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Square Indefinite LQ-Problem: Existence of a Unique Solution[1]

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Abstract. In this paper, we consider discrete-time systems. We study conditions under which there is a unique control that minimizes a general quadratic cost functional. The system considered is described by a linear time-invariant recurrence equation in which the number of inputs equals the number of states. The cost functional differs from the usual one considered in optimal control theory, in the sense that we do not assume that the weight matrices considered are semipositive definite. For both a finite planning horizon and an infinite horizon, necessary and sufficient solvability conditions are given. Furthermore, necessary and sufficient conditions are derived for the existence of a solution for an arbitrary finite planning horizon.

Key Words. Linear-quadratic control, discrete-time systems, indefinite cost functions.

1. Introduction

In this paper, we consider the minimization of the performance function

$$J(N) := \sum_{k=1}^{N-1} \{y^T(k)Qy(k) + u^T(k)Ru(k)\} + y^T(N)Q_f y(N), \qquad (1)$$

s.t. the linear finite-dimensional time-invariant difference equation

$$y(k+1) = Ay(k) + Bu(k) + Cx(k), \qquad k = 1, 2, \ldots, \qquad (2)$$

where the matrices Q, Q_f, R are symmetric, but not necessarily semipositive definite.

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In case the weight matrices are semipositive definite, the solution of this problem is well known; see, e.g., Refs. 1–6. Moreover, most of these references also treat the case of an infinite planning horizon. This last case is of special importance, because it can be shown that, under some smoothness conditions, the resulting solution to the optimal control problem stabilizes the system. However, in case the definiteness assumption is dropped, it is clear that in general problem (1)–(2) will not have a solution. So, the question arises under which conditions on the weight matrices there exists a solution.

Applications which typically fit into this generalized framework come from variational problems, game-theoretic problems, problems in H_∞-control and filtering theory. In variational situations, the goal is to increase the gains (measured by a quadratic function of the state of the system, expressed by $y^T Q y$, with not necessarily $Q \geq 0$) by using as little as possible control efforts (measured by $u^T R u$); see, e.g., Ref. 7.

A well-known game-theoretic application which fits into this framework is the two-person linear-quadratic zero-sum dynamic game (see, e.g., Ref. 8, p. 247, Theorem 4), described by the state equation

$$y(k+1) = Ay(k) + B_1 u_1(k) + B_2 u_2(k)$$

and the objective functional

$$L(u_1, u_2) = \sum_{k=1}^{N-1} \{y^T(k)\tilde{Q}y(k) + u_1^T(k)u_1(k) - u_2^T(k)u_2(k)\} + y^T(N)\tilde{Q}_f y(N).$$

with both $\tilde{Q}$ and $\tilde{Q}_f$ semipositive definite, which player 1 wishes to minimize and player 2 attempts to maximize. This game admits a unique open-loop saddle-point solution if problem (1)–(2) has a solution with

$$B := B_2, \quad Q := -\tilde{Q}, \quad Q_f := -\tilde{Q}_f, \quad R := I, \quad C := 0.$$

The infinite planning horizon problem naturally occurs in finding a solution for H_∞-control problems. Consider the following H_∞-control problem. Find a compensator

$$u(k) = F_1 y(k) + F_2 x(k),$$

such that the closed-loop system

$$y(k+1) = Ay(k) + \tilde{B}u(k) + \tilde{C}x(k), \qquad z(k) = y(k),$$

is internally stable and its l_2-induced operator norm from the disturbance x to the state y is less than one. As shown by Ref. 9, this problem has a solution if, among other conditions, the equation

$$K=A^{\mathrm{T}}\left\{K-K(\tilde{B}\tilde{C})\left\{\begin{bmatrix}0 & 0\\ 0 & -I\end{bmatrix}+(\tilde{B}\tilde{C})^{\mathrm{T}}K(\tilde{B}\tilde{C})\right\}^{-1}(\tilde{B}\tilde{C})^{\mathrm{T}}K\right\}A+I$$

has a semipositive-definite solution K. By introducing

$$Q:=I, \qquad Q_f:=I, \qquad R:=\begin{bmatrix}0 & 0\\ 0 & -I\end{bmatrix}, \qquad B:=(\tilde{B}\tilde{C})$$

in the (ARE) in Section 2, we see that the solvability conditions that we present are also necessary conditions for solvability of this H_∞-control problem. Finally, we note that the (ARE) also naturally occurs in the study of minimal stationary Gaussian reciprocal processes; see Ref. 10, Section 4.

In this paper, we give necessary and sufficient solvability conditions for problem (1)–(2) for both a finite planning horizon and infinite planning horizon under the assumption that matrix B in Eq. (2) is invertible. Moreover, for simplicity reasons, we drop the exogenous variable $x(k)$ in Eq. (2). It will become clear from our analysis that this variable does not play any role in deriving the solvability conditions. Consequently, under the conditions that we will derive, algorithms obtained in the literature which incorporate this exogenous variable (see, e.g., Refs 1, 6) are also optimal for the indefinite case.

