The Markov Consumption Problem

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April 12, 2010

Abstract

The paper derives the solution to a simple stochastic continuous-time dynamic control problem in which a consumer determines consumption and saving while moving between employment and unemployment according to a Markov process. The results differ from the permanent income hypothesis and some of Hall's 1978 results based on autoregressive income shocks.

*The author is indebted to Michael Jerison, John Jones, Adrian Masters, Silvio Rendon, David Sattinger, Fang Yang and Kwan Koo Yun for helpful comments. Remaining errors are the responsibility of the author.

JEL codes: C61,D91,E21,J64

Keywords: Consumption, unemployment, liquidity constraints, optimal control, Markov process

1. Introduction

This paper describes the solution to a stochastic dynamic control problem in which the only source of randomness is transitions between two states of a Markov chain process. A consumer moves between employment and unemployment in continuous time, earning different incomes (each constant over time) in the two states. At each point in time the consumer decides the level of consumption and the level of accumulation or decumulation of an asset paying a constant interest rate. The solution provides an alternative to computational methodologies and is relevant to bankruptcy, liquidity constraints and precautionary saving. The solution takes the form of the differential equations for consumption while employed and unemployed. This problem will be called the Markov Consumption Problem (hereafter MCP).

A major objective in consumption literature is to explain how saving behavior responds to income uncertainty (see surveys by Attanasio, 1999; Deaton, 1992; Carroll, 2001; and Hayashi, 1997). Schechtman (1976) and Schechtman and Escudero (1977) establish that individuals facing income uncertainty in an infinite horizon optimal consumption problem will accumulate assets that smooth out consumption (see also Miller, 1974). Huggett (1993) and Aiyagari (1994, Section

III) relate precautionary saving to uninsurable risks. Kimball (1990) establishes conditions of the utility function that generate precautionary saving. Gourinchas and Parker (2002) decompose saving empirically into precautionary and life cycle components.

The MCP differs from theoretical models in the above papers by having income shocks that are not autoregressive. An individual who becomes unemployed can expect to remain unemployed for some period of time before experiencing a transition to employment. Being unemployed at one point in time reduces the likelihood of being employed at any future point in time, although the effect diminishes over time (Karlin and Taylor, 1975, p. 154). Then having a low income at the current point in time (because of unemployment) increases the likelihood of having a low income in the future. An unemployed individual will have a lower expected growth rate of future income than if the individual were employed, everything else the same.

Since the MCP generates a different simultaneous optimization problem for employment and unemployment, it generates a two-state solution to the dynamic control problem. (The two-state nature of the solution can be seen from the two consumption functions in Figure 3.1, one for each state.) In contrast, the analysis of consumption and saving has been dominated by the use of discrete time single state solutions. In these single state solutions, the consumption behavior of the individual depends only on a single scalar state variable such as assets or permanent income. Uncertainty can be introduced into the consumption saving problem without departing from the single state apparatus. If the income shocks are independently and identically distributed, then the shocks only move the individual to a different point on the same consumption versus assets (or consumption versus permanent income) relation. In contrast, in the MCP, the movement between employment and unemployment causes a shift from one consumption asset relation to another.

A discussion of results for discrete time processes in Caballero (1990) and Hall (1978) will highlight the differences between the MCP and the conventional literature. In Caballero's paper, a consumer faces uninsured uncertain income in discrete time and can borrow and save at a fixed interest rate. Income follows an autoregressive moving average (ARMA) process. Caballero shows in Proposition 1 (p. 117) that the effect of an income shock or innovation affects consumption through a change in permanent income, i.e., the income shock is annuitized and is equivalent to a change in wealth. Although the income shocks are not independently and identically distributed, Caballero obtains a single state solution that is consistent with the permanent income hypothesis. Income shocks simply move the consumer to a different point on the same relation between consumption and permanent income (or consumption and a measure of wealth) and do not shift the consumer to a new relation. In contrast, the Markov process in the MCP is not autoregressive and shocks (movements between employment and unemployment) would move the consumer to a different consumption-wealth relation. Specifically, the consumer's shock can only be negative when employed and positive when unemployed, unlike the income processes considered in Caballero's Tables 1 and 2 (pp. 121, 122).

