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I. Garcia-Jurado & S.H. Tijs

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I. Garcia-Jurado*& S.H. Tijs[†]

Abstract. In this paper we consider situations where a finite number of bimatrix games are going to be played once. We suppose that the column player is the same and has the same strategies in all those bimatrix games and, moreover, that he must play identically in all of them. We study several properties concerning the equilibria arising in such situations. Problems of tax control and inspection can be modeled in the way described above.

Keywords. Stack of games, matrix and bimatrix games, equilibrium.

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*Assistant Professor, Department of Mathematics, University of Santiago de Compostela, Spain.

[†] Professor, Department of Mathematics, Universities of Nijmegen and Limburg, The Netherlands

1 Introduction

In this paper we study a special kind of conflicts where a player T must play s games $\Gamma^1 =< \{1,T\}, X^1, Y, K^1, L^1 >, \ldots, \Gamma^s =< \{s,T\}, X^s, Y, K^s, L^s >$ against players $1, \ldots, s$ respectively in the following way (which is common knowledge). Each game Γ^i is played once. Player T has to choose a $y \in Y$ that he will use in all games meanwhile, for every i, player i elects an $x^i \in X^i$. Such decisions are taken simultaneously and independently, resulting that $\sum_{i=1}^{s} L^i(x^i, y)$ is the payoff to player T and $K^i(x^i, y)$ is the payoff to player i for every $i \in \{1, \ldots, s\}$. This situation is called the *stack game* based on $\Gamma^1, \ldots, \Gamma^s$.

In this work we concentrate on stack games based on bimatrix games and study the equilibria for this class of conflicts. Such a class is especially interesting because it models those situations where an inspector has to adopt a uniform behavior for his inspection task, meanwhile every person who could be audited must decide to act according to the law or not. To illustrate this we consider the following example.

Example 1.1 (Tax Game). Suppose there is a population of s individuals who must pay their taxes after declaring their incomes. Each of them can cheat or can be honest. At the same time, the tax inspector must design a policy for the control of the declarations. The tax payer i earns c_i dollars if he cheats and the inspector does not check his declaration, but he must pay a fine of f_i dollars if the inspector discovers his cheating. For the inspector, to check a declaration involves a cost c. This situation can be modeled by a stack game based on $\Gamma^1, \ldots, \Gamma^s$ where, for every i, Γ^i is the bimatrix game below.

		Player T	
		Check	Not Check
Player i	Cheat	$(-f_i, f_i - c)$	$(c_i, -c_i)$
	Be honest	$(0, -c)$	$(0, 0)$

Figure 1. Tax game

The aim of this paper is to study the rational behavior of players in stacks of bimatrix games. In section 2 we are concerned with their equilibria and derive some results about them. We devote section 3 to stacks of matrix games. Finally, we include some considerations and concluding remarks in section 4.

Notation. For every $n \in I\!N$, we denote by N the set of the n first natural numbers and by $\Delta(N)$ the set

$$\Delta(N) := \left\{(x_1, \ldots, x_n) \in I\!R^n \middle| \sum_{k=1}^n x_k = 1, x_k \geq 0 \text{ for all } k \in \{1, \ldots, n\}\right\}.$$

If $A, B \subset I\!R^n$, we write $A + B$ for the set $\{x + y | x \in A,\ y \in B\}$. We denote by conv$(A)$ the convex hull of $A \subset I\!R^n$. For any two-person zero-sum game Γ with a value, we write $v(\Gamma)$ for its value. Given the sets $A_1, \ldots, A_n$, we denote its Cartesian product $A_1 \times \ldots \times A_n$ by $\prod_{i=1}^n A_i$. The elements of the canonical basis of $I\!R^n$ are denoted by $e_1, \ldots, e_n$. For any game Γ, we represent by $E(\Gamma)$ its set of equilibria.

2 Equilibria of stacks of bimatrix games

In this section we look at stack games based on the s $m_i \times n$ bimatrix games (A^1, B^1), ..., (A^s, B^s). For every i, (A^i, B^i) is the game $< \{i, T\}$, $\Delta(M_i)$, $\Delta(N)$, $K^i, L^i >$ such that:

a) i and T are the players of the game.

b) $\Delta(M_i)$ and $\Delta(N)$ are the strategy spaces of player i and T respectively.

c) K^i and L^i are the payoff functions of player i and T respectively, given by

$$K^i(x^i, y) = x^i A^i y, L^i(x^i, y) = x^i B^i y, (x^i \in \Delta(M_i), y \in \Delta(N)).$$

