Annuities, Bequests and Portfolio Diversification*

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Abstract: In this article, the diversification motives of the demand for annuities is analyzed. Using a model allowing for the uncertainty of both the human life length and the interest rate, the Decision Maker is supposed to choose an optimal portfolio to maximize a bequest. Conditions under which an increase in the risk of bond returns increase the demand for annuities are proposed and discussed. Moreover, it is shown that, contrary to previous claims, more risk adversion is associated with a lower demand for annuities.

Keywords: Annuities, uncertain longevity, risk aversion.

JEL codes: D11, D81, G11, G22.

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1   Introduction

Since Yaari (1965), the demand for annuities has become the cornerstone of the theory of consumption under uncertain lifetime. Provided that annuities are fairly priced, it has been shown that their demand should be relatively high to finance the last period's consumption. The remainder of the portfolio may then be composed of regular bonds, which, if the Decision Maker (DM) exhibits some bequest motives, are intended for her heirs. As shown in Davidoff et al. (2005), this result holds under quite general specifications and is therefore at odds with most empirical studies. This is known to be the annuity puzzle.

In most of the literature, a rather specific financial environment with no other uncertainty than life duration is assumed. The annuity is therefore a risky asset, whose stochastic yield is compared to a risk free interest rate. We claim, that introducing a stochastic interest rate strongly modifies the demand for annuities, as this becomes an instrument for portfolio diversification. The problem we consider is the following. At each period of life, annuities provide deterministic returns if the DM is alive and zero returns if she is dead. Conversely, bonds (or stocks) yield an uncertain return, which is independent of their owner's survival probability. We propose the simplest possible model to analyze the diversification problem of a DM that faces these two types of risk. The simplicity lies on the assumption that only the bequest yields some utility, or equivalently, that there is no consumption in the case of survival. This eases the comparison with the literature, since the optimal demand for annuities is zero if bond returns are deterministic. In such a setting, a positive demand for annuities is only due to the uncertainty of the interest rate.

We show that, in this stochastic financial environment, the demand for annuities may be positive. Annuities are purchased to diversify the portfolio, but their demand does not necessarily increase with the risk of bond returns. We show that a mean-preserving spread on bond returns increases the optimal demand for annuities if the

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DM is prudent, but not too prudent. We therefore extend the initial result proposed by Hadar and Seo (1990) to demand for annuities. Within this framework, we also study the impact of risk aversion on the demand for annuities. We provide a sufficient condition, under which the demand for annuities is reduced when risk aversion is increased. The condition, based only on prices, simply ensures that the optimal utility if alive is greater than the utility if not alive. This is not true for all sets of parameters, since the bequest yields some utility. Under that condition, a more risk adverse DM chooses to purchase fewer annuities to reduce the gap between the utilities in the two states of nature, life and death. This result puts into perspective the previous studies results (Blake, 1999; Friedman and Warshawsky, 1990; Milevsky, 1998; and Warshawsky, 1998) that claim that demand for annuities increases with risk aversion.

Section 2 presents the basic static framework, while the results lie in Section 3. In Section 4, the model is extended to a dynamic setting with savings.

2   The basic framework

We consider a portfolio choice model under uncertain lifetime in which the Decision Maker (DM) can invest in two assets to maximize her bequest utility. To focus on the diversification issue, the problem is static and there is no utility derived from consumption. In Section 4, this basic framework is extended to a dynamic problem with endogenous savings.

The DM faces two independent risks: the survival and the return of one of the two assets. The length of life is at most, two periods, but only the second one is uncertain. Life uncertainty is characterized using the random variable $\tilde{x}$ which follows a Bernoulli law whose expectancy is $E_x(\tilde{x}) = p \in (0,1)$. The bequest might therefore happen at the end of Periods 1 and 2.

At the first period, the DM is endowed with a positive initial income ω_1 that can

be shared between bonds[1] and annuities. The returns in Period 2 of bonds purchased Period 1 are given by the random variable $\tilde{r}$ with support on $[r_-, r_+]$, where $0 < r_- \leq r_+ < \infty$, and whose expectancy is $E_r(\tilde{r}) = \bar{r}$. Bond returns are paid to the DM if alive and to her heir if not alive. Conversely, the annuities return is $\bar{r}/p$ in Period 2 if the DM is alive and nothing if she is not alive.

Following Yaari (1965), annuities returns are fair. However, we suppose that annuity sellers can eliminate risk at no cost. It follows that annuities and bonds are two risky assets that have the same expected return:

$$E_x\left(\tilde{x}\frac{\bar{r}}{p}\right) = E_r(\tilde{r}) = \bar{r}.$$

Conditional to survival, investments in annuities on average are more profitable that investments in bonds. This is nevertheless not true when the realizations of $\tilde{r}$ are larger than $\bar{r}/p$. As a consequence, holding annuities may permit an increase in wealth that will be bequeathed: annuities can thus be purchased even if the DM does not derive utility from her consumption. Remark that it could be possible to extend our framework by assuming that bonds have a higher expected return than annuities, e.g., due to diversification costs supported by annuity sellers. If the difference in returns is small enough, our results are not modified.

If alive in Period 2, the DM receives a non-negative income ω_2 and bequeaths her entire wealth to her heir. Since death is certain at the end of Period 2, the bequest is exclusively a demand for bonds. More simply, it is supposed that, between Periods 2 and 3, there is no uncertainty on bond returns, which are normalized to 1. In addition, whatever the length of the DM's life, bequests are received in Period 3. We denote respectively by a and $\omega_1 - a$ the demand for annuities and the demand for bonds in Period 1. As of Period 1, the wealth that will be bequeathed in Period 3 hence satisfies:

$$\tilde{x}\left(\frac{\bar{r}}{p}a + \omega_2\right) + \tilde{r}(\omega_1 - a).$$

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[1]For simplicity, we call the "bond" the risky asset.

Note that there is no consumption in this basic setting, or equivalently, that consumptions in Periods 1 and 2 are fixed. Then, ω_1 and ω_2 represent the exogenous difference between income and consumption in each period. Note also that the case of endogenous savings is considered in Section 4.

