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Balanced games arising from infinite linear models

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Abstract. Kalai and Zemel introduced a class of flow-games showing that these games have a non-empty core and that a minimum cut corresponds to a core allocation.

We consider flow-games with a finite number of players on a network with infinitely many arcs: assuming that the total sum of the capacities is finite, we show the existence of a maximum flow and we prove that this flow can be obtained as limit of approximating flows on finite subnetworks.

Similar results on the existence of core allocations and core elements are given also for minimum spanning network models (see Granot and Huberman) and semi-infinite linear production models (following the approach of Owen).

Key words: Balancedness, semi-infinite linear models

Introduction

Much work has been done in cost and reward sharing, e.g. in network problems and linear production situations, where we minimize costs or maximize rewards.

These problems can be cast into the framework of cooperative games (with side payments): there are three relevant cases in which it is known that these games are balanced (i.e. have a non-empty core).

We are referring in particular to Kalai and Zemel (1982), Granot and

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Huberman (1981 and 1984), Owen (1975) and Curiel, Derks and Tijs (1989); a survey is offered in Tijs (1992).

Our main goal is to see whether the balancedness conditions still hold when some of the finiteness conditions are dropped: this paper considers problems where one of the factors in the problem is countably infinite (the number of players is always considered to be finite).

In particular we deal with the following three questions:

Q1 What can be said about balancedness if there is an infinite number of arcs in flow situations?
Q2 What can be said about balancedness if there is an infinite number of points to be connected to a source at a minimum cost?
Q3 What can be said about balancedness if there is an infinite number of possible products in linear production situations?

It turns out that all these problems give rise (under certain conditions) to balanced games; it is also indicated how to find (approximate) core elements.

For Q1, due to the presence of infinitely many arcs in the network, the problem of max-flow and min-cut is not a standard one and we give a set of conditions allowing the non-ambiguous definition of a flow.

Concerning Q2 we solve the problem not by looking at a minimum spanning tree but by allowing the presence of cycles at low costs.

In Q3 we have the problem of a possible duality gap in semi-infinite programming. We give sufficient conditions in order that such a gap does not appear and the result is a construction of a core element à la Owen.

Also an approximation with finite programs is possible.

Basic elements

An *n-person game with transferable utility* (TU-game) in characteristic form is a couple $\langle N, v\rangle$ where $N = \{1, \dots, n\}$ is the set of players and $v : \wp(N) \to R$ is the characteristic function that assigns to each subset $S \subseteq N$, the value $v(S)$ that represents the worth of the *coalition* S; in particular $v(\varnothing) = 0$. When the values assigned to the coalitions represent the cost of the coalition we denote it by $c(S)$ and call it the *cost game* $\langle N, c\rangle$. For every game $\langle N, v\rangle$ it is possible to define the corresponding cost game $\langle N, c\rangle$, where $c(S) = -v(S)$.

Given a game $\langle N, v\rangle$ an *imputation* is a vector $x \in R^n$ that satisfies the following conditions:

$$x_i \geq v(\{i\}) \quad \forall i \in N$$

$$\sum_{i \in N} x_i = v(N)$$

The *core* of a game $\langle N, v\rangle$ is the subset of the imputation set that satisfy the following conditions:

$$\sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N$$

For a cost game $\langle N, c \rangle$ the conditions for an imputation can be written as:

$$x_i \leq c(\{i\}) \quad \forall i \in N$$

$$\sum_{i \in N} x_i = c(N)$$

and the further conditions for the core can be represented as

$$\sum_{i \in S} x_i \leq c(S) \quad \forall S \subseteq N$$

Flow Games

As said in the introduction we are interested in flows on networks with an infinite number of nodes and arcs, considering a finite set of players who own the arcs. To fix the setting in which we are working, we give some definitions.

Definition 1. *A* **network**, *with privately owned arcs, is given by a 7-tuple:*

$$H = (M, A, f, N, p, S_0, P_0)$$

where: M is a countable set whose elements are called nodes;
A is a collection of elements of the set $\{(i,j) \in M \times M \mid i \neq j\}$ called arcs;
$f : A \to]0, +\infty[$ is a map that assigns to each arc $a \in A$ its capacity $f(a)$;
N is a finite set whose elements are called players;
$p : A \to N$ is a map describing the ownership of the arcs, i.e. $p(a)$ is the player that owns arc a;
$S_0, P_0 \in M$, with $\omega(S_0)_- = \varnothing$ and *$\omega(P_0)_+ = \varnothing$ are special nodes called source and sink respectively.* ◇

Remark 1.