Then, we define a *stack of s bimatrix games S* as the game

$$< \{1, \ldots, s, T\}, \Delta(M_1), \ldots, \Delta(M_s), \Delta(N), K^1, \ldots, K^s, L > \qquad (1)$$

such that:

a) $\{1, \ldots, s, T\}$ is the set of players of the game.

b) $\Delta(M_i)$ and K^i are respectively the strategy space and the payoff function of player $i (i \in \{1, \ldots, s\})$, and $\Delta(N)$ and L are respectively the strategy space and the payoff function of player T, with

$$\begin{aligned} K^i &: \Delta(M_i) \times \Delta(N) \to I\!R, K^i(x^i, y) = x^i A^i y \ (i \in \{1, \ldots, s\}) \\ L &: \prod_{i=1}^{s} \Delta(M_i) \times \Delta(N) \to I\!R, L(x, y) = \sum_{i=1}^{s} x^i B^i y, \end{aligned}$$

where $x = (x^1, \ldots, x^s) \in \prod_{i=1}^{s} \Delta(M_i)$.

An $(x, y) \in \prod_{i=1}^{s} \Delta(M_i) \times \Delta(N)$ is an equilibrium of S if and only if

$$K^i(x^i, y) \geq K^i(\overline{x}^i, y), \forall \overline{x}^i \in \Delta(M_i), \forall i \in \{1, \ldots, s\},$$

and

$$L(x, y) \geq L(x, \overline{y}), \forall \overline{y} \in \Delta(N).$$

Remark 2.1. Let us consider a stack S like in (1). Now, take the $s+1$-person game $\Gamma(S)$ given by

$$< \{1, \ldots, s, T\}, \Delta(M_1), \ldots, \Delta(M_s), \Delta(N), H^1, \ldots, H^s, L >,$$

where, for all $i \in \{1, \ldots, s\}$, H^i is defined by

$$H^i : \prod_{i=1}^{s} \Delta(M_i) \times \Delta(N) \to I\!R, H^i(x, y) := K^i(x^i, y).$$

Clearly, (x, y) is an equilibrium of S if and only if it is an equilibrium of $\Gamma(S)$. Then, as $\Gamma(S)$ is the mixed extension of a finite $s+1$-person normal form game, according to the Nash theorem (Ref. 1), S has at least one equilibrium.

Moreover, it is immediate that we can identify the set of stacks of s bimatrix games with the set of mixed extensions of finite $s+1$-person games in normal form which verify that there is a player T such that the payoff function for any other player i different form T depends only on the strategies that i and T have chosen.

Next, we demonstrate the following theorem which relates the equilibira of a stack S with the equilibria of a certain two-person game.

Theorem 2.1. Consider a stack of bimatrix games S like in (1) and define $K(x, y) = \sum_{i=1}^{s} K^i(x^i, y)$ for every $(x, y) \in \prod_{i=1}^{s} \Delta(M_i) \times \Delta(N)$. Now take the two-person game $TP(S) =< \{P,T\}, \prod_{i=1}^{s} \Delta(M_i), \Delta(N), K, L >$. Then (x,y) is an equilibrium of S if and only if it is an equilibrium of $TP(S)$.

Proof. From the definition of K, it is clear that, if (x,y) is an equilibrium of S, it is also an equilibrium of $TP(S)$. Conversely, take an equilibrium (x,y) of $TP(S)$. Then, $K(x,y) \geq K(x^1, \ldots, x^{i-1}, x'^i, x^{i+1}, \ldots, x^s, y)$, $\forall x'^i \in \Delta(M_i)$, $\forall i \in \{1, \ldots, s\}$

$$L(x,y) \geq L(x,y'), \forall y' \in \Delta(N).$$

Now, looking at the definition of K, it is immediate that (x,y) is an equilibrium of S. □

In the following of this section we go further and relate the equilibria of a stack S to the ones of a certain bimatrix game. Namely, let S be a stack of bimatrix games like in (1) and consider the $(m_1 \times \ldots \times m_s) \times n$ matrices

$$\begin{aligned} C &= (c_{\sigma j})_{\sigma \in \prod_{i=1}^{s} M_i,\ j \in N} \\ D &= (d_{\sigma j})_{\sigma \in \prod_{i=1}^{s} M_i,\ j \in N} \end{aligned}$$

such that, if $\sigma = (i_1, \ldots, i_s)$,

$$\begin{aligned} c_{\sigma j} &= a^1_{i_1 j} + \ldots + a^s_{i_s j} \\ d_{\sigma j} &= b_{i_1 j} + \ldots + b^s_{i_s j} \end{aligned}$$

where $A^i = (a^i_{kl})_{k \in M_i, l \in N}$, $B^i = (b^i_{kl})_{k \in M_i, l \in N}$, for all $i \in \{1, \ldots, s\}$.