In the dynamic programming solution to the problem of optimal consumption in discrete periods, Hall (1978) concludes that the expected value of the marginal utility of consumption in the next period will be a constant multiple of the current marginal utility of consumption. An analogous result holds for continuous time models. However, some of Hall's corollaries to this result do not hold, even in a discrete time model, when the income shocks arise from Markov movements between employment and unemployment. In corollary 2, using Hall's notation, marginal utility satisfies a regression relation

$$u'(c_{t+1}) = \gamma u'(c_t) + \varepsilon_{t+1} \tag{1.1}$$

where $E(\varepsilon_{t+1}) = 0$. In a Markov process, the distribution of ε_{t+1} depends on which state the individual is in, so that the distribution of ε_{t+1} depends on c_t if assets are known. The expected value of the error term would be positive if the individual is unemployed and negative if employed. Hall's corollaries 3, 4 and 5 would also fail to hold for a Markov process if they require $E(\varepsilon_{t+1}) = 0$. Hall's reasoning leads to the conclusion that the relation

$$c_t = \lambda c_{t-1} + \varepsilon_t \tag{1.2}$$

approximates the stochastic behavior of consumption given the life cycle-permanent income hypothesis, i.e. consumption itself follows a random walk. However, in a Markov process, the expected value of ε_t at time $t-1$ in this relation will vary depending on the state and consumption level at time $t-1$. If the individual is unemployed, it can only be positive, and if the individual is employed, it can only be negative. The departure in this paper from Hall's results arises as a consequence of the necessary conditions for optimal consumption in the MPC taking the form of a differential equation for each state rather than a single equation relating consumption over time, as in Ljungqvist and Sargent's Euler equation 1.3.3 (2004, page 4).

Deaton (1991), citing Tauchen (1986), has developed computational methods in discrete time to examine the consequences of income shocks that would lead to multi-state systems (see also Deaton, 1992; Aiyagari, 1994, Section IV; Ljungqvist and Sargent, 2004, chapter 4; and Judd, 1998, chapter 12). Deaton represents a normally distributed serially correlated income generation process as movements among income intervals with transition rates from intervals in one period to intervals in the next period corresponding to the serial correlation. This discretization of the normal distribution, with transition rates among intervals, is a version of the MCP discussed here. Deaton concludes that serial correlation reduces the scope for income smoothing for liquidity constrained consumers.

Conventional dynamic programming models of consumption behavior over discrete periods do not permit the use of functional analysis that can be applied to relevant features of the consumption problem. In the MCP, bankruptcy (appropriately defined for a two state system) can serve as the initial condition for the location of the solutions to the differential equations. However, bankruptcy occurs at a singularity where derivatives cannot be determined by ordinary means. This paper develops methods for analyzing consumer behavior near bankruptcy. Another singularity, occurring at a break-even point, can be studied using methods of functional analysis applied in this paper. These methods are unavailable in the period analysis used for conventional consumption models. The application of functional analysis in the MCP, by providing the consumption function for the two states, generates alternative explanations for phenomena in consumption theory, including liquidity constraints, precautionary saving, and concavity.

Following Merton's analysis of portfolio selection (1971), optimal consumption and saving have also been modeled in continuous time. Merton assumes a geometric Brownian mechanism for income shocks (see also Koo, 1998), which generates a single state solution. Briys (1986) and Gollier (1994), in models of insurance versus precautionary saving, assume Poisson wealth shocks that move the individual to a different point on a single state relation between consumption and wealth. Toche (2005) considers precautionary saving in continuous time when the individual may move permanently into unemployment according to a Poisson process. Sheshinski (1989) had earlier considered a model in which income can move into an absorbing state as a result of a stochastic process, for example because of disability, and considered consequences for changes in consumption. Kimball and Mankiw (1989) consider an individual with stochastic income following a Markov process in continuous time. Their formulation of the optimal consumption and saving problem would lead to a multi-state solution but they simplify the problem by dropping out the interaction terms among state value functions in the Bellman equation (1989, equation 2, p. 866).

As in Aiyagari and Hayashi, liquidity constraints and borrowing constraints can be defined and differentiated as follows. Assume that the individual cannot default on debt (and therefore cannot engage in a Ponzi scheme) and cannot consume negative amounts. The non-negative consumption requirement imposes a borrowing constraint on the individual since the individual could not borrow so much that interest on the debt exceeds income. For example, if the lowest income is 0.5, and the interest rate on debt is .05, then the borrowing constraint on debt would be $0.5/.05 = 10$, i.e., the individual's assets must be greater than or equal to -10. A liquidity constraint arises when the individual cannot borrow beyond a limit even though the individual would always be able to pay interest on that debt. For example, if the lowest income is 0.5 and the individual cannot borrow any assets, then the liquidity constraint would be that the individual's assets must be greater than or equal to zero. Since the liquidity constraint at 0 exceeds the borrowing constraint, the liquidity constraint prevents the individual from engaging in some consumption and saving decisions that would be allowable with just the borrowing constraint, i.e., the liquidity constraint is more restrictive.

Liquidity constraints have been proposed as an explanation for why some households vary consumption in response to income fluctuations more than would be expected on the basis of the permanent income hypothesis (see discussions by Deaton, 1992, Chapter 6; Carroll, 2001; Hayashi, 1997, Part I; Zeldes, 1989). A household facing a binding liquidity constraint would be unable to smooth income by borrowing when income is low. Consumption would then fluctuate in response to fluctuations in income. In the cases considered by Deaton (with and without autoregressive income shocks), the individual consumes all cash on hand at sufficiently low asset levels, so that consumption would afterwards track income. It has also been argued that liquidity constraints generate a precautionary motive for saving since individuals would accumulate assets to avoid episodes of binding liquidity constraints that reduced their ability to smooth income. This paper will consider the consequences of liquidity constraints in the context of the MCP.