The paper is organized as follows. We first deal with the infinite planning horizon case. Using results from (Ref. 11, in Section 2, necessary and sufficient conditions are derived for the existence of a unique stabilizing solution for the problem. Furthermore, a numerical algorithm to approximate this solution and an algebraic algorithm to calculate the exact solution are provided. Next, in Section 3, we recall the solvability conditions for a fixed finite planning horizon and relate these conditions to the conditions obtained for the infinite planning horizon. Using the results from Sections 2 and 3, we derive in Section 4 necessary and sufficient conditions to conclude that the problem has a unique solution for every finite planning horizon. The paper ends with some concluding remarks. Parts of Section 4 of this paper were also reported in Ref. 12.

2. Infinite Planning Horizon Case

The basic problem treated in this section is Problem P1 below.

Problem P1. Find necessary and sufficient conditions under which the optimal control problem

$$\inf_{u(0,\cdot)} \lim_{N\to\infty} J(N), \quad \text{s.t. } y(k+1)=Ay(k)+Bu(k), \qquad y(0)=\bar{y},$$

with J given by (1) and matrix B invertible, has a unique solution for every $\bar{y}$, under the additional constraint $\lim_{N\to\infty} y(N)=0$. In the sequel, we will abbreviate this infimum, if it exists, by J^*.

For continuous-time systems, generalizations of this problem have been studied by many people; see, e.g., Refs. 13–14 and, more recently, Refs. 15–16. For discrete-time systems, the theory concerning this problem, where positivity of the weight matrices Q, Q_f, R is not assumed a priori, is much less developed. Jonckheere et al. (Refs. 17–18) study this problem for stable, controllable systems and give a solution in terms of a frequency-domain condition for positive semidefiniteness of a bounded self-adjoint Hilbert space operator, together with the frequency-domain characterization of its spectrum. Lancaster et al. show in Ref. 19, Theorems 2.4 and 2.5, that under the additional assumption that the matrix A is invertible, the problem has a solution iff the rational matrix function

$$\psi(z):=R+B^T(z^{-1}-A^T)^{-1}Q(z-A)^{-1}B$$

is positive semidefinite on the unit circle. In more recent times, Ref. 20, showed that, under the assumptions that A is invertible, ψ is positive definite at some point of the unit circle, and the system is controllable, the algebraic Riccati equation corresponding to this problem has an appropriate solution. This result was used by Ran et al. (Ref. 21) to solve the above-mentioned problem, where the additional constraint (that the state variable must converge to zero) is replaced by the more general requirement that the state variable must converge to an a priori given subspace.

In all these references, we see that assumptions are made w.r.t. the matrix A. Note that the invertibility assumption on the matrix A can be partly avoided, by using a prefeedback $u(k)=Ky(k)$, which places the eigenvalues of the closed-loop matrix in the annulus $\{z|0<|z|<1\}$. However, the solvability conditions depend in that case explicitly on the matrix K used, which is something that we like to avoid here; see Ref. 19. Noteworthy in this context is also the pencil approach taken by Pappas et al. in Ref. 22 to solve the singular case. As we will see later on in this section, we do not make more assumptions than the invertibility assumption on the matrix B. So, in a certain sense, our results are complementary to existing results on this topic.

In the proof of our main result, we use two results which are both worth mentioning separately. Our first result is from Engwerda et al. (Ref. 11, Theorems 2.1 and 3.4). We will see later on in Section 4 that this lemma also plays a crucial role in deriving results there.

Lemma 2.1. Suppose that the matrix $\bar{Q}$ is positive definite. Then, the matrix equation $X+\bar{A}^TX^{-1}\bar{A}=\bar{Q}$ has a positive-definite solution X (i.e., $X>0$) if and only if:

(i) $\tilde{\psi}(\lambda):=\bar{Q}+\lambda\bar{A}+\lambda^{-1}\bar{A}^T\geq 0$, for all λ on the unit circle;
(ii) $\tilde{\psi}(\lambda)$ is nonsingular.

Moreover, if the equation has a positive-definite solution, there exists also a largest solution X_L and a smallest solution X_S. Here, X_L is the unique solution for which $X+\lambda A$ is invertible for all $|\lambda|<1$, and X_S is the unique solution for which $X+\lambda A^T$ is invertible for all $|\lambda|>1$.

Our second result states the key property that the optimal cost function $J^*(\bar{y})$ is a quadratic form in $\bar{y}$. The proof that we give is a more direct equivalent of the proof given by Molinari in Ref. 14, Lemma 3 for the continuous-time problem.

Lemma 2.2. Let (A, B) be reachable, i.e., $\text{rank}[BAB\cdots A^{n-1}B]=n$. Then, if $J^*(\bar{y})$ exists (i.e., is finite for all $\bar{y}$), it is a quadratic form; i.e., $J^*(\bar{y})=\bar{y}^TK\bar{y}$ for some symmetric matrix K.

Proof. According to Molinari (Ref. 23, Lemmas 3–5), it suffices to show that $J^*(\bar{y})$ satisfies the two following conditions:

(i) $|J^*(\bar{y})|\leq c\bar{y}^T\bar{y}$, for some constant c;
(ii) $J^*(\bar{y}+\bar{z})+J^*(\bar{y}-\bar{z})=2(J^*(\bar{y})+J^*(\bar{z}))$, i.e., $J^*(\cdot)$ satisfies the parallelogram identity.

The second condition follows analogously to the proof of Molinari (Ref. 14, Lemma 3).

