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of Threshold Effects - The Shallow Lake Model**

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Abstract

In this article we analyze how the presence of thresholds influences multi agent decision making situations. We introduce a class of discounted autonomous optimal control problems with threshold effects and discuss tools to analyze these problems. Later, using these results we investigate two types of threshold effects; namely, simple and hysteresis switching, in the canonical model of the shallow lake. We solve the optimal management and open loop Nash equilibrium solutions for the shallow lake model with threshold effects. We establish a bifurcation analysis of the optimal vector field. Further, we observe that modeling with threshold effects simplifies this analysis. To be precise, the bifurcation scenarios rely on simple rules (inequalities) which can be verified easily. However, the qualitative behavior of the switching vector field is similar to the smooth case.

Keywords Optimal control, Differential games, Threshold effects, Discontinuous dynamics, Shallow lake

JEL Codes C61, Q57

1 Introduction

Most of the optimal decision making problems studied in economics and ecology are complex in nature. These complexities generally arise while modeling the inherent behavior of the dynamic environment, which includes agents interacting with the system. Modeling with hybrid systems [11, 29] capture some of these complex situations. The behavior of such systems is described by the integration of continuous and discrete dynamics. An abrupt change in the discrete state of the system is called a switch. If a decision maker influences a switch then it is said to be controlled/external, whereas an internal switch generally results when the continuous state variable satisfies some constraints. Threshold effects are autonomous switches that happen when the continuous state variable hits a boundary. Some examples in this direction are, a firm going bankrupt when its equity is negative and regime shifts in ecology [24] etc. Optimal control of hybrid systems received considerable interest in control engineering, see for instance [31, 28, 3, 26, 23, 25]. These works include formulation of different versions of the necessary conditions. For computational issues related to optimal control of hybrid systems, see [15, 2, 26].

In this article, we study optimal management and differential games in a pollution control model called the shallow lake problem, see [16, 14, 30, 4]. The production function of this model is nonlinear, convex-concave in particular. As a result, the optimal vector field displays several interesting qualitative behaviors

such as existence of multiple steady states, Skiba points¹ and bifurcations due to parameter variations. A complete bifurcation analysis of this vector field is provided, recently, in [13]. The inflection point of this convex-concave function acts as a soft threshold. In this article, we approximate this nonlinear effect with simple and hysteresis switching, by using deterministic hard thresholds, and study optimal management and open loop Nash equilibrium policies. Some literature incorporating threshold effects include [20], [21] and references cited in those papers. Though in [20], the author uses necessary conditions, in the line of [25], the objective function is quadratic and the state variable admits jumps, whereas, the present article deals with discontinuous dynamics. Recently, in [21] the authors consider an optimal management problem, with probabilistic thresholds and simple dynamics, and use dynamic programming to derive optimal policies.

In this article we do not attempt to solve the optimal controls or equilibrium strategies for a generic class of (hybrid) differential games. Instead, we study the shallow lake problem with switching approximation and highlight the key differences with the smooth case, the classical shallow lake model. This article is organized as follows. In section 2, we introduce a class of discounted autonomous infinite horizon optimal control problems with internal switching. We review the necessary conditions for these class of problems. Further, for one dimensional problems we develop some methods to analyze the optimality equations of a specific class of switched systems. In section 3, we introduce a class of differential games, that arise in pollution control, with threshold effects. We study optimal management and open loop Nash equilibrium policies related to the shallow lake model in section 4. In section 5 we analyze the shallow lake problem with quadratic benefit functions. Finally, section 6 concludes.

2 Optimal control of switched systems

In this section we review necessary conditions to solve optimal control problems for a specific switched system. The switching system that we have in mind has the following description:

Definition 1 (Switched System). A switched system is a triple $\mathcal{S} = (\mathcal{I}, \mathcal{F}, \mathcal{E})$ where

- \mathcal{I} is a finite set, called the set of discrete states representing the vertices of a graph.
- $\mathcal{F} = \{f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n, i \in \mathcal{I}\}$ is a collection of vector fields. We denote $\dot{x}(t) = f_i(x(t), u(t))$ to be the vector field at location $i \in \mathcal{I}$.
- \mathcal{E} is a finite set of edges called transitions. A transition $(i, j) \in \mathcal{E}$ is triggered by events (internal or external) resulting in abrupt change in dynamics from f_i to f_j .

In this article we consider internal switchings i.e., transitions from vertex i to j happen when certain state constraints, say $\phi_{ij}(x(t)) = 0$, are satisfied. Let $\Phi_{ij} \triangleq \left\{x \in \mathbb{R}^n : \phi_{ij}(x) = 0\right\}$ be the switching surface associated with transition i to j and $\Phi \triangleq \cup \Phi_{ij}$. Now, we introduce a class of discounted autonomous infinite horizon optimal control problems with internal switching dynamics, described by $\mathcal{S} = (\mathcal{I}, \mathcal{F}, \Phi)$, as follows.

$$\max J, \quad J = \int_0^{\infty} e^{-rt} g(x(t), u(t)) dt \quad (1)$$

$$\dot{x}(t) = f_i(x(t), u(t)), \quad i \in \mathcal{I}, \quad f_i \in \mathcal{F} \quad (2)$$

$$x(0) = x_0 \in \mathbb{R}^n, u(\cdot) \in \mathcal{U}. \quad (3)$$

Assumption 1. The real-valued functions $f_i(\cdot), i \in \mathcal{I}$ and $g(\cdot)$ are continuous, $\frac{\partial f_i(\cdot)}{\partial x}$ and $\frac{\partial g(\cdot)}{\partial x}$ exist and are continuous. The control space \mathcal{U} consists of piecewise continuous functions with $u(t) \in U$, where U is

¹Starting from such a point the optimal control problem has more than one optimal solution, and as a result the decision maker is indifferent to a particular solution, see [27].

a bounded set included in \mathbb{R}^m . Further, we assume the left- and right-hand limits for $u(\cdot)$ exist and $x(\cdot)$ is continuous and piecewise continuously differentiable, which satisfies (2) for all points t where $u(\cdot)$ is continuous. The initial state satisfies $x_0 \notin \Phi$. The velocity vector satisfies $\dot{x}(\tau) \neq 0$ during a transition where τ is the switching instant, i.e., the switching surface should necessarily be crossed.

We call a pair $(x(\cdot), u(\cdot))$ *admissible* for the problem (1-3) if assumption 1 is satisfied. Let $k(\cdot)$ represent the switching sequence associated with \mathcal{S} , i.e., when the system is in mode i at time t we have $k(t) = i$. Since the switchings happen internally we see that $u(\cdot)$ induces a switching sequence $k(\cdot)$. The necessary condition for a pair $(x^*(\cdot), u^*(\cdot))$ to be optimal for the problem (1-3) is given by the following theorem², see theorem 2.3 of [23] or theorem 1 of [22] or theorem 3 of [25] or theorem 2.2 and theorem 2.3 of [26] for more details.

Theorem 1. *If $(x^*(t), u^*(t))$ represent an optimal admissible pair for the problem (1-3), there exists a piecewise absolutely continuous function $\lambda_{k^*(\cdot)}(\cdot)$ and a constant $\lambda_0 \geq 0$, $(\lambda_0, \lambda_{k^*(t)}) \neq 0$ on $[0, \infty)$ such that:*

a) *let Hamiltonian be defined as*

$$H_{k(t)}(t, x(t), u(t), \lambda_{k(t)}(t), \lambda_0) \triangleq \lambda_0 e^{-rt} g(x(t), u(t)) + \lambda'_{k(t)} f_{k(t)}(x(t), u(t)) \quad (4)$$

then for a given $(\lambda_{k^(t)}(t), \lambda_0, x^*(t))$ at a given time t , except at the switching instants, the following maximum condition holds*

$$H_{k^*(t)}(t, x^*(t), u^*(t), \lambda_{k^*(t)}(t), \lambda_0) \geq H_{k^*(t)}(t, x^*(t), v, \lambda_{k^*(t)}(t), \lambda_0) \quad \forall v \in U,$$

and if $u^(t)$ is an interior solution then $\frac{\partial H_{k^*(t)}(\cdot)}{\partial u} \Big|_{u(t)=u^*(t)} = 0$.*

b) *the adjoint process $\lambda_{k^*(t)}(t)$ satisfies $\dot{\lambda}_{k^*(t)}(t) = -\frac{\partial H_{k^*(t)}(\cdot)}{\partial x} \Big|_{x(t)=x^*(t)}$ for all $t \geq 0$ except at the switching instants.*

c) *if τ is a switching instant then the following conditions hold true:*

1. $x^*(\tau) \in \Phi$, $\tau \in [0, \infty)$

2. (adjoint jump condition)

$$\text{there exists a } \beta \in \mathbb{R} \text{ such that } \lambda_{k^*(\tau^-)} = \lambda_{k^*(\tau^+)} + \beta \frac{\partial \phi_{k^*(\tau^-)k^*(\tau^+)}(x)}{\partial x} \Big|_{x(t)=x^*(\tau)}$$

3. (Hamiltonian continuity)

$$H_{k^*(\tau^-)}(\tau^-, x^*(\tau^-), u^*(\tau^-), \lambda(\tau^-), \lambda^0) = H_{k^*(\tau^+)}(\tau^+, x^*(\tau^+), u^*(\tau^+), \lambda(\tau^+), \lambda^0).$$

2.1 Consequence of (switched) maximum principle

The above necessary conditions, when solved, generally give a large number of candidates and the optimal solution is obtained by comparing the objective, denoted as Ψ , evaluated along the candidate trajectories. The non-switched analog of problem (1-3) is the classical discounted autonomous infinite horizon optimal control problem, which is a well studied problem in economics literature. For this class of problems an additional necessary condition called as *asymptotic Hamiltonian property* is satisfied, i.e., the limit of the maximized Hamiltonian, along the candidate trajectory, is zero when t goes to infinity, see [18]. Further, when necessary conditions hold true in normal form, i.e., $\lambda_0 = 1$, the objective evaluated along a candidate trajectory is given by $\Psi = \frac{1}{r} H(0, x^*(0), u^*(0), \lambda(0), 1)$, see proposition 3.75 of [12] for a proof. We show in the next lemma, for the problem (1-3), that the objective evaluated along the optimal candidates with a finite number of switchings also satisfies this property when $\lambda_0 = 1$. The proof of the lemma, given in the appendix, hinges upon the Hamiltonian continuity property.

²There exist several variations of the theorem in a more general setting, for instance refer [26, 25, 23, 28]. Here, we consider a specific system \mathcal{S} where switchings happen internally.

Lemma 1. Let $(x^*(\cdot), u^*(\cdot))$ be an optimal admissible candidate pair with a finite number of switchings for the problem (1-3), then the objective evaluated along the optimal candidate trajectory is given by $\Psi = \frac{1}{r} H_{k^*(0)}(0, x^*(0), u^*(0), \lambda_{k^*(0)}(0), 1)$.

In the smooth version of the problem, when the state and control variable are one dimensional, several tools can be developed to analyze the optimality equations. These results, which are a consequence of the smooth maximum principle, see section 3.2 of [30], are helpful in analyzing the qualitative properties of the optimal vector field. In the following discussion we show, under some assumptions, that similar results can be obtained for one dimensional optimal control problems of the type (1-3) as a result of the switched maximum principle. The two lemmas given below and their proofs as provided in the appendix, are inspired by section 3 of [30]. We make the following assumption.

Assumption 2. The maximum principle holds in normal form, i.e., $\lambda_0 = 1$. The partial derivatives of f_i with respect to u are strictly positive for all $i \in \mathcal{I}$, i.e., $\frac{\partial f_i}{\partial u} > 0$. The current value Hamiltonian defined as $H_i^c(\cdot) = e^{rt} H_i(\cdot)$ ³ attains its maximum at u^* in the interior of U for all $i \in \mathcal{I}$. Further, at this point the second derivative of $H_i^c(\cdot)$ is strictly negative, i.e., $\frac{\partial^2 H_i^c(x(t), u(t), \lambda_{k^*(t)}^c(t), 1)}{\partial u^2} \Big|_{u(t)=u^*(t)} < 0$.