Now we consider the bimatrix game $BI(S)$ given by (C,D), i.e.

$$BI(S) =< \{P,T\}, \Delta(\prod_{i=1}^{s} M_i), \Delta(N), O, Q >$$

where $O(z,y) = zCy$ and $Q(z,y) = zDy$, being $z = (z_\sigma)_{\sigma \in \prod_{i=1}^{s} M_i}$ an element of $\Delta(\prod_{i=1}^{s} M_i)$. Observe that we can identify $\prod_{i=1}^{s} \Delta(M_i)$ with a subset of $\Delta(\prod_{i=1}^{s} M_i)$. Namely, for an $x = (x^1, \ldots, x^s)$, we define

$$i(x) := z \in \Delta(\prod_{i=1}^{s} M_i),$$

with

$$z_\sigma = x^1_{i_1} \cdot \ldots \cdot x^s_{i_s} \quad (\forall \sigma = (i_1, \ldots, i_s) \in \prod_{i=1}^{s} M_i).$$

Then we prove the following theorem.

Theorem 2.2. With the notation above, $(x, y) \in \prod_{i=1}^{s} \Delta(M_i) \times \Delta(N)$ is an equilibrium of $TP(S)$ if and only if $(i(x), y) \in \Delta(\prod_{i=1}^{s} M_i) \times \Delta(N)$ is an equilibrium of $BI(S)$.

Proof. We denote $R(A^i) := \{e_1 A^i, \ldots, e_{m_i} A^i\}$. Then, observe that, for any $(x, y) \in \prod_{i=1}^{s} \Delta(M_i) \times \Delta(N)$,

$$K(x,y) = \sum_{i=1}^{s} x^i A^i y = (\sum_{i=1}^{s} x^i A^i) y = ay,$$

where $a := \sum_{i=1}^{s} x^i A^i \in \sum_{i=1}^{s} \mathrm{conv}(R(A^i))$. Note also that, for every $(z, y) \in \Delta(\prod_{i=1}^{s} M_i) \times \Delta(N)$,

$$O(z,y) = zCy = cy,$$

where $c := zC \in \mathrm{conv}(\sum_{i=1}^{s} R(A^i))$. Moreover, it is clear that, for any $a \in \sum_{i=1}^{s} \mathrm{conv}(R(A^i))$ and $c \in \mathrm{conv}(\sum_{i=1}^{s} R(A^i))$, there exist $x \in \prod_{i=1}^{s} \Delta(M_i)$ and $z \in \Delta(\prod_{i=1}^{s} M_i)$ such that, for every $y \in \Delta(N)$,

$$K(x,y) = ay \text{ and } O(z,y) = cy.$$

Observe that $\sum_{i=1}^{s} \mathrm{conv}(R(A^i)) = \mathrm{conv}(\sum_{i=1}^{s} R(A^i))$. Besides, it is clear that $K(x,y) = O(i(x), y)$ and $L(x,y) = O(i(x), y)$ for every $(x, y) \in \prod_{i=1}^{s} \Delta(M_i) \times \Delta(N)$. Now, it is straightforward to prove the theorem. □

Note that, from theorems 2.1 and 2.2, we can state that, in order to find all the equilibria of a stack of bimatrix games, it is enough to obtain all the equilibria of a certain bimatrix game whose second player has the same number of pure strategies as the second player in the stack. This fact is important. For example, it allows us to use the geometric methods described in Borm et al. (Ref. 2) and in Fiestras-Janeiro and García-Jurado (Ref. 3) to solve stacks of bimatrix games when T has two pure strategies. These methods are particularly useful in this context because,when T is an inspector (as we have suggested it could be in many practical situations), he will probably have two pure strategies: check or not check.

3 Stacks of matrix games

In this section we consider stack games S like in (1) when $B^i = -A^i$ for all $i \in \{1, \ldots, s\}$.