- $G(M, A)$ is an oriented graph with a countable set of nodes and without loops.
- $\omega(m)_-$ and $\omega(m)_+$ denote respectively the set of arcs entering and leaving the node $m \in M$.
- In the standard situation a network has a finite number of nodes and arcs; in this new situation we have to redefine some related concepts. ◇

Given a network H, a (feasible) *flow* φ on H is a map:

$$\varphi : A \to [0, +\infty[$$

satisfying the conditions:

$$\varphi(a) \leq f(a) \qquad \forall a \in A \qquad (*)$$

$$\sum_{a \in \omega(m)_-} \varphi(a) = \sum_{a \in \omega(m)_+} \varphi(a) \quad \forall m \in M \backslash \{S_0, P_0\} \qquad (**)$$

Given a network H and a flow φ on H, the *value of* φ, denoted by $w(\varphi)$, is:

$$w(\varphi) = \sum_{a \in \omega(S_0)_+} \varphi(a)$$

The value of a maximum flow on a network H is denoted by $w^*(H)$.

Hypothesis 1. We assume that the sum of the capacities over all of the arcs, or the total capacity, is finite, i.e.:

$$\sum_{a \in A} f(a) < +\infty \tag{1}$$

Proposition 1. *Given a network H, satisfying (1),* $w(\varphi) < +\infty$ *for every flow* φ *on H.*

Proof. $w(\varphi) = \sum_{a \in \omega(S_0)_+} \varphi(a) \leq \sum_{a \in \omega(S_0)_+} f(a) \leq \sum_{a \in A} f(a) < +\infty.$ ◇

Remark 2. It is possible to prove (by a standard approximation argument) that $\sum_{a \in \omega(S_0)_+} \varphi(a) = \sum_{a \in \omega(P_0)_-} \varphi(a)$. So, as in the finite case, we could equivalently have defined the value of the flow as the total flow entering the sink. ◇

Proposition 2. *Given a network H satisfying (1), let us denote by* Φ *the set of feasible flows* φ *on H. Then there exists a flow* $\varphi_0 \in \Phi$ *s.t.* $w(\varphi_0) \geq w(\varphi)$ *for all* $\varphi \in \Phi$.

Proof. Note first that (1) and condition $(*)$ in the definition of the flow guarantee that Φ may be identified with a subset of $\ell^1 = \ell^1(N)$, where ℓ^1 is the set of infinite sequences $(a_1, a_2, \ldots)$ of real numbers with $\sum_{i \geq 1} |a_i| < +\infty$. Conditions $(*)$ and $(**)$ guarantee that it is a closed subset of ℓ^1. Moreover condition (1) ensures that Φ is compact (see exercise 3, page 338 in Dunford and Schwartz, 1958).

Furthermore $w : \Phi \to R$ is a Lipschitz continuous function. Hence, by Weierstrass theorem there exists a maximum flow. ◇

Now let $S \subseteq N$. If we consider only arcs "owned" by S (i.e. belonging to $p^{-1}(S)$), we get a subnetwork H_S (clearly $H_N = H$).

We define a side-payment game $\langle N, v_H \rangle$ assigning to each coalition $S \subseteq N$ the maximum value of the flow on the subnetwork H_S, i.e. $v_H(S) = w^*(H_S)$. We shall prove the following theorem.

Theorem 1. *Given a network H satisfying (1), the game* $\langle N, v_H \rangle$ *has a nonempty core.*

Proof. We consider for every $\varepsilon > 0$ the subnetwork H_ε constructed in the following way.

Order A in a decreasing way w.r.t. capacity (ties are broken "ad libitum"). Take $\varepsilon > 0$ and consider the network consisting of the minimum number of arcs, according with the order introduced above, s.t. their total capacity is greater than $\sum_{a \in A} f(a) - \varepsilon$. This network has a finite number of arcs. Delete the nodes (different from source and sink) of this network that are isolated. This will give us the finite network H_ε.