Consider that the above assumption holds true. Then for any candidate (x^*, u^*) corresponding to the one dimensional optimal control problem of the type (1-3) the following lemma holds true as a consequence of the switched maximum principle.

Lemma 2. Consider the portion of trajectory (x^*, u^*) in the interior of mode $i \in \mathcal{I}$. Then for every fixed x^* on this trajectory, the function $\eta_i(u^*; x^*) = -\frac{\frac{\partial g(x^*, u^*)}{\partial u} \Big|_{u=u^*}}{\frac{\partial f_i(x^*, u^*)}{\partial u} \Big|_{u=u^*}}$ is differentiable and strictly monotone as a function of u^* , and hence it has a differentiable inverse.

Notice, that the dynamics in the switched maximum principle is defined only during non transition time instants. The above lemma allows to transform the optimality equations formulated in the state-adjoint system to a state-input system⁴. The objective along an optimal candidate starting at (x_0, u_0) can be obtained using lemma 1. The next lemma captures the variation of this objective with respect to u_0 for a fixed x_0 . This result is helpful in the selection of an optimal solution from the candidate trajectories.

Lemma 3. Assume $x_0 \notin \Phi$ and let $i(x_0)$ denote the mode of the system at $t = 0$. Then, the signs of $f_{i(x_0)}$ and $\frac{\partial \Psi}{\partial u} \Big|_{u=u^*}$ are the same.

3 A class of differential games with threshold effects

In this section we discuss briefly a differential game model in pollution control, taken from [14], and modify the dynamics to include threshold effects (and as a result belongs to the class of problems introduced in the previous section). Assume a situation where N economic agents, sharing a natural system, take actions $a_i(t)$, $i = 1, 2, \dots, N$ at time t , and as a result affect the state $x(t)$ of a natural system. The economic agents could be societies, dealing with eutrophication of a lake that they manage, or countries, worried about climate change. The stock of pollutant in the natural system admits a dynamics described by:

$$\dot{x}(t) = f(x(t), a_1(t), a_2(t), \dots, a_N(t)) = \sum_{i=1}^N a_i(t) - bx(t) + h(x(t)), \quad x(0) = x_0 \geq 0. \quad (5)$$

³The above necessary conditions can be reformulated in the current value form by first defining the current value adjoint variable as $\lambda_{k^*(t)}^c(t) = e^{rt} \lambda_{k^*(t)}(t)$ and writing the current value Hamiltonian as $H_i^c(x^*(t), u^*(t), \lambda_{k^*(t)}^c(t), \lambda_0) = e^{rt} H_i(t, x^*(t), u^*(t), \lambda_{k^*(t)}(t), \lambda_0)$.

⁴We use this result to analyze the shallow lake system with threshold effects.

The single state variable $x(t)$ could be interpreted as accumulated greenhouse gases or accumulated phosphorous in a lake. Besides the activity of economic agents, the sources that promote $x(t)$ are the nonlinear internal dynamics captured by the term $h(x(t))$. The sinks that contribute to the reduction of $x(t)$ are abstracted as the linear decay rate $b > 0$. The second part of the model deals with economic analysis of the agents. An agent i , with action a_i , generates benefits according to a strictly increasing and concave utility function $B(a_i)$. The stock of pollutants $x(t)$ causes damage to the natural system according to a strictly increasing and convex damage function $D(x)$, sometimes referred as disutility of agents. The net profit that an agent i receives at a point of time t is then given by $B(a_i(t)) - D(x(t))$. Each agent uses a strategy $a_i(\cdot)$ to maximize the present value of net benefits over an infinite time horizon, i.e.,

$$\max_{a_i(\cdot)} \int_0^{\infty} e^{-rt} (B(a_i(t)) - D(x(t))) dt, \quad i = 1, 2, \dots, N, \quad (6)$$

subject to (5), where $r > 0$ is a discount rate. Here, the quantities $B(\cdot)$, $D(\cdot)$ and r are assumed to be the same for all the agents. In this article, the function $h(x)$ that determines the nonlinear internal dynamics is assumed to be a convex-concave function; that is for lower stocks of $x(t)$ there is relatively low marginal return to the system, whereas for the higher stocks this marginal return first increases and then decreases again. When the maximal feedback rate is greater than the decay rate, i.e., $\max(h'(x)) > b$, the system (5) exhibits three equilibria for a certain values of inputs $a = \sum_i a_i$. There will be two stable steady states, one corresponding to low x ‘clear state’ which is highly valued by concerned users of the natural system⁵, but also a relatively high x ‘polluted state’ which is valued by the agents due to economic interest. This nonlinear positive feedback effect is a potential source for complex qualitative behaviors in optimal solutions in the model (5-6). The region of stock near the inflection point of $h(x)$ acts as a soft threshold distinguishing the clear and polluted regions. A piecewise approximation of the nonlinearity in $h(x)$ results

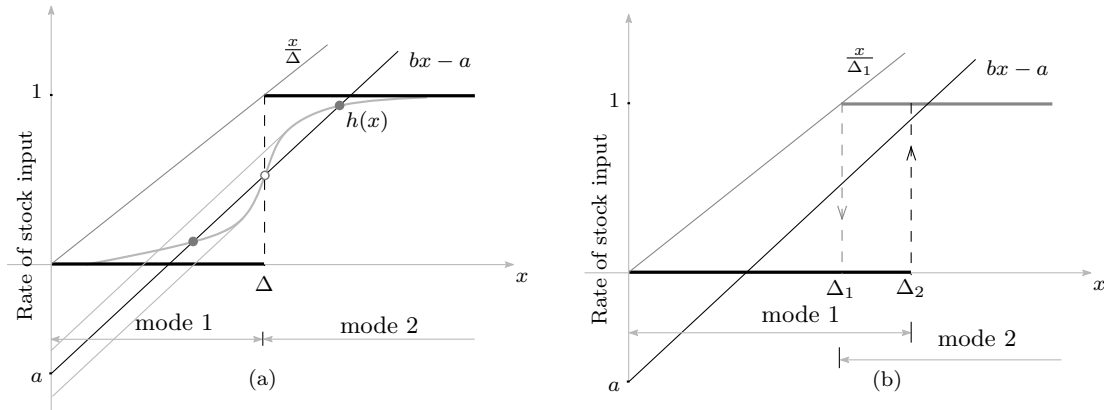


Figure 1 – Approximation of $h(x)$ with hard thresholds

in a dynamics with discontinuous right hand side; that is when the amount of stock reaches a pre-specified threshold the dynamics changes abruptly. In this article we consider two approximations of $h(x)$, firstly with simple switching and later with ideal hysteresis switching. Hereafter, we address ‘clean state’ as mode 1 and ‘polluted state’ as mode 2. Let $a(t) \geq 0$ denote total agents’ activities i.e., $a(t) \triangleq \sum_i a_i(t)$.

⁵Could be people using a lake for recreation etc.

3.1 Simple switching

Figure 1(a) illustrates the situation when $h(x)$ is approximated by a Heaviside step function. Thus, the dynamics of the system (5) is given by:

$$\begin{aligned} \mathcal{S} &= \{\mathcal{I}, \mathcal{F}, \mathcal{E}\} \text{ where} \\ \mathcal{I} &= \{1, 2\}, \mathcal{F} = \{f_1(x, a) = a - bx, f_2(a, x) = a - bx + 1\} \\ \mathcal{E} &= \{\phi_{12}(x(t)) = \phi_{21}(x(t)) = x(t) - \Delta = 0\} \\ \dot{x}(t) &= a(t) - bx(t), \text{ for } x(t) < \Delta \\ \dot{x}(t) &= a(t) - bx(t) + 1, \text{ for } x(t) > \Delta. \end{aligned}$$

For certain values of a the above system exhibits two steady states, one each in mode 1 and mode 2. If the decay rate, b , is larger than $\frac{1}{\Delta}$ it is possible to reach the steady state in mode 1, from mode 2, by lowering the external loading a . So, if $b\Delta < 1$ a steady state in mode 1 cannot be reached from mode 2 even by setting $a = 0$.

3.2 Hysteresis switching

Figure 1(b) illustrates the situation when $h(x)$ is approximated by ideal hysteresis⁶. The dynamics of the system (5) is given by:

$$\begin{aligned} \mathcal{S} &= \{\mathcal{I}, \mathcal{F}, \mathcal{E}\} \text{ where} \\ \mathcal{I} &= \{1, 2\}, \mathcal{F} = \{f_1(x, a) = a - bx, f_2(a, x) = a - bx + 1\} \\ \mathcal{E} &= \{\phi_{12}(x(t)) = x(t) - \Delta_2 = 0, \phi_{21}(x(t)) = \Delta_1 - x(t) = 0, \Delta_2 > \Delta_1\} \\ \dot{x}(t) &= a(t) - bx(t), \text{ for } x(t) < \Delta_2 \\ \dot{x}(t) &= a(t) - bx(t) + 1, \text{ for } x(t) > \Delta_1. \end{aligned}$$

Here, the dynamics admit history dependence. Again, if $b\Delta_1 < 1$, we see that a transition from mode 2 to mode 1 is not possible even when a is set to zero.

The above models, with threshold effects, can be useful in analyzing several problems that arise in pollution management. In the next section, we consider a particular example, the shallow lake model, and analyze the optimal management and open loop Nash equilibrium policies.

4 The shallow lake model

The shallow lake model has received considerable interest over the last two decades. The essential dynamics [16] of the eutrophication process can be modeled by the differential equation

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t) + 1}, x(0) = x_0,$$

where $x(t)$ is the amount of phosphorus in the lake at time t , $a(t)$ is the total input of phosphorus washed into the lake due to farming activities, b is the rate of loss of phosphorus due to sedimentation, and the last term captures internal biological processes for the production of phosphorus. An agent i receives benefits, by an action $a_i(t)$, as $B(a_i(t)) = \ln a_i(t)$ ⁷ and incurs a cost, towards cleaning, as $D(x(t)) = cx^2(t)$. Here,

⁶Hysteresis effects can be modeled using smooth nonlinear functions, for instance, in [1], the author formulates dynamic programming methods for some optimal control problems with hysteresis.

⁷Notice that since $B(a_i(t)) = \ln a_i(t)$ we implicitly assume $a_i : [0, \infty) \rightarrow (0, \infty)$.

the parameter c models the relative cost of pollution. Next, we study the above canonical model with threshold effects by approximating the nonlinearity with simple switching and with hysteresis. Using the shallow lake terminology, modes 1 and 2 correspond to oligotrophic and eutrophic regions respectively. We first consider the optimal management problem with simple switching.