Due to uncertain life spans, it is not possible to borrow by selling bonds. It is also assumed that short sale positions on annuities are not possible. Thus, the demand for annuities and the demand for bonds are supposed to be non negative:

$$0 \leq a \leq \omega_1. \tag{1}$$

Finally, the utility derived from the bequest is computed using the function $u\left(.\right)$, which is C^3, satisfies $u' > 0$, $u'' < 0$, as well as the usual limit conditions: $\lim_{y \to 0} u'\left(y\right) = +\infty$ and $\lim_{y \to +\infty} u'\left(y\right) = 0$. The DM is an expected utility maximizer and her objective is therefore given by:

$$\max_a E_x E_r \left[u\left(\tilde{x} \left(\frac{\bar{r}}{p} a + \omega_2 \right) + \tilde{r}\left(\omega_1 - a\right) \right) \right]. \tag{2}$$

The DM's problem is to solve (2) subject to (1).

3   Annuities and portfolio diversification

This section is devoted to the analysis of the optimal portfolio. The first proposition establishes the condition under which the DM invests in annuities. After which, the impact of risk and risk aversion are studied.

Proposition 1 − *There exists a unique optimal demand for annuities.*

If $\omega_2 = 0$, the optimal demand for annuities is positive.

If $\omega_2 > 0$, the optimal demand for annuities decreases with ω_2 when the DM is prudent. Moreover, there exists $\bar{\omega}_2 > 0$ such that the optimal portfolio is composed of bonds and annuities, if $\omega_2 \leq \bar{\omega}_2$ and only of bonds if $\omega_2 > \bar{\omega}_2$.

PROOF – As a preliminary, we define the function $\phi(a)$ as the first derivative of (2) with respect to a, such that:

$$\phi(a) = E_x E_r \left[\left(\tilde{x} \frac{\bar{r}}{p} - \tilde{r} \right) u' \left(\tilde{x} \left(\frac{\bar{r}}{p} a + \omega_2 \right) + \tilde{r} (\omega_1 - a) \right) \right]. \tag{3}$$

This function is well defined for a, such that, whatever the state of nature, the realization $\tilde{x}(\bar{r}a/p + \omega_2) + \tilde{r}(\omega_1 - a)$ is non-negative. As a result, a is lower than the positive threshold $a_{max} = \omega_1$ which is necessary and sufficient to guarantee the positivity of $\tilde{r}(\omega_1 - a)$. This is larger than the negative threshold $a_{min} = -(\omega_2 + r_-\omega_1)/(\bar{r} - pr_-)$, which is necessary and sufficient to guarantee the positivity of $\bar{r}a/p + \omega_2 + \tilde{r}(\omega_1 - a)$. Subsequently, $\phi(.)$ is well defined for $a \in (a_{min}, a_{max})$.

By construction of these thresholds and using the Inada conditions we have:

$$\lim_{a \to a_{min}} \phi(a) = \lim_{\varrho \to 0} (\bar{r} - pr_-)u'(\varrho) = +\infty,$$

$$\lim_{a \to a_{max}} \phi(a) = \lim_{\varrho \to 0} E_r[-\tilde{r}u'(\tilde{r}\varrho)] = -\infty.$$

Consequently, $\phi(.)$ has a real root $\hat{a} \in (a_{min}, a_{max})$. Since

$$\phi'(a) = E_x E_r \left[\left(\tilde{x} \frac{\bar{r}}{p} - \tilde{r} \right)^2 u'' \left(\tilde{x} \left(\frac{\bar{r}}{p} a + \omega_2 \right) + \tilde{r} (\omega_1 - a) \right) \right]$$

is negative, $\hat{a}$ is the unique real root of $\phi(.)$.

The solution of the DM's problem is thus denoted as a^* and satisfies:

$$a^* = \left| \begin{array}{ll} \hat{a}, & \text{if } \hat{a} \geq 0, \\ \\ 0 & \text{otherwise.} \end{array} \right. \tag{4}$$

We now establish the positivity of $\hat{a}$, when $\omega_2 = 0$. Since $u(.)$ is concave and $\lim_{\varrho \to 0} u'(\varrho) = +\infty$, we have:

$$\lim_{a \to 0} \phi(a)|_{\omega_2=0} = E_r[(\bar{r} - \tilde{r})u'(\tilde{r}\omega_1)] = cov[\bar{r} - \tilde{r}, u'(\tilde{r}\omega_1)] > 0,$$

$$\lim_{a \to a_{max}} \phi(a)|_{\omega_2=0} = \lim_{\varrho \to 0} E_r[-\tilde{r}u'(\tilde{r}\varrho)] = -\infty.$$

The continuity of $\phi(.)$ implies that the unique root $\hat{a}$ belongs to $(0, \omega_1)$ when $\omega_2 = 0$.

When ω_2 is positive, by applying the implicit function theorem to (3), we can establish that:

$$\frac{d\hat{a}}{d\omega_2} = \frac{E_r\left[(\bar{r} - p\tilde{r})\, u''\, (\bar{r}\hat{a}/p + \omega_2 + \tilde{r}\,(\omega_1 - \hat{a}))\right]}{-\phi'\,(\hat{a})}.$$

The numerator of the RHS can now be rewritten as:

$$p\, cov\left(\bar{r} - \tilde{r},\, u''\left(\frac{\bar{r}}{p}\hat{a} + \omega_2 + \tilde{r}\,(\omega_1 - \hat{a})\right)\right) + (1 - p)\,\bar{r} E_r\left[u''\left(\frac{\bar{r}}{p}\hat{a} + \omega_2 + \tilde{r}\,(\omega_1 - \hat{a})\right)\right].$$

This numerator is negative if $u'''\,(.) > 0$, or equivalently, if the DM is prudent. The denominator is positive. Hence, the optimal demand for annuities decreases with ω_2 (i.e., $d\hat{a}/d\omega_2 < 0$).

Finally, using Inada conditions, we can prove the existence of the positive threshold $\bar{\omega}_2$ by computing the following limit:

$$\lim_{\omega_2 \to +\infty} \phi\,(a) = -\,(1 - p)\, E_r\left[\tilde{r}u'\,(\tilde{r}\,(\omega_1 - a))\right].$$

Since this limit is negative, there exists, by continuity, a threshold $\bar{\omega}_2$ above which the optimal portfolio is only composed of bonds (i.e., $a^* = 0$). $\square$

The optimal portfolio always includes bonds, as it is the only possibility to bequeath in case of death after the first period. The assumption of an infinite marginal utility when the bequest goes to zero indeed ensures a positive demand for bonds.