To prove the first condition, we show that

$$\bar{y}^TQ_r\bar{y}\leq J^*(\bar{y})\leq\bar{y}^TQ_c\bar{y},$$

for some symmetric matrices Q_r and Q_c, respectively. The inequality

$$J^*(\bar{y})\leq\bar{y}^TQ_c\bar{y}$$

is the easiest one (see Ref. 14 again). For, using the control sequence $(u[0, n-1], 0, 0, \ldots)$ with

$$u^T[0, n-1]:=-S^T(SS^T)^{-1}A^n\bar{y},$$

where S is the reachability matrix $[BAB\cdots A^{n-1}B]$, it is clear that we get a control sequence which majorizes $J^*(\bar{y})$ in the advertised way, and moreover has the property that

$$y(n+k)=0, \qquad \text{for all } k\in\mathbb{N}.$$

Now, we consider the other inequality. To prove this inequality, following, e.g., Molinari (Ref. 14, proof of Lemma 2.1), we first note that, from the above paragraph, it is obvious that $J^*(0) \leq 0$. On the other hand, it is clear that, if the infimum $J^*(0)$ exists, it cannot be negative. For, if some control sequence $u^*[0, \cdot\,]$ yields $J^*(0) < 0$, then using a control sequence obtained by multiplying this control sequence $u^*[0, \cdot\,]$ by an arbitrary positive scalar $k > 1$ would yield a lower cost than $J^*(0)$. So, $J^*(0) = 0$ whenever the infimum exists. Consequently, using the fact that the problem is time invariant, we obtain the following inequality:

$$0 = \inf_{\substack{u[-n,\cdot)\\ \text{s.t. } y(-n)=0}} \lim_{N\to\infty} \sum_{k=-n}^{N-1} \{y^T(k)Qy(k) + u^T(k)Ru(k)\} + y^T(N)Q_f y(N)$$

$$\leq \inf_{\substack{u[-n,\cdot)\\ \text{s.t. } y(0)=\bar{y} \text{ and } y(-n)=0}} \lim_{N\to\infty} \sum_{k=-n}^{N-1} \{y^T(k)Qy(k) + u^T(k)Ry(k)\} + y^T(N)Q_f y(N)$$

$$= \inf_{\substack{u[-n,0)\\ \text{s.t. } y(0)=\bar{y} \text{ and } y(-n)=0}} \left\{ \sum_{k=-n}^{-1} \{y^T(k)Qy(k) + u^T(k)Ru(k)\} + \inf_{\substack{u[0,\cdot)\\ \text{s.t. } y(0)=\bar{y}}} \lim_{N\to\infty} J(N) \right\}.$$

Now, due again to our reachability assumption, we have that, with

$$u^T[-1, -n] := S^T(SS^T)^{-1}\bar{y},$$

we get a control sequence majorizing

$$\inf \sum_{k=-n}^{-1} \{y^T(k)Qy(k) + u^T(k)Ru(k)\}$$

by $-\bar{y}^T Q_r \bar{y}$ for some symmetric matrix Q_r. So, we obtain the estimate

$$0 \leq -\bar{y}^T Q_r \bar{y} + \inf_{\substack{u[0,\cdot)\\ \text{w.r.t. } y(0)=\bar{y}}} \lim_{N\to\infty} J(N),$$

which yields the stated result.

The main theorem of this section now reads as follows.

Theorem 2.1. Problem (P1) has a unique solution iff:

(i) $\psi(z) := B^TQB + (I + z^*B^TA^TB^{-T})R(I + zB^{-1}AB)$ is regular;

(ii) $\psi(z) > 0$, for $|z| = 1$.

Proof. ($\Rightarrow$) Since the problem has a solution, we know from Lemma 2.2 that

$$J^*(\bar{y}) = \bar{y}^T K \bar{y},$$

for some symmetric matrix K. So, we have

$$\begin{aligned}
\bar{y}^T K \bar{y} &= \inf_{\substack{u[0,\cdot) \\ \text{s.t. } y(0)=\bar{y}}} \lim_{N\to\infty} J(N) \\
&= \inf_{\substack{u[0,\cdot) \\ \text{s.t. } y(0)=\bar{y}}} \lim_{N\to\infty} \Bigg\{ y^T(0)Qy(0) + u^T(0)Ru(0) \\
&\qquad + \sum_{k=1}^{N-1} \{y^T(k)Qy(k) + u^T(k)Ru(k)\} \\
&\qquad + y^T(N)Q_f y(N) \Bigg\} \\
&= \inf_{\substack{u(0) \\ \text{s.t. } y(0)=\bar{y}}} \Bigg\{ y^T(0)Qy(0) + u^T(0)Ru(0) \\
&\qquad + \inf_{\substack{u[1,\cdot) \\ \text{s.t. } y(1)=A\bar{y}+Bu(0)}} \lim_{N\to\infty} \\
&\qquad \Bigg\{ \sum_{k=1}^{N-1} \{y^T(k)Qy(k) + u^T(k)Ru(k)\} \\
&\qquad + y^T(N)Q_f y(N) \Bigg\}\Bigg\}.
\end{aligned}$$

Now, due to the time-invariance property of the problem, we can rewrite this last expression as follows:

$$\inf_{u(0)} \{\bar{y}^T Q \bar{y} + u^T(0)Ru(0) + (A\bar{y} + Bu(0))^T K (A\bar{y} + Bu(0))\}.$$