4.1 Optimal management

Following theorem 1, the necessary conditions for $(x^*(t), a_1^*(t), \dots, a_N^*(t))$ to be optimal for the optimal management problem with simple switching are given as follows:

$$\begin{aligned}
\text{mode 1} \quad H_1(\cdot) &= \lambda_0 e^{-rt} \left(\sum_i \ln a_i(t) - Ncx^2(t) \right) + \lambda_1(t) \left(\sum_i a_i(t) - bx(t) \right) \\
\frac{\partial H_1(\cdot)}{\partial a_i} \Big|_{a_i=a_i^*} = 0 &\text{ gives } \lambda_0 e^{-rt} \frac{1}{a_i^*(t)} + \lambda_1(t) = 0, \quad i = 1, 2, \dots, N \\
x^*(t) &= \sum_i a_i^*(t) - bx^*(t), \quad x^*(0) = x_0 \\
\dot{\lambda}_1(t) &= - \frac{\partial H_1(\cdot)}{\partial x} \Big|_{x=x^*} = b\lambda_1(t) + 2cN\lambda_0 e^{-rt} x^*(t) \\
(\lambda_0, \lambda_1(t)) &\neq 0, \forall t \geq 0
\end{aligned}$$

$$\begin{aligned}
\text{mode 2} \quad H_2(\cdot) &= \lambda_0 e^{-rt} \left(\sum_i \ln a_i(t) - Ncx^2(t) \right) + \lambda_2(t) \left(\sum_i a_i(t) - bx(t) + 1 \right) \\
\frac{\partial H_2(\cdot)}{\partial a_i} \Big|_{a_i=a_i^*} = 0 &\text{ gives } \lambda_0 e^{-rt} \frac{1}{a_i^*(t)} + \lambda_2(t) = 0, \quad i = 1, 2, \dots, N \\
x^*(t) &= \sum_i a_i^*(t) - bx^*(t) + 1, \quad x^*(0) = x_0 \\
\dot{\lambda}_2(t) &= - \frac{\partial H_2(\cdot)}{\partial x} \Big|_{x=x^*} = b\lambda_2(t) + 2cN\lambda_0 e^{-rt} x^*(t) \\
(\lambda_0, \lambda_2(t)) &\neq 0, \forall t \geq 0.
\end{aligned}$$

Here, $x(t) \geq 0$ and $a_i : [0, \infty) \rightarrow (0, \infty)$, i.e., interior solutions are considered. If $\lambda_0 = 0$ we see that $\lambda_i(t) = 0$, $i = 1, 2, \forall t \geq 0$ in the above necessary conditions. So, necessary conditions hold in normal form, i.e., $\lambda_0 = 1$. We still cannot assume the natural transversality condition to hold true. During a switching instant $\tau > 0$ we have $x^*(\tau) = \Delta$ and the adjoint jump condition given by $\lambda_i(\tau^-) = \lambda_j(\tau^+) + \beta$, $i \neq j$, $i, j = 1, 2$, $\beta \in \mathbb{R}$ holds true. Further, if $k^*(\tau^-) = i$ and $k^*(\tau^+) = j$, i.e., during a transition from mode i to mode j , we have $H_i(\tau^-, x^*(\tau^-), a_1^*(\tau^-), \dots, a_N^*(\tau^-), \lambda_i(\tau^-)) = H_j(\tau^+, x^*(\tau^+), a_1^*(\tau^+), \dots, a_N^*(\tau^+), \lambda_j(\tau^+))$. So, the Hamiltonian continuity conditions are given by:

$$H_1(\tau^-, \Delta, a_1^*(\tau^-), \dots, a_N^*(\tau^-), \lambda_1(\tau^-)) = H_2(\tau^+, \Delta, a_1^*(\tau^+), \dots, a_N^*(\tau^+), \lambda_2(\tau^+)) \quad (7)$$

$$H_2(\tau^-, \Delta, a_1^*(\tau^-), \dots, a_N^*(\tau^-), \lambda_2(\tau^-)) = H_1(\tau^+, \Delta, a_1^*(\tau^+), \dots, a_N^*(\tau^+), \lambda_1(\tau^+)). \quad (8)$$

The above optimality equations, excepting equations (7) and (8), remain unaltered even for ideal hysteresis approximation. For the latter case, the parameter Δ in (7) and (8) should be replaced with Δ_2 and Δ_1 respectively. Following lemma 2, the obtained optimal dynamics and switching conditions in $(x^*(t), \lambda_c(t))$

space are transformed in $(x^*(t), a_1^*(t), \dots, a_N^*(t))$ space as follows:

$$\dot{x}^*(t) = \sum_i a_i^*(t) - bx^*(t) + \alpha, \quad x^*(0) = x_0, \quad \alpha = \begin{cases} 0 & x^*(t) < \Delta \\ 1 & x^*(t) > \Delta \end{cases}$$

$$\dot{a}_i^*(t) = -(r+b)a_i^*(t) + 2Nca_i^{*2}(t)x^*(t), \quad i = 1, \dots, N \text{ (except at the switching instants).}$$

During the switching instant $t = \tau$, we have $x^*(\tau) = \Delta$ and the Hamiltonian continuity conditions given by (7) and (8) are satisfied. The above necessary conditions lead to an $N + 1$ dimensional optimal vector field. In order to analyze the optimal dynamics we consider symmetric strategies, i.e., $a_i^*(t) = a^*(t)/N$. The symmetry assumption results in a 2 dimensional vector field which can be analyzed using the phase plane diagram. The necessary conditions with symmetry are now given as:

$$\dot{x}^*(t) = a^*(t) - bx^*(t) + \alpha, \quad x^*(0) = x_0, \quad \alpha = \begin{cases} 0 & x^*(t) < \Delta \\ 1 & x^*(t) > \Delta \end{cases} \quad (9)$$

$$\dot{a}^*(t) = -(r+b)a^*(t) + 2ca^{*2}(t)x^*(t) \text{ (except at the switching instants).} \quad (10)$$

Again, during the switching instant $t = \tau$ the Hamiltonian continuity conditions lead to the following equations.

$$\text{If } k^*(\tau^-) = 1 \text{ and } k^*(\tau^+) = 2, \text{ then } \ln a^*(\tau^-) + \frac{b\Delta}{a^*(\tau^-)} = \ln a^*(\tau^+) + \frac{b\Delta - 1}{a^*(\tau^+)} \quad (11)$$

$$\text{If } k^*(\tau^-) = 2 \text{ and } k^*(\tau^+) = 1, \text{ then } \ln a^*(\tau^-) + \frac{b\Delta - 1}{a^*(\tau^-)} = \ln a^*(\tau^+) + \frac{b\Delta}{a^*(\tau^+)}. \quad (12)$$

The equations (9-12) constitute the optimal vector field of the shallow lake model with simple switching. Due to symmetry, we have the following observation (see appendix for the proof):

Lemma 4. *The symmetric open loop Nash equilibrium problem can be solved as a symmetric optimal management problem with c replaced by $\frac{c}{N}$.*

Due to lemma 4, the symmetric open loop Nash equilibrium problem can be solved from the necessary conditions of the symmetric optimal management problem with parameter c replaced by $\frac{c}{N}$. So, the symmetric open loop Nash equilibrium problem is a potential game⁸. In the following discussion we study the optimal management problem in detail.

4.1.1 Phase plane analysis

The equilibrium points of the optimal vector field are:

$$\text{mode 1} \quad (x_{eq}, a_{eq}) = \left\{ (0, 0), \left(\sqrt{\frac{r+b}{2cb}}, \sqrt{\frac{b(r+b)}{2c}} \right), \left(-\sqrt{\frac{r+b}{2cb}}, -\sqrt{\frac{b(r+b)}{2c}} \right) \right\}$$

⁸A potential game [19] facilitates to compute Nash equilibria as an optimization problem instead of a fixed point problem.

$$\text{mode 2} \quad (x_{eq}, a_{eq}) = \left\{ \left(\frac{1}{b}, 0 \right), \left(\frac{1}{2b} + \sqrt{\frac{1}{4b^2} + \frac{r+b}{2cb}}, \sqrt{\frac{1}{4} + \frac{b(r+b)}{2c}} - \frac{1}{2} \right), \right. \\ \left. \left(\frac{1}{2b} - \sqrt{\frac{1}{4b^2} + \frac{r+b}{2cb}}, -\sqrt{\frac{1}{4} + \frac{b(r+b)}{2c}} - \frac{1}{2} \right) \right\}.$$

Here, it should be noted that the presence of equilibrium points in the above optimal vector field depends

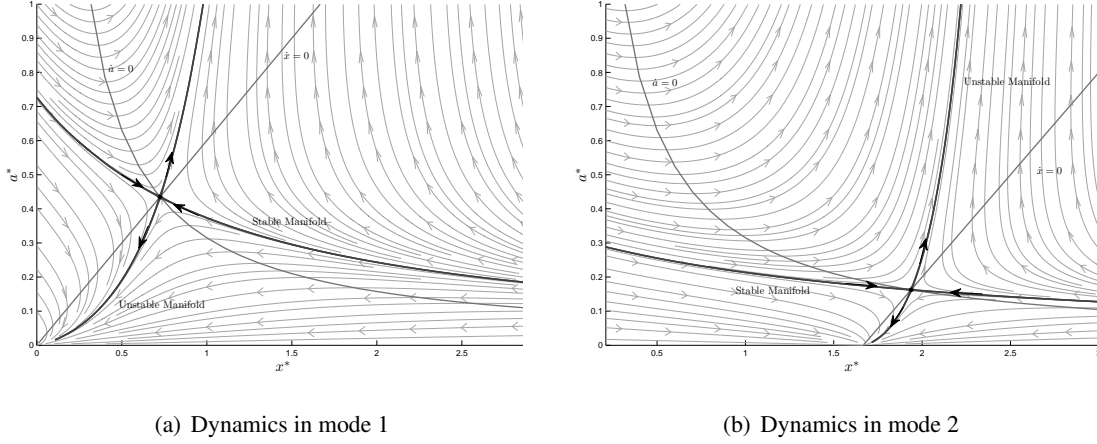


Figure 2 – Phase plane analysis

on the location of the threshold. We discuss these issues related to bifurcations in section 4.1.3.

Since the cost term involves $\ln(a)$, the nutrient loading should satisfy $a(t) > 0$. So, only the second equilibrium point is chosen for each of the modes. Let x_{eq}^1 and x_{eq}^2 denote these equilibrium points. Then we have, $0 < x_{eq}^1 < x_{eq}^2$ and $x_{eq}^2 > \frac{1}{b}$. The eigenvalues of the Jacobian matrix, for the linearized dynamics near the equilibrium points, are:

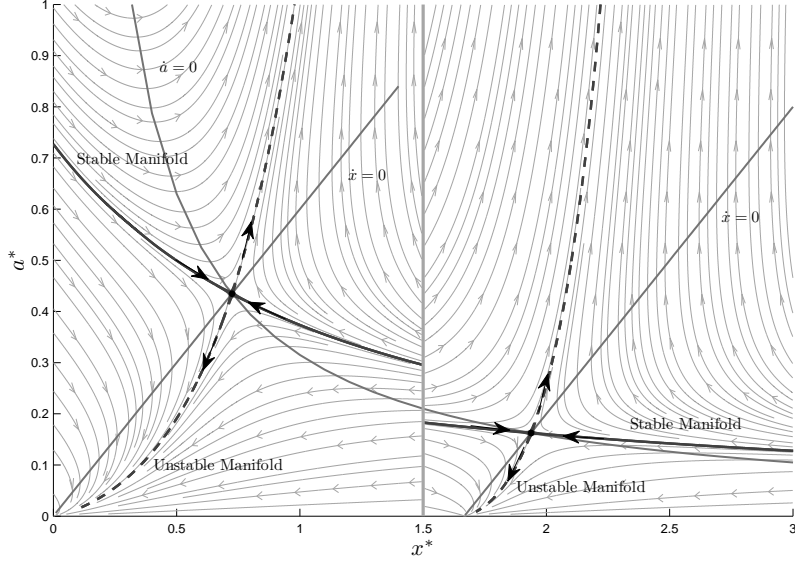
$$\text{mode 1} \quad \frac{r}{2} \pm \frac{\sqrt{8b^2 + 8br + r^2}}{2}$$

$$\text{mode 2} \quad \frac{r}{2} \pm \frac{\sqrt{r^2 + 8b^2 + 8br + 4c - 4\sqrt{c^2 + 2brc + 2b^2c}}}{2}.$$

By inspection, the equilibrium point in mode 1 is clearly a saddle point. Now we have:

$$8b^2 + 8br + 4c - 4\sqrt{c^2 + 2brc + 2b^2c} = 4 \left(c + 2b(r+b) - \sqrt{c(c + 2b(r+b))} \right) \\ = 4\sqrt{c + 2b(r+b)} \left(\sqrt{c + 2b(r+b)} - \sqrt{c} \right) > 0.$$

So, the equilibrium point in mode 2 is also a saddle point. Figures 2(a), 2(b) and 3(a) illustrate the phase portrait of the optimal dynamics in mode 1 and mode 2 with simple switching respectively. The chosen parameters are $b = 0.6$, $c = 0.5$, $r = 0.03$ and $\Delta = 1.5$. Any trajectory approaching the surface at $x = \Delta$ undergoes a switching according to the rules (11) and (12). Next, we analyze these switching rules in



(a) Switching dynamics with $\Delta = 1.5$

Figure 3 – Phase plane analysis

detail.