By Kalai and Zemel (1982) the game $\langle N, v_{H_\varepsilon} \rangle$ has non-empty core. Furthermore, there is a minimum cut with capacity $w^*(H_\varepsilon)$ that gives us an allocation $(x_1^\varepsilon, \ldots, x_n^\varepsilon)$ belonging to Core (v_{H_ε}), where x_i^ε is the sum of the capacities of the arcs owned by player i in the minimum cut.

We have:

$$x_i^\varepsilon \geq 0 \quad \forall \varepsilon > 0$$

$$\sum_{i=1}^{n} x_i^\varepsilon = w^*(H_\varepsilon) \leq w^*(H)$$

So $\{(x_1^\varepsilon, \ldots, x_n^\varepsilon) \mid \varepsilon > 0\}$ lies in a compact set and we can find a subsequence $(x_1^{\varepsilon_k}, \ldots, x_n^{\varepsilon_k})$ converging to some $(\bar{x}_1, \ldots, \bar{x}_n) \in R^n$.

We shall prove that $(\bar{x}_1, \ldots, \bar{x}_n) \in \text{Core}(v_H)$.

Note that $\sum_{i=1}^{n} x_i^\varepsilon = w^*(H_\varepsilon)$ and $w^*(H_\varepsilon) \leq w^*(H) \leq w^*(H_\varepsilon) + \varepsilon$, that is:

$$w^*(H) - \varepsilon \leq w^*(H_\varepsilon) \leq w^*(H) \tag{2}$$

Taking the limit for $\varepsilon \to 0$, we get: $\sum_{i=1}^{n} \bar{x}_i = w^*(H)$.

Because a relation analogous to (2) holds also for the subnetworks H_S (and $H_{\varepsilon,S}$), and $\sum_{i \in S} x_i^\varepsilon \geq w^*(H_{\varepsilon,S})$, we get in the same way: $\sum_{i \in S} \bar{x}_i \geq w^*(H_S)$.

So, $(\bar{x}_1, \ldots, \bar{x}_n) \in \text{Core}(v_H)$. ◇

In general, we cannot guarantee that the whole sequence provided by our method of proof converges to an allocation in $\text{Core}(v_H)$. The example of Fig. 1 shows that it can be necessary to consider a subsequence $(x_1^{\varepsilon_k}, \ldots, x_n^{\varepsilon_k})$ of allocations:

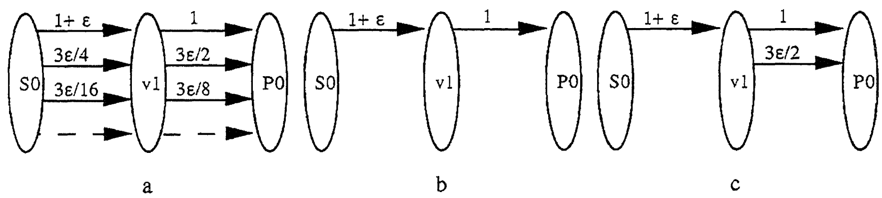

Fig. 1.

In Fig. 1-a is the network H, where player 1 is the owner of the arcs from S_0 to v_1, and player 2 is the owner of the arcs from v_1 to P_0.

The minimum cut of the subnetwork in Fig. 1-b contains the arc (v_1, P_0) and the corresponding allocation is $(0, 1)$.

The minimum cut of the subnetwork in Fig. 1-c contains the arc (S_0, v_1) and the corresponding allocation is $(1 + \varepsilon, 0)$.

In the subsequent subnetworks the minimum cut contains alternatively the arcs between (v_1, P_0) or between (S_0, v_1) and the corresponding allocations assign a null payoff to player 1 or 2 alternatively.

So, the sequence that we get in our proof does not converge.

Remark 3. Without Hypothesis 1, for our infinite network we cannot guarantee a behaviour similar to the finite case, as it is shown in the following examples.

– The conclusion of Proposition 1 does not hold, as is shown in the example of Fig. 2:

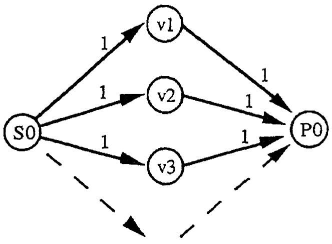

Fig. 2.