It is well known that this last problem has a unique solution $u^*(0)$ for an arbitrary $\bar{y}$ iff $R + B^T KB$ is positive definite; moreover, this solution is given by

$$u^*(0) = -(R + B^T KB)^{-1} B^T KA\bar{y}.$$

Substitution of this optimal control into the last expression gives

$$\bar{y}^T K \bar{y} = \bar{y}^T Q \bar{y} + \bar{y}^T A^T \{K - KB(R + B^T KB)^{-1} B^T K\} A\bar{y}.$$

Since $\bar{y}$ is arbitrary, we conclude that K satisfies the algebraic Riccati equation,

$$\text{(ARE)} \quad K = A^T\{K - KB(R + B^TKB)^{-1}B^TK\}A + Q.$$

Furthermore we recall the property that $R + B^TKB$ is positive definite in this equation. Some elementary rewriting yields (see Ref. 24) that this (ARE) can be rewritten as

$$X + \bar{A}^TX^{-1}\bar{A} = \bar{Q},$$

where

$$\bar{A} := RB^{-1}AB,$$

$$\bar{Q} := B^TA^TB^TRB^{-1}AB + R + B^TQB.$$

Therefore, Lemma 2.1 yields condition (i) together with the condition $\psi(z) \geq 0$, for $|z| = 1$. So, what is left to be shown is that $\psi(z)$ has no roots on the unit circle. To prove this, we first note that application of the above-mentioned optimal control sequence yields the closed-loop system

$$y(k+1) = (I - B(R + B^TKB)^{-1}B^TK)Ay(k), \qquad y(0) = \bar{y}.$$

Since, by assumption, $y(k) \to 0$ for every $\bar{y}$, we conclude that the spectrum of

$$(I - B(R + B^TKB)^{-1}B^TK)A$$

$$[\text{i.e., } \sigma((I - B(R + B^TKB)^{-1}B^TK)A)]$$

is contained in the open unit disk.

Now,

$$\begin{aligned} &\sigma((I - B(R + B^TKB)^{-1}B^TK)A) \\ &= \sigma((I - B(R + B^TKB)^{-1}(B^TKB + R - R)B^{-1})A) \\ &= \sigma(B(R + B^TKB)^{-1}RB^{-1}A) \\ &= \sigma((R + B^TKB)^{-1}RB^{-1}AB). \end{aligned}$$

Consequently, we have that

$$R + B^TKB + zRB^{-1}AB \text{ is invertible for } |z| \leq 1.$$

Noting finally that

$$\begin{aligned} \psi(z) = {} & (R + B^TKB + (1/z)B^TA^TB^{-T}R) \\ & \times (R + B^TKB)^{-1}(R + B^TKB + zRB^{-1}AB), \end{aligned}$$

we see that $\psi(z)$ has no roots on the unit circle, i.e.,

$$\psi(z) > 0, \qquad \text{for } |z| = 1.$$

($\Leftarrow$) Both conditions imply (see Lemma 2.1) that (ARE) has a largest solution K_L satisfying

$$R + B^T K_L B > 0,$$

which has the additional property that

$$R + B^T K_L B + zRB^{-1}AB \text{ is invertible for } |z| < 1.$$

Moreover, since

$$\psi(z) = (R + B^T K_L B + (1/z) B^T A^T B^{-T} R) \times (R + B^T K_L B + zRB^{-1}AB) > 0, \qquad \text{for } |z| = 1,$$

we conclude that

$$\sigma((R + B^T K_L B)^{-1} RB^{-1}AB)$$

is contained in the open unit disc. So, with

$$F = -(R + B^T K_L B)^{-1} B^T K_L A,$$

we have that all eigenvalues of the matrix $A + BF$ are located in the open unit disc.

To prove now that Problem P1 has a unique solution, one can use standard arguments, like, e.g., completion of the square; see, e.g., Engwerda (Ref. 24, Theorem 14). □

Remark 2.1. In fact, we showed above that, under the stated condition, the optimal control solving Problem P1 is given by

$$u(k) = -(R + B^T K_L B)^{-1} B^T K_L A x(k),$$

where K_L is the largest solution of (ARE).

From Remark 2.1, it is clear that the largest solution K_L of (ARE) plays a crucial role in calculating the optimal control. Moreover, we will see in Section 4 that the smallest solution K_S of (ARE) plays a crucial role in the problems to be solved there. Therefore, the question arises how these solutions can be determined.

We provide here two algorithms to calculate them. One algorithm gives a recurrence relation which approximates the exact solution. The other algorithm describes how the exact solution can be calculated by using a factorization approach. Detailed proofs of both algorithms can be deduced from existing results elsewhere in the literature; see Engwerda et al. (Ref. 11, Section 4) and Rozanov (Ref. 25, Theorem 10.1).

Algorithm A1. Consider the recurrence equation

$$K(0)=Q+A^TB^{-T}RB^{-1}A,$$

$$K(n+1)=A^T\{K(n)-K(n)B(R+B^TK(n)B)^{-1}B^TK(n)\}A+Q.$$

If (ARE) has a solution, then $K(n)$ converges, monotonically decreasing to K_L.

Algorithm A2. Consider the following algorithm to calculate the smallest solution of (ARE).

Step 1. See (i) to (iv) below.

(i) If $A_{11}:=RB^{-1}AB$ is invertible, then go to Step 2 of this algorithm.