4.1.2 Switching rules

Before proceeding with the actual switching analysis we discuss solvability of the equation

$$s(x, m) = \ln(x) + \frac{m}{x} = n, \quad m, n \in \mathbb{R}, x > 0. \quad (13)$$

If $m = 0$ then $x = e^n$. We consider the case $m \neq 0$. After rearranging terms the above equation can be written as $ye^y = l$, $y = -\frac{m}{x}$, $l = -me^{-n}$. Solution of the reformulated equation is given by $y = \mathcal{W}(l)$, where $\mathcal{W}(\cdot)$ is the Lambert W function [6]. Here, $\mathcal{W}(z)$ is single valued for $\{z \geq 0\} \cup \{-\frac{1}{e}\}$, multiple valued for $-\frac{1}{e} < z < 0$, and not defined for $z < -\frac{1}{e}$. Figure 4(a) shows two branches of $\mathcal{W}(z)$ denoted as $\mathcal{W}_0(z)$ and $\mathcal{W}_{-1}(z)$. Thus, the solution of (13) is given by $x = -\frac{m}{\mathcal{W}(-me^{-n})}$.

mode 1 to mode 2: When the optimal system switches from mode 1 to mode 2, following (11), the jump in the control satisfies:

$$s(a^-, b\Delta) = s(a^+, b\Delta - 1).$$

Here, $s(\cdot, b\Delta) : (0, \infty) \rightarrow [\ln b\Delta + 1, \infty)$. If $b\Delta = 1$, then $a^+ = e^{-s(a^-, b\Delta)}$. If $b\Delta \neq 1$, then $a^+ = -\frac{b\Delta - 1}{\mathcal{W}(-(b\Delta - 1)e^{-s(a^-, b\Delta)})}$. For $b\Delta < 1$, we have $-(b\Delta - 1)e^{-s(a^-, b\Delta)} > 0$. So, a jump results in a^+ in the interval $[a_{12}, \infty)$, $a_{12} = \frac{1 - b\Delta}{\mathcal{W}_0((\frac{1}{b\Delta} - 1)^{\frac{1}{2}})}$. Here, $\mathcal{W}(z)$ increases for $z > 0$. For $b\Delta > 1$, we have $-(b\Delta - 1)e^{-s(a^-, b\Delta)} < 0$. So, a jump results in a_i^+ in the interval $(0, a_{12}^l]$ and a_h^+ in the interval $[a_{12}^h, \infty)$ $a_{12}^l = \frac{1 - b\Delta}{\mathcal{W}_0((\frac{1}{b\Delta} - 1)^{\frac{1}{2}})}$ and

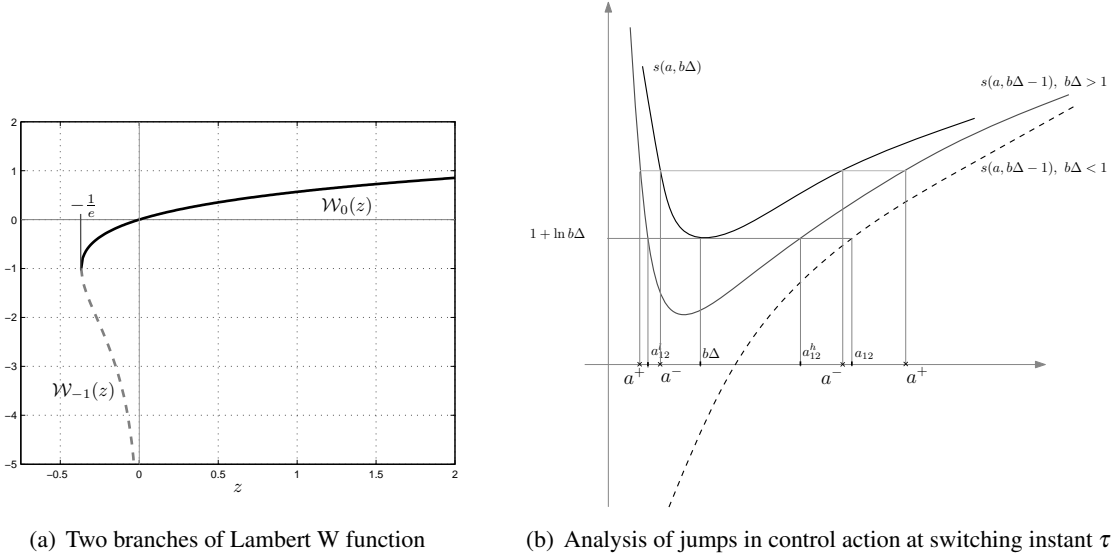


Figure 4 – Switching analysis

$a_{12}^h = \frac{1-b\Delta}{\mathcal{W}_{-1}\left(\left(\frac{1}{b\Delta}-1\right)\frac{1}{e}\right)}$. Here, $\mathcal{W}(z)$ decreases for $z < 0$. Superscripts l and h denote the lower and higher values which are computed at different branches of $\mathcal{W}(z)$ for $-\frac{1}{e} < z < 0$.

mode 2 to mode 1: When the optimal system switches from mode 2 to mode 1, following (12), the jump in the control satisfies:

$$s(a^-, b\Delta - 1) = s(a^+, b\Delta).$$

Then, $a^+ = \frac{b\Delta}{\mathcal{W}(-b\Delta e^{-s(a^-, b\Delta-1)})}$. We know a^+ is well defined only if $0 < b\Delta e^{-s(a^-, b\Delta-1)} \leq \frac{1}{e}$ which implies $s(a^-, b\Delta - 1) \geq \ln b\Delta + 1$. Further, for $b\Delta e^{s(a^-, b\Delta-1)} = \frac{1}{e}$ which implies $s(a^-, b\Delta - 1) = \ln b\Delta + 1$. So, a^- should satisfy $s(a^-, b\Delta - 1) \geq \ln b\Delta + 1$ for a jump to happen from mode 2 to mode 1. In such a case, a jump results in two points, namely $a_l^+ \in (0, b\Delta]$ and $a_h^+ \in [b\Delta, \infty)$.

A graphical illustration of the switchings is given in figure 4(b). Here, a^- is called a predecessor of a^+ , so a^+ is a successor of a^- . All switchings happen on the surface $x = \Delta$. Notice, there always exists a successor during transitions from mode 1 to mode 2, whereas some points on the switching surface may not have predecessors in mode 1. Further, in some cases there exist more than one successor or predecessor. These characteristics of the optimal vector field should be considered while analyzing the optimal candidates. We discuss these issues in the next section.

4.1.3 Analysis of optimal candidates

In this subsection we use the results from subsections 4.1.1 and 4.1.2 to analyze the optimal system (9-12) and arrive at conclusions regarding the optimal candidates and control actions. First, we notice that a solution trajectory, say $\gamma(t)$, of the optimal system (9-12) starting at a point (x_0, a_0) , with $x_0 > 0$ and $a_0 > 0$, either

1. converges to one of the equilibrium points as $t \rightarrow \infty$, or
2. leads to a control $a^*(t)$ that goes to infinity in a finite time, or
3. converges to a closed orbit.

A) Solutions approaching stable equilibrium points

First, we notice that a trajectory $\gamma(t)$ approaching any equilibrium point admits a finite number of switchings. As a result, the truncated candidate trajectory in the last interval satisfies necessary conditions similar to a classical problem. So, the transversality condition, given by $\lim_{t \rightarrow \infty} -\frac{Ne^{-rt}}{a^*(t)} = 0$, holds true⁹. Next, we show that the trajectory $\gamma(t)$ approaching the stable equilibrium points $(0,0)$ and $(\frac{1}{b}, 0)$ fails to satisfy the transversality condition. First, consider a linearization around the stable equilibrium point $(0,0)$. The eigenvectors associated with the eigenvalues are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -r \end{bmatrix}$. Thus, the trajectory $\gamma(t)$ approaching the stable equilibrium points can be approximated as $\gamma(t) = \begin{bmatrix} x^*(t) \\ a^*(t) \end{bmatrix} = c_1 e^{-bt} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-(r+b)t} \begin{bmatrix} 1 \\ -r \end{bmatrix} + \begin{bmatrix} o(e^{-bt}) \\ o(e^{-(r+b)t}) \end{bmatrix}$. We see that the transversality condition is violated for trajectories approaching stable equilibrium points, i.e., $\lim_{t \rightarrow \infty} -\frac{e^{-rt}}{-c_2 r e^{-(r+b)t} + o(e^{-(r+b)t})} = \lim_{t \rightarrow \infty} \frac{e^{bt}}{c_2 r - o(1)} \neq 0$. Following the same reasoning it can be shown that trajectories approaching the other stable equilibrium point also fail to satisfy the transversality condition.

B) Solutions going to infinity

In this discussion, similar to sections A.2 and A.3 of [30], we show two properties for the solutions that grow without bound. Firstly we show that it is not possible for solutions going to infinity that $a(t)$ remains bounded. For solutions going to infinity we have $x(t) \geq \delta > 0$ for all t . As a result, we have $\dot{a}(t) = -(r+b)a(t) + 2cx(t)a^2(t) \geq -(r+b)a(t) + 2c\delta a^2(t)$. Since the solution grows unbounded there exists a t_* such that $a(t_*) = \frac{r+b}{c\delta}$. So, we have $\dot{a}(t) \geq c\delta a^2(t)$ for $t \geq t_*$. Now, setting $v(t_*) = a_*$, the equation $\dot{v}(t) = c\delta v^2(t)$ has a solution $v(t) = \frac{u_*}{1 - a_* c \delta (t - t_*)}$ for all $t \geq t_*$ which goes to infinity in finite time. By Gronwall's inequality we have $a(t) \geq v(t)$ for all $t \geq t_*$. So, $a(t)$ goes to infinity in finite time as well.

We show that it is not possible for a trajectory $\gamma(t)$ to grow unbounded while $x^*(t)$ remains bounded. If the latter condition holds, then we have $x^*(t) < M$ for $t > 0$, which implies $x^* = a^* - bx^* > a - bM$ for dynamics in mode 1 and $\dot{x}^* = a^* - bx^* + 1 > a^* - bM + 1$ for dynamics in mode 2. However, since $(x^*(t), a^*(t)) \rightarrow \infty$, there exists $T_0 > 0$ such that $a^*(t) > bM + 2$ for all $t > T_0$. So, for $T = T_0 + M$, $x^*(T) \geq M$, which contradicts the assumption $x^*(t) < M$ for all t .

The solutions with finite escape time are not admissible, see assumption 1. So, we are left with candidates that approach saddle node equilibria and closed orbits. The divergence of the non-switched analogue of the optimal system, in state-adjoint coordinates, is equal to $r > 0$. So, using Poincaré - Bendixson criterion the existence of closed orbits is ruled out, see [30]. In a switched system, however, it is unclear as to how the notion of divergence should be defined. Numerical simulations suggest (see also section 4.2) that closed orbits do not exist. For that reason we make the following assumption.

Assumption 3. We only consider optimal candidates that reach saddle node equilibria as $t \rightarrow \infty$.