The next section poses the MCP formally, derives the Hamiltonian and adjoint equations, and solves for the differential equations describing the optimal consumption paths. Section 3 describes solutions to the differential equations derived numerically and analyzes the break-even point that occurs in the favorable state if the discount rate is greater than the interest rate. Section 4 concludes with a discussion of consequences of liquidity constraints in the MCP.

2. Differential Equations

Consider an individual moving between two states of a continuous time Markov process. Let p_1 be the transition rate from state 1 to state 2, and let p_2 be the transition rate from state 2 to state 1. Suppose the individual earns income at the rate y_i when in state i. Without loss of generality, assume $y_1 > y_2$. State 1 can be regarded as employment and state 2 as unemployment. Let $A[t]$ be the individual's assets at time t. Suppose the individual earns income from assets at the rate $rA[t]$, where r is positive and constant over time. Let $C_i[A[t]]$ be the consumption rate chosen if the individual is currently in state i with assets

$A[t]$. Since time only enters through discounting, the MCP is autonomous, and consumption levels will depend only on the state and the state variable, assets. If there is no ambiguity, the consumption levels will be written C_1 and C_2. Then the consumer's rate of asset accumulation in state i would be:

$$\left(\frac{dA}{dt}\right)_i = rA[t] + y_i - C_i[A[t]]$$

Suppose the consumer's instantaneous, time-separable utility at time t has constant relative risk aversion and takes the form

$$U[C] = (C)^{\gamma}/\gamma,\ \gamma < 1,\ \gamma \neq 0$$

The case where $U[C] = Log[C]$ corresponds to $\gamma = 0$ and can be treated using the same methodology. Suppose future utility is discounted at the rate b. Let $V_i[A, t]$ be the value function for the individual in state i with assets A. If there is no ambiguity, the value functions will be written V_1 and V_2. The consumer may borrow against future income if no liquidity constraint is binding (so that A could be negative) but may not default on borrowed funds.

Applying a dynamic programming argument (Sethi and Thompson, 2000, pp.

27-30), it can be shown that[1]

$$0 = Max_{C_1} \left\{ \begin{array}{c} U[C_1]e^{-bt} + p_1(V_2[A[t], t] - V_1[A[t], t]) \\ +(rA[t] + y_1 - C_1)V_{1A} + V_{1t} \end{array} \right\} \tag{2.1}$$

$$0 = Max_{C_2} \left\{ \begin{array}{c} U[C_2]e^{-bt} + p_2(V_1[A[t], t] - V_2[A[t], t]) \\ +(rA[t] + y_2 - C_2)V_{2A} + V_{2t} \end{array} \right\} \tag{2.2}$$

where V_{iA} refers to the partial derivative of V_i with respect to A and V_{it} refers to the partial derivative of V_i with respect to time assuming the individual remains in state i.

Since $p_1(V_2[A[t], t] - V_1[A[t], t]) + V_{1t}$ does not depend on C_1 or C_2, the Hamiltonian for the individual in state 1 at time t is formed as

$$H_1[A, C_1, \lambda_1, t] = U[C_1]e^{-bt} + \lambda_1(rA[t] + y_1 - C_1) \tag{2.3}$$

where $\lambda_1 = V_{1A}$ is the adjoint variable. Analogously, the Hamiltonian for the individual in state 2 is

$$H_2[A, C_2, \lambda_2, t] = U[C_2]e^{-bt} + \lambda_2(rA[t] + y_2 - C_2) \tag{2.4}$$

With differentiability, the first order conditions for the levels of consumption

[1]The procedure is to express $V_1[A[t], t]$ and $V_2[A[t], t]$ in terms of values at a small period of time τ in the future using a Taylor series expansion and taking into account the likelihood of transitions into the other state. Then subtracting V_i from both sides of the expression, dividing by τ and taking the limit yields 2.1 and 2.2.

the individual would choose in each state are

$$\frac{\partial H_i}{\partial C_i} = C_i^{\gamma-1}e^{-bt} - \lambda_i = 0,\ i = 1, 2 \tag{2.5}$$

or

$$C_i = \left(\lambda_i e^{bt}\right)^{1/(\gamma-1)},\ i = 1, 2 \tag{2.6}$$

Derivation of the differential equations for consumption requires the derivatives of the adjoint variables λ_1 and λ_2 with respect to time. These can be constructed using the adjoint equations generated by differentiating 2.1 and 2.2 with respect to $A:$ [2]

$$0 = Max_A \left\{ \begin{array}{c} U[C_1]e^{-bt} + p_1(V_2[A,t] - V_1[A,t]) \\ +(rA + y_1 - C_1)V_{1A} + V_{1t} \end{array} \right\} \tag{2.7}$$

$$0 = Max_A \left\{ \begin{array}{c} U[C_2]e^{-bt} + p_2(V_1[A,t] - V_2[A,t]) \\ +(rA + y_2 - C_2)V_{2A} + V_{2t} \end{array} \right\} \tag{2.8}$$