Conversely, investing in annuities may increase the bequest value in case of survival, but this is not necessary. Proposition 1 states that diversification using annuities arise if ω_2, the income received if alive in the second period, is low enough.

The intuition is the following. For a given realization of $\tilde{r}$, an increase in ω_2 reduces the marginal utility if alive. At the optimum, this is compensated by an increase in the demand for bonds, meaning a reduction of the marginal utility if not alive. This additional demand for bonds corresponds to a reduction in the demand for annuities.

Nevertheless, the concavity of the utility function is not sufficient to obtain our result. Indeed, for large realizations of $\tilde{r}$, i.e. those which are greater than the annuities

return $\bar{r}/p$, a reduction in the demand for annuities increases the utility if alive. A prudent behavior, as characterized by Kimball (1990), is then a sufficient condition for this latter effect to be dominated. It indeed implies the concavity of the marginal utility if alive with respect to $\tilde{r}$.

Further assumptions on the utility function permit us to derive additional results. In the following corollary, this is notably the case, when the preferences of the DM are homothetic. Such preferences can be represented by a homogeneous utility function, such as the CRRA function.[2]

Corollary 1 – *When preferences are homothetic, the demand for annuities is linearly increasing in ω_1 and linearly decreasing in ω_2.*

PROOF – Homothetic preferences can be represented by a concave utility function $u(y) = y^\lambda$ with $0 < \lambda < 1$. According to Mas-Colell et al (1995, p. 50) this is a necessary and sufficient condition for representing homothetic and continuous preferences. Moreover, for homothetic preferences, the DM's problem can be simplified by a change in variable. Hence, we define k such that:

$$a = \frac{k\omega_1 - \omega_2}{\bar{r}/p + k},$$

with $k \neq -\bar{r}/p$. Then, (3) can be rewritten as a function of k as follows:

$$\phi\left(\frac{k\omega_1 - \omega_2}{\bar{r}/p + k}\right) = E_x E_r \left[\left(\tilde{x}\frac{\bar{r}}{p} - \tilde{r}\right) u'\left((\tilde{x}k + \tilde{r})\frac{\bar{r}\omega_1 + p\omega_2}{\bar{r} + pk}\right)\right].$$

As the function $u'(.)$ is homogeneous of a given degree, the real roots of $\phi(.)$ are those of $\varphi(.)$, where:

$$\varphi(k) = E_x E_r \left[\left(\tilde{x}\frac{\bar{r}}{p} - \tilde{r}\right) u'(\tilde{x}k + \tilde{r})\right]. \tag{5}$$

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[2]When the indifference curves are homothetic with respect to the origin, we say that the preferences are homothetic. Such preferences can be represented by the composition of a function homogeneous of degree 1 with an increasing function.

Using the Inada conditions we have:

$$\lim_{k \to +\infty} \varphi(k) = (1 - p)E_r[-\tilde{r}u'(\tilde{r})] < 0.$$

Importantly, we also have:

$$\varphi(0) = E_r[(\bar{r} - \tilde{r})u'(\tilde{r})] = cov[\bar{r} - \tilde{r}, u'(\tilde{r})] > 0.$$

Concerning the derivative of $\varphi(.)$ we find that:

$$
\begin{aligned}
\varphi'(k) &= E_r[(\bar{r} - p\tilde{r})u''(k + \tilde{r})] \\
&= p \, cov\,(\bar{r} - \tilde{r}, u''\,(k + \tilde{r})) + (1 - p)\,\bar{r}E_r\,[u''\,(k + \tilde{r})]\,.
\end{aligned}
$$

Since $u'''(y) = \lambda(\lambda - 1)(\lambda - 2)y^{\lambda - 3} > 0$, $\varphi(.)$ is a decreasing function of k. As $\varphi(0) > 0$, $\varphi'(k) < 0$ and $\lim_{k \to +\infty} \varphi(k) < 0$, the function $\varphi(.)$ has a unique real positive root, $\hat{k}$. Then, the optimal demand for annuities, denoted a^*, satisfies:

$$
a^* = \left|
\begin{array}{ll}
\dfrac{\hat{k}\omega_1 - \omega_2}{\bar{r}/p + \hat{k}}, & \text{if } \hat{k} \geq \omega_2/\omega_1, \\[3mm]
0 & \text{otherwise.}
\end{array}
\right.
\tag{6}
$$

Since $\hat{k}$ is independent of ω_1 and ω_2, one can conclude that a^* is linearly increasing in ω_1 and linearly decreasing in ω_2. $\square$

Proposition 1 and Corollary 1 reveal that the demand for annuities may be positive, even if there is no utility derived from the consumption in Period 2. As shown in the two following propositions, this demand can be explained by the uncertainty on bond returns.

Proposition 2 – *If the DM is prudent, the demand for annuities increases with the survival probability.*

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PROOF – Suppose that the demand for annuities a^* is positive. According to the proof of Proposition 1, we have $a^* = \hat{a}$. To perform comparative statics, it is useful to rewrite (3) as follows:

$$\phi(a) = E_r \left[(\bar{r} - p\tilde{r}) u' \left(\frac{\bar{r}}{p} a + \omega_2 + \tilde{r}(\omega_1 - a) \right) - (1 - p)\tilde{r} u' (\tilde{r}(\omega_1 - a)) \right], \quad (7)$$

which will be denoted $E_r[\Gamma(\tilde{r})]$ for convenience.

From (7), and using the implicit function theorem, we have:

$$\frac{d\hat{a}}{dp} = \left\{ - E_r \left[\tilde{r} \left(u'(\tilde{r}(\omega_1 - \hat{a})) - u' \left(\frac{\bar{r}}{p}\hat{a} + \omega_2 + \tilde{r}(\omega_1 - \hat{a}) \right) \right) \right] \right.$$
$$\left. + \frac{\bar{r}\hat{a}}{p^2} E_r \left[(\bar{r} - p\tilde{r}) u'' \left(\frac{\bar{r}}{p}\hat{a} + \omega_2 + \tilde{r}(\omega_1 - \hat{a}) \right) \right] \right\} \bigg/ \phi'(\hat{a}).$$

The first term of the numerator is negative by the concavity of $u(.)$, while the second one is negative if $u'''(.) > 0$. Thus, if the DM is prudent, the numerator is negative and we have $da^*/dp > 0$. $\square$

Proposition 2 is rather intuitive. An increase in the survival probability reduces the annuities return if alive but not its expected return, which is still equal to the return of the bonds. However, increasing the survival probability provides more weight to the marginal utilities if alive which is compensated by an increase in the demand for annuities.