The optimal solution is an unitary flow on each arc, so we have:

$$w(\varphi) = \sum_{a \in \omega(S_0)_+} \varphi(a) = \sum_{i=1}^{+\infty} \varphi(S_0, v_i) = +\infty$$

The value of the flow as previously defined may not correspond to the intuitive idea of a flow from S_0 to P_0. Consider a unitary flow on each arc in the example of Fig. 3:

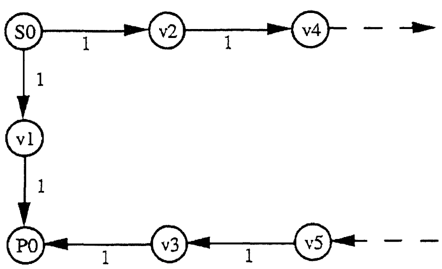

Fig. 3.

In this case $\sum_{a \in \omega(S_0)_+} \varphi(a) = \sum_{a \in \omega(P_0)_-} \varphi(a) = 2$, while we would expect a unitary flow on the path $S_0 - v_1 - P_0$. ◇

Connection problems and games

Now we devote our attention to the problem of spanning graphs in a weighted undirected graph with an infinite number of vertices and edges.

Definition 2. *A* **weighted undirected graph** *is described by a triple:*

$$U = (X, E, g)$$

where: X is a countable set whose elements are called vertices;
E is a collection of subsets of X with two different elements; the elements of E are called edges;
$g : E \longrightarrow [0, +\infty[$ is a map that assigns to each edge $e \in E$ its weight or cost $g(e)$. ◇

Remark 4.

- We consider graphs without loops.
- In the standard situation a graph has a finite number of vertices and edges and the costs may also be non positive. ◇

Given a weighted undirected graph U we identify a special vertex S_0 and consider a finite set of players that own the remaining vertices; we are interested in finding a subset of edges (i.e. a spanning graph) such that all of the vertices in X are connected, while the total cost of the edges in the subset is minimal.

Definition 3. *A* **minimum cost spanning graph (m.c.s.g.) problem,** *with privately owned vertices, is given by a 6-tuple:*

$$K = (X, E, g, N, q, S_0)$$

where: X, E, g are as in definition 2;
N is a finite set whose elements are called players;
$q : X \longrightarrow N$ is a map describing the ownership of the vertices, i.e. $q(v)$ is the player that owns vertex v;
$S_0 \in X$ is a special vertex called source. ◇

As in the previous section we make a finiteness assumption.

Hypothesis 2 We assume that the graph U is connected and that the sum of the costs over all of the edges is finite, i.e.:

$$\sum_{e \in E} g(e) < +\infty$$ ◇

Remark 5. Contrary to Granot and Huberman (1981) we do not assume that the graph U is complete; clearly, adding "missing" edges with sufficiently high costs $\left(\text{e.g. } 2N \sum_{e \in E} g(e)\right)$, we could obtain a complete graph, without altering the problem that we are considering, except that Hypothesis 2 could be no longer valid. ◇

Given $W \subseteq X \backslash \{S_0\}$ we shall consider U_W, i.e. the set of all connected subgraphs (X', E'), where $X' = W \cup \{S_0\}$ and E' is a subset of E containing edges with vertices belonging to X'.

We define $c^*(W) = \inf\{k(E') | (X', E') \in U_W\}$, where $k(E')$ is the cost of the spanning graph (X', E'), i.e. $\sum_{e \in E'} g(e)$.

The meaning of $c^*(W)$ should be clear: it is the "minimum" cost for connecting all of the vertices in W with the source S_0.

It is possible that there exists no subgraph connecting all the vertices of X' because we do not assume that U is complete; in this case $c^*(W)$ will be infinite.

We shall now introduce the games $\langle N, c\rangle$ and $\langle N, \underline{c}\rangle$ as in Granot and Huberman (1981).

Given a non empty subset $S \subseteq N$, we define $\underline{c}(S) = \inf\{c^*(W) | W \supseteq q^{-1}(S)\}$ and $c(S) = c^*(q^{-1}(S))$, where $q^{-1}(S)$ is the set of vertices owned by the players in S.

Remark 6.