First observe that, from theorem 2.1, S can be identified with the corresponding two-person game $TP(S)$ (which now is also zero-sum) and that $E(S) = E(TP(S)) \neq \emptyset$. Besides, $TP(S)$ has a value and $E(TP(S))$ is a rectangular set (i.e., if (x, y) and (x', y') are in $E(TP(S))$, then (x', y) and (x, y') are also in $E(TP(S))$), and then we can speak of optimal strategies of P and T in $TP(S)$. Moreover, as K is a continuous function and $\prod_{i=1}^{s} \Delta(M_i)$ and $\Delta(N)$ are compact sets,

$$v(TP(S)) = \max_x \min_y K(x, y) = \min_y \max_x K(x, y).$$

Then,

$$\begin{aligned} v(TP(S)) &= \min_y \max_x \sum_{i=1}^{s} K^i(x^i, y) = \min_y \sum_{i=1}^{s} \max_{x^i} K^i(x^i, y) \geq \\ & \sum_{i=1}^{s} \min_y \max_{x^i} K^i(x^i, y) = \sum_{i=1}^{s} v(\Gamma^i). \end{aligned}$$

Now, from theorem 2.2, we know that, in order to solve a stack of matrix games, it is enough to solve a certain matrix game. But here we can go further. Namely, for a given S we can construct the matrix X like in section 2 and consider the corresponding matrix game $BI(S)$. Then,

$$\begin{aligned} & v(TP(S)) = \min_y \sum_{i=1}^{s} \max_{x^i} K^i(x^i, y) = \min_y \sum_{i=1}^{s} \max_{k^i} e_{k^i} A^i y = \\ & \min_y \max_i e_i C y = \min_y \max_z z C y = v(BI(S)). \end{aligned}$$

Moreover, in a matrix game, the set of equilibria is rectangular and, according to theorem 2.2, $(x, y) \in E(TP(S))$ if and only if $(i(x), y) \in E(BI(S))$. Then, y is an optimal strategy of T in $TP(S)$ if and only if so it is in $BI(S)$.

Next, let us see some examples which will show some interesting facts about stacks of matrix games.

Example 3.1 (A stack of matching pennies games). Suppose that players 1, 2 and T have to choose an integer greater than zero. If the sum of the numbers chosen by 1 (2) and T is odd (even), then 1 (2) pays one dollar to T. In the other case, T pays one dollar to 1 (2). This situation can be modeled by the stack S_i based on the two following matrix games.

Player T

		E	O
Player 1	E	1	-1
	O	-1	1

Player T

		E	O
Player 2	E	-1	1
	O	1	-1

Figure 2. The stack S_1

To analyse this stack observe that $BI(S_1)$ is the matrix game below

Player T

		E	O
Player P	EE	0	0
	EO	2	-2
	OE	-2	2
	OO	0	0

Figure 3. $BI(S_1)$

whose equilibria are the elements of the set

$$\left\{(x,y) \in \Delta(4) \times \Delta(2) | x \in \operatorname{conv}\{e_1, \frac{1}{2}e_2 + \frac{1}{2}e_3, e_4\}, y = \frac{1}{2}e_1 + \frac{1}{2}e_2\right\}$$

and then, according to theorem 2.2,

$$E(S_1) = \left\{(x^1, x^2, y) \in \Delta(2) \times \Delta(2) \times \Delta(2) |\right.$$

$$\left.(x^1, x^2) \in \operatorname{conv}\{(e_1, e_1), (e_2, e_2)\}, y = \frac{1}{2}e_1 + \frac{1}{2}e_2\right\}.$$

From example 3.1 above we can assert that, in general, given a stack of matrix games S, (x^1, x^2, y), $(\overline{x}^1, \overline{x}^2, y) \in E(S)$ does not imply that $(x^1, \overline{x}^2, y)$,

$(\overline{x}^1, x^2, y) \in E(S)$. However, we know that, if (x, y), $(\overline{x}, \overline{y}) \in E(S)$, then $(x, \overline{y})$, $(\overline{x}, y) \in E(S)$, because $E(TP(S))$ is a rectangular set. From all this we say that $E(S)$ verifies the *weak exchange property* for any S in the conditions above. Observe that this property suggests that there should be a sort of coordination between players 1 to s to reach an equilibrium.

Note also that, in example 3.1, $E(S_1)$ is a convex set. This fact is not a coincidence as we see in theorem 3.1 below.

Theorem 3.1. For any stack of matrix games S, $E(S)$ is a convex set.

Proof. From theorem 2.1., it is enough to prove that $E(TP(S))$ is convex. To do it, observe that

$$x \text{ is optimal of } P \text{ in } TP(S) \Leftrightarrow K(x, y') \geq v(TP(S)) \quad \forall y' \in \Delta(N)$$
$$y \text{ is optimal of } T \text{ in } TP(S) \Leftrightarrow K(x', y) \leq v(TP(S)) \quad \forall x' \in \prod_{i=1}^{s} \Delta(M_i).$$

Then, taking into account that K is a bilinear function, it is straightforward to prove the theorem. □

Now, let us study another example.

Example 3.2. Consider the stack S_2 based on the two matrix games Γ^1 and Γ^2 below.