As in Proposition 1, the assumption of prudence is sufficient to eliminate counter-intuitive behaviors. Moreover, the limit cases of determinisitic survival and death are useful to understand why annuities are purchased.

In the limit case, such that $p = 1$, (7) becomes:

$$\phi(a)|_{p=1} = cov \left[\bar{r} - \tilde{r}, u'(\bar{r}a + \omega_2 + \tilde{r}(\omega_1 - a)) \right] > 0.$$

As the concavity of $u(.)$ ensures its positivity, $\phi(.)|_{p=1}$ has no real root and the optimal solution is a corner solution, $a^* = a_{max} = \omega_1$.

In the limit case, such that $p = 0$, there is no annuity market and the objective function (2) can be rewritten as: $E_r \big[u \left(\tilde{r} \left(\omega_1 - a \right) \right) \big]$. In this case, the optimal solution is a corner solution: $a^* = 0$.

In both limit cases, there is by definition no annuity market, but they permit to understand the logic behind Proposition 2. If survival is certain, the "annuity" appears to be a standard non-risky asset whose return equals the expected return of the bonds. Because of risk aversion, the optimal portfolio is then composed of non-risky assets only. Conversely, if death is certain, there is no non-risky asset and all the initial endowment are invested in bonds, whose minimal return, r_-, is greater than zero.

The next proposition studies the effect of uncertainty on bond returns. To begin, we define the relative prudence as:

$$RP = -\frac{y u''' \left(y \right)}{u'' \left(y \right)}.$$

Proposition 3 $-$ *If $r_+ = r_-$, the optimal demand for annuities is zero.*

If $r_+ > r_-$, a mean-preserving spread on bond returns increases the optimal demand for annuities if the relative prudence is positive and less than 2.

PROOF $-$ When there is no risk on bond returns, equation (3) can be rewritten as:

$$\phi \left(a \right)|_{\tilde{r} = \bar{r}} = \left(1 - p \right) \bar{r} \left[u' \left(\frac{\bar{r}}{p} a + \omega_2 + \bar{r} \left(\omega_1 - a \right) \right) - u' \left(\bar{r} \left(\omega_1 - a \right) \right) \right].$$

The unique root of $\phi \left(. \right)|_{\tilde{r} = \bar{r}}$, $\hat{a} = -p\omega_2 / \bar{r}$, is non-positive and, according to (4), the optimal solution is then $a^* = 0$.

Suppose now that $\hat{a} > 0$. Consider a mean-preserving spread on $\tilde{r}$. Following Rothschild and Stiglitz (1971) and using (7), it increases $\hat{a}$ if $\Gamma(\tilde{r})$ is convex. Differentiating twice $\Gamma(.)$ and rearranging the equation yields:

$$
\begin{aligned}
\Gamma''(\tilde{r}) \;=\; & -2 \left(\omega_1 - a \right) E_x \left[u'' \left(\tilde{x} \left(\frac{\bar{r}}{p} a + \omega_2 \right) + \tilde{r} \left(\omega_1 - a \right) \right) \right] \\
& - \left(\omega_1 - a \right) E_x \left[\left(\tilde{x} \left(\frac{\bar{r}}{p} a + \omega_2 \right) + \tilde{r} \left(\omega_1 - a \right) \right) u''' \left(\tilde{x} \left(\frac{\bar{r}}{p} a + \omega_2 \right) + \tilde{r} \left(\omega_1 - a \right) \right) \right] \\
& + \left(\omega_1 - a \right) \left(\omega_1 \bar{r} + p \omega_2 \right) u''' \left(\frac{\bar{r}}{p} a + \omega_2 + \tilde{r} \left(\omega_1 - a \right) \right).
\end{aligned}
$$

Conclude that $\Gamma(\tilde{r})$ is convex when, for all $y > 0$, $u'''(y) \geq 0$ and $RP \leq 2$. $\square$

Whatever the uncertainty of the length of life, there is no demand for annuities if there is no risk on bond returns. In this case, the optimal behavior simply aims at equalizing the utilities in the two states of nature: life and death. Since there is no consumption if alive, the optimal demand for annuities is zero. Uncertainty on bond returns is, in this framework, a necessary condition for annuitization. This motive complements the one traditionally studied in the annuity literature, which relies on consumption.

The impact of a change in risk on the demand for annuities was also discussed in Proposition 3. As bonds become more risky, the DM optimally diversifies her portfolio by increasing the share of annuities. To obtain such a behavior, sufficient conditions on preferences are exhibited. The DM has to be prudent, but not relatively too prudent, with a benchmark value at 2. This result is similar to Hadar and Seo (1990) who studied an optimal portfolio problem with two risky assets. It is also generalized in Gollier (2001). Interestingly, Eeckhoudt et al. (2007) proposed an interpretation of the benchmark value in terms of preferences for disaggregating harms.

Finally, note that the conditions exhibited in Proposition 3 are satisfied if the relative risk aversion is both lower than one and non-decreasing. The following example reveals that the demand for annuities may decrease with the risk on bonds if the relative risk aversion is greater than one.

Assume that the random variable $\tilde{r}$ has only two realizations: $r_+ \equiv \bar{r} + \varepsilon/q$ with probability q and $r_- \equiv \bar{r} - \varepsilon/(1-q)$ with probability $1 - q$. Hence, $\bar{r}$ is the expected return and $\varepsilon \geq 0$ is a mean-preserving spread measure. Assume that the utility function is CRRA: $u(y) = y^{1-\alpha}/(1-\alpha)$, where $\alpha > 0$ stands for the relative risk aversion coefficient. As $u(.)$ is homothetic, the optimal demand for annuities a^* is defined by (6) where $\hat{k}$ is given by (5) (see Corollary 1).