(ii) Else, apply a unitary transformation T such that

$$A_{11}=T^T\begin{bmatrix}\tilde{A}_{11} & 0\\ \tilde{A}_{21} & 0\end{bmatrix}T.$$

(iii) If $\tilde{A}_{11}=0$, then

$$X_S:=T\begin{bmatrix}I-\tilde{A}_{21}^T\tilde{A}_{21} & 0\\ 0 & I\end{bmatrix}T,$$

and the algorithm stops.

(iv) Else,

$$X_S:=T^T\begin{bmatrix}Y_S & 0\\ 0 & I\end{bmatrix}T,$$

with $Y_S>0$ determined by repeating the algorithm in (i), with A_{11} replaced by

$$(I-\tilde{A}_{21}^T\tilde{A}_{21})^{-1/2}\,\tilde{A}_{11}(I-\tilde{A}_{21}^T\tilde{A}_{21})^{-1/2}.$$

Step 2. Consider the recurrence equation

$$Y(0):=A_{11}A_{11}^T,$$

$$Y(n+1):=A_{11}(I-Y(n))^{-1}A_{11}^T.$$

Then, $Y(n)$ converges, monotonically increasing to X_S. If (ARE) has a solution, then

$$K_S=B^{-T}(X_S-R)B^{-1}.$$

Algorithm A3. Consider $\psi(z)$. Factorize $\psi(z)$ as $(Q_0+Q_1 z)(Q_0^*+Q_1^*(1/z))$ using the following procedure:

(i) Make a $L(z)D(z)L^*(1/z)$ factorization of $\psi(z)$.
(ii) Use the factorization in (i) to factorize $\psi(z)$ as $Q_1(z)Q_1^*(1/z)$, where $Q_1(z):=n(z)L(z)D_1(z)$ is analytic in the open unit disc; here, $n(z)$ = product of all denominators of $L(z)$ and $D_1(z)$ is such that $D(z)/n(z)n^*(1/z)=D_1(z)D_1^*(1/z)$.
(iii) Cancel the zeros of $Q_1(z)$ which are inside the open unit disc by factorizing $\psi(z)$ as $Q_2(z)Q_2^*(1/z)$, where $Q_2(z)=Q_1(z)U\Lambda(z)$ is analytic in the open unit disc; here, $U\in\mathbb{C}^{n\times n}$ is a unitary matrix and $\Lambda(z)$ is an appropriate rational matrix satisfying $\Lambda(z)\Lambda^*(1/z)=I$.

Then,

$$K_L=B^{-T}(Q_0Q_0^*-R)B^{-1}.$$

The smallest solution K_S can be obtained by following the same procedure. Step (iii) has to be replaced by the next step:

(iii′) Cancel all zeros of $Q_1(z)$ which are outside the closed unit disc. This yields a factorization of $\psi(z)$ into $(\tilde{Q}_0+\tilde{Q}_1 z)(\tilde{Q}_0^*+\tilde{Q}_1^*(1/z))$. Then,

$$K_S=B^{-T}(\tilde{Q}_0\tilde{Q}_0^*-R)B^{-1}.$$

3. Finite Planning Horizon Case

As an introduction to the following section, we consider here the following problem.

Problem P2. Find necessary and sufficient conditions under which the optimal control problem

$$\inf_{u[0,N-1]} J(N), \qquad \text{s.t. } y(k+1)=Ay(k)+Bu(k), \quad y(0)=\bar{y},$$

and matrix B invertible, has a unique solution for every $\bar{y}$.

From, e.g., Engwerda (Ref. 26, Theorem 4), the following result is easily obtained [see also Rappaport et al. (Ref. 27)].

Theorem 3.1. Problem P2 has a unique solution iff

$$R+B^T K(k)B>0, \qquad \text{for } k=1,\ldots,N.$$

Here, $K(k)$ is given by the backward recurrence equation:

$$\text{(RRE)} \quad K(k-1)=A^T\{K(k)-K(k)B(R+B^T K(k)B)^{-1}B^T K(k)\}A+Q,$$
$$K(N)=Q_f.$$

The disadvantage of this solution is that, in particular for a large planning horizon, the verification of these conditions is a cumbersome job. So, the question arises whether it is possible to present arguments from which we can directly conclude whether the solution for a given planning horizon exists. The next example shows that the condition that the infinite planning horizon problem has a solution (either stabilizing the closed-loop system or not) is not sufficient to conclude that this problem is solvable.

Example 3.1. Consider the scalar Problem P2 with a planning horizon

$$N=1, \qquad A=4, \qquad B=2, \qquad Q=Q_f=-1, \qquad R=1.$$

According to Theorem 3.1, this problem has a solution iff $R+B^TQB$ is positive definite. Obviously, this condition is not satisfied here. On the other hand, it is easily verified that $Y(z)$ is regular and positive on the unit circle. So, the infinite planning horizon problem has a solution, whereas the finite planning horizon problem has no solution. So, the solution to this problem is far from trivial.

Note that, again due to the time-invariance property of the problem, we have that, whenever a problem with a planning horizon N has a solution, all problems with a smaller planning horizon will also be solvable. This implies that, whenever we are able to find conditions which guarantee that a problem with a large planning horizon has a solution, these conditions are also sufficient to conclude that any problem with a smaller planning horizon has a solution.

