Taxation and Rotation Age under Stochastic Forest Stand Value

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Abstract

The paper uses both the single rotation and ongoing rotation framework to study the impact of yield tax, lump-sum tax, cash flow tax and tax on interest rate earnings on the privately optimal rotation period when forest value growth is stochastic and forest owners are either risk neutral or risk averse. In the case of risk-neutral forest owner higher yield tax raises the optimal harvesting threshold and thereby prolongs the expected rotation period. The same qualitative result holds for lump-sum tax and for the tax on interest rate earnings, while the cash flow tax is neutral. Under risk aversion the optimal harvesting threshold is lower and the expected rotation period shorter than under risk neutrality both in the single and ongoing rotation cases. Comparative statics of taxes are similar as under risk neutrality with the exception of cash flow tax, which may not be neutral anymore. Numerical results indicate that the optimal harvesting threshold both as a function of the yield tax and the forest value volatility increases more rapidly under risk neutrality than under risk aversion.

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1   Introduction

The behavioral impacts of several kinds of forest taxes on privately optimal rotation age have been studied both in the ongoing rotation Faustmann model, where the forest owner maximizes the net present value of harvest revenue (see e.g. Chang 1982, 1983, Johansson and Löfgren 1985, and Camponia and Mendelsohn 1987) and also in the Hartman type model (see Hartman 1986 and Samuelson 1976), where private landowners are interested not only in the present value of net harvest revenue, but also in the private amenity services provided by forest stands (for behavioral analysis, see Koskela and Ollikainen 2001 and for welfare analysis, see Koskela and Ollikainen 2003a, 2003b). If the amenity services matter for forest owners, then the conventional wisdom concerning the effect of forest taxes on privately optimal rotation age, distilled from the basic Faustmannian ongoing rotation framework, predominantly ceases to hold. This is simply because in the Hartmannian framework with amenity services forest taxes affect the relative benefits of timber revenue and amenity production in a way, which depends on the precise nature of amenity valuation. An additional important aspect in terms of welfare analysis is the potential role of externalities of amenity services of forest stands (see Englin and Klan 1990 and Koskela and Ollikainen 2003a, 2003b). All these studies mentioned above, however, have made the heroic assumptions of constant timber prices, constant total cost of clear-cutting and replanting as well as constant interest rate, perfect capital markets and perfect foresight.

The perfect foresight assumption has been relaxed in studies focusing the implications of stochastic timber prices (see e.g. Plantinga 1998, Insley 2002, and Gjolberg and Guttormsen 2002), risk of forest fire (see e.g. Reed 1984), and stochastic forest growth on optimal Faustmannian rotation age (see e.g. Willassen 1998, Sødal 2002, and Alvarez 2003) and as well as in the study focusing the implications of variable and stochastic interest rate on Wicksellian single rotation age (see Alvarez and Koskela 2003, 2004). But to our knowledge neither the behavioral impacts of various types of forest taxes on privately optimal rotation age nor the optimal design of forest taxes have not been studied in this literature with stochastic timber prices, risk of forest fire, stochastic forest growth and interest rate. Thorsen 1998 has studied the case where forest owners have two sources of taxable income, namely stochastic farm income and a controllable income represented by the harvesting of a stock of timber. But in his paper land value is set to zero. In this paper we analyze the unexplored issue of the behavioral effects of various types of forest taxes on the private rotation decision under stochastic forest value growth. More precisely, we study the impacts of (i) yield tax, i.e. tax levied on timber revenues, (ii) lump-sum tax, (iii) cash flow tax and (iv) tax on interest rate earnings on privately optimal rotation age both in the single rotation and in the ongoing rotation frameworks. Moreover, and importantly, we study both the case when harvesters are either risk-neutral or when they are risk-averse. The risk averse case has not to our knowledge been previously studied in stochastic rotation models.

We show in the case of a risk neutral forest owner that both in the single and ongoing rotation frameworks increased yield tax and higher lump-sum tax increase the optimal harvesting threshold and thereby lengthen the expected rotation period. The same qualitative result holds in the case of interest rate earnings taxation, while the optimal harvesting threshold is shown to be independent of the cash flow tax. If forest owners are risk averse, then the optimal harvesting threshold is lower and the expected rotation period shorter than under risk neutrality both in the single and in the ongoing rotation cases. Under risk aversion the effective forest value uncertainty in the future is higher than under risk neutrality. Hence, risk averse harvesters typically want to shorten the length of the rotation period. In the case of risk aversion higher lump-sum tax increases the optimal harvesting threshold. As for the impact of other taxes increased yield tax increases, while higher interest rate earnings tax decreases the optimal harvesting threshold. It is also worth pointing out that under risk aversion the optimal harvesting threshold is not typically independent of a cash flow tax. The reason for this result is that at the optimal threshold the marginal expected net harvesting return has to be equal to the elasticity of the utility function. Hence, the optimal threshold can be independent of the cash flow tax only if the elasticity is a

constant which does not depend on the tax. As an interesting implication of this observation, we find that a cash flow tax can be neutral even under risk aversion whenever the utility function is iso-elastic (for example, of the HARA-type).

We proceed as follows: In section 2 we characterize the optimal rotation problem in the presence of stochastic forest value growth and various types of taxes and explore the impact of taxes on the optimal harvesting threshold and the value of the single rotation problem. Section 3 characterizes the optimal ongoing rotation problem in terms of the optimal harvesting threshold and the expected cumulative net present value of future harvests the focus being in the potential role of various types of taxes in terms of the harvesting threshold and the value of the ongoing harvesting opportunity when harvesters are risk neutral. In section 4 we study the impact of risk aversion on the relationship between various types of taxes on the harvesting threshold and on the value of the ongoing harvesting opportunity and the optimal rotation policy in both frameworks. In section 5 we present some numerical calculations concerning the relationship between the harvesting threshold, taxes, volatility, and risk aversion. Finally, there is a brief concluding section.

2   The Optimal Single Rotation Policy under Taxation and Risk Neutrality

In this section we characterize the unexplored issue of the optimal rotation problem under various types of taxes when risk-neutral forest owners are interested in the expected cumulative harvest revenue in the presence of stochastic forest stand value. We proceed as follows: First, we specify the dynamics of the system and define both the ongoing rotation and the single rotation problems. Second, we characterize the optimal single rotation policy and its relationship both to forest value volatility and to the following taxes: yield tax, lump-sum tax, tax on interest rate earnings, and cash flow tax.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space, and assume that the dynamics of the controlled *stochastic forest stand value growth* are described by the generalized Itô-equation (cf. Willasen 1998)

$$X_t^\nu = x + \int_0^t \mu(X_s^\nu)ds + \int_0^t \sigma(X_s^\nu)dW_s - \sum_{\tau_k \leq t} \zeta_k, \quad 0 \leq t \leq \tau^\nu(0), \tag{2.1}$$