- Since we did not assume that the graph U is complete, clearly $c(S)$ can be infinite for some $S \neq N$, because the vertices of $q^{-1}(S)$ may generate a non connected subgraph. On the contrary, $\underline{c}(S)$ will be always a real number because the players in the game $\langle N, \underline{c}\rangle$ can use every vertex and every edge of the given graph.
- With the same arguments as in Granot and Huberman (1981) it is possible to prove that the game $\langle N, \underline{c}\rangle$ is monotonic (we call it m.m.c.s.g. game). ◇

Granot and Huberman (1981) proved that an imputation x in the core of the minimum cost spanning tree game is given by the rule of Bird (1976):

$$x_i = k(E_{i^-})$$

where an edge $e \in E_{i^-}$ if it is the edge entering in a vertex owned by i in the unoriented path originating from the source in the spanning tree.

$\langle N, c\rangle$ can be considered as a kind of generalized cooperative game with side payments (since $c(S)$ can be equal to $+\infty$), to which the usual definition of the core can be immediately extended: it turns out, furthermore, that as in Granot and Huberman (1981) the core of $\langle N, \underline{c}\rangle$ is contained in the core of $\langle N, c\rangle$. So, it will be sufficient to prove that core of $\langle N, \underline{c}\rangle$ is non-empty.

Given $\varepsilon > 0$, we shall use in the proof the same argument of Theorem 1. We order the edges in a decreasing way w.r.t. costs and, according to this order, we delete "expensive" edges until we are left with a set of edges E_ε s.t. $\sum_{e \in E_\varepsilon} g(e) < \varepsilon$.

Due to the connectedness assumption on the graph $U(X, E)$, the graph U_ε with edges set E_ε cannot have an infinite number of connected components (because the number of edges left out is finite). Consider these connected components $Y_1, \ldots, Y_{k_\varepsilon}$ of he graph U and contract each subset Y_i into a vertex y_i; so, any edge that had one vertex of Y_i as an endpoint (and the other endpoint outside Y_i) will now have this vertex y_i as endpoint; the ownership of y_i will be defined as the set of all the owners of some vertex in Y_i. Notice that in this way we have contracted a set of vertices $\left(\text{i.e. } \bigcup_{i=1}^{k_\varepsilon} Y_i\right)$ that may be spanned at a cost less than ε (the connected components are generated by a set of edges whose total cost is less than ε).

In this way we obtain a graph K_ε and the associated m.m.c.s.g. game $\langle N, c^\varepsilon \rangle$ which will have only a finite number of vertices and edges.

However, K_ε will present new features with respect to the model considered in Granot and Huberman (1981):

1) vertices may have more than one owner;
2) there may be multiple edges;
3) a player may own more than one vertex.

It is easy, however, to extend the result of Granot and Huberman (1981) to this class of finite graphs, as is shown here on.

1) The case of multiple owners of a vertex is easily treated just splitting the vertex into as many vertices as the number of owners, connecting them with edges of null cost, and duplicating the edges that had this vertex as an endpoint (the idea is shown in the example of Fig. 4).
2) If there are multiple edges connecting the same pair of vertices we retain just one of the edges of minimum cost and delete all of the remaining ones. In such a way, the original finite graph is transformed into another one that is equivalent to it from the point of view of connection at minimum cost.
3) When a player owns more than one vertex, it is easy to check that the proof of Theorem 3 in Granot and Huberman (1981) can be adapted to this context. Taking into account the way in which we have defined $\langle N, \underline{c} \rangle$, instead of looking at a coalition $S \supset R$ s.t. $\underline{c}(R) = c(S)$ as done in Granot and Huberman (1981), we have to consider a set of vertices W s.t. $\underline{c}(R) = c^*(W)$.

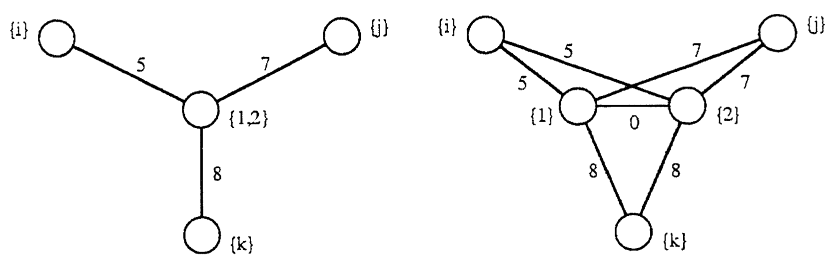

Fig. 4.