This motivates the problem that we study in the next section, that is, to find necessary and sufficient conditions such that Problem P2 has a solution for an arbitrarily long planning horizon.

4. Arbitrarily Long Finite Planning Horizon Case

As motivated in the previous section, we consider here the following problem.

Problem P3. Find necessary and sufficient conditions such that, $\forall N \in \mathbb{N}$,

$$\inf_{u[0,N-1]} J(N), \qquad \text{s.t. } y(k+1) = Ay(k) + Bu(k), \quad y(0) = \bar{y},$$

and matrix B invertible, has a unique solution for every $\bar{y}$.

Using the time-invariance property of the problem, the problem can be reformulated algebraically as Problem P4 below.

Problem P4. Find necessary and sufficient conditions such that

$$R + B^T K(k) B > 0, \qquad \text{for all } k \in \mathbb{N},$$

where $K(k)$ satisfies the following recurrence equation:

$$\text{(RRE)}' \quad K(k+1) = A^T\{K(k) - K(k)B(R + B^T K(k) B)^{-1} B^T K(k)\} A + Q,$$
$$K(0) = Q_f.$$

In order to simplify the analysis, we reformulate the problem once again. For this purpose, we need two preliminary results.

Lemma 4.1. Two necessary conditions for Problem P3 to have a solution are:

(i) $R + B^T Q_f B > 0$;
(ii) $R + B^T Q B + B^T A^T B^{-T} R B^{-1} A B > 0$.

Proof. The first condition follows immediately by considering the case $k = 0$ in the problem statement.

The second condition follows by considering $R + B^T K(1) B$. From the recurrence equation, we have that

$$\begin{aligned}
R + B^T K(1) B &= R + B^T Q B + B^T A^T Q_f A B \\
&\quad - B^T A^T Q_f B(R + B^T Q_f B)^{-1} B^T Q_f A B \\
&= R + B^T Q B + B^T A^T Q_f A B \\
&\quad - B^T A^T B^{-T}(-R + R + B^T Q_f B) \\
&\quad \times (R + B^T Q_f B)^{-1}(B^T Q_f B + R - R) B^{-1} A B \\
&= R + B^T Q B + B^T A^T B^{-T} R B^{-1} A B \\
&\quad - B^T A^T B^{-T} R(R + B^T Q_f B)^{-1} R B^{-1} A B.
\end{aligned}$$

Since by assumption both $R+B^TQ_fB$ and $R+B^TK(1)B$ are positive definite, the result follows now directly. □

Proposition 4.1. Using this result, we have that Problem P3 has a solution iff:

(i) $R+B^TQ_fB>0$;
(ii) $M:=R+B^TQB+B^TA^TB^{-T}RB^{-1}AB>0$;
(iii) $X(k)>0$, for all $k\in\mathbb{N}$, where $X(k)$ satisfies the recurrence equation

$$X(k+1)=I-F^TX^{-1}(k)F,$$

$$X(0)=M^{-1/2}(R+B^TQ_fB)M^{-1/2},$$

$$F:=M^{-1/2}RB^{-1}ABM^{-1/2}.$$

Proof. The first two conditions are obvious from Lemma 4.1.

To show the third one, we note that, by substitution of (RRE)′, we have

$$\begin{aligned}R+B^TX(k+1)B&=R+B^TQB+B^TA^T\\&\quad\times\{K(k)-K(k)B(R+B^TK(k)B)^{-1}B^TK(k)\}AB\\&=R+B^TQB+B^TA^T\\&\quad\times\{K(k)-B^{-T}(-R+R+B^TK(k)B)\\&\qquad\times(R+B^TK(k)B)^{-1}\\&\qquad(R+B^TK(k)B-R)B^{-1}\}AB\\&=M-B^TA^TB^{-T}R(R+B^TK(k)B)^{-1}RB^{-1}AB.\end{aligned}$$

Using the fact that $M>0$, it is clear now that

$$X(k)=M^{-1/2}(R+B^TK(k)B)M^{-1/2}$$

satisfies the third condition.

The "if" part of the theorem follows similarly by showing via induction that

$$\tilde{K}(k):=B^{T^{-1}}(M^{1/2}X(k)M^{1/2}-R)B^{-1}$$

satisfies (RRE)′. □

Since we are considering an arbitrary planning horizon, it is not surprising that, in solving this problem, the algebraic equation corresponding to this problem,

$$X+F^TX^{-1}F=I, \tag{3}$$

plays a crucial role. Note that this equation is of the type that we studied in Lemma 2.1.

To prove the main result of this section, we need the following lemma.

Lemma 4.2. Let $Z(k)$, $k\in\mathbb{N}$, satisfy the recurrence equation:

$$Z(k+1)=I-F^TZ^{-1}(k)F, \qquad Z(0)=\bar{Z}.$$

Assume that $\bar{Z}>0$ and $Z(k)>0$, for all $k\in\mathbb{N}$. Then, the following results hold:

(i) $Z(k)>FF^T$, $\forall k\in\mathbb{N}$;
(ii) $Z(k)>\beta I$, for some $\beta>0$, $\forall k\in\mathbb{N}$.

Proof.