where $\tau^\nu(0) = \inf\{t \geq 0 : X_t^\nu \leq 0\} \leq \infty$ denotes the potentially finite date at which the forest stand value vanishes. The drift $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and the volatility coefficient $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are assumed to be sufficiently smooth (at least continuous) mappings for guaranteeing the existence of a solution for (2.1), and $\nu = (\{\tau_k\}_{k \leq N}; \{\zeta_k\}_{k \leq N})_{N \leq \infty}$ is an admissible cutting strategy for the system (2.1). It is worth pointing out that N denotes the potentially finite number of rotation cycles prior the forest stand value vanishes. Moreover, the rotation dates $\{\tau_k\}_{k \leq N}$ constitute an increasing sequence of $\mathcal{F}_t$-stopping times and the cutting values $\{\zeta_k\}_{k \leq N}$ constitute a sequence of non-negative impulses. As usually, we assume that the system is instantaneously driven to a known generic initial state $x_0 \in \mathbb{R}_+$ whenever the irreversible harvesting decision is exercised. Naturally, this means that if the forest stand is cut when the system is in the state $y \in \mathbb{R}_+$, it is instantaneously driven to the new lower state x_0 implying that $\zeta = y - x_0$. We denote as $\mathcal{V}$ the class of admissible harvesting strategies $\nu = (\tau_1, \tau_2, \ldots, \ldots; \zeta_1, \zeta_2, \ldots)_{k \leq N}$ and assume that $\tau_k \to \tau^\nu(0)$ almost surely for all $\nu \in \mathcal{V}$ and $x \in \mathbb{R}_+$. We assume that the upper boundary ∞ is natural for X_t^ν in the absence of regulation. Thus, although the value of the forest stand may be expected to increase intertemporally, it is never expected to become infinitely high in finite time. We also assume that if the lower boundary is regular for the diffusion in the absence of harvesting, then it is killing (implying that the considered model is defined up to the first time the forest value vanishes). As usually, we denote the differential operator associated to the

controlled diffusion as

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}.$$

Before proceeding in our analysis we first denote as X_t the diffusion X_t^ν in the absence of interventions. As is well-known from the classical theory of diffusions, there are two linearly independent fundamental solutions $\psi(x)$ and $\varphi(x)$ satisfying a set of appropriate boundary conditions based on the boundary behavior of the underlying diffusion X and spanning the set of solutions of the ordinary differential equation $(\mathcal{A}u)(x) = (1-s)ru(x)$ (cf. Borodin and Salminen 2002, pp. 18 - 19). Moreover, $\psi'(x)\varphi(x) - \varphi'(x)\psi(x) = BS'(x)$, where $S'(x)$ denotes the density of the scale function of X and $B > 0$ denotes the constant Wronskian of the fundamental solutions $\psi(x)$ and $\varphi(x)$. As we will later observe, it is the increasing fundamental solution $\psi(x)$ which plays a key role in the determination of the optimal rotation policy.

Given the stochastic forest stand value growth described in (2.1) and our assumptions, we now determine the harvesting strategy which will maximize the expected cumulative net present value of all future harvests from the present up to a potentially infinite future. Given our interest on the impact of taxation on the optimal rotation policy and its value, we will assume that taxes are constant so that each time the harvesting opportunity is exercised, the decision maker receives an exercise payoff $(1-t)x - (c+T)$, where $t \in [0,1)$ measures the yield tax rate at which timber revenues are taxed, $c > 0$ is a sunk cost capturing the direct costs of the irreversible decision (i.e. the harvesting and replanting costs), and T is an exogenously determined parameter satisfying the condition $T > -c$. Thus, if $-c < T < 0$ the term T can be interpreted as a lump-sum subsidy. It is worth observing that if $T = -tc$, then the tax regime can be interpreted as a cash flow tax. We will also assume that $(1-t)x_0 < c+T$ and that interest earnings are taxed at a rate $s \in [0,1]$. Consequently, given the description of the revenues associated to the irreversible harvesting decision, our objective is to study the following optimal rotation problem

$$V(x) = \sup_{\nu \in \mathcal{V}} \mathbf{E}_x \left[\sum_{k=1}^{N} e^{-(1-s)r\tau_k}((1-t)X_{\tau_k}^\nu - (c+T)) \right], \quad x \in \mathbb{R}_+ \tag{2.2}$$

and to determine an admissible cutting policy $\nu^* \in \mathcal{V}$ for which this maximum is attained. Hence, we consider the rotation policy which will maximize the expected cumulative net present value of the future harvests from the present up to a potentially arbitrarily distant future.

Given our assumptions above, we now define the value of the associated single rotation problem as

$$J(x) = \sup_{\tau < \tau(0)} \mathbf{E}_x \left[e^{-(1-s)r\tau}((1-t)X_\tau - (c+T)) \right], \tag{2.3}$$

where τ is an arbitrary $\mathcal{F}_t$-stopping time satisfying the condition $\tau < \tau(0)$, where $\tau(0) = \inf\{t \geq 0 : X_t \leq 0\}$. It is at this point worth defining the mapping $\theta : \mathbb{R}_+ \mapsto \mathbb{R}$ which measures the net rate at which the timber revenues are appreciating as

$$\theta(x) = (1-t)(\mu(x) - (1-s)rx) + (1-s)r(c+T).$$

It is worth observing that $\theta(x)$ can alternatively be interpreted as the rate at which the net harvesting yield is expected to increase by postponing the irreversible harvesting decision for an infinitesimally short period of time. We shall assume throughout this study the absence of speculative bubbles condition

$$\mathbf{E}_x \int_0^{\tau(0)} e^{-rs}|\theta(X_s)|ds < \infty.$$

This guarantees the existence of the expected cumulative present value of the flow $\theta(x)$ meaning that the value of the optimal policy is finite. Our main result characterizing the optimal single rotation policy and its value under risk neutrality are now summarized in our next lemma.

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Lemma 2.1. *Assume that there is a threshold $\check{x} \in \mathbb{R}_+$ such that $\theta(x) \gtreqqless 0$ when $x \lesseqqgtr \check{x}$. Then under risk neutrality there is a harvesting threshold*

$$x^* = \operatorname{argmax}\left\{\frac{(1-t)x - (c+T)}{\psi(x)}\right\}$$

satisfying the ordinary first order condition $(1-t)\psi(x^*) = \psi'(x^*)((1-t)x^* - (c+T))$. Especially, $\tau^* = \inf\{t \geq 0 : X_t \notin (0, x^*)\}$ is the optimal rotation date and the value of the single rotation problem reads as

$$J(x) = \psi(x)\sup_{y \geq x}\left[\frac{(1-t)y - (c+T)}{\psi(y)}\right] = \begin{cases} (1-t)x - (c+T) & x \geq x^* \\ (1-t)\frac{\psi(x)}{\psi'(x^*)} & x < x^*. \end{cases} \tag{2.4}$$

Proof. See Appendix A. □

Lemma 2.1 states a set of weak conditions under which the associated Wicksellian single rotation problem is solvable and under which the optimal rotation policy can be described in terms of a single optimal harvesting threshold at which the harvesting opportunity should be exercised. It is worth noticing that at the optimal harvesting threshold x^* we have

$$(1-t)x^* + \max(-T, 0) = J(x^*) + c + \max(T, 0). \tag{2.5}$$

This means that at the optimum the project value (after-tax timber revenues + subsidy) has to coincide with its full cost (lost option value + costs + lump sum tax payments). As we will later observe, it is this balance equation which differs in the ongoing rotation case, since in that case the harvesting opportunity is valuable even after the harvesting opportunity has been exercised. A key implication of Lemma 2.1 is now summarized in the following.