C) Optimal candidates and objective

Let E_i^u and E_i^s denote the unstable and stable manifolds in the mode i . If $x_0 \in \mathbb{R}_+$ is the initial state of the lake then the optimal candidates are obtained by first tracing the trajectories backwards starting at the equilibrium points. Let $\gamma(t) \in \mathbb{R}_+^2$ be one such candidate, then the initial nutrient loading, i.e., $a(0) = a_0$, is obtained as the intersection of $\gamma(t)$ with the line $x = x_0$, and as a result multiple candidates, starting at x_0 , are possible. Due to assumption 3, these candidates undergo only a finite number of switchings and as

⁹The transversality condition $\lim_{t \rightarrow \infty} e^{-rt} \lambda(t) = 0$ is satisfied if $\liminf_{t \rightarrow \infty} x(t) > 0$, see [13].

a result lemma 1 can be used to compare the objectives along the candidate trajectories.

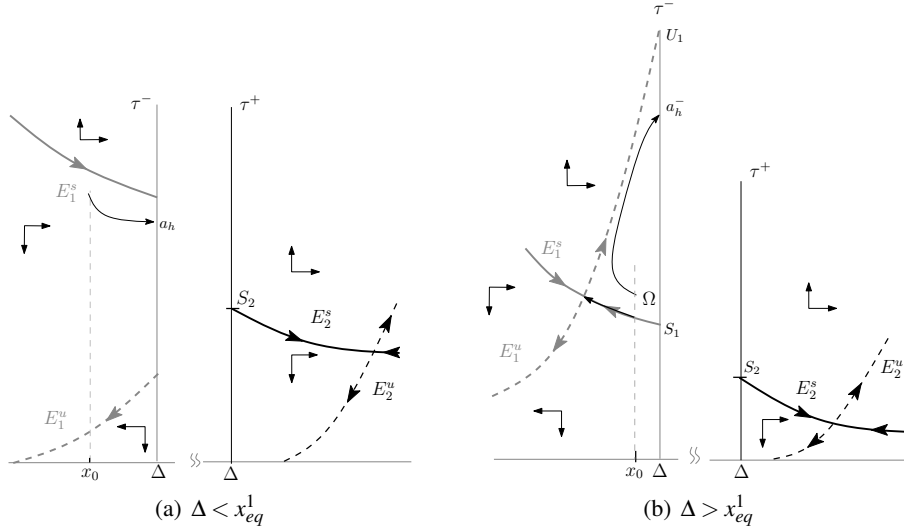


Figure 5 – Bifurcation analysis for $b\Delta < 1$

The optimal vector field of the classical shallow lake problem with smooth nonlinearities admits complex qualitative behaviors such as multiple steady states, existence of indifference or Skiba points and bifurcations due to variations in the parameters b , c and r , refer [13] for a complete analysis. In the present model with deterministic thresholds, we consider bifurcations due to variations in the switching surface. Further, we make the following assumption.

Assumption 4. The switching surface does not coincide with the equilibrium points, i.e., $\Delta \notin \{0, \frac{1}{b}, x_{eq}^1, x_{eq}^2\}$.

D) Bifurcations due to switching surface

The qualitative behavior of the optimal dynamics depends upon the position of the switching surface. Let S_i and U_i be points where the stable and unstable manifolds in mode i touch the switching surface. As described in section 3, the critical decay rate $b = \frac{1}{\Delta}$ plays a crucial role. We consider the following situations:

$b\Delta < 1$: First, we notice that if $b\Delta < 1$ any trajectory approaching the switching surface from mode 1, after entering mode 2 satisfies $\dot{x} = a - bx + 1$. Near the switching surface in mode 2 we have $\dot{x}(\tau^+) = a - (b\Delta - 1) > 0$. So, the trajectory never returns to mode 1, i.e., if the lake switches to turbid state it can never return to clear state.

- (a) Consider the case with $\Delta < x_{eq}^1$ and the related phase diagram in figure 5(a). For any $x_0 < \Delta$, the optimal candidate is the one that switches to the point S_2 from mode 1. If S_2 does not have a predecessor in mode 1 then there is no optimal solution¹⁰. If $x_0 > \Delta$, then the optimal candidate is the trajectory starting at $(x_0, E_2^s(x_0))$. So, the admissible candidates converge to the steady state in mode 2.
- (b) Next, we consider the case $\Delta > x_{eq}^1$ and the related phase diagram in figure 5(b). Following the discussion in 4.1.3B there always exists a closed region Ω such that a trajectory originating in Ω leaves Ω in a finite time. A detailed analysis includes finding the predecessors of the point S_2 . The optimal candidates can reach the steady states either in mode 1 or in mode 2. However, starting in

¹⁰Transversality condition allows for trajectories with $a(t)$ going to ∞ .

mode 2 the steady state in mode 1 cannot be reached, whereas the steady state in mode 2 can be reached starting in mode 1.

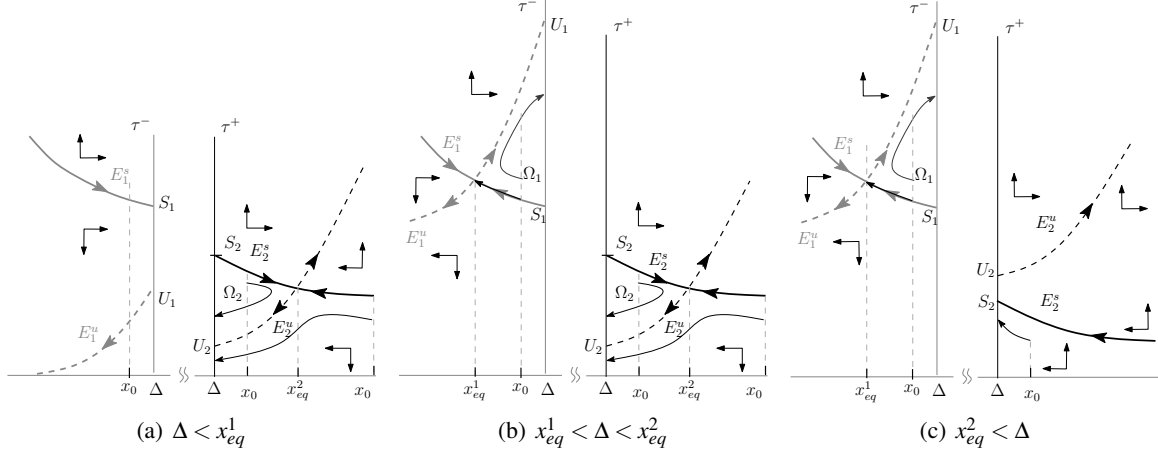


Figure 6 – Bifurcation analysis for $b\Delta > 1$

$b\Delta > 1$: The following observations can be made about the optimal candidates if $b\Delta > 1$. The economic agents, by lowering the nutrient loading, can reverse the lake to mode 1 from mode 2, i.e., $\dot{x}(\tau^+) < 0$ near the switching surface. We have the following three cases.

- (c) Consider the case with $\Delta < x_{eq}^1$ and the related phase diagram in figure 6(a). A trajectory starting in mode 1 either switches to mode 2 or approaches the origin. Trajectories reaching the equilibrium point in mode 2 are optimal. The existence of the closed region Ω_2 follows from the discussion in section 4.1.3A. A detailed analysis includes tracing the predecessors for the point S_2 on the surface $x = \Delta$ at τ^- .
- (d) Consider the case with $x_{eq}^1 < \Delta < x_{eq}^2$ and the related phase diagram in figure 6(b). The optimal candidates can reach either of the steady states in mode 1 and mode 2 by first reaching the points S_1 and S_2 . So, the optimal candidates are obtained by finding the predecessors of these points using the switching rules devised in section 4.1.2.
- (e) Consider the case with $\Delta > x_{eq}^2$ and the related phase diagram in figure 6(c). Optimal candidates reaching the steady state in mode 1 are optimal and these are obtained by using the switching rules for finding the predecessors.

Next, we demonstrate the subtleties in finding the optimal candidates for a specific choice of the above mentioned possibilities. We consider the case 4.1.3D.(e), i.e, when $b\Delta > 1$ and $\Delta > x_{eq}^2$, and explain in detail about the optimal candidates for all initial states x_0 . Towards that end, we have the following algorithm/procedure to generate points on the switching surface which eventually reach the steady state in mode 1.

Algorithm 1. Construct sequences $d_k, e_k, g_k, h_k, k = 0, 1, 2, \dots$ using the following steps:

1. For $k = 0$, set $d_0 = S_1$ and obtain e_0, g_0, h_0 by solving the equation (follows from section 4.1.2)

$$s(d_0, b\Delta) = s(e_0, b\Delta) = s(g_0, b\Delta - 1) = s(h_0, b\Delta - 1)$$

$$s.t \quad d_0 < b\Delta < e_0, \quad g_0 < a_{12}^l < b\Delta - 1 < a_{12}^h < h_0.$$

2. If $e_0 < U_1$ go to step 3 else STOP.

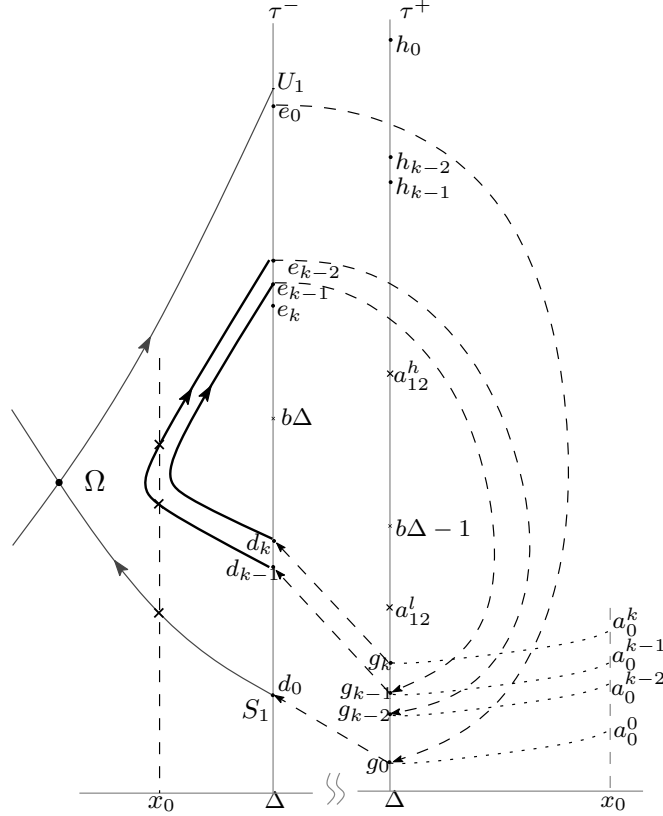


Figure 7 – Graphical illustration of algorithm 1.

3. For, $k \geq 1$, solve the boundary value problem to obtain $a(0)$ (solution exists due to the property of the region Ω)

$$\dot{x} = a - bx, \dot{a} = -(r+b)a + 2ca^2x, x(0) = \Delta, x(\tau) = \Delta, a(\tau) = e_{k-1}.$$

Set $d_k = a(0)$

4. Solve $s(d_k, b\Delta) = s(e_k, b\Delta) = s(g_k, b\Delta - 1) = s(h_k, b\Delta - 1)$, $d_k < b\Delta < e_k$, $g_k < a_{12}^l < b\Delta - 1 < a_{12}^h < h_k$.
5. Set $k = k + 1$ and go to step 3.