[2] The basis for the adjoint equations can be presented briefly using the development of Sethi and Thompson (2000, pp. 31-33). Suppose C_1^* is the optimal control that maximizes the right-hand side of (2.1) given the optimal path of assets, A^*. Then the right-hand side takes the value of zero. Consider a perturbation of A^*, say A_P. In general C_1^* will not be optimal for A_P, so the right-hand side would be less than zero. Thus A^* maximizes the right-hand side of (2.1) given C_1^*. This argument yields the adjoint equation for state 1 and an analogous argument holds for state 2.

With differentiability,

$$
\begin{aligned}
0 &= p_1(V_{2A} - V_{1A}) + V_{1AA}(rA + y_1 - C_1) + rV_{1A} + V_{1tA} && (2.9)\\
0 &= p_2(V_{1A} - V_{2A}) + V_{2AA}(rA + y_2 - C_2) + rV_{2A} + V_{2tA} && (2.10)
\end{aligned}
$$

In the above, $V_{iA} = \lambda_i$. Since time only enters the problem through discounting, the system is autonomous. The effect of time on the value function, V_{it}, is the same as moving back the point in time to which utility is discounted. Thus $V_{it} = -bV_i$ and

$$V_{itA} = \partial V_{it}/\partial A = \partial(-bV_i)/\partial A = -b\lambda_i, \; i = 1, 2 \tag{2.11}$$

Then by solving the adjoint equation (2.9),

$$V_{1AA} = \lambda_{1A} = \frac{(b - r)\lambda_1 + p_1(\lambda_1 - \lambda_2)}{rA + y_1 - C_1} \tag{2.12}$$

Let

$$\left(\frac{d\lambda_1}{dt}\right)_i = \left(\frac{\partial\lambda_1}{\partial t} + \frac{\partial\lambda_1}{\partial A}\frac{dA}{dt}\right)_i \tag{2.13}$$

be the total derivative of λ_1 with respect to time when the consumer is in state i. That is, the subscript i outside the parentheses indicates the consumer's state for which the derivative is calculated. These derivatives can be found from the foregoing results. In state 1, using 2.12,

$$\left(\frac{d\lambda_1}{dt}\right)_1 = -b\lambda_1 + \lambda_{1A}(rA + y_1 - C_1) \tag{2.14}$$

Then from (2.9) or from (2.12),

$$\left(\frac{d\lambda_1}{dt}\right)_1 = -r\lambda_1 - p_1(\lambda_2 - \lambda_1) \tag{2.15}$$

In state 2,

$$\begin{aligned}\left(\frac{d\lambda_1}{dt}\right)_2 &= -b\lambda_1 + \lambda_{1A}(rA + y_2 - C_2) \\ &= -b\lambda_1 + \frac{(b-r)\lambda_1 + p_1(\lambda_1 - \lambda_2)}{rA + y_1 - C_1}(rA + y_2 - C_2) \end{aligned} \tag{2.16}$$

Analogous procedures yield V_{2AA} and $(d\lambda_2/dt)_j,\ j = 1, 2$.

Differentiating consumption in each state with respect to time in each state yields:

$$\left(\frac{dC_i}{dt}\right)_j = \frac{\partial C_i}{\partial t} + \frac{\partial C_i}{\partial \lambda_i}\left(\frac{d\lambda_i}{dt}\right)_j,\ i = 1, 2,\ j = 1, 2 \tag{2.17}$$

Substituting $(d\lambda_i/dt)_j$ derived above and $\lambda_i = C_i^{\gamma-1}e^{-bt}$ from 2.5 yields differential equations in time for consumption in each state in terms of consumption in each state and assets.

Since the MCP is an autonomous system, consumption in each state depends only on assets and not on time. Differential equations in terms of consumption levels and assets can be derived as:

$$\frac{dC_i}{dA} = \frac{(dC_i/dt)_1}{(dA/dt)_1} = \frac{(dC_i/dt)_2}{(dA/dt)_2} \tag{2.18}$$

That is, the differential equations for consumption with respect to assets are the same whether calculated from time derivatives in state 1 or state 2. These calculations yield the following result.