Corollary 2.2. *Assume that the conditions of Lemma 2.1 are satisfied, that the net appreciation rate $\mu(x) - rx$ is non-increasing, and that $\lim_{x \downarrow 0} \mu(x) \leq 0$ whenever the lower boundary 0 is attainable for the forest value process X_t. Then the value of the optimal Wicksellian single rotation policy $J(x)$ is increasing and convex. Moreover, increased volatility increases its value and postpones rational harvesting by expanding the continuation region where harvesting is suboptimal.*

Proof. See Appendix C. □

Corollary 2.2 states a set of typically satisfied conditions under which increased forest value volatility increases both the value and the harvesting threshold of the optimal rotation policy. As usually, the main reason for this finding is that higher forest value volatility increases the value of waiting while leaving the expected net revenues from harvesting unchanged. Put somewhat differently, as is clear from the balance equation (2.5), increased volatility increases the full costs of the harvesting opportunity while leaving the project value unchanged and, therefore, tend to increase the optimal harvesting threshold.

Now we are in position to explore the impact of taxes on the harvesting threshold x^* and the value $J(x)$ of the single rotation problem. These findings are characterized in the following.

Corollary 2.3. *When Lemma 2.1, characterizing the optimal single rotation policy, holds*

(A) higher yield tax increases the optimal harvesting threshold and decreases the value of the harvesting opportunity;

(B) higher lump-sum tax increases the optimal harvesting threshold and decreases the value of the harvesting opportunity;

(C) higher taxation of interest rate earnings increases both the optimal harvesting threshold and the value of the harvesting opportunity by decreasing the after tax discount rate.

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Moreover, if $T = -tc$ then the optimal harvesting threshold is independent of the cash flow tax.

Proof. See Appendix D. □

These results are natural and their interpretation goes as follows. In the case of yield tax, lump-sum tax and interest rate tax, higher taxes will increase the optimal harvesting threshold. A rise in the yield tax decreases the net return from harvesting while leaving the costs unchanged. Therefore, a higher yield tax decreases the value of the optimal harvesting policy and increases the optimal rotation threshold. A higher lump-sum tax is equivalent to a higher harvesting and replanting cost. Thus, a higher lump-sum tax increases the costs associated with harvesting while leaving the timber revenues unchanged. Hence, it decreases the value of the optimal policy and increases the optimal harvesting threshold. Finally, a cash flow tax is neutral since while an higher cash flow tax decreases the net timber revenues it simultaneously decreases the harvesting costs in the same proportion (cf. Smith (1963)).

3 The Optimal Ongoing Rotation Strategy under Taxation and Risk Neutrality

We now consider the ongoing rotation problem (2.2) when harvesters are risk neutral. We proceed as follows. First, we characterize the optimal rotation policy in terms of the optimal harvesting threshold and the expected cumulative present value of future harvests. We provide conditions under which the problem is solvable and under which it can be described in terms of a unique harvesting threshold. Second, we study the impact of the same taxes as earlier on the harvesting threshold and the value of future harvesting opportunities.

Instead of tackling the stochastic impulse control problem via dynamic programming arguments, we now consider the admissible ongoing rotation strategy characterized by a known constant harvesting threshold at which the irreversible harvesting policy is exercised. Once the harvesting policy has been exercised the forest value process is instantaneously driven to an exogenously given generic initial state at which the forest value process is restarted. More precisely, we now plan to consider the admissible harvesting policy ν characterized by the sequence of harvests $\zeta_k = \max(x, y) - x_0$, $k \geq 1$, and the sequence of harvesting dates $\tau_1 = \inf\{t \geq 0 : X_t^\nu \geq y\}$ and $\tau_{k+1} = \inf\{t \geq \tau_k : X_t^\nu \geq y\}$, $k \geq 1$. Having characterized the considered harvesting strategy, it is now clear that for all $x < y$ we have the finding that the value of such a potentially suboptimal cutting strategy reads as (cf. Alvarez 2004a)

$$
\begin{aligned}
V_y(x) &= \mathbf{E}_x\left[e^{-(1-s)r\tau_y}\left((1-t)X_{\tau_y} - (c+T) + V_y(x_0)\right)\right] \\
&= \left((1-t)y - (c+T) + V_y(x_0)\right)\frac{\psi(x)}{\psi(y)},
\end{aligned} \tag{3.1}
$$

where $\tau_y = \inf\{t \geq 0 : X_t \geq y\}$ denotes the first date at which the underlying value process X_t exceeds the threshold y. Letting x tend to y yields

$$
(1-t)y + V_y(x_0) + \max(-T, 0) = V_y(y) + c + \max(T, 0). \tag{3.2}
$$

This states that for any harvesting threshold the value of the considered potentially suboptimal ongoing rotation policy satisfies the familiar balance equation stating that *the project value (current after-tax timber revenues + future harvesting potential + subsidy) has to be equal to its full cost (direct costs + lost option value + lump-sum tax payments)*. It is useful to compare this equation with (2.5) which characterizes the optimal policy in the single rotation case. As one can directly observe from (3.2) the main difference is the term $V_y(x_0)$ capturing the value of the

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future remaining harvesting opportunities which are naturally absent in the single rotation case. Letting $x \downarrow x_0$ in (3.1) and reordering terms then yields that

$$V_y(x_0) = \frac{\psi(x_0)((1-t)y - (c+T))}{\psi(y) - \psi(x_0)}. \tag{3.3}$$

Plugging now (3.3) into (3.1) then yields that for all $x \in (0, y)$ we have

$$V_y(x) = \frac{\psi(x)((1-t)y - (c+T))}{\psi(y) - \psi(x_0)}. \tag{3.4}$$

Consequently, we find that for all potentially suboptimal rotation policies described in the beginning of this section the expected cumulative present value of the future harvests from the present up to a potentially infinite future reads as

$$V_y(x) = \begin{cases} (1-t)x - (c+T) + \psi(x_0)g(y) & x \geq y \\ \psi(x)g(y) & x < y, \end{cases} \tag{3.5}$$

where

$$g(y) = \frac{((1-t)y - (c+T))}{\psi(y) - \psi(x_0)}.$$

Standard differentiation now yields

$$g'(y) = \frac{[(1-t)(\psi(y) - \psi(x_0)) - \psi'(y)((1-t)y - (c+T))]}{(\psi(y) - \psi(x_0))^2}. \tag{3.6}$$

Therefore, if a threshold y^* maximizing the mapping $g(y)$ on the set (x_0, ∞) exists, it has to satisfy the ordinary first order condition

$$(1-t)(\psi(y^*) - \psi(x_0)) = \psi'(y^*)((1-t)y^* - (c+T)). \tag{3.7}$$

Our main result characterizing the optimal ongoing rotation policy and its value under risk neutrality is now summarized in the following.