We notice that the above procedure generates sequences that satisfy the following condition:

$$g_0 < g_1 < g_2 < \dots < g_k < \dots < a_{12}^l < b\Delta - 1 < a_{12}^h < \dots < h_k < \dots < h_2 < h_1 < h_0$$

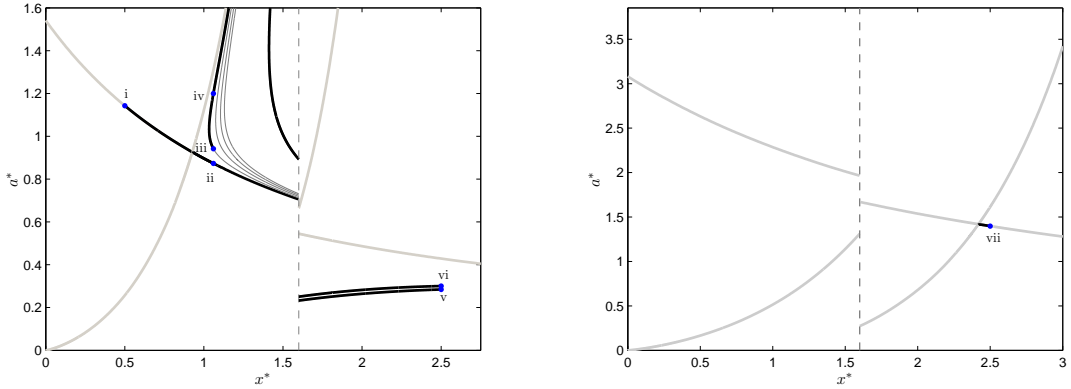
and $d_0 < d_1 < d_2 < \dots < d_k < \dots < b\Delta < \dots < e_k < \dots < e_2 < e_1 < e_0$.

A graphical illustration of the algorithm is given by figure 7. Any trajectory starting at (Δ, d_k) , (Δ, e_k) and (Δ, g_k) will eventually reach the steady state in mode 1. Now, we consider the three situations w.r.t. choice of initial state x_0 and find the optimal candidates for each one of them.

- First, if $x_0 > \Delta$, i.e., starting in mode 2, let (x_0, a_0^k) denote the initial state of the trajectory which reaches the point (Δ, g_k) (shown as dotted lines in figure 7). Then a candidate trajectory starting at (x_0, a_0^k) will undergo k cycles, that spiral out, before reaching the stable manifold E_1^s starting at d_0 .

So, we have countably infinite candidates that satisfy the necessary conditions. The objective along each path can be calculated using lemma 1.

- $x_{eq}^1 < x_0 < \Delta$. If $e_0 > U_1$, then there may exist two candidates that approach steady state in mode 1. The first one is the stable manifold. The other one may lie above the unstable manifold E_1^u which approaches e_0 , then switches to mode 2 at g_0 and switches back to mode 1 at d_0 . For $e_0 < U_1$, there always exists one candidate that lies on the stable manifold E_1^s . Further, we observe that trajectories starting at d_{k-1} and ending at e_k intersect the line $x = x_0$ at two points or at one point (tangential intersection). So, depending upon the location of x_0 we have either $2L$ or $2L + 1$ candidates, where L represents the number of paths that intersect the section $x = x_0$.
- For $x_0 < x_{eq}^1$, if $e_0 < U_1$ then there exists one candidate that lies on the stable manifold E_1^s . Further, if $e_0 > U_1$ there may exist an additional candidate that reaches e_0 switches to mode 2 at g_0 and returns to mode 1 at d_0 .



(a) Phase portrait with $b = 1$, $c = 0.6$, $r = 0.03$ and $\Delta = 1.6$ (b) Phase portrait with $b = 1$, $c = 0.15$, $r = 0.03$ and $\Delta = 1.6$

Figure 8 – Optimal candidates for various initial states. Thick gray lines indicate stable and unstable manifolds, dashed line indicates the switching surface. Thick dark lines indicate the optimal candidates.

Remark 1. Notice, that the qualitative behavior of the optimal vector field alters when $b\Delta - 1$ changes sign. The scenarios discussed above consider all the possibilities that can arise due to parameter variations. These situations can be easily checked once the parameter values are known. However, for the classical shallow lake problem one has to resort to extensive numerical simulations to analyze the bifurcations, see figure 4 of [30]. The situations, where there exists only one equilibrium point in the phase plane, illustrated in figures 5(a), 6(a) and 6(c) are similar to plots (i), (vii) and (viii) in figure 4 of [30]. Similarly, the situations, where there exist two equilibrium points in the phase plane, illustrated in figures 5(b) and 6(b) are similar to plots (ii)-(vi) in figure 4 of [30].

E) Numerical illustration

Consider the shallow lake system with parameter values $b = 1$, $c = 0.6$, $r = 0.03$, $\Delta = 1.6$ and $N = 4$. For this choice of parameters, we have $b\Delta = 1.6 > 1$, $U_1 = 5.03493$ and $e_0 = 4.9259$. The optimal candidates are obtained by following the previous discussion. Optimal candidates starting at various initial states are illustrated, in small roman letters, and the benefit of each player along these trajectories is calculated according to equations (14), in cooperation, and (15), in noncooperation. First, we consider the optimal management case and the results are illustrated in figure 8(a). For initial state $x_0 = 1.06$, we obtain three candidates labeled as (ii), (iii) and (iv). When following the paths (iii) and (iv), the agents increase

Initial State	Candidate			Objective $N = 4$ optimal management
	starting at (x_0, a_0)	# of cycles	reaches steady state in mode #	
0.50	(0.50, 1.1425) i	0	1	-65.5150
1.06	(1.06, 0.8733) ii	0	1	-66.0710
	(1.06, 0.9420) iii	1	1	-66.4980
	(1.06, 1.1998) iv	1	1	-66.4940
2.50	(2.50, 0.2844) v	0	1	-70.6235
	(2.50, 0.2851)	1	1	-71.0171
	(2.50, 0.2859)	2	1	-71.3826
	⋮	⋮	⋮	⋮
	(2.50, 0.3003)	38	1	-78.1329
	(2.50, 0.3005) vi	39	1	-78.2385
	⋮	⋮	⋮	⋮

Table 1 – Performance of optimal candidates for the parameters $b = 1$, $c = 0.6$, $r = 0.03$ and $\Delta = 1.6$.

the level of nutrient loading till the lake switches to mode 2 and instantaneously drop the levels to be able to switch back to mode 1 along the stable manifold E_1^s . We observe that $f_1(x_0, a_0) < 0$ for candidates (ii) and (iii). So, from lemma 3, we see that following the trajectory (ii) results in higher benefits, see table 1. When the agents start in mode 2, i.e., $x_0 > 1.6$, there exist countably infinite candidates each undergoing a finite number of cycles before reaching the steady state in mode 1. For instance, on path (vi) the agents can alter the nutrient levels, i.e., a decrease and increase cycle, 39 times before reaching the steady state in mode 1. We observe that $f_2(x_0, a_0) < 0$ for the candidates (v) and (vi). Again, from lemma 3 it can be inferred that following trajectory (v) results in higher benefits, see table 1.

The results can alter once the costs associated with switchings are considered. For the open loop Nash equilibrium, following lemma 4, we analyze the optimal field with $c = 0.6$ replaced with $\frac{c}{N} = 0.15$. The phase plane for the optimal vector field with $c = 0.15$, ceteris paribus, is illustrated in figure 8(b). We notice that only steady state in mode 2 can be achieved by the agents. So, the trajectory (vii) corresponds to the open loop Nash equilibrium path with $x_0 = 2.5$ and the welfare parameter set to $c = 0.6$. We notice that the mode of play induces a bifurcation in the optimal vector field; that is when players play cooperatively steady state in mode 1 is attained whereas a noncooperative behavior leads to the steady state in mode 2. Further, it is clear from the tables 1 and 2 that each player receives greater benefits in cooperation. Now, consider the effect of reducing the c from 0.6 to 0.15 on optimal management. Since, in the later case the players incur less costs, towards cleaning activities, there is an incentive for increasing the nutrients and as a result the optimal vector field results in the steady state in mode 2.

4.2 Hysteresis switching

In the previous section we notice that some candidates undergo multiple switches before reaching a steady state in mode 1. During each cycle, a candidate leaving mode 1 reenters the same mode instantaneously

¹¹The candidate starting at (0.5, 3.5) switches to mode 2 at time $t = 0.4732$ and the system (9-12) has a finite escape time at $t = 0.7844$. So, the objective is computed in the interval $[0, 0.7844)$. We discard this candidate as assumption 1 is violated. In section 5 we consider quadratic benefit functions and observe that the optimal candidates do not display finite escape time.

Initial State	Candidate			Objective	
	starting at (x_0, a_0)	# of cycles	reaches steady state in mode #	$N = 4$ optimal management	$N = 4$ ONLE for $c = 0.6$
0.50	(0.50, 3.500000)	-	-	-11.1676 ¹¹	
1.70	(1.70, 1.635331)	0	2	-63.3300	
2.50	(2.50, 1.398072) vii	0	2	-63.8598	-150.3192

Table 2 – Performance of optimal candidates for the parameters $b = 1$, $c = 0.15$, $r = 0.03$ and $\Delta = 1.6$

following the switching sequence $e_k \rightarrow g_k \rightarrow d_k$, $k \geq 0$. These control actions are still admissible in the class \mathcal{U} . The successors of g_k , in figure 7, are e_k and d_k . If we consider the path $g_k \rightarrow e_k$ instead of $g_k \rightarrow d_k$ we obtain a limit cycle $\cdots g_k \rightarrow e_k \rightarrow g_k \rightarrow e_k \cdots$. However, all these multiple (indefinite) switchings happen simultaneously at the switching instant τ and these (ill posed) candidates are not considered in our analysis. These candidates, though they remain on the switching surface, are a result of switching rules obtained from the necessary conditions. The conditions formulated in theorem 1 do not include Filippov type solutions [10] due to assumption 1. A hysteresis approximation, as given in section 3, would be useful to avoid the simultaneous multiple switchings. Sometimes, a hysteresis behavior could be inherent to the economic or ecosystem dynamics (5), see [7] and [24]. The analysis of the optimal vector field remains unaltered except for the switching rules where Δ in (11) and (12) should be replaced with Δ_2 and Δ_1 respectively. For initial states in the range $\Delta_1 < x_0 < \Delta_2$, the objective along a particular candidate depends upon whether x_0 is in mode 1 or mode 2.

Next, we discuss the issue of existence of closed orbits in the optimal dynamics with hysteresis. The Poincaré - Bendixson criterion is generally used to analyze the existence of closed orbits in a smooth vector field, and it involves computation of divergence. At present, it is unclear as to how the notion of divergence of a switching vector field ¹² can be defined. It may be noted however that a closed orbit cannot occur within either of the modes 1 and 2 as the divergence of the optimal vector field, in state-adjoint coordinates, is equal to $r > 0$. So, a closed orbit may occur during mode transitions. Let us denote the map $\varphi_i(\cdot)$ to be an orbit of the vector field in mode i . Let the adjoint jump and Hamiltonian continuity properties, see item (c) of theorem 1, be abstracted as a switching map $Sw_{ij}(\cdot)$ during a transition from mode i to mode j . To see if there exists a closed orbit, we construct the forward and backward maps as $n = Sw_{12}(\varphi_1(m))$ and $m = Sw_{21}(\varphi_2(n))$ and look for intersection points in the (m, n) plane. Figure 9(b) shows one such instance for parameter values $b = 1$, $c = 0.15$, $r = 0.03$, $\Delta_1 = 2.1$ and $\Delta_2 = 2.2$. The forward and backward maps never intersect hinting that there may not be a closed orbit. At this point we do not have conclusive evidence, however.

5 Quadratic objective

We consider the analysis with quadratic benefit given by $B(a_i(t)) = a_i(t) - \frac{k}{2}a_i^2(t)$ and $a_i(t) : [0, \infty) \rightarrow [0, \frac{1}{k}]$. With straightforward calculations it can be shown that $\lambda_0 = 1$ and lemma 4 holds true. Setting $a(t) = Na_i(t)$ and considering interior solutions the symmetric optimal management problem is solved as

¹²See [17] for some work in this direction.