(i) This follows immediately from the identities

$$Z^{-1}(k)+Z^{-1}(k)FZ^{-1}(k+1)F^TZ^{-1}(k)$$
$$=Z^{-1}(k)-Z^{-1}(k)F(F^TZ^{-1}(k)F-I)^{-1}F^TZ^{-1}(k)$$
$$=[Z(k)-FF^T]^{-1}.$$

The last identity follows from the matrix inversion lemma; see, e.g., Kailath (Ref. 28, pp. 656).

(ii) In case F is invertible, this result follows immediately from (i). If F is not invertible, the claim can be proved by using Algorithm A2, with $A_{11}:=F$, to reduce the problem to a recurrence equation with a nonsingular matrix A_{11}. From the algorithm and the fact noted above that, for a nonsingular matrix F, the claim holds, the result follows then directly. □

Theorem 4.1. Problem P3 has a unique solution iff

(i) $\psi(z):=B^TQB+(I+z^*B^TA^TB^{-T})R(I+zB^{-1}AB)$ is regular;
(ii) $\psi(z)\geq 0$, for $|z|=1$;
(iii) $Q_f\geq K_S$, where K_S is the smallest solution of (ARE).

The proof of this theorem can be found in Appendix A (Section 6).

Note that condition (iii) in this theorem can always be analytically verified, since we can calculate the exact solution K_S of (ARE) using Algorithm A3. We believe, however, that it is more elegant to replace this condition by one or more solvability conditions, which are expressed in direct terms of the system parameters.

For SISO systems (i.e., all system parameters are scalars), this problem was solved by Engwerda in Ref. 7 by elaborating condition (iii) analytically, which is possible since we can calculate the explicit analytical solutions of (ARE). It was shown that, in case $Q_f = Q$, the necessary and sufficient conditions for existence of a unique solution are:

Case $|A| < 1$: $\psi(z)$ satisfies conditions (i) and (ii) of Theorem 4.1;
Case $|A| = 1$: $R + B^T QB > 0$, $B^T QB > 0$, $4R + B^T QB \geq 0$;
Case $|A| > 1$: $\psi(z)$ satisfies conditions (i) and (ii) of Theorem 4.1 and $B^T QB \geq 0$.

The generalization for MIMO systems is, however, unclear.

Finally note that, whenever $R \leq 0$, condition (iii) in Theorem 4.1 is trivially satisfied in case $Q_f \geq Q$. Indeed, since

$$R + B^T K_S B = -B^T A^T B^{-T} R (R + B^T K_S B)^{-1} R B^{-1} AB + B^T A^T B^{-T} R B^{-1} AB + R + B^T QB,$$

we have that

$$R \leq 0 \quad \text{and} \quad R + B^T K_S B > 0$$

imply that

$$R + B^T K_S B \leq R + B^T QB.$$

Together with the fact that, whenever

$$R \geq 0, \qquad Q \geq 0, \qquad R + B^T QB > 0,$$

the problem is solvable, we conclude that condition (iii) in Theorem 4.1 becomes effective iff either the weight matrix R is really indefinite, the matrix R is semipositive, and Q is indefinite on $Q_f \ngeq Q$.

5. Conclusions

In this paper, we presented necessary and sufficient conditions under which three different indefinite LQ-problems have a unique solution.

The first problem was to find a unique control infimizing a quadratic cost functional over an infinite planning horizon under the additional restriction that the control should also stabilize the system. The presented results correspond completely with existing results in this area. The solvability conditions were presented in the frequency domain. It was shown that the problem has a solution iff some frequency function, which depends only on the system parameters, is both regular and positive definite on the unit circle. We showed that, under these conditions, the optimal control is given by a state feedback control in which the largest solution of the algebraic Riccati equation plays a crucial role. Two algorithms were presented to calculate this solution, one which approximates this largest solution and one to calculate this solution exactly.

The second problem was in fact a preamble for the third problem. Here, we characterized conditions under which the infimization of a quadratic cost functional over a finite planning horizon yields a unique control. Furthermore, we showed in an example that, whenever the infinite planning horizon problem has a solution, this does not yet imply that the finite planning horizon problem also has a solution.

Since in particular for a large planning horizon, the verification of the solvability conditions is a cumbersome job, we considered in the third problem conditions under which the finite planning horizon problem has a unique solution for an arbitrary planning horizon. After some reformulation, we saw that the solvability conditions for this problem are closely related to those of our first problem.

Again, the solvability of the algebraic Riccati equation plays a crucial role. However, contrary to the solution of our first problem, it is not a condition on the largest solution, but one on the smallest solution of the Riccati equation which determines whether or not the problem is solvable. We concluded the section on this problem by considering a number of special cases in which this condition on the smallest solution either is trivially satisfied or is reformulated in terms of just the system parameters.

All results presented in this paper were derived under the assumption that matrix B is invertible. Obviously, this is a rather stringent condition, which is usually not satisfied, and therefore should be relaxed. It will be clear from the analysis that this is not a trivial job. However, we believe that both the presented analysis and the obtained solvability conditions may be helpful in solving the general problem.

As already pointed out at the end of Section 4, another open problem is to reformulate the solvability conditions for the third problem in more direct terms of the system parameters. This would probably give us more insight into the basics of the problem and might lead to conditions which are easier to verify.

6. Appendix A: Proof of Theorem 4.1

In this appendix, we use the notation of Proposition 4.1.

(⇒) Consider the recurrence equation

$$P(k+1)=I-F^TP^{-1}(k)F, \qquad P(0)=\alpha I, \tag{4}$$

where $\alpha \geq 1$ is such that $X(0) \leq \alpha I$. We now show by induction that $X(k) \leq P(k)$.