Proposition 3.1. *Assume that the conditions of Lemma 2.1 are satisfied and that the net appreciation rate $\mu(x) - rx$ is non-increasing, and that $\lim_{x \downarrow 0} \mu(x) \leq 0$ whenever the lower boundary 0 is attainable for the forest value process X_t. Then, there is a unique optimal harvesting threshold y^* satisfying the optimality condition (3.7). The expected cumulative present value of the future harvests accrued by following the optimal harvesting policy reads as*

$$V(x) = \begin{cases} (1-t)x - (c+T) + \psi(x_0)g(y^*) & x \geq y^* \\ \psi(x)g(y^*) & x < y^* \end{cases} \tag{3.8}$$

and the optimal ongoing rotation strategy is characterized by the pair (τ_k^*, ζ_k^*), where $\zeta_k^* = \max(x, y^*) - x_0$ is the optimal rotation size, $\tau_1^* = \inf\{t \geq 0 : X_t^\nu \geq y^*\}$, and $\tau_{k+1}^* = \inf\{t \geq \tau_k^* : X_t^\nu \geq y^*\}$, $k \geq 1$, denotes the sequence of optimal rotation dates. Moreover, $V(x)$ is increasing and convex and the marginal expected present value of the future harvests can be expressed as

$$V'(x) = (1-t)\min\left(1, \frac{\psi'(x)}{\psi'(y^*)}\right) = (1-t)\left(1 - \left(1 - \frac{\psi'(x)}{\psi'(y^*)}\right)^+\right). \tag{3.9}$$

Proof. See Appendix E. □

Proposition 3.1 states a set of typically satisfied conditions under which the considered ongoing rotation problem is solvable and under which the optimal ongoing rotation policy can be described in terms of an unique harvesting threshold at which the harvesting opportunity should be exercised. As intuitively is clear, the value of the optimal rotation policy is increasing and convex as a function of the current forest value. Interestingly, our results indicate that considered rotation problem is strongly connected to traditional capital theoretic models since the marginal value $V'(x)$ can be interpreted as the Tobin's q associated with the considered harvesting problem (cf. Abel 1990 who provides an excellent survey of the classical q-theory of investment). According to (3.9) the marginal value is dominated by the constant $1 - t$, that is, $V'(x) \leq 1 - t$. Thus, the optimal rotation rule actually states that harvesting is suboptimal as long as the value accrued from harvesting yet another marginal falls short of its marginal return $1 - t$. It is worth noticing that the second term in (3.9) (the option-like term) can be interpreted as the value of waiting associated with the ongoing rotation policy in the sense that it measures the gain which can be accrued from postponing the exercise of the harvesting opportunity. As intuitively is clear, this gain vanishes at the optimal threshold when harvesting becomes optimal.

Next we ask: What is the effect of higher forest stand value volatility on the optimal harvesting policy? We can show the following.

Proposition 3.2. *Assume that the conditions of Proposition 3.1 are satisfied. Then, increased volatility increases the value of the optimal rotation policy and prolongs the expected rotation period by increasing the rotation threshold at which harvesting is optimal.*

Proof. See Appendix F. □

Proposition 3.2 extends the findings of Corollary 2.2 to the ongoing rotation case. Again we find that although higher forest value volatility increases the expected net revenues it simultaneously raises the value of waiting by increasing the expected cumulative net present value of the future harvesting opportunities. Since the latter effect dominates the former, we find that higher forest value volatility unambiguously increases both the value of the optimal ongoing rotation policy and the optimal rotation threshold.

Now we are again in the position to study the effect of taxes on the optimal harvesting threshold y^* and the value of the harvesting opportunity in the ongoing rotation framework. Our qualitative results are now presented in

Proposition 3.3. *Assume that the conditions of Proposition 3.1 are satisfied. Then,*

(A) *increased yield tax increases the optimal harvesting threshold and decreases the value of the ongoing harvesting opportunity;*

(B) *an increase in the lump sum tax increases the optimal harvesting threshold and decreases the value of the ongoing harvesting opportunity;*

(C) *higher taxation of interest rate earnings increases both the value and the optimal rotation threshold of the ongoing harvesting opportunity.*

Moreover, if $T = -tc$ then the optimal ongoing rotation threshold is independent of the cash flow tax.

Proof. See Appendix G. □

According to Proposition 3.3 the qualitative effects of taxes are similar as in the case of the single rotation problem (cf. Corollary 2.3). The main reason for this argument is essentially the similarity of the first order conditions characterizing the optimal harvesting threshold (see Lemma 2.1 and the first order condition (3.7)). The interpretation of this result goes in an analogous way we presented after Corollary 2.3.

4 Stochastic Forest Value, Risk Aversion, and Taxation

Having considered the optimal rotation policy and the impact of taxes under risk neutrality, it is our purpose in this section to analyze the impact of taxation on the optimal rotation policy in the presence of risk aversion. We want to notice that to our knowledge this is an issue which has not been previously analyzed in stochastic rotation models even in the absence of taxation. To this end, we denote as $U : \mathbb{R} \mapsto \mathbb{R}$ the utility function of the harvester and assume that $U(x)$ is twice continuously differentiable, monotonically increasing and strictly concave. For the sake of simplicity, we will also assume in this section that the lower boundary 0 is unattainable for the underlying stand value dynamics X_t. Given these assumptions, define the mapping $\Lambda : \mathbb{R} \mapsto \mathbb{R}$ measuring the growth rate of the expected net present value of utility as

$$\Lambda(x) = \frac{1}{2}(1-t)^2\sigma^2(x)U''(\pi(x)) + (1-t)\mu(x)U'(\pi(x)) - (1-s)rU(\pi(x)),$$

where $\pi(x) = (1-t)x - (c+T)$, and assume that $\Lambda \in \mathcal{L}^1(\mathbb{R}_+)$. It is our purpose now to consider both the Wicksellian single rotation problem

$$J(x) = \sup_{\tau < \tau(0)} \mathbf{E}_x \left[e^{-(1-s)r\tau} U((1-t)X_\tau - (c+T)) \right], \tag{4.1}$$

and the Faustmannian ongoing rotation problem

$$V(x) = \sup_{\nu \in \mathcal{V}} \mathbf{E}_x \left[\sum_{k=1}^{N} e^{-(1-s)r\tau_k} U((1-t)X_{\tau_k}^\nu - (c+T)) \right], \quad x \in \mathbb{R}_+. \tag{4.2}$$

In the case of single rotation problem we can now establish the following

Proposition 4.1. *Assume that there is a unique threshold $\check{x}_1 > 0$ such that $\Lambda(x) \gtreqqless 0$ when $x \lesseqqgtr \check{x}_1$. Then, under risk aversion there is a unique optimal harvesting threshold*

$$x^* = \mathrm{argmax} \left\{ \frac{U((1-t)x - (c+T))}{\psi(x)} \right\}$$

satisfying the ordinary first order condition $(1-t)U'((1-t)x^* - (c+T))\psi(x^*) = \psi'(x^*)U((1-t)x^* - (c+T))$. Especially, $\tau^* = \inf\{t \geq 0 : X_t \notin (0, x^*)\}$ is the optimal rotation date and the value of the single rotation problem reads as

$$J(x) = \psi(x) \sup_{y \geq x} \left[\frac{U((1-t)y - (c+T))}{\psi(y)} \right] = \begin{cases} U((1-t)x - (c+T)) & x \geq x^* \\ \frac{\psi(x)(1-t)U'((1-t)x^* - (c+T))}{\psi'(x^*)} & x < x^*. \end{cases} \tag{4.3}$$

Proof. See Appendix H. □

Proposition 4.1 states a set of conditions under which the associated Wicksellian single rotation problem in the presence of risk aversion is solvable and under which the optimal rotation policy can be described in terms of a single harvesting threshold. Assuming that the net appreciation rate $\mu(x) - rx$ is non-increasing we can now establish the following.