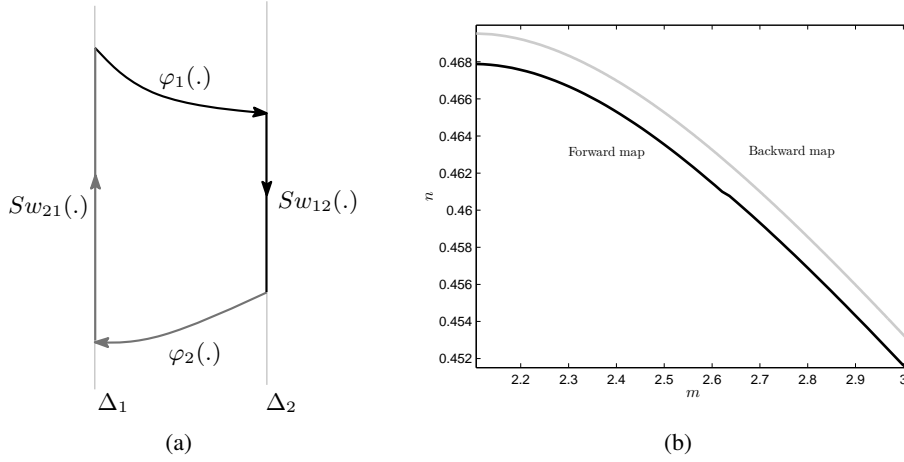


Figure 9 – Forward and backward maps to analyze closed orbits

follows:

Dynamics:
$$\begin{bmatrix} \dot{x} \\ \dot{a} \end{bmatrix} = A \begin{bmatrix} x \\ a \end{bmatrix} + B, \quad x(0) = x_0, \quad a(t) \in \left(0, \frac{N}{k}\right), \quad \alpha = \begin{cases} 0 & x < \Delta \\ 1 & x > \Delta \end{cases}$$

where
$$A = \begin{bmatrix} -b & 1 \\ \frac{2cN^2}{k} & (r+b) \end{bmatrix}, \quad B = \begin{bmatrix} \alpha \\ -\frac{(r+b)N}{k} \end{bmatrix}$$

Switching rules: mode 1 to mode 2

$$a^2(\tau^-) - 2b\Delta \left(a(\tau^-) - \frac{N}{k} \right) = a^2(\tau^+) - 2(b\Delta - 1) \left(a(\tau^+) - \frac{N}{k} \right)$$

mode 2 to mode 1

$$a^2(\tau^-) - 2(b\Delta - 1) \left(a(\tau^-) - \frac{N}{k} \right) = a^2(\tau^+) - 2b\Delta \left(a(\tau^+) - \frac{N}{k} \right)$$

Eigenvalues of A:
$$\lambda_s = \frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{2cN^2}{k} + b^2 + rb} < 0$$

$$\lambda_u = \frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{2cN^2}{k} + b^2 + rb} > r > 0$$

Equilibrium points:
$$(x^{eq}, a^{eq}) = \left(\frac{(N + k\alpha)(r+b)}{k\lambda_u\lambda_s}, \frac{N(b(r+b) - 2cN\alpha)}{k\lambda_u\lambda_s} \right)$$

$\alpha = 0$ in mode 1 and $\alpha = 1$ in mode 2 and (x^{eq}, a^{eq}) are saddle points

5.1 Switching analysis

During a transition instant at $t = \tau > 0$ there exists a jump in $a(t)$ from $a(\tau^-)$ to $a(\tau^+)$ and depending upon the parameter β (below) we have the following three possibilities.

a) If $\beta = 1 + \frac{2N}{k} - 2b\Delta > 0$, then for transitions

mode 1 to mode 2

$$a(\tau^+) = b\Delta - 1 \pm \sqrt{(a(\tau^-) - b\Delta)^2 + \beta}, \quad a(\tau^-) \in \left(0, \frac{N}{k}\right)$$

mode 2 to mode 1

$$a(\tau^+) = \begin{cases} b\Delta \pm \sqrt{(a(\tau^-) - (b\Delta - 1))^2 - \beta} & a(\tau^-) \in [0, (b\Delta - 1) - \sqrt{\beta}] \cup [(b\Delta - 1) + \sqrt{\beta}, \frac{N}{k}) \\ \text{no switching} & \text{otherwise} \end{cases}$$

b) If $\beta = 1 + \frac{2N}{k} - 2b\Delta = 0$, then for transitions

mode 1 to mode 2

$$a(\tau^+) = b\Delta - 1 \pm (a(\tau^-) - b\Delta), \quad a(\tau^-) \in \left(0, \frac{N}{k}\right)$$

mode 2 to mode 1

$$a(\tau^+) = b\Delta \pm (a(\tau^-) - (b\Delta - 1)), \quad a(\tau^-) \in \left(0, \frac{N}{k}\right)$$

c) If $\beta = 1 + \frac{2N}{k} - 2b\Delta < 0$, then for transitions

mode 1 to mode 2

$$a(\tau^+) = \begin{cases} b\Delta - 1 \pm \sqrt{(a(\tau^-) - b\Delta)^2 + \beta}, & a(\tau^-) \in (0, b\Delta - \sqrt{-\beta}] \cup [b\Delta + \sqrt{-\beta}, \frac{N}{k}) \\ \text{no switching} & \text{otherwise} \end{cases}$$

mode 2 to mode 1

$$a(\tau^+) = b\Delta \pm \sqrt{(a(\tau^-) - (b\Delta - 1))^2 - \beta} \quad a(\tau^-) \in \left(0, \frac{N}{k}\right)$$

5.2 Key observations

In the following discussion we list some key observations when the player's benefits vary quadratically with their efforts.

- The quadratic objective leads to simple linear dynamics whereas the logarithmic benefit function results in nonlinear dynamics. As a result, the candidates, in the quadratic objective context, do not exhibit finite escape time behavior.
- The equilibrium point in mode 1, i.e., with $\alpha = 0$, always exists, whereas the equilibrium point in mode 2, i.e., with $\alpha = 1$, exists only if $b(r + b) \geq 2cN$. So, (x_2^{eq}, a_2^{eq}) can vanish with an increase in

the number of players.

- Again, it is not clear if there exist limit cycles within this quadratic objective context. So, assuming that candidates undergo just a finite number of switchings there exists $\tau > 0$ such that the candidate does not undergo further switches in the interval $[\tau^+, \infty)$. The optimal candidate at $t \in [\tau^+, \infty)$ is solved as:

$$x(t) = \frac{1}{(\lambda_u - \lambda_s)} \left[\left((\lambda_u + b)x(\tau^+) - a(\tau^+) + \frac{1}{\lambda_s} \left((\lambda_u + b)\alpha + \frac{(r+b)N}{k} \right) \right) e^{\lambda_s t} + \left(-(\lambda_s + b)x(\tau^+) + a(\tau^+) - \frac{1}{\lambda_u} \left((\lambda_s + b)\alpha + \frac{(r+b)N}{k} \right) \right) e^{\lambda_u t} \right] + x^{eq}$$

$$a(t) = \frac{1}{(\lambda_u - \lambda_s)} \left[- \left(\frac{2cN^2}{k} x(\tau^+) + (\lambda_s + b)a(\tau^+) + \frac{1}{\lambda_s} \left(\frac{2cN^2}{k} \alpha - (\lambda_s + b) \frac{(r+b)N}{k} \right) \right) e^{\lambda_s t} + \left(\frac{2cN^2}{k} x(\tau^+) + (\lambda_u + b)a(\tau^+) + \frac{1}{\lambda_u} \left(\frac{2cN^2}{k} \alpha - (\lambda_u + b) \frac{(r+b)N}{k} \right) \right) e^{\lambda_u t} \right] + a^{eq}.$$

The stable manifold associated with the equilibrium point (x^{eq}, a^{eq}) is obtained by setting

$$a(\tau^+) = a^s(x(\tau^+)) = (\lambda_s + b)x(\tau^+) + \frac{1}{\lambda_u} \left((\lambda_s + b)\alpha + \frac{(r+b)N}{k} \right).$$

So, any optimal candidate not reaching the point $(x(\tau^+), a^s(x(\tau^+)))$ at $t = \tau^+$ grows unbounded and thus violates the condition $a(t) \in [0, \frac{N}{k}]$. The asymptotic Hamiltonian property along any optimal candidate is given by $\lim_{t \rightarrow \infty} e^{-rt} H(\cdot) = e^{-rt} \left(\frac{ka^2}{2N} - \left(\frac{ka}{N} - 1 \right) (bx - \alpha) - Ncx^2 \right) = 0$. Only those candidates which reach the stable manifold at $t = \tau^+$ satisfy this property. So, starting at $x(0) = x_0$, the trajectories reaching the stable manifold of an equilibrium point are optimal and lemma 1 is used to select the optimal solution, by comparing net benefits along each trajectory.

6 Conclusions

In this article we introduce a class of discounted autonomous infinite horizon optimal control problems with threshold effects. When the state and control variables are one dimensional, additional results are obtained as a consequence of the (switched) maximum principle. Using these tools we analyze the shallow lake model in the presence of threshold effects. We approximate the nonlinearities in the shallow lake dynamics with simple and hysteresis switching, which results in a switching vector field. Assuming symmetry in agents' actions, we solve the associated optimal management problem using relevant necessary conditions. Further, we provide a bifurcation analysis of the vector field. We observe that the variation of switching surface induces bifurcations in this vector field. Further, we notice, that agents' mode of play can also induce bifurcations; that is, cooperation can lead to 'clean' steady state (oligotrophic state) and noncooperation can lead to 'turbid' steady state (eutrophic state).

The main objective of this article was to highlight, with an example, the key observations when smooth nonlinear models are approximated with simple discontinuous functions. This approximation leads to simple dynamics within each mode and a complex jump rule near the switching surface. The dynamic behavior of the switched vector field is similar, qualitatively, to the smooth version. However, there are some differences. The bifurcations in the present analysis are governed by simple rules, a set of inequalities, which

can be checked/verified once the parameters are given. In the previous works these bifurcation scenarios were established numerically though their existence was proved using tools from ordinary differential equations.

There are several open issues that require considerable attention. It was shown in [30], that existence of Skiba points is closely related to heteroclinic connections¹³ in the optimal vector field. A piecewise approximation of the convex-concave production function [27] also results in multiple steady states, see figures 5(b) and 6(b), then to see whether Skiba points exist in the optimal switching dynamics would be interesting. A closed orbit, if it exists, would result in policies where players increase and decrease the nutrient levels in a sustainable way. So, to characterize the switched optimal control problems, of the type (1-3), that admit closed orbits as an optimal solution would be insightful¹⁴. For noncooperation, the present analysis is restricted to symmetric open loop Nash equilibrium policies. Using feedback policies for the optimal management problem may involve some computational burden, for instance, see [5].

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¹³This happens when a branch of an unstable manifold of an equilibrium point coincides with a branch of a stable manifold of a different equilibrium point.

¹⁴See [9, 8] for references in this direction with smooth dynamics.