Theorem 2.1. *In the MCP, optimal consumption levels satisfy*

$$\frac{dC_1}{dA} = \frac{C_1}{1-\gamma}\frac{r-b-p_1\left(1-\left(\frac{C_1}{C_2}\right)^{1-\gamma}\right)}{rA+y_1-C_1} \tag{2.19}$$

$$\frac{dC_2}{dA} = \frac{C_2}{1-\gamma}\frac{r-b-p_1\left(1-\left(\frac{C_2}{C_1}\right)^{1-\gamma}\right)}{rA+y_2-C_2} \tag{2.20}$$

3. Description of Solutions

Description of the solutions to the differential equations in Theorem 2.1 will be facilitated by reference to examples generated by numerical solution. For comparison with results from the literature on precautionary saving, assume that $b > r$ and that there is a liquidity constraint at $A = 0$ (i.e., assets must be greater than or equal to zero). It is unnecessary to impose a transversality condition (equivalent to a bankruptcy constraint) on the solution since a binding liquidity constraint is more restrictive. Differential equations such as the pair in the theorem above describe a family of curves rather than a single specific solution. A specific solution can be found by requiring that the consumption curves satisfy particular initial conditions given by C_1 and C_2 at some asset level. Given the initial conditions, it is then usually possible to solve the differential equations for consumption levels at other asset levels. Since C_2 would equal y_2 at the liquidity constrained asset level $A = 0$, it would appear to be convenient to specify the initial conditions at that asset level. However, the differential equation for dC_2/dA in 2.20 cannot be used to find consumption at different asset levels because the denominator in 2.20 is then zero and the derivative is infinite at $A = 0$.

Instead, the following procedure can be applied. The initial conditions for the differential equations determine the asset level at which consumption just equals income in the employed state. Let A_s be the break-even asset level at which this occurs, so that $C_1 = rA_s + y_1$. Instead of finding the break-even asset level for a given set of initial conditions, it is possible to specify the initial conditions in terms of the break-even asset level A_s. Since the slope of the derivative dC_1/dA will be positive and finite at A_s, the numerator in 2.19 will be zero at the same time the denominator is zero, i.e. the differential equation will have a singularity at A_s.(Singularities are treated in Courant,1936, p. 551, and Knopp, 1945, Section IV.) Setting the numerator of 2.19 equal to zero and solving yields the following result.

Proposition 3.1. *If the differential equation for dC_1/dA has a singularity at A_s, then*

$$C_2 = C_1 \left(\frac{p_1}{p_1 + b - r} \right)^{1/(1-\gamma)} \tag{3.1}$$

Applying L'Hospital's rule by differentiating the numerator and denominator of 2.19 with respect to A yields an expression that can be solved for dC_1/dA. Then using 3.1 and the derivatives for consumption with respect to assets yields values of C_1 and C_2 just below and just above A_s. These initial conditions can be used