For $k=0$, the inequality holds by construction. Next, assume that

$$X(k) \leq P(k).$$

Note that this implies that $P(k)>0$ and, more in particular, that

$$X^{-1}(k)-P^{-1}(k) \geq 0.$$

The rest of the induction argument follows now immediately using the definition of $X(k+1)$ and $P(k+1)$, respectively.

Note that the above inequality in particular implies [see Lemma 4.2(ii)] that

$$P(k)>\beta I, \qquad \text{for some } \beta>0, \quad \forall k \in \mathbb{N}.$$

We next show that $P(k)$ is a monotonically decreasing sequence. From both these observations, we can then conclude that $P(k)$ converges to a positive-definite limit. Since $P(k)$ satisfies the recurrence equation (4), we obtain that its limit satisfies the equation

$$P=I-F^TP^{-1}F.$$

Conditions (i) and (ii) follow then directly from Lemma 2.1.

The monotonicity of $P(k)$ is proved again by induction. Since

$$P(1)-P(0)=(1-\alpha)I-(1/\alpha)F^TF,$$

the initialization part is obvious. The induction argument follows using the definition of $P(k)$, i.e.,

$$\begin{aligned} P(k+1)-P(k) &= I-F^TP^{-1}(k)F-(I-F^TP^{-1}(k-1)F) \\ &= F^T(P^{-1}(k-1)-P^{-1}(k))F \leq 0. \end{aligned}$$

Condition (iii) of the theorem is equivalent to the assertion that

$$X(0) \geq X_S,$$

where X_S is the smallest solution of Eq. (3). To prove this result, we use the algorithm presented in Algorithm A2 with

$$A_{11}=F.$$

We show that, with this choice of A_{11},

$$Y(k)\leq X(0), \qquad \forall k\in\mathbb{N}.$$

Since $Y(k)$ converges, monotonically increasing to X_S we automatically obtain $X(0)\geq X_S$. First, we note from Algorithm A2 that the basic problem reduces to a case in which the matrix A_{11} is invertible. So, without loss of generality, we may assume that the matrix F is invertible.

So, assume that $Y(k)$ is given by the recurrence equation

$$Y(k+1)=F(I-Y(k))^{-1}F^T, \qquad Y(0)=FF^T.$$

We now show that the assertion

$$Y(k+1)\leq X(0)$$

is equivalent to the claim that

$$Y(0)\leq X(k+1).$$

To prove this, we first show that, for a fixed N and $0\leq i\leq N$,

$$Y(N-i)\leq X(i), \qquad \text{iff } Y(N-i-1)\leq X(i+1).$$

From the definition of $Y(N-i)$, we have that

$$Y(N-i)\leq X(i), \qquad \text{iff } F(I-Y(N-i-1))^{-1}F^T\leq X(i).$$

Since F is invertible, obviously the last inequality holds iff

$$(I-Y(N-i-1))^{-1}\leq F^{-1}X(i)F^{-T},$$

or equivalently,

$$F^TX^{-1}(i)F\leq I-Y(N-i-1).$$

Rewriting this last expression yields the advertised statement that

$$Y(N-i-1)\leq I-F^TX^{-1}(i)F=X(i+1).$$

That the statement

$$Y(k+1)\leq X(0)$$

is equivalent to the statement

$$Y(0)\leq X(k+1)$$

follows now easily using this relation inductively. This part of the proof of the theorem is completed by finally noting that the assertion

$$Y(0) \leq X(k+1)$$

holds according to Lemma 4.2(i).

($\Leftarrow$) We show by induction that

$$X(k) - X_S \geq 0.$$

Since $X_S > 0$ (see Lemma 2.1), we then immediately have that

$$X(k) \geq X_S.$$

That $X(0) \geq X_S$ follows directly from condition (iii). So, let us assume now that $X(k) \geq X_S$. Then, $X(k+1) - X_S$ equals $I - F^T X^{-1}(k) F - (I - F^T X_S^{-1} F)$, which can be rewritten as $F^T[X_S^{-1} - X^{-1}(k)]F$. Obviously, this last expression is semipositive definite. □

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Globally and Superlinearly Convergent Trust-Region Algorithm for Convex SC^1-Minimization Problems and Its Application to Stochastic Programs[1,2]

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Abstract. A function mapping from $\mathscr{R}^n$ to $\mathscr{R}$ is called an SC^1-function if it is differentiable and its derivative is semismooth. A convex SC^1-minimization problem is a convex minimization problem with an SC^1-objective function and linear constraints. Applications of such minimization problems include stochastic quadratic programming and minimax problems. In this paper, we present a globally and superlinearly convergent trust-region algorithm for solving such a problem. Numerical examples are given on the application of this algorithm to stochastic quadratic programs.

Key Words. Trust-region algorithms, global convergence, superlinear convergence, stochastic quadratic programs.

1. Introduction

In this paper, we consider the following linearly constrained convex minimization problem:

$$\min f(x), \qquad \text{s.t. } x \in X, \tag{1}$$

where $f: \mathscr{R}^n \to \mathscr{R}$ is a convex differentiable function and $X = \{x: Ax \leq b\}$. For convenience of notation, we use both g and ∇f to denote the derivative of

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