Proposition 4.2. *Assume that the conditions of Proposition 4.1 are satisfied and that the net appreciation rate $\mu(x) - rx$ is non-increasing. Then, increased forest stand value volatility increases the optimal harvesting threshold x^* and, therefore, prolongs the expected length of the optimal Wicksellian single rotation period.*

Proof. See Appendix I. □

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Proposition 4.2 states a set of conditions under which the optimal rotation threshold is an increasing function of forest value volatility. It is worth noticing that the effect of volatility is essentially based on the convexity of the value function on the continuation region where harvesting is suboptimal. Since this observation is closely related to our findings in the risk neutral case as well, we next ask an important new question: How does risk aversion affect the optimal harvesting threshold and thereby the expected length of the rotation period? We can now establish the following.

Proposition 4.3. *Assume that the conditions of Proposition 4.1 are satisfied and that the utility function satisfies the condition $U(\pi(x^*)) \geq U'(\pi(x^*))\pi(x^*)$. Then, the optimal harvesting threshold is lower under risk aversion than under risk neutrality.*

Proof. The first order optimality condition can be re-expressed as

$$(1-t)\psi(x^*) = \psi'(x^*)\frac{U(\pi(x^*))}{U'(\pi(x^*))}.$$

Therefore, condition $U(\pi(x^*)) \geq U'(\pi(x^*))\pi(x^*)$ implies that $(1-t)\psi(x^*) \geq \psi'(x^*)\pi(x^*)$ from which the alleged result follows. $\square$

Hence risk aversion will shorten the expected length of the rotation period compared with the risk neutral case by lowering the optimal harvesting threshold. This result is of importance, since it emphasizes the intertemporal aspect of risk arising from the time dependence of the driving Brownian motion. Our result simply indicates that a risk averse harvester prefers less uncertain returns. Therefore, the optimal harvesting threshold has to decrease.

Next we study the effects of the same taxes analyzed earlier in the case of risk neutrality. Our main results on the comparative static properties of the single rotation problem are characterized in the following.

Proposition 4.4. *Assume that the conditions of Proposition 4.1 are satisfied. Then,*

(A) increased yield tax decreases the value of the single harvesting opportunity and if the condition of Proposition 4.3 is also satisfied, then increased yield tax increases the optimal harvesting threshold as well;

(B) an increase in the lump sum tax increases the optimal harvesting threshold and decreases the value of the single harvesting opportunity;

(C) higher taxation of interest rate earnings increases the optimal harvesting threshold and decreases the value of the single harvesting opportunity by decreasing the after tax discount rate.

A cash flow tax is not neutral.

Proof. The proof is analogous with the proof of Corollary 2.3. $\square$

According to Proposition 4.4 a higher yield tax decreases the value of a single rotation policy. The reason for this clear since a higher yield tax decreases the net timber revenues. The monotonicity of the utility function then implies that it decreases the expected utility as well and, therefore, the value of the optimal policy. A higher lump sum tax decreases the value of the optimal policy and increases the optimal rotation threshold by increasing the harvesting costs and, therefore, by decreasing the expected utility and increasing the break-even level at which net harvesting revenues are equal to harvesting costs. Finally, higher taxation of interest rate earnings increases the optimal harvesting threshold by decreasing the after-tax discount rate. As usually, we observe that a cash flow tax is not neutral when forest owners are risk averse. This result is essentially based on the observation that at the optimal threshold the marginal

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expected net harvesting return has to coincide with the elasticity of the utility function. Therefore, the optimal harvesting threshold is independent of the cash flow tax only if the elasticity is a constant.

Next we study the impact of taxes on the harvesting threshold and the value of the harvesting opportunity in the presence of risk aversion under the Faustmannian ongoing rotation problem. These findings are presented in the following.

Proposition 4.5. *Assume that $\Lambda(x)$ is non-increasing, that there is a unique threshold $\check{x}_1 > 0$ such that $\Lambda(x) \gtreqqless 0$ when $x \lesseqqgtr \check{x}_1$, and that $\lim_{x \downarrow \tilde{x}}(1-t)U'(\pi(x))(\psi(x)-\psi(x_0)) - \psi'(x)U(\pi(x)) > 0$, where $\tilde{x}$ denotes the break-even state satisfying the condition $(1-t)\tilde{x} = c + T$. Then, under risk aversion there is a unique optimal harvesting threshold*

$$y^* = \arg\max \left\{ \frac{U((1-t)x - (c+T))}{\psi(x) - \psi(x_0)} \right\} \in (\tilde{x}, x^*)$$

satisfying the ordinary first order condition $(1-t)U'((1-t)y^* - (c+T))(\psi(y^*) - \psi(x_0)) = \psi'(y^*)U((1-t)y^* - (c+T))$. The expected cumulative present value of the future harvests accrued by following the optimal harvesting policy reads as

$$V(x) = \begin{cases} U((1-t)x - (c+T)) + \psi(x_0)G(y^*) & x \geq y^* \\ \psi(x)G(y^*) & x < y^* \end{cases} \tag{4.4}$$

where

$$G(x) = \frac{U((1-t)x - (c+T))}{\psi(x) - \psi(x_0)}.$$

In this case, the optimal ongoing rotation strategy is characterized by the pair (τ_k^*, ζ_k^*), where $\zeta_k^* = \max(x, y^*) - x_0$, $\tau_1^* = \inf\{t \geq 0 : X_t^\nu \geq y^*\}$, and $\tau_{k+1}^* = \inf\{t \geq \tau_k^* : X_t^\nu \geq y^*\}$, $k \geq 1$.

Proof. See Appendix J. □

Proposition 4.5 states a set of conditions under which the Faustmannian ongoing rotation problem is solvable under risk aversion. It is worth noticing that, as intuitively is clear, the conditions of Proposition 4.5 are stronger than the conditions of Proposition 4.1 guaranteeing the existence of an optimal Wicksellian single rotation policy. More precisely, although the growth rate of the expected net present value of the utility has to change sign on the state space of the underlying forest value process in this case as well, it also has to be monotonic as a function of the underlying forest value process in order to guarantee the uniqueness of the optimal harvesting threshold.

Having established a set of sufficient conditions guaranteeing the existence and uniqueness of an optimal ongoing rotation policy under risk aversion we next consider the impact of higher forest value volatility on the optimal ongoing rotation policy and its value.

Proposition 4.6. *Assume that the conditions of Proposition 4.5 are satisfied and that the net appreciation rate $\rho(x)$ is non-increasing. Then, increased forest stand value volatility increases the optimal harvesting threshold y^* and, therefore, prolongs the expected length of the optimal Faustmannian ongoing rotation period.*

Proof. The proof is analogous with the proof of 3.2. □

Proposition 4.6 states a set of conditions under which higher forest value volatility unambiguously increases the optimal harvesting threshold and, therefore, under which the required rate of return is an increasing function of volatility. It is worth pointing out that as in the case of Proposition 4.2, the net appreciation rate $\mu(x) - rx$ of the forest value is the principal determinant of the sign of the relationship between forest value volatility and the optimal rotation policy and, therefore, that the form of the utility function is of importance only in the determination

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of the optimal harvesting threshold. This observation is of interest since it demonstrates that the sign of the relationship between forest value volatility and the optimal rotation policy is a process-specific, not a payoff-specific, property (cf. Alvarez 2003). The effect of risk aversion on the optimal harvesting threshold and thereby on the expected length of the rotation period is now summarized in the following.