Γ_1

0	1
2	0

Γ_2

1	0
0	2

Figure 4. The stack S_2

$BI(S)$ is the following matrix game.

1	1
0	3
3	0
2	2

Figure 5. $BI(S_2)$

Using theorem 2.2 and, for example, a geometric method to solve $m \times 2$ matrix games it is obtained that $v(\Gamma^1) = v(\Gamma^2) = \frac{2}{3}$, $v(BI(S_2)) = 2$ and

$$E(\Gamma_1) = \left\{ \left(\frac{2}{3}e_1 + \frac{1}{3}e_2, \frac{1}{3}e_1 + \frac{2}{3}e_2\right) \right\}$$

$$
\begin{aligned}
E(\Gamma_2) &= \left\{(\frac{2}{3}e_1+\frac{1}{3}e_2,\frac{2}{3}e_1+\frac{1}{3}e_2)\right\} \\
E(BI(S_2)) &= \left\{(e_4,y)\in\Delta(4)\times\Delta(2)|y\in\right. \\
&\qquad \left.\text{conv}\{\frac{1}{3}e_1+\frac{2}{3}e_2,\frac{2}{3}e_1+\frac{1}{3}e_2\}\right\} \\
E(S_2) &= \left\{(e_2,e_2,y)\in\Delta(2)\times\Delta(2)\times\Delta(2)|y\in\right. \\
&\qquad \left.\text{conv}\{\frac{1}{3}e_1+\frac{2}{3}e_2,\frac{2}{3}e_1+\frac{1}{3}e_2\}\right\}.
\end{aligned}
$$

We had already proved that, for any stack S based on the matrix games $\Gamma^1,\ldots\Gamma^s$, $v(TP(S))\geq\sum_{i=1}^{s}v(\Gamma^i)$. From example 3.2, we can assert that the equality does not always hold. Namely,

$$v(TP(S_2))=v(BI(S_2))=2>v(\Gamma^1)+v(\Gamma^2)=\frac{4}{3}.$$

Besides, it is clear that, for any S in the conditions above, $K(x,y)=K(x',y')$ and $L(x,y)=L(x',y')$ for all (x,y), (x',y') in $E(TP(S))=E(S)$. However, it is not true in general that $K^i(x^i,y)=K^i(x'^i,y')$ for all $i\in\{1,\ldots,s\}$ and (x,y), (x',y') in $E(TP(S))=E(S)$ (as we can see in example 3.2). From this we say that $E(S)$ verifies the *weak equal payoff property*.

Finally, observe that the strategies that all equilibria in S_2 propose for player 1 and 2 are not optimal for such players in Γ^1 and Γ^2 respectively. However, $K^i(x^i,y)\geq v(\Gamma^i)$ for all $i\in\{1,2\}$ and all $(x,y)\in E(S_2)$. In general, we can state the following result.

Theorem 3.2. For any stack S based on s matrix games $\Gamma^1,\ldots,\Gamma^s$, $K^i(x^i,y)\geq v(\Gamma^i)$ for all $i\in\{1,\ldots,s\}$ and $(x,y)\in E(S_2)$.

Proof. If there exist $i\in\{1,\ldots,s\}$ and $(\overline{x},\overline{y})\in E(S_2)$ such that $K^i(\overline{x}^i,\overline{y})<v(\Gamma^i)$, then

$$K^i(\overline{x}^i,\overline{y})<v(\Gamma^i)=\max_{x^i}\min_{y}K^i(x^i,y)\leq\max_{x^i}K^i(x^i,\overline{y})$$

and hence $(\overline{x},\overline{y})$ is not in $E(S)$, which is a contradiction. □

4 Concluding remarks

Our aim in this paper was to establish the initial theoretical basis for the study of stacks of games, but we know that now many interesting questions arise. Let us enumerate some of them:

a) More general models of stacks could be studied.

b) Many refinements of the equilibrium concept have been introduced to avoid some unsatisfactory properties of such a concept (for an interesting survey on refinements see Van Damme's book, Ref. 4). One possible object of study is the behavior of those refined concepts in the context of stacks.

c) It is easily seen that a stack of bimatrix games is a special polymatrix game in the sense of Quintas (Ref. 5). Then, we know that $E(S)$ is a finite union of polytopes. However, easier proofs of such a result (or of stronger ones) can perhaps be given for this particular class of polymatrix games.

d) Much research work has been made to prove the ordered field property for different classes of games (for a recent paper on the topic see Vrieze et al., Ref. 6). Using theorem 2.2, the ordered field property could be proved for stacks of matrix games, but it could be studied for other classes of stacks.

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