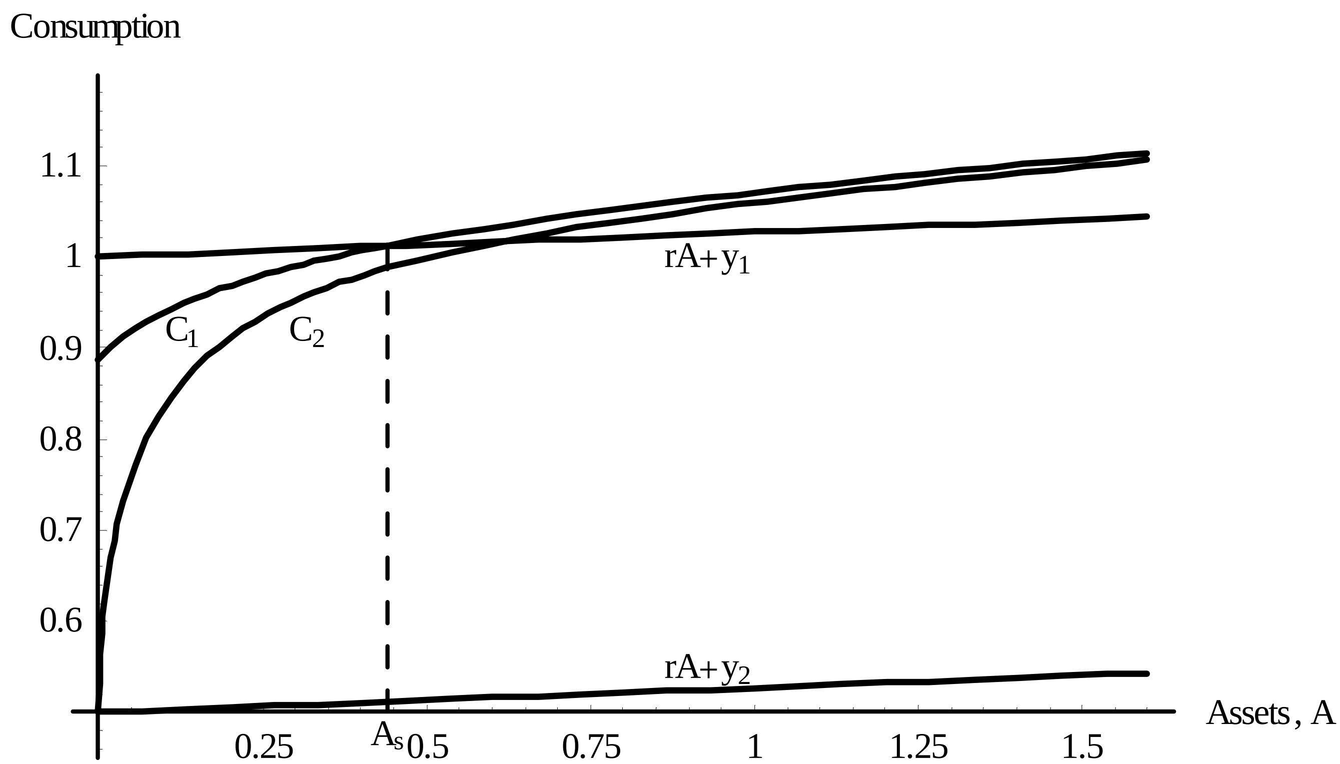

Figure 3.1: Consumption with Liquidity Constraint

to solve the differential equations for C_1 and C_2 below and above the singularity point. By varying A_s, it is possible to determine a solution such that the limit of C_2 as A_t approaches zero is y_2, satisfying the liquidity constraint. The solution is shown in Figure 3.1, with the singularity for the differential equation for C_1 occurring at $A_s = .439$. The solution assumes $p_1 = .2447$, $p_2 = 3.828$, $y_1 = 1$, $y_2 = .5$, $\gamma = .5$, $r = .027$ and $b = .03$.

In state 2 (unemployment), the individual always dissaves because consumption exceeds income. As assets decline during unemployment, consumption remains substantially above income but then declines sharply as assets approach the liquidity constrained level. Since the slope of the consumption curve is infinite at the liquidity constraint, the individual's assets would decline to the liquidity constrained level $A = 0$ in a finite amount of time. This corresponds to Deaton's discrete time result (1991) where the individual consumes all cash on hand in the current period for low asset levels.

In state 1, employment, the individual dissaves at asset levels above A_s but saves when $A < A_s$. At asset levels below A_s, the asset level alternatively increases and decreases as the individual moves between employment and unemployment. The break-even asset level A_s determines where the individual switches from saving to dissaving and corresponds to Carroll's buffer stock target level of wealth (1997).

Figure 3.2 compares consumption in state 2 with and without a liquidity constraint. In the absence of a liquidity constraint, individual borrowing is limited by the requirement that consumption should always be nonnegative and that the individual cannot default on debt. Then the budget constrained lower limit on debt is $-y_2/r$, at which consumption during unemployment would reach zero. The individual never reaches asset level $-y_2/r$ since dissaving slows as the individual approaches bankruptcy. Define $A_{\min} = -y_2/r$ as the budget constrained asset level. Without a liquidity constraint, consumption in state 2 (unemployment) is greater at each asset level, although the two consumption levels approach each other as assets increase. Similarly, consumption in state 1 is higher without a liquidity constraint. The break-even point in state 1 occurs at a much lower asset level, $A_s = -17.7$ for the case shown in Figure 3.2. Despite appearances, the derivative of consumption with respect to assets in state 2 is not infinite at $A_{\min}$, as indicated in the following result.

Proposition 3.2. *In the absence of a liquidity constraint, at the budget constrained minimum asset level* $A_{\min} = -y_2/r$,

$$\frac{dC_2}{dA} = \frac{b + p_2 - r\gamma}{1 - \gamma} \tag{3.2}$$

Proof. *Since* C_2 *approaches zero at* $A_{\min} = -y_2/r$, *both the numerator and denominator of* dC_2/dA *are zero and the differential equation has a singularity at that asset level. Applying L'Hospital's rule and solving yields 3.2.*

The reason for the different behavior of C_2 at $A_{\min} = -y_2/r$ compared to the solution with a liquidity constraint is that the Inada condition holds at $A_{\min}$, i.e., the marginal utility of consumption $C^{\gamma-1}$ becomes infinite at that asset level but not at $A = 0$. As a consequence of the finite slope of C_2 at $A_{\min}$, assets would never decline to that level and consumption would remain positive for any positive amount of time in state 2.

