{with } x(0) = x_0.$$

Now, consider the system

$$\dot{z}(t) = Mz(t);\ y(t) = Cz(t),\ \text{with } C := \begin{bmatrix} I & 0 & 0 \\ 0 & -R_1^{-1}B_1 & 0 \\ 0 & 0 & -R_2^{-1}B_2 \end{bmatrix}.$$

Since $(A, B_i),\ i = 1,2$, is stabilizable, it is easily verified that the pair (C, M) is detectable. Consequently, due to our assumption that $x(t)$ and $u_i^*(t),\ i = 1,2$, converge to zero, we have from Lemma [20, Lemma 14.1] that $[x^T(t),\ \psi_1^T(t),\ \psi_2^T(t)]$ converges to zero. Therefore, $[x^T(0),\ \psi_1^T(0),\ \psi_2^T(0)]$ has to belong to the stable subspace of M. However, as we argued above, for every x_0 there is exactly one vector $\psi_1(0)$ and vector $\psi_2(0)$ such that $[x^T(0),\ \psi_1^T(0),\ \psi_2^T(0)] \in E^s$. So we conclude that for every x_0 there exists exactly one Nash equilibrium.

Finally notice that by Theorem 3.3 the game has an equilibrium for every initial state given by (10). Since for every initial state the equilibrium actions are uniquely determined, it follows that the equilibrium actions $u_i^*,\ i = 1,2$, have to coincide with (10). $\square$

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