Proposition 4.7. *Assume that the conditions of Proposition 4.5 are satisfied and that the utility function satisfies the condition $U(\pi(y^*)) \geq U'(\pi(y^*))\pi(y^*)$. Then, the optimal harvesting threshold is lower under risk aversion than under risk neutrality.*

Proof. The proof is analogous with the proof of Proposition 4.3. $\square$

Proposition 4.7 extends the results of Proposition 4.3 to the ongoing rotation case. More precisely, it states a set of typically satisfied conditions under which the optimal harvesting threshold is lower under risk aversion than under risk neutrality. The main reason for this observation naturally is that the effective forest value uncertainty in the future under risk aversion is higher than under risk neutrality. Hence, rational risk averse harvesters typically want to shorten the expected length of the rotation period by lowering the optimal harvesting threshold. Finally, the comparative static properties of the optimal rotation policy and its value as functions of the considered taxes are now summarized in the following.

Proposition 4.8. *Assume that the conditions of Proposition 4.5 are satisfied. Then,*

(A) *increased yield tax decreases the value of the ongoing harvesting opportunity and if the condition of Proposition 4.7 is also satisfied, then increased yield tax increases the optimal harvesting threshold as well;*

(B) *an increase in the lump sum tax increases the optimal harvesting threshold and decreases the value of the single harvesting opportunity;*

(C) *higher taxation of interest rate earnings decreases the optimal harvesting threshold and increases the value of the single harvesting opportunity by decreasing the after tax discount rate.*

A cash flow tax is not neutral.

Proof. The proof is analogous with the proof of Proposition 3.3. $\square$

According to Proposition 4.8 the qualitative effects of taxes on the optimal ongoing rotation policy and its value under risk aversion are similar as in the case of the single rotation problem (cf. Proposition 4.4). Again, this argument is essentially based on the similarity of the first order conditions characterizing the optimal harvesting thresholds.

Note: It is worth observing that if the utility function of the decision maker is of the standard HARA-type (cf. Merton 1971) $U(x) = x^\gamma/\gamma$, where $\gamma \in (0,1)$ is a known exogenously determined constant and $1 - \gamma$ measures the relative risk aversion of the forest owner, then the optimality condition characterizing the Wicksellian rotation threshold reads as

$$(1-t)\psi(x^*) = \frac{1}{\gamma}\psi'(x^*)((1-t)x^* - (c+T))$$

and the optimality condition characterizing the Faustmannian rotation threshold reads as

$$(1-t)(\psi(y^*) - \psi(x_0)) = \psi'(y^*)\frac{1}{\gamma}((1-t)y^* - (c+T)).$$

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Hence, we observe that if $T = -tc$ then the optimality conditions read as $\gamma\psi(x^*) = \psi'(x^*)(x^*-c)$ and $\gamma(\psi(y^*)-\psi(x_0)) = \psi'(y^*)(y^*-c)$ demonstrating that a cash flow tax is neutral in the HARA-utility case. A second important observation is that

$$\frac{\partial x^*}{\partial \gamma} = \frac{\psi'(x^*)\pi(x^*)}{\gamma(\psi''(x^*)\pi(x^*) + (1-t)(1-\gamma)\psi'(x^*))} > 0$$

and that

$$\frac{\partial y^*}{\partial \gamma} = \frac{\psi'(y^*)\pi(y^*)}{\gamma(\psi''(y^*)\pi(y^*) + (1-t)(1-\gamma)\psi'(y^*))} > 0$$

proving that increased relative risk aversion (i.e. a decreased γ) decreases the optimal rotation threshold x^* and, therefore, shortens the expected length of the optimal rotation cycle both in the Wicksellian single rotation and in the Faustmannian ongoing rotation framework.

5   Explicit Example

Having considered theoretically the relationship between various taxes and the privately optimal rotation policy both in the single rotation and in the ongoing rotation frameworks when forest owners are either risk neutral or risk averse we now illustrate our results more explicitly under parametric specifications concerning the forest value process and utility function.

5.1   Risk Neutrality

In order to illustrate our results in the risk neutral case explicitly, we assume that the underlying value process evolves according to an ordinary geometric Brownian motion described by the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t \quad X_0 = x \in \mathbb{R}_+,$$

where μ and σ are exogenously determined constants (cf. Clarke and Reed 1989). It is well-known that in this case, the increasing fundamental solution, discussed in Section 3, reads as $\psi(x) = x^\kappa$, where

$$\kappa = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(1-s)r}{\sigma^2}}$$

denotes the positive root of the characteristic equation $\sigma^2\kappa(\kappa-1) - 2\mu\kappa - 2(1-s)r = 0$.

Given these assumptions, we observe that if $r > \mu$, which guarantees the finiteness of the value of the optimal policy, then the value of the optimal single rotation policy reads as

$$J(x) = x^\kappa \sup_{y \geq x} \left[y^{-\kappa}((1-t)y - (c+T))\right] = \begin{cases} (1-t)x - (c+T) & x \geq x^* \\ \frac{(1-t)}{\kappa}x^\kappa x^{*1-\kappa} & x < x^*, \end{cases} \quad (5.1)$$

where

$$x^* = \left(1 + \frac{1}{\kappa-1}\right)\frac{(c+T)}{(1-t)}$$

denotes the optimal harvesting threshold. As one can see the optimal threshold depends on yield and lump sum taxes, harvesting and replanting cost, as well as on the net interest rate as well as drift and volatility parameters of the underlying forest value process.

On the other hand, we observe from Proposition 3.1 that if $r > \mu$ then the value of the optimal ongoing rotation policy reads as

$$V(x) = \begin{cases} (1-t)x - (c+T) + \frac{(1-t)}{\kappa}x_0^\kappa y^{*1-\kappa} & x \geq y^* \\ \frac{(1-t)}{\kappa}x^\kappa y^{*1-\kappa} & x < y^* \end{cases} \quad (5.2)$$

where the optimal harvesting threshold $y^* < x^*$ constitutes the unique root of the equation

$$(1-t)(y^{*\kappa} - x_0^\kappa) = \kappa y^{*\kappa-1}((1-t)y^* - (c+T)),$$

which can be re-expressed as

$$y^* = \frac{\kappa(c+T)}{(\kappa-1)(1-t)} - \frac{x_0^\kappa y^{*1-\kappa}}{\kappa-1}.$$

In order to compare the quantitative difference between the optimal thresholds x^* and y^* we first observe that if $(1-s)r = k\mu + \frac{1}{2}\sigma^2 k(k-1)$, where $k \in \mathbb{R}_+$, then $\kappa = k$. Therefore, if $(1-s)r = 2\mu + \sigma^2$ then $\kappa = 2$,

$$x^* = \frac{2(c+T)}{(1-t)},$$

and

$$y^* = \frac{(c+T)}{(1-t)} + \frac{\sqrt{(c+T)^2 - (1-t)^2 x_0^2}}{1-t} \in \left(\frac{(c+T)}{(1-t)}, x^*\right).$$

It is, therefore, clear that in this special case the difference $x^* - y^*$ between the single rotation and ongoing rotation thresholds is a decreasing function of the yield tax t and the lump-sum tax T.