Obviously we have $da^*/d\varepsilon = da^*/d\hat{k} \times d\hat{k}/d\varepsilon$. Since the first term of the RHS is positive for all $a^* > 0$, the sign of the effect of an increase in risk on the demand for annuities can be evaluated with $d\hat{k}/d\varepsilon$. According to (5):

$$\frac{d\hat{k}}{d\varepsilon} = -\frac{\varphi'_\varepsilon}{\varphi'_k} = \left\{ E_x \left[u'\left(\tilde{x}k + r_+\right) - \left(\tilde{x}\frac{\bar{r}}{p} - r_+\right) u''\left(\tilde{x}k + r_+\right) \right] \right.$$
$$\left. -E_x \left[u'\left(\tilde{x}k + r_-\right) - \left(\tilde{x}\frac{\bar{r}}{p} - r_-\right) u''\left(\tilde{x}k + r_-\right) \right] \right\} \Big/ E_r\left[(\bar{r} - p\tilde{r})u''(k + \tilde{r})\right].$$

If the individual is prudent and exhibits a relative risk aversion greater than one, the demand for annuities increases with the realization of $\tilde{r}$. The effect of a larger r_+ is then the opposite of the effect of a larger r_-. If relative prudence is strong enough, it is not clear which effect dominates.

This is illustrated in Figure 1, which plots a^* as a function of ε for the following parameter values: $\alpha = 4$, $p = 0.9$, $q = 0.8$, $\bar{r} = 1.5$, $\omega_1 = 1$ and $\omega_2 = 0$. Figure 1 shows that the demand for annuities is reduced by an increase in risk when the mean-preserving spread is sufficiently high. This example has been computed for a relative risk aversion of 4, which is standard. Nevertheless, it is a crucial parameter.
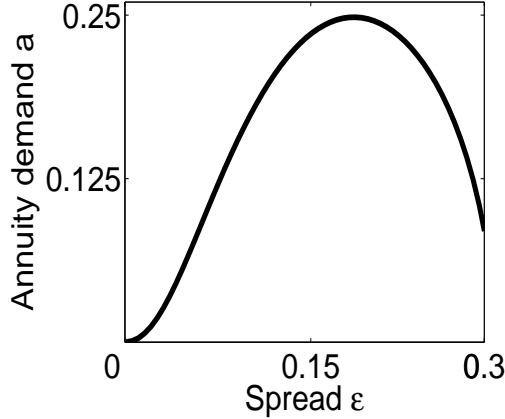

Figure 1

In Figure 2, a^* as a function of ε is plotted for various values of α.

The impact of risk aversion on the demand for annuities will be discussed in Proposition 4.
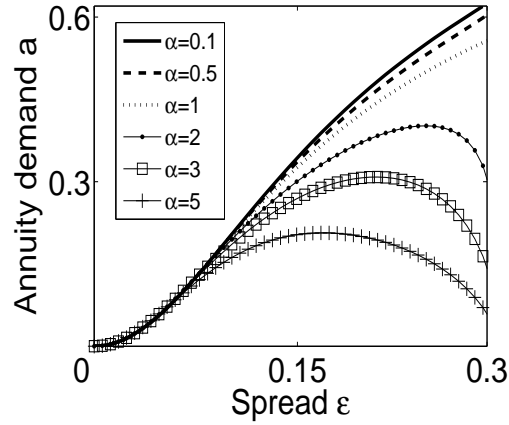
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Figure 2

Proposition 4 − *If $\omega_2/\omega_1 \geq r_+ - r_-$, the demand for annuities reduces with risk aversion.*

PROOF − Consider two DMs, namely A and B. We assume that the utility function of A is $u^A(.)$, whereas the utility function of B is $u^B(.) \equiv T \circ u^A(.)$ where $T(.)$ is an increasing and concave function. In this case, the DM B is more risk averse than the DM A.[3] Each DM i $(i = A, B)$ maximizes their expected utility $E_x E_r[u^i(\tilde{x}(\bar{r}a/p + \omega_2) + \tilde{r}(\omega_1 - a))]$ subject to (1). We denote by $a_\star^i$ $(i = A, B)$ their optimal choices, which are the real roots of $\phi^i(a)$ if this latter is positive, or zero otherwise. One has:

$$\phi^i(a) = E_x E_r \left[\left(\tilde{x}\frac{\bar{r}}{p} - \tilde{r} \right) u^{i\prime} \left(\tilde{x} \left(\frac{\bar{r}}{p}a + \omega_2 \right) + \tilde{r}(\omega_1 - a) \right) \right].$$

Assume that $a_\star^A$ and $a_\star^B$ are both positive. We now exhibit a condition such that $\phi^B(a_\star^A) < 0$. Such a condition implies that $a_\star^B < a_\star^A$ because $\phi^B(.)$ is a decreasing function satisfying $\phi^B(a_\star^B) = 0$.

Observe first that $\phi^B(a_\star^A)$ can be rewritten as follows:

$$\phi^B(a_\star^A) = E_r \left[T' \left(u^A \left(\frac{\bar{r}}{p}a_\star^A + \omega_2 + \tilde{r}(\omega_1 - a_\star^A) \right) \right) \psi_1^A(\tilde{r}) \right]$$
$$- E_r \left[T' \left(u^A(\tilde{r}(\omega_1 - a)) \right) \psi_2^A(\tilde{r}) \right],$$

---

[3]This result was stated in Pratt (1988) who showed that concavity is preserved under mixture of independent risks (see also Finkelshtain et al (1999)).

where:

$$\psi_1^A(\tilde{r}) = (\bar{r} - p\tilde{r}) u^{A\prime}(\bar{r}a_\star^A/p + \omega_2 + \tilde{r}(\omega_1 - a_\star^A)),$$

$$\psi_2^A(\tilde{r}) = (1 - p)\tilde{r} u^{A\prime}(\tilde{r}(\omega_1 - a_\star^A)).$$

As $\psi_2^A(\tilde{r})$ is positive we have:

$$\phi^B(a_\star^A) < E_r\left[T'\left(u^A\left(\frac{\bar{r}}{p}a_\star^A + \omega_2 + \tilde{r}(\omega_1 - a_\star^A)\right)\right)\psi_1^A(\tilde{r})\right]$$
$$- T'\left(u^A(r_+(\omega_1 - a))\right)E_r[\psi_2^A(\tilde{r})]. \tag{8}$$