It is possible to solve for the levels of the value functions with and without liquidity constraints. The effect of liquidity constraints is to reduce the levels of the value functions, as one would expect from any restriction placed on optimal behavior. In the absence of a liquidity constraint, the optimal behavior for the

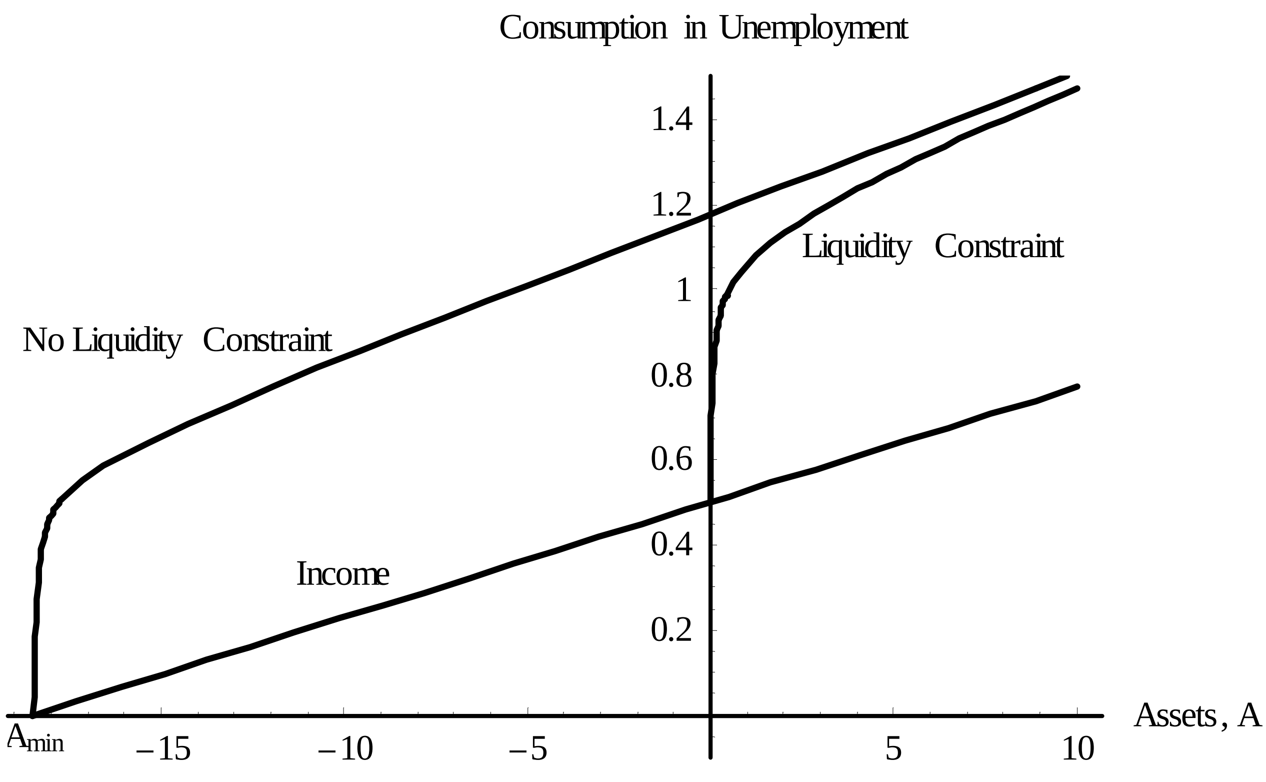

Figure 3.2: Consumption in Unemployment with and without Liquidity Constraint

unemployed individual is to go into deeper and deeper debt and rely on an eventual transition to employment to reduce the debt level.

Three different marginal propensities to consume out of income can be identified in the MCP:

$$MPC_1[A_t] = (1/r)\,\frac{dC_1[A_t]}{dA_t} \tag{3.3}$$

$$MPC_2[A_t] = (1/r)\frac{dC_2[A_t]}{dA_t} \tag{3.4}$$

$$MPC_3[A_t] = \frac{C_1[A_t] - C_2[A_t]}{y_1 - y_2} \tag{3.5}$$

In 3.3 and 3.4, the change in income occurs because of a change in assets, and the marginal propensity to consume is derived from the derivative of consumption with respect to assets in 2.19 and 2.20 for states 1 and 2, respectively. Note that 2.19 and 2.20 provide the marginal propensities to consume at the same asset level but not at the same income level. A comparison at the same income level can be obtained by using $MPC_1[A_t]$ and $MPC_2[A_t + (y_1 - y_2)/r]$. In 3.5, the change in income arises from a transition from one state to another, and the marginal propensity to consume is calculated as the change in consumption divided by the change in income.

Since the curve for $C_2[A_t]$ is not simply the curve $C_1[A_t]$ shifted to the left by $(y_1 - y_2)/r$, the two marginal propensities to consume in 3.3 and 3.4 calculated for the same income level will in general be unequal. The asset changes that generate 3.3 and 3.4 affect income indefinitely into the future and correspond roughly to long run marginal propensities to consume. They start out high, much greater than one, at asset levels close to minimum levels (0 or $-y_2/r$, depending on whether there is a liquidity constraint), and decline towards the risk-free level at very high asset levels, b/r. In the case shown in the figures, $b/r > 1$, so the first two marginal propensities to consume are always greater than one. The third marginal propensity to consume is generated by a change in income from a transition between employment and unemployment, and corresponds roughly to the marginal propensity to consume out of transitory income in the permanent income hypothesis since the transition alters expected future income much less than current income. The third marginal propensity to consume is very low relative to the first two marginal propensities to consume, and is much less than one. Additional marginal propensities to consume could be generated from unanticipated parameter changes, e.g. y_1, y_2, or r.