5.2 Risk Aversion

In order to illustrate our general results explicitly in the case of risk aversion, assume that the utility function of the decision maker is of the standard HARA-type (cf. Merton 1971) $U(x) = x^\gamma/\gamma$, where $\gamma \in (0,1)$ is a known exogenously determined constant measuring the relative risk aversion of the forest owner. In that case, we observe that if $(1-s)r > \gamma\mu + \frac{1}{2}\sigma^2\gamma(\gamma-1)$, which guarantees the finiteness of the value of the optimal policy, then the optimal harvesting threshold in the single rotation case satisfies the ordinary first order condition

$$(1-t)\gamma x^* = \kappa((1-t)x^* - (c+T))$$

implying that

$$x^* = \frac{\kappa(c+T)}{(\kappa-\gamma)(1-t)}. \tag{5.3}$$

Comparing this optimal harvesting threshold to the one in the case of risk neutrality suggests that in the single rotation case risk aversion decreases the harvesting threshold thereby shortening the the expected length of the optimal rotation period, ceteris paribus. It is also worth noticing that the optimal rotation threshold is an increasing function of the parameter γ. Consequently, our results unambiguously indicate that increased risk aversion decreases the optimal rotation threshold and, therefore, shortens the expected length of the rotation period. Moreover, it is also worth emphasizing that (5.3) clearly indicates that in the HARA-utility case a cash flow tax is neutral. More precisely, if $T = -tc$ then $x^* = \kappa c/(\kappa-\gamma)$ which is independent of t.

In the ongoing rotation case the optimality condition reads as

$$(1-t)(y^{*\kappa} - x_0^\kappa) = \frac{\kappa}{\gamma} y^{*\kappa-1}((1-t)y^* - (c+T)). \tag{5.4}$$

Unfortunately, determining the root of this equation is impossible in the general case. However, if we again assume that if $(1-s)r = 2\mu + \sigma^2$ then $\kappa = 2$ and the optimality condition becomes

$$(1-t)(y^{*2} - x_0^2) = \frac{2}{\gamma} y^*((1-t)y^* - (c+T)).$$

This implies that

$$y^* = \frac{(c+T)}{(1-t)(2-\gamma)} + \frac{\sqrt{(c+T)^2 - (1-t)^2(2-\gamma)\gamma x_0^2}}{(1-t)(2-\gamma)} \in \left(\frac{(c+T)}{(1-t)(2-\gamma)}, x^* \right)$$

Moreover, since the HARA-utility mapping satisfies for all $x \geq \tilde{x}$ the inequality

$$\frac{U(\pi(x))}{U'(\pi(x))} = \left(1 + \frac{1-\gamma}{\gamma} \right) \pi(x) \geq \pi(x),$$

we observe in accordance with the findings of Proposition 4.7 that risk aversion decreases the optimal rotation threshold and, therefore, shortens the expected length of the optimal rotation period in the ongoing rotation case as well. It is also worth noticing that as in the Wicksellian single rotation case, a cash flow tax is again neutral. More precisely, if $T = -tc$, then the optimality condition (5.4) reads as $\gamma(y^{*\kappa} - x_0^\kappa) = \kappa y^{*\kappa-1}(y^* - c)$ which, in turn, implies that the optimal rotation threshold is independent of t. Now we turn to look at numerical results.

In the ongoing rotation case the optimal harvesting threshold is illustrated in Figure 1 as a function of the yield tax by using the assumptions: $c = 1, T = 0, \mu = 0.01, r = 0.03, \sigma = 0.01, s = 0$, and $x_0 = 0.01$. In accordance with our general findings, the exercise threshold increases non-linearly as a function of the yield tax. Moreover, higher relative risk aversion, i.e. a lower γ, will decrease the optimal harvesting threshold thereby shortening the expected length of the optimal rotation period.

In Figure 2, we illustrate the optimal harvesting threshold as a function of the forest value volatility and explore how it depends on the attitude towards the riskiness of forest stand value. We use the following assumptions: $c = 1, T = 0, \mu = 0.01, r = 0.03, t = 0.1, s = 0$, and $x_0 = 0.01$. As intuitively is clear, the optimal exercise threshold is a non-linear function of forest stand value volatility both in the risk neutral and in the risk averse case. As was established in our general analysis, the optimal rotation threshold is lower under risk aversion than under risk neutrality. Moreover, as in the case of a yield tax, we find that higher relative risk aversion decreases the optimal harvesting threshold and, therefore, shortens the expected length of the optimal rotation cycle. However, it is worth mentioning that Figure 2 clearly indicates that the monotonicity of the impact of higher forest value volatility depends heavily on the relative risk aversion of forest owners. More precisely, Figure 2 indicates that the rate at which the optimal rotation threshold increases as a function of forest value volatility decreases as the relative risk aversion of forest owners increases.

6 Conclusions

In this paper we have used both the Wicksellian single rotation and the Faustmannian ongoing rotation frameworks to explore for the first time the impact of several taxes on the private optimal harvesting threshold and the expected rotation period when the forest stand value is stochastic and forest owners are either risk neutral or risk averse. In stochastic rotation models the risk averse case has not to our knowledge been studied even in the absence of taxation. More precisely we have analyzed the effects of the following taxes on the harvesting threshold and thereby on the expected length of rotation period: yield tax, lump-sum tax, cash flow tax and the rate on interest rate earnings. First we characterized the optimal rotation problem in terms of its solvability and presented weak conditions under which the optimal rotation policy can be described in terms of a single harvesting threshold at which the optimal harvesting opportunity should be exercised. Then we explored the impact of taxes on the optimal harvesting threshold and thereby the expected length of rotation period in both frameworks when forest owners are either risk neutral or risk averse.

When forest owners are risk neutral, we have shown the following results both in the single rotation and in the ongoing rotation frameworks. Higher yield tax increases the optimal harvesting threshold and thereby decreases the value of harvesting opportunity. The impact of higher
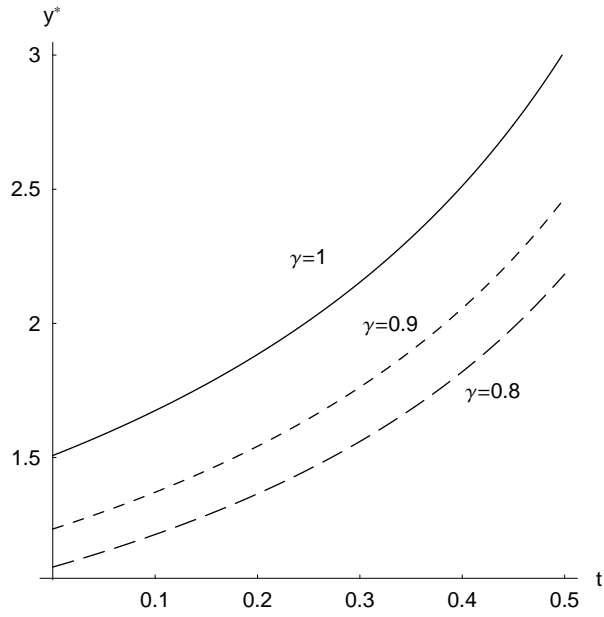
Figure 1: The optimal harvesting threshold y^* as a function of the yield tax rate t
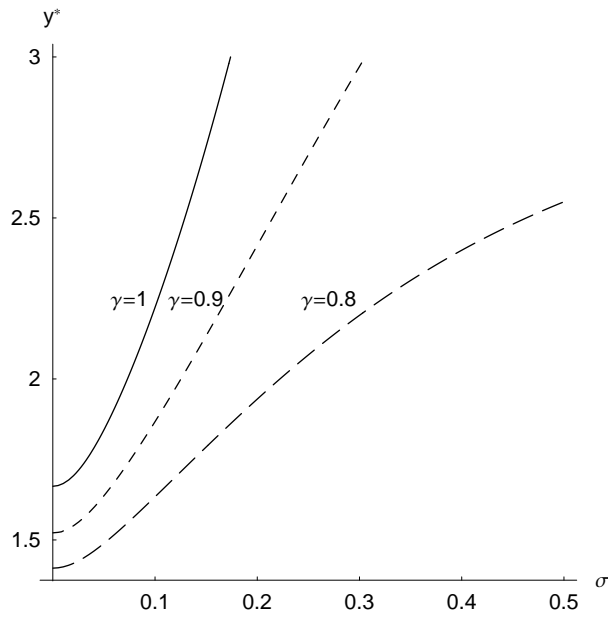
Figure 2: The optimal harvesting threshold y^* as a function of volatility σ