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A Appendix

Proof of Lemma 1. Since $(x^*(\cdot), u^*(\cdot))$ is an optimal admissible pair, it satisfies the necessary conditions given in theorem 1. As the number of switchings is finite, say M , there exists a sequence of switching instants associated with $k^*(\cdot)$, which we denote as $\tau_1, \tau_2, \dots, \tau_j, \dots, \tau_M$. Taking the total derivative of the Hamiltonian $H_{k^*(t)}(\cdot)$ in the interval $t \in (\tau_j^+, \tau_{j+1}^-)$ we have:

$$\begin{aligned} \frac{dH_{k^*(t)}}{dt} &= \frac{\partial H_{k^*(t)}}{\partial t} + \frac{\partial H_{k^*(t)}}{\partial x^*} \dot{x}^* + \frac{\partial H_{k^*(t)}}{\partial \lambda_{k^*}} \dot{\lambda}_{k^*} + \frac{\partial H_{k^*(t)}}{\partial u^*} u^* \\ &= \frac{\partial H_{k^*(t)}}{\partial t} \quad (\text{last three terms vanish due to necessary conditions}) \\ &= -re^{-rt} g(x^*(t), u^*(t)) \\ H_{k^*(\tau_{j+1}^-)} - H_{k^*(\tau_j^+)} &= -r \int_{\tau_j^+}^{\tau_{j+1}^-} e^{-rt} g(x^*(t), u^*(t)) dt. \end{aligned}$$

Again from the necessary conditions we notice that in the last interval, i.e., $t \in [\tau_M^+, \infty)$, $(x^*(t), u^*(t))$ maximizes the objective $\int_{\tau_M^+}^{\infty} e^{-rt} g(x(t), u(t)) dt$. The truncated trajectory $(x^*(t), u^*(t))$, $t \in [\tau_M^+, \infty)$ is an optimal admissible pair for the classical discounted infinite horizon optimal control problem

$$\begin{aligned} \max \int_{\tau_M^+}^{\infty} e^{-rt} g(x(t), u(t)) dt \\ \dot{x}(t) = f_{k^*(\tau_M^+)}(x(t), u(t)), \quad x(\tau_M^+) = x^*(\tau_M^+) \\ u(t) \in U, \quad t \in [\tau_M^+, \infty). \end{aligned}$$

The truncated candidate in the last interval satisfies the additional necessary condition that the maximized Hamiltonian tends to zero when t goes to infinity. As a result, the objective along the truncated trajectory is given by $\frac{1}{r} H_{k^*(\tau_M^+)}(\tau_M^+, x^*(\tau_M^+), u^*(\tau_M^+), \lambda_{k^*(\tau_M^+)}(\tau_M^+))$. The objective along $(x^*(t), u^*(t))$, $t \in [0, \infty)$ is then given by:

$$\begin{aligned} \int_0^{\infty} e^{-rt} g(x^*(t), u^*(t)) dt &= \int_0^{\tau_1^-} e^{-rt} g(x^*(t), u^*(t)) dt + \sum_{j=1}^{M-1} \int_{\tau_j^+}^{\tau_{j+1}^-} e^{-rt} g(x^*(t), u^*(t)) dt \\ &\quad + \int_{\tau_M^+}^{\infty} e^{-rt} g(x^*(t), u^*(t)) dt \\ &= \frac{1}{r} \left(H_{k^*(0)}(\cdot) - H_{k^*(\tau_1^-)}(\cdot) + H_{k^*(\tau_1^+)}(\cdot) \cdots - H_{k^*(\tau_M^-)}(\cdot) + H_{k^*(\tau_M^+)}(\cdot) \right) = \frac{1}{r} H_{k^*(0)}(\cdot). \end{aligned}$$

In the above result the costs associated with switching are assumed to be zero. □

Proof of Lemma 2. Since the trajectory (x^*, u^*) satisfies the optimality conditions there exists a portion of this trajectory in the interior of a mode $i \in \mathcal{I}$ such that $\frac{\partial H_i^c(x^*, u, \lambda_i^c, 1)}{\partial u} \Big|_{u=u^*} = \lambda_i^c \frac{\partial f_i(x^*, u)}{\partial u} \Big|_{u=u^*} + \frac{\partial g(x^*, u)}{\partial u} \Big|_{u=u^*} = 0$. Fixing the value of x^* on this trajectory (interior portion), the concavity condition of the current value Hamiltonian (see assumption 2) leads to $\frac{\partial^2 H_i^c(x^*, u, \lambda_i^c, 1)}{\partial u^2} \Big|_{u=u^*} = \lambda_i^c \frac{\partial^2 f_i(x^*, u)}{\partial u^2} \Big|_{u=u^*} + \frac{\partial^2 g(x^*, u)}{\partial u^2} \Big|_{u=u^*} < 0$. Let λ_i^c

be denoted as a function $\eta_i(x^*, u^*)$. From the above, we have $\eta_i(x^*, u^*) = -\frac{\frac{\partial g(x^*, u)}{\partial u}|_{u=u^*}}{\frac{\partial f_i(x^*, u)}{\partial u}|_{u=u^*}}$ and the partial derivative of $\eta_i(x^*, u)$ with respect to u , for a fixed x^* , computed at u^* is given as:

$$\frac{\partial \eta_i(x^*, u)}{\partial u}|_{u=u^*} = \frac{\frac{\partial g(x^*, u)}{\partial u}|_{u=u^*} \frac{\partial^2 f_i(x^*, u)}{\partial u^2}|_{u=u^*} - \frac{\partial^2 g(x^*, u)}{\partial u^2}|_{u=u^*} \frac{\partial f_i(x^*, u)}{\partial u}|_{u=u^*}}{\left(\frac{\partial f_i(x^*, u)}{\partial u}|_{u=u^*}\right)^2} = -\frac{\frac{\partial^2 H^c(x^*, u, \eta_i(x^*, u^*), 1)}{\partial u^2}|_{u=u^*}}{\frac{\partial f_i(x^*, u)}{\partial u}|_{u=u^*}}.$$

From assumption 2, the above quantity is nonzero (strictly positive) for all the trajectories in the interior of mode i , and the statement of the lemma holds true. \square

Proof of Lemma 3. If $x_0 \notin \Phi$, there exists a portion of trajectory (x^*, u^*) starting at x_0 in the interior of a mode i . Here $i(x_0)$ denotes the mode corresponding to x_0 . From lemma 1 we have Ψ as a function of (x_0, u_0) and is well defined for all the points on the trajectory. Starting at any point (\tilde{x}, \tilde{u}) on this trajectory, and using lemma 2, we have:

$$r \frac{\partial \Psi(\tilde{x}, u)}{\partial u}|_{u=\tilde{u}} = H_{i(x_0)}^c(\tilde{x}, \tilde{u}, \eta_{i(x_0)}(\tilde{x}, \tilde{u}), 1) = \eta_{i(x_0)}(\tilde{x}, \tilde{u}) f_{i(x_0)}(\tilde{x}, \tilde{u}) + g(\tilde{x}, \tilde{u}).$$

Now, taking the partial derivative with respect to u at (\tilde{x}, \tilde{u}) gives:

$$r \frac{\partial \Psi(\tilde{x}, u)}{\partial u}|_{u=\tilde{u}} = \frac{\partial g(\tilde{x}, u)}{\partial u}|_{u=\tilde{u}} + \frac{\partial \eta_{i(x_0)}(\tilde{x}, u)}{\partial u}|_{u=\tilde{u}} f_{i(x_0)}(\tilde{x}, \tilde{u}) + \eta_{i(x_0)}(\tilde{x}, \tilde{u}) \frac{\partial f_{i(x_0)}(\tilde{x}, u)}{\partial u}|_{u=\tilde{u}}.$$

Following the proof of lemma 2 the above quantity is equal to $\frac{\partial \eta_{i(x_0)}(\tilde{x}, u)}{\partial u}|_{u=\tilde{u}} f_{i(x_0)}(\tilde{x}, \tilde{u})$ (from the maximum principle the sum of first and last terms is equal to zero). Since $\frac{\partial \eta_{i(x_0)}(\tilde{x}, u)}{\partial u}|_{u=\tilde{u}} > 0$, the statement of the lemma holds true. \square

Proof of Lemma 4. Optimal management involves solving the following optimization problem

$$\dot{x}(t) = \sum_i a_i(t) - bx(t) + \alpha, \quad \alpha = \begin{cases} 0 & x(t) < \Delta \\ 1 & x(t) > \Delta \end{cases}, \quad x(0) = x_0,$$

$$\max_{a_1, a_2, \dots, a_N} J, \quad J = \int_0^\infty e^{-rt} \left(\sum_i \ln a_i(t) - Ncx^2(t) \right) dt.$$

The necessary conditions in the current value form are given by:

$$\text{mode } j \quad H_j^c(\cdot) = \left(\sum_i \ln(a_i) - Ncx^2 \right) + \lambda_j^c \left(\sum_i a_i - bx + j - 1 \right)$$

$$\frac{1}{a_i^*} + \lambda_j = 0, \quad i = 1, 2, \dots, N \text{ (follows from maximum condition)}$$

$$\dot{\lambda}_j(t) = (r + b)\lambda_j + 2cNx^*$$

$$\dot{a}_i^* = -(r + b)a_i^* + 2cNa_i^{*2}x^*.$$

Assuming symmetry, i.e., $a_i^* = \frac{a^*}{N}$, $i = 1, 2, \dots, N$, the optimal system is given by

$$\begin{aligned} \text{mode } j \quad \dot{x}^* &= a^* - bx^* + j - 1 \\ \dot{a}^* &= -(r+b)a^* + 2ca^{*2}x^*, \end{aligned}$$

and the optimal cost for a candidate starting at (x_0, a_0) is given by (denoting the $j(x_0)$ as the mode corresponding to x_0):

$$\begin{aligned} J^{\text{opt}}(x_0, a_0) &= \frac{1}{r} H_j^c(a_0, x_0) = \sum_i \ln\left(\frac{a_0}{N}\right) - Ncx_0^2 + \left(-\frac{N}{a_0}\right) (a_0 - bx_0 + j(x_0) - 1) \\ &= \frac{1}{r} \left(N \ln\left(\frac{a_0}{N}\right) - Ncx_0^2 + N \frac{bx_0 - j(x_0) + 1}{a_0} - N \right). \end{aligned}$$

So, each player receives a benefit of J_i^{opt} given by:

$$J_i^{\text{opt}}(x_0, a_0) = \frac{J^{\text{opt}}(x_0, a_0)}{N} = \frac{1}{r} \left(\ln\left(\frac{a_0}{N}\right) + \frac{bx_0 - j(x_0) + 1}{a_0} - cx_0^2 - 1 \right). \quad (14)$$

Let $(a_1^*, a_2^*, \dots, a_N^*)$ denote the open loop Nash equilibrium, then agent i solves the following optimization problem:

$$\dot{x}(t) = a_i + \sum_{k \neq i} a_k^*(t) - bx(t) + \alpha, \quad \alpha = \begin{cases} 0 & x(t) < \Delta \\ 1 & x(t) > \Delta \end{cases}, \quad x(0) = x_0$$

$$\max_{a_i} J_i, \quad J_i = \int_0^\infty e^{-rt} (\ln a_i(t) - cx^2(t)) dt$$

$$\text{mode } j \quad H_j^c(\cdot) = \ln a_i - cx^2 + \lambda_j^i \left(a_i + \sum_{k \neq i} a_k^* - bx + j - 1 \right)$$

$$\frac{1}{a_i^*} + \lambda_j^i = 0, \quad i = 1, 2, \dots, N \text{ (follows from maximum condition)}$$

$$\dot{\lambda}_j^i = (r+b)\lambda_j^i + 2cx^*$$

$$\dot{a}_i^* = -(r+b)a_i^* + 2ca_i^{*2}x^*.$$

Again, assuming symmetry we have $a_i^* = a^*/N$, $i = 1, 2, \dots, N$. So, the optimal system is given by

$$\begin{aligned} \text{mode } j \quad \dot{x}^* &= a^* - bx^* + j - 1 \\ \dot{a}^* &= -(r+b)a^* + 2\left(\frac{c}{N}\right)a^{*2}x^*, \end{aligned}$$

and the benefit for the candidate, at equilibrium, starting at (x_0, a_0) is given by:

$$J_i^{\text{olnc}}(a_0, x_0) = \frac{1}{r} H_j^{ci}(x_0)(a_0, x_0) = \frac{1}{r} \left(\ln \left(\frac{a_0}{N} \right) + \frac{bx_0 - j(x_0) + 1}{\frac{a_0}{N}} - cx_0^2 - N \right). \quad (15)$$

□