These consumption responses correspond roughly but not exactly to the responses in Deaton's analysis (1991). In Deaton's analysis, in the range of assets where all cash on hand is spent, the marginal propensity to consume out of income in response to a change in assets is $1/r$, a very large response. This case corresponds to $MPC_2[A_t]$ in 3.4 with assets near the minimum level determined by either the liquidity constraint or the budget constraint. In the liquidity constrained case, as shown in Figure 3.1, $MPC_2[A_t]$ increases indefinitely as assets decline to zero while in the unemployed state. If instead the change is in non-asset income, then in Deaton's analysis the marginal propensity to consume equals 1 when the individual has assets in the interval where all cash on hand is spent.

This corresponds to $MPC_3[A_t]$ in the MPC, arising from a transition between employment and unemployment. From Figure 3.1, $MPC_3[A_t]$ approaches about .8 as assets decline to the liquidity constrained level.

In the permanent income hypothesis, consumption depends on a measure of permanent income that incorporates the expected present value of future income streams. In the MCP, it is possible to calculate the expected present value of labor incomes from the Markov movements between employment and unemployment. The difference in the expected present values of future labor incomes between employment and unemployment is

$$\frac{b(y_1 - y_2)}{b + p_1 + p_2} \tag{3.6}$$

If the permanent income hypothesis were strictly valid for the MCP, the difference in consumption between the two states would be a constant proportion of the difference in present values in 3.6. Then the consumption functions for the two states in Figure 3.1 would differ by a constant vertical amount. However, the figure shows that the difference in consumption is large near bankruptcy and then declines to a much smaller difference as financial assets increase. The MCP is therefore inconsistent with a formulaic version of the permanent income hypothesis.

The solution with $b < r$ can be briefly described. There will be a break-even asset level A_s in state 2 instead of state 1. At asset levels above A_s, the individual saves in both states instead of dissaving. Below A_s, the consumer dissaves in state

2, unemployment, and saves in state 1, as in the case where $b > r$.

4. Conclusions

From the foregoing results, it is possible to examine the consequences of liquidity constraints. First, consider whether liquidity constraints generate a qualitatively different solution to the optimal consumption problem. In Deaton's analysis, liquidity constraints lead the individual to consume all cash on hand for sufficiently low asset levels. In the MCP, a corresponding phenomenon occurs in which unemployed individuals reach zero assets and consume exactly their income y_2 a positive proportion of the time. This suggests that liquidity constraints in both the MCP and computational approaches track income part of the time. However, Hayashi notes that there will be a budget constraint in the individual's consumption problem that is equivalent to a liquidity constraint. The effect of a liquidity constraint would then be to shift the constraint on the individual's consumption rather than to generate a constraint where none existed. This appears to be the case in the MCP because, in Figure 3.1, the liquidity constraint shifts the asset level at which consumption equals income while unemployed from $-y_2/r$ to 0.

Nevertheless, the solution without a liquidity constraint differs qualitatively from the solution with a liquidity constraint. Without a liquidity constraint, at $A = -y_2/r$ (where the individual's income while unemployed just equals interest on the individual's debt and consumption would be zero), the slope of the consumption function $C_2[A]$ is positive and finite from Proposition 2. This contrasts with the infinite slope at $A = 0$ in the liquidity constraint case. Then the individual never reaches $A_{\min} = -y_2/r$ in a finite amount of time. The proportion of time that an unemployed worker spends entirely their income would be zero. Also, since individuals never reach the break-even point while employed in a finite amount of time, virtually no individual consumes entirely their income in the absence of a liquidity constraint.

A point that arises from being able to view the consumption functions in Figures 3.1 and 3.2 is that significant features of consumption occur at low asset levels near either bankruptcy or the liquidity constraint. The conventional economic literature on consumption has mostly used discrete period analysis (in order to provide empirically relevant conclusions). This paper, by using continuous time, has been able to apply methods of analysis to the singularities occurring at bankruptcy or break-even points, thereby providing limiting results at those points.

The methodology developed in this paper provides deterministic consumption functions for each state because the stochastic process generating uncertainty consists entirely of movements between the two states. The consumption functions in Figures 3.1 and 3.2 show directly changes in consumption behavior between employment and unemployment, with and without liquidity constraints, and in the limit as assets approach minimum levels. This paper has described consequences of liquidity constraints. Other consumption phenomena that can be studied include bankruptcy in the absence of a liquidity constraint and the determination of the break-even level of assets (where consumption equals income in one of the states).

The methodology can be extended to describe uncertainty generated by other Markov process transitions, such as movements in the interest rate between two levels.

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