Importantly, $\phi^A(a_\star^A) = 0$ implies that:

$$E_r(\psi_1^A(\tilde{r})|\tilde{r} < \bar{r}/p)\Pr(\tilde{r} < \bar{r}/p) = E_r\psi_2^A(\tilde{r}) - E_r(\psi_1^A(\tilde{r})|\tilde{r} \geq \bar{r}/p)\Pr(\tilde{r} \geq \bar{r}/p).$$

Moreover, the concavity of $T(.)$ implies that $E_r[T'(u^A(\bar{r}a_\star^A/p + \omega_2 + \tilde{r}(\omega_1 - a_\star^A)))\psi_1^A(\tilde{r})]$
is lower than:

$$T'(u^A(r_- a_\star^A/p + \omega_2 + r_-(\omega_1 - a_\star^A)))E_r[\psi_1^A(\tilde{r})|\tilde{r} < \bar{r}/p]\Pr(\tilde{r} < \bar{r}/p)$$

$$+ T'(u^A(r_+ a_\star^A/p + \omega_2 + r_+(\omega_1 - a_\star^A)))E_r[\psi_1^A(\tilde{r})|\tilde{r} \geq \bar{r}/p]\Pr(\tilde{r} \geq \bar{r}/p).$$

Then, according to (8), we have:

$$\phi^B(a_\star^A) < \left[T'\left(u^A\left(\frac{\bar{r}}{p}a_\star^A + \omega_2 + r_-(\omega_1 - a_\star^A)\right)\right) - T'\left(u^A(r_+(\omega_1 - a_\star^A))\right)\right]E_r\left[\psi_2^A(\tilde{r})\right]$$
$$+ \left[T'\left(u^A\left(\frac{\bar{r}}{p}a_\star^A + \omega_2 + r_+(\omega_1 - a_\star^A)\right)\right) - T'\left(u^A\left(\frac{\bar{r}}{p}a_\star^A + \omega_2 + r_-(\omega_1 - a_\star^A)\right)\right)\right]$$
$$\times E_r\left[\psi_1^A(\tilde{r})\big|\tilde{r} \geq \bar{r}/p\right]\Pr(\tilde{r} \geq \bar{r}/p).$$

Since $T(.)$ is concave, the second term of the RHS of this inequality is negative. A sufficient condition for $\phi^B(a_\star^A) < 0$ is therefore:

$$T'\left(u^A\left(\frac{\bar{r}}{p}a_\star^A + \omega_2 + r_-(\omega_1 - a_\star^A)\right)\right) < T'\left(u^A(r_+(\omega_1 - a_\star^A))\right),$$

which can be rewritten as:

$$\frac{\bar{r}}{p}a_\star^A + \omega_2 - (r_+ - r_-)(\omega_1 - a_\star^A) > 0.$$

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Since the RHS of this inequality is increasing with $a_\star^A$, condition $\omega_2 \geq (r_+ - r_-)\omega_1$ is sufficient. $\square$

Proposition 4 states that more risk aversion may increase the optimal share of the portfolio invested in bonds. The sufficient condition given only relies on parameters and is therefore true for any increasing and concave utility function.

The condition is simply that the endowment growth factor should not be lower than the support of the distribution of the interest factor. Were bonds supposed to be riskless, the condition would be satisfied. More precisely, this condition ensures that, when computed at the optimal point, the utility if alive in Period 2 is always greater than the utility if not alive. And consequently, living with the lowest possible interest rate bears more utility than being dead with the highest bond returns. Note that this is not necessarily the case since it is the bequest which yields some utility.

If our condition is satisfied, more risk aversion induces to reduce the difference in utility for each realization of $\tilde{r}$. A more risk adverse DM therefore lowers her demand for annuities, which increases the utility if dead and decreases the utility if alive. Note finally that our condition is sufficient and, as shown in the following example, that a positive income in Period 2 is not necessary.
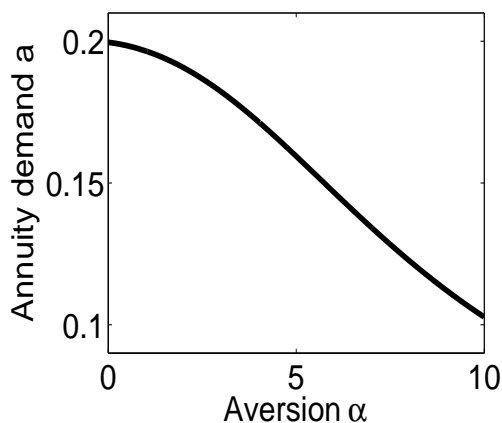

Figure 3

We now consider the example used previously and plot the demand for annuities a^* as a function of the relative risk aversion coefficient α, for the following parameter

values: $p = 0.9$, $q = 0.8$, $\bar{r} = 1.5$, $\omega_1 = 1$, $\omega_2 = 0$, and an interest rate spread of $\varepsilon = 0.1$.

Despite the assumption of $\omega_2 = 0$, the demand for annuities typically monotonically decreases with relative risk aversion. However, this is not always true. For instance, by reducing the interest rate spread, the relationship may be reversed for some value of α.
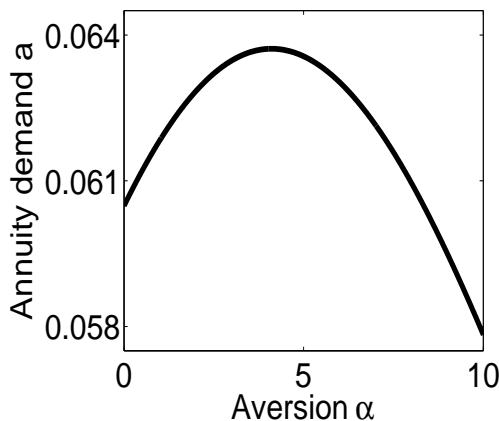

Figure 4

As shown in Proposition 3, a reduction of the spread ε may indeed lower the demand for annuities, and therefore imply, for some realizations of $\tilde{r}$, that the utility if not alive is higher than the utility if alive. As shown in Figure 4, the relationship between the demand for annuities and the risk aversion may become positive (case where $\varepsilon = 0.05$).