For what concerns the original problem, it is easy now to prove the following theorem.

Theorem 2. *Given a m.c.s.g. problem with a countable number of vertices, satisfying Hypothesis 2, the game* $\langle N, c\rangle$ *has non-empty core.*

Proof. We prove first that the m.m.c.s.g. game $\langle N, \underline{c}\rangle$ has non-empty core.

Given $\varepsilon > 0$, we prove that $\underline{c}(S) \leq \underline{c}^{\varepsilon}(S) + \varepsilon$ for each coalition S.

Actually, consider the m.m.c.s.g. in the finite graph $K_\varepsilon(S)$, i.e. the connected subgraph of K_ε containing the vertices owned by the players of S, whose cost may be denoted by $\underline{c}^{\varepsilon}(S)$.

Then, adding to $K_\varepsilon(S)$ all of the edges of the connected components Y_i (whose total cost is less than ε), we get a connected subgraph whose cost is less than $\underline{c}^{\varepsilon}(S) + \varepsilon$ (by definition it is the infimum of all of the costs allowing the connection to S_0 of all of the vertices owned by the players of S).

Conversely we have to prove that $\underline{c}^{\varepsilon}(S) \leq \underline{c}(S)$.

Consider a set of edges connecting all of the vertices owned by the players of S to the source in the given graph; put down to zero the cost of the edges in the shrunk connected components. The resulting graph connects all the vertices in the finite graph $K_\varepsilon(S)$. So, $\underline{c}^{\varepsilon}(S) \leq \underline{c}(S)$.

So, we have proved that for every coalition S:

$$\lim_{\varepsilon \to 0} \underline{c}^{\varepsilon}(S) = \underline{c}(S)$$

Due to the upper semicontinuity of the core multifunction and to the fact that the core of the game $\langle N, \underline{c}^{\varepsilon}\rangle$ is non-empty for any $\varepsilon > 0$, we get that also the game $\langle N, \underline{c}\rangle$ has a non-empty core.

That the game $\langle N, c\rangle$ has non-empty core follows immediately by the remark of Granot and Huberman (1981) that $c(N) = \underline{c}(N)$ and $c(S) \geq \underline{c}(S)$ for any coalition S. ◇

Under our hypothesis of finite total cost of the edges, the case in which the given graph is complete turns out to be a trivial one; in fact the infimum of the cost of the spanning graphs is zero. Namely, let be given $\varepsilon > 0$. We assign an order to the vertices of the graph, then we can connect the source with a vertex at cost less than $\varepsilon/2$, choosing the first vertex w.r.t. this order; next we can add an edge with cost less than $\varepsilon/4$ connecting the first remaining vertex and so on. In such a way we get a spanning graph whose cost is less than ε.

Linear production games

In this section we consider a situation in which we have a finite number of resources that can be used in infinitely many production processes, each one resulting in one product, with a finite number of players each owning a bundle of resources. Note that we consider that different processes result in different products, even if the products are equivalent.

Let us define first the finite production problem.

Definition 4. *A* **production problem** *is a 5-tuple:*

$$P = (m, n, A, b, c)$$

where: m *is the number of available resources;*
n *is the number of possible products;*
$A \in M_{m,n}$ *is the technology matrix;*
$b \in R^m$ *is the resource vector;*
$c \in R^n$ *is the price vector.*
A*,* b*,* c *have non negative entries.* ◇

Remark 7. A_{ij} is the quantity of resource i needed to produce one unit of product j; b_i is the available quantity of resource i; c_j is the market price of product j. ◇

We are interested in finding a production plan z using the available resources, with the given technology, in order to maximize the total revenue, i.e. a solution of the linear problem:

$$\max\{c^T z | Az \leq b, z \geq 0\}$$

In the related dual problem we look for a shadow-price of each resource, i.e. a solution of the linear problem:

$$\min\{b^T y | A^T y \geq c, y \geq 0\}$$

According to the duality theorem the maximum total revenue equals the minimum total shadow-value.

As in the previous section we associate to the production problem a cooperative game with side payments with a finite set N of players, assuming that each player i owns a vector of resources b^i.