4 Extension to savings

This section introduces a dynamic behavior allowing for a first period consumption. The assumption of no consumption if alive in the second period (or, equivalently, of a fixed consumption) is kept to show that the demand for annuities is still positive, provided that bond returns are risky.

The framework is the one described Section 2 extended to a first period consumption. Before the lotteries, the DM may allocate her initial income $\omega_1 > 0$ between consumption and savings, which are respectively denoted $\omega_1 - s$ and s. The utility

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derived from consumption is described by the function $v\,(.)$ that satisfies the same assumptions as those of function $u\,(.)$. The DM's program can now be rewritten as:

$$\max_{s,a}\left\{v\,(\omega_1 - s) + E_x E_r\left[u\left(\tilde{x}\left(\frac{\bar{r}}{p}a + \omega_2\right) + \tilde{r}\,(s - a)\right)\right]\right\} \tag{9}$$

subject to $0 \leq s \leq \omega_1$ and $0 \leq a \leq s$.

Proposition 5 $-$ *If $r_+ = r_-$, there exists a unique optimum (s^*, a^*) where $s^* \in (0, \omega_1)$ and $a^* = 0$. If $r_+ > r_-$, there exists $\hat{\omega}_2 > 0$ such that if the DM is prudent and if $\omega_2 < \hat{\omega}_2$, there exists a unique optimum $(s^*, a^*) \in (0, \omega_1)^2$.*

PROOF $-$ First, we establish the existence and uniqueness of the optimum (s^*, a^*), when the bond return is random. The first order conditions of problem (9) are:

$$-v'\,(\omega_1 - s^*) + E_x E_r\left[\tilde{r}u'\left(\tilde{x}\left(\frac{\bar{r}}{p}a^* + \omega_2\right) + \tilde{r}\,(s^* - a^*)\right)\right] = 0, \tag{10}$$

$$E_x E_r\left[\left(\tilde{x}\frac{\bar{r}}{p} - \tilde{r}\right)u'\left(\tilde{x}\left(\frac{\bar{r}}{p}a^* + \omega_2\right) + \tilde{r}\,(s^* - a^*)\right)\right] = 0. \tag{11}$$

From (10), we can define the following function $\xi_a(s)$:

$$\xi_a(s) = -v'\,(\omega_1 - s) + E_x E_r\left[\tilde{r}u'\left(\tilde{x}\left(\frac{\bar{r}}{p}a + \omega_2\right) + \tilde{r}\,(s - a)\right)\right].$$

The function $\xi_a(.)$ is well defined for all s, such that $\tilde{x}\,(\bar{r}a/p + \omega_2) + \tilde{r}\,(s - a)$ is non-negative. $\xi_a(.)$ is well defined for $s \in (a, \omega_1)$ and using the Inada conditions:

$$\lim_{s \to a} \xi_a(s) = \lim_{\varrho \to 0} (1 - p)E_r[\tilde{r}u'(\tilde{r}\varrho)] = +\infty,$$

$$\lim_{s \to \omega_1} \xi_a(s) = \lim_{\varrho \to 0} -v'(\varrho) = -\infty.$$

As a result, $\xi_a(.)$ has at least one root s^* that belongs to (a, ω_1). According to Proposition 1, it is straightforward that for any root s^*, there exists a positive threshold $\hat{\omega}_2$ such that (11) has one real root a^* that belongs to $(0, s^*)$, if $\omega_2 < \hat{\omega}_2$.

To prove the uniqueness of the pair (s^*, a^*), we first combine (10) and (11) to obtain:

$$\gamma(s^*, a^*) = -v'\,(\omega_1 - s^*) + \bar{r}E_r\left[u'\left(\frac{\bar{r}}{p}a^* + \omega_2 + \tilde{r}\,(s^* - a^*)\right)\right] = 0.$$

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From $\gamma(s^*, a^*) = 0$, we can use the implicit function theorem to define a continuous application $a^* = \eta(s^*)$ where:

$$\eta'(s^*) = -\frac{\gamma'_s(s^*, a^*)}{\gamma'_a(s^*, a^*)} = -\frac{v''(\omega_1 - s^*) + \bar{r}E_r\left[\tilde{r}u''(\bar{r}a^*/p + \omega_2 + \tilde{r}(s^* - a^*))\right]}{\bar{r}E_r\left[(\bar{r} - p\tilde{r})u''(\bar{r}a^*/p + \omega_2 + \tilde{r}(s^* - a^*))\right]/p}, \quad (12)$$

As $\gamma'_a(.) = \bar{r}\,cov[\bar{r} - \tilde{r}, u''(\bar{r}a^*/p + \omega_2 + \tilde{r}(\omega_1 - a^*))] + (1/p - 1)\bar{r}^2 E_r\left[u''(\bar{r}a^*/p + \omega_2 + \tilde{r}(\omega_1 - a^*))\right]$, the denominator of (12) is negative when $u'''(.) > 0$. Since $\gamma'_s(.) < 0$, $\eta(.)$ is a decreasing function when the DM is prudent.

Now, replacing $\eta(s)$ in (10), we can define the function $\xi(s)$ as follows:

$$\xi(s) = \xi_{\eta(s)}(s) = -v'(\omega_1 - s) + E_x E_r\left[\tilde{r}u'\left(\tilde{x}\left(\frac{\bar{r}}{p}\eta(s) + \omega_2\right) + \tilde{r}(s - \eta(s))\right)\right].$$

We have:

$$\begin{aligned}
\xi'(s) &= v''(\omega_1 - s) + E_x E_r\left[\tilde{r}^2 u''\left(\left(\tilde{x}\frac{\bar{r}}{p} - \tilde{r}\right)\eta(s) + \tilde{x}\omega_2 + \tilde{r}s\right)\right] \\
&\quad + \eta'(s)E_x E_r\left[\tilde{r}\left(\tilde{x}\frac{\bar{r}}{p} - \tilde{r}\right)u''\left(\left(\tilde{x}\frac{\bar{r}}{p} - \tilde{r}\right)\eta(s) + \tilde{x}\omega_2 + \tilde{r}s\right)\right].
\end{aligned}$$

Next, a sufficient condition for $\xi'(s) < 0$ is:

$$-\eta'(s)E_r[\tilde{r}(\bar{r} - p\tilde{r})u''(z)] > v''(\omega_1 - s) + pE_r[\tilde{r}^2 u''(z)],$$

or, equivalently, using (12):

$$\frac{E_r\left[\tilde{r}(\bar{r} - p\tilde{r})u''(z)\right]}{\bar{r}E_r\left[(\bar{r} - p\tilde{r})u''(z)\right]/p} < \frac{v''(\omega_1 - s) + pE_r\left[\tilde{r}^2 u''(z)\right]}{v''(\omega_1 - s) + \bar{r}E_r\left[\tilde{r}u''(z)\right]},$$

where $z = \bar{r}\eta/p + \omega_2 + \tilde{r}(s - \eta)$.