Definition 5. *A* **linear production game** $\langle N, v\rangle$ *is defined as follows:*

$$v(S) = \max\{c^T z | Az \leq b^S, z \geq 0\} \quad \forall S \subseteq N$$

where b^S *is the vector of the resources owned by* S $\left(\text{i.e. } b^S = \sum_{i \in S} b^i\right)$*; in particular* $b^N = b$. ◇

Owen (1975) shows that an imputation x, given by:

$$x_i = (b^i)^T y^* \quad \forall i \in N$$

where y^* is a solution of the dual problem for the grand coalition N, is in the core.

By the duality theorem the primal and dual problem have both no optimal solution or have both an optimal solution and in this case the optimal values are the same; this is necessary in order to guarantee that Owen's method gives us an efficient allocation.

In the semi-infinite case we have a countable set of products and a finite number of resources, i.e. we consider now $P = (m, N, A, b, c)$. As in the finite case above, we can define the game $\langle N, v\rangle$.

When we deal with semi-infinite linear programming a duality gap may arise, i.e. the primal and dual problem have both optimal solution but their optimal values may be different.

Example (Tijs, 1979). Given:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} & \cdots \end{bmatrix}; \quad b^T = (1,0); \quad c^T = (1,2,2,2,\ldots)$$

(P) $\sup\{c^T z | Az \leq b, z \geq 0, z \in R^N\}$

(D) $\inf\{b^T y | A^T y \geq c, y \geq 0, y \in R^2\}$

The feasible solutions of the two problems are respectively:

$\{z \in R^N | 0 \leq z_1 \leq 1; z_i = 0, i \geq 2\}$

$\{y \in R^2 | y_1 \geq 2; y_2 \geq 0\}$

The optimal solutions of the two problems are respectively:

$z^* = (1,0,0,0,\ldots); \quad c^T z^* = 1$

$y^* = (2,0); \quad b^T y^* = 2$

We have different optimal values $c^T z^*$ and $b^T y^*$ so we have a duality gap. Other examples of duality gaps are presented in Tijs (1979). ◇

If a duality gap arises we do not get anymore a core element using the previous method: namely $v(N)$ is equal to the optimal value of the primal problem, and this is different from the optimal value of the dual problem.

We need further conditions in order to avoid a duality gap. To this aim we state the following theorem.

Theorem 3. *Given a semi-infinite production problem P, assume that:*

a) $\sup_j\{c_j\} = \gamma < +\infty$

b) $\sup_i\{A_{ij}\} \geq \alpha > 0 \quad \forall j$

Then the associated game $\langle N, v \rangle$ *has a non-empty core.*

Proof. By a) and b) the dual feasible region is non-empty since $y^T = \frac{\gamma}{\alpha}(1,1,1,\ldots)$ is a feasible dual solution and the primal problem has finite total revenue.

Consider then the additional assumption:

$b_i > 0 \quad \forall i$

By this assumption there is no duality gap (Tijs, 1979) and an optimal dual solution exists: so it is possible to apply Owen's method in order to obtain a core allocation.

If assumption $b_i > 0$ doesn't hold for every i, i.e. one or more resources are available at level zero, we may eliminate these resources and all the products ɑeed a non-zero quantity of them (it is impossible to produce such prod-, getting a reduced problem that satisfies the hypotheses; to obtain a ion of the given problem and an allocation in Core(v) we may give value to the primal variables corresponding to the eliminated products and to ual variables corresponding to the eliminated resources. ◇

he reduced problem is not a standard one. From a computational point iew, consider a problem with finite number of products P_0; let), x_{B_0}, y_0) be an optimal solution of P_0, where I is the set of indices of variables, B_0 is the basis, x_{B_0} is the vector of basic variables and y_0 is the sponding dual solution.

dd a dual constraint j, i.e. a primal variable, such that:

$$A_j^T y_0 + c_j \geq \sup\{-A_k^T y_0 + c_k\} - \varepsilon$$

solve the new problem P_1.

ontinuing in this way, we obtain a sequence of dual solutions $\{y_i\}$ with osequence $\{y_{i_k}\}$ that converges to the optimal solution y^* of the dual. equently there exists a corresponding subsequence $\{x_{B_{i_k}}\}$ converging to ptimal solution x^* of the primal.

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