The condition can be rewritten as:

$$-\frac{1}{p}v''(\omega_1 - s)E_r[(\bar{r}/p - \tilde{r})^2 u''(z)] < (\bar{r}/p)^2\{E_r[\tilde{r}^2 u''(z)]E_r[u''(z)] - (E_r[\tilde{r}u''(z)])^2\}.$$

According to the Cauchy-Schwartz inequality, the RHS of this inequality is positive. Subsequently, the condition is satisfied and $\xi'(s) < 0$. Consequently, $\xi(s)$ has at most one real root s^* and the pair (s^*, a^*) is unique.

Finally, we focus on the riskless case. When there is no risk of bond returns (i.e. for $\tilde{r} = \bar{r}$), (11) can be rewritten as follows:

$$(1 - p)\left[u'\left(\frac{\bar{r}}{p}a + \omega_2 + \bar{r}(s - a)\right) - u'\left(\bar{r}(s - a)\right)\right] = 0.$$

Then, one may explicitly compute the solution, $\hat{a} = -p\omega_2/\bar{r}$, which is negative. The optimal demand for annuities is therefore: $a^* = 0$. $\square$

Proposition 5 shows that the results contained in Propositions 1 and 3 still hold when a saving motive is introduced. The following numerical simulations also show that, as claimed in Proposition 4, the demand for annuities may still decrease with risk aversion.

We extend the basic example studied in Section 3 with the following utility function for the first period consumption: $v(y) = y^{0.6}/0.6$. Using the parameter values of Figure 3, Figure 5 plots the demand for annuities a^* and the savings as functions of the relative risk aversion coefficient α.
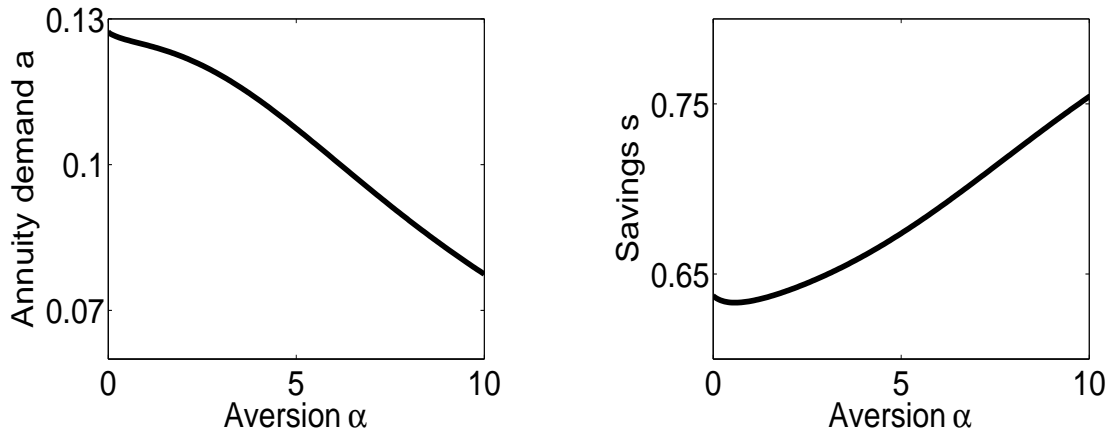

Figure 5

The demand for annuities still monotonically decreases with relative risk aversion, despite an increase in saving.

However, considering a lower spread ε (0.05 rather than 0.1) is sufficient to obtain a non-monotonic relationship. Figure 6 plots the demand for annuities a^* and the savings as functions of the relative risk aversion coefficient α, with a lower spread.
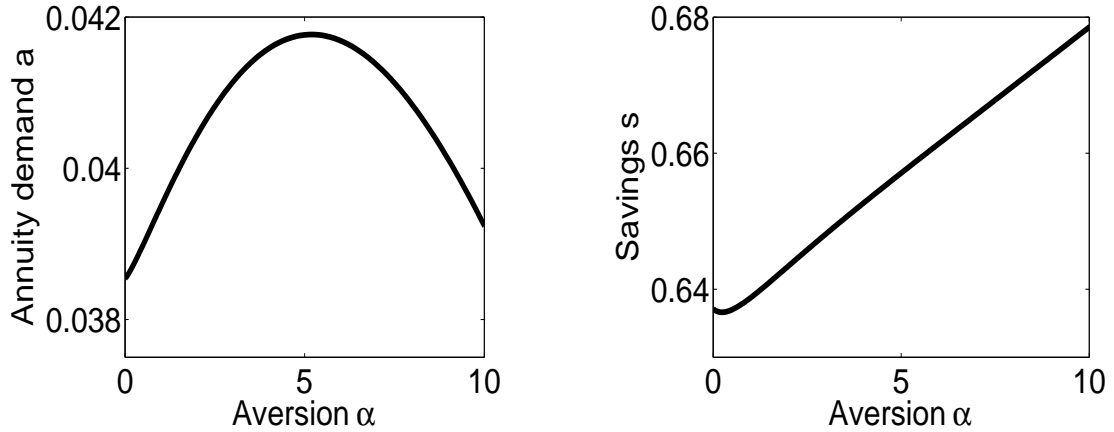
Figure 6

5   Conclusions

In this article, we have studied the demand for annuities when the alternative asset has a risky return. We have provided conditions under which the demand for annuities increases with the financial risk and reduces with risk aversion. An important assumption of this work is the DM's perfect knowledge of the probability distributions. It would be interesting to instead suppose some ambiguity in the probabilities and study the DM's attitude when facing those two different sources of uncertainty.

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