lump sum tax is qualitatively similar. On the other hand, higher taxation of interest rate earnings increases both the optimal harvesting threshold and the value of harvesting opportunity by decreasing the after tax discount rate. Finally, the optimal harvesting threshold has been shown to be independent of cash flow tax so that it is neutral in terms of rotation decisions. Interpretations go as follows: higher yield tax decreases the net return from harvesting while

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higher lump-sum tax is equivalent to larger harvesting and replanting cost so that in both cases the value of optimal policy goes down and the optimal harvesting threshold goes up. Higher tax rate on interest rate earnings is equivalent to lower after tax discount rate so that the optimal harvesting threshold goes up. Hence, in all these case taxes affect positively the expected length of rotation period. The neutrality of cash flow tax is clear cut since changes in cash flow tax do not affect the difference between the net timber revenues and the total harvesting and replanting costs.

When forest owners are risk averse, we first have shown that the optimal harvesting threshold is lower than in the risk neutral case both in the single rotation and ongoing rotation frameworks. This is simply because uncertainty goes up over time and a risk averse harvester prefers less uncertain returns. Thereby the expected length of rotation period is shorter under risk aversion. In terms of the impact of taxes on the optimal harvesting threshold and the expected length of rotation period we have shown that in the case of yield tax, lump-sum tax and taxation of interest rate earnings the qualitative results are similar as under risk neutrality and interpretation of results goes along similar lines. But the cash flow tax is no longer typically neutral in this case. The reason for this observation is that at the optimal rotation threshold the marginal expected net harvesting return has to be equal to the elasticity of the utility function of the land owner. Therefore, the independence of the optimal rotation threshold can be guaranteed only if the elasticity of the utility function is a constant. In line with this argument, we found that a cash flow tax is neutral even under risk aversion as long as the utility function is iso-elastic (for example, of the HARA-type). Finally, we have illustrated our results under parametric specifications concerning the forest value process and the utility function of risk averse harvester. We have shown that the optimal harvesting threshold as a function of the yield tax increases nonlinearly, while higher relative risk aversion decreases the threshold. Moreover, the optimal harvesting threshold as a function of forest value volatility increases more rapidly under risk neutrality than under risk aversion.

An interesting and important further research question is, among others, the following: what would be the optimal design of taxation from the point of view of the society in order to keep the expected tax revenue from forests constant? Clearly our results partly suggest that forest owners' behavior in terms of risk attitude matters. Also tax reform issues under stochastic forest stand value are currently open and our new results will provide a good background to study these policy questions. An important research topic is also to study the impacts of taxation in an extended framework where harvesters are interested not only in timber revenues but also in amenity services provided by forests. From policy point of view this allows for possibility to analyze the implications of public good aspects of forests in the stochastic rotation frameworks. This is also a completely open issue under the stochastic framework.

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A   Proof of Lemma 2.1

Proof. By following the proof of Lemma 2.3 in Alvarez 2004a, we can establish that

$$(1-t)\frac{\psi(x)}{S'(x)} - \frac{\psi'(x)}{S'(x)}((1-t)x - (c+T)) = I(x),$$

where

$$I(x) = \int_0^x \psi(y)\theta(y)m'(y)dy + \begin{cases} \frac{(c+T)B}{\varphi(0)} & \text{if } 0 \text{ is attainable for } X_t \\ 0 & \text{if } 0 \text{ is unattainable for } X_t. \end{cases}$$

Our assumptions imply that $I(x) > 0$ for all $x \leq \check{x}$. If $x > K > \check{x}$, then the identity

$$\frac{\psi'(x)}{S'(x)} - \frac{\psi'(a)}{S'(a)} = (1-s)r \int_a^x \psi(y)m'(y)dy$$

implies that

$$I(x) \leq I(K) + \frac{\sup_{y\in[K,x]}\theta(y)}{(1-s)r}\left(\frac{\psi'(x)}{S'(x)} - \frac{\psi'(0)}{S'(0)}\right).$$

However, since $\theta(x) < 0$ for all $x > \check{x}$ and $\psi'(x)/S'(x) \to \infty$ as $x \to \infty$ since ∞ was assumed to be a natural boundary, we find that $I(x) \downarrow -\infty$ as $x \to \infty$. Combining this observation with the continuity and monotonicity of $I(x)$ on $[\check{x}, \infty)$ then finally implies that there is a unique threshold x^* at which the equation $I(x^*) = 0$ is satisfied. Moreover, since

$$\frac{d}{dx}\left[\frac{(1-t)x - (c+T)}{\psi(x)}\right] = \frac{S'(x)}{\psi^2(x)}I(x)$$

we observe that

$$x^* = \text{argmax}\left\{\frac{(1-t)x - (c+T)}{\psi(x)}\right\}.$$

Having proved the existence and uniqueness of the threshold x^* it now remains to establish that the proposed value function indeed constitutes the value of the optimal rotation policy. To this end, denote the proposed value as $J_p(x)$ and observe that

$$J_p(x) = \mathbf{E}_x\left[e^{-(1-s)r\tau^*}((1-t)X_{\tau^*} - (c+T))\right]$$

implying that $J(x) \geq J_p(x)$. To establish the opposite inequality, we first observe that $J_p \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+\backslash\{x^*\})$, that $J_p(x) \geq \max((1-t)x-(c+T), 0)$, and that $(\mathcal{A}J_p)(x) = (1-s)rJ_p(x)$ on $(0, x^*)$. However, since $(\mathcal{A}J_p)(x) - (1-s)rJ_p(x) = \theta(x) < 0$ on (x^*, ∞) we find that the proposed value constitutes a r-excessive majorant of the exercise payoff $(1-t)x - (c+T)$. Since $J(x)$ is the least of such majorants, we find that $J_p(x) \geq J(x)$ and, therefore, that $J_p(x) = J(x)$. $\square$

B   Proof of Corollary 2.2

Proof. By Lemma 2.1 the value of the optimal rotation policy can be expressed as

$$J(x) = \begin{cases} (1-t)x - (c+T) & x \geq x^* \\ (1-t)\frac{\psi(x)}{\psi'(x^*)} & x < x^*. \end{cases}$$

On the other hand, as was established in Alvarez (2004a) the monotonicity of the appreciation rate $\mu(x) - rx$ implies that $\psi(x)$ is strictly convex on $\mathbb{R}_+$ and, therefore, that $J(x)$ is convex as well.

Denote now as $\hat{J}(x)$ the value of the optimal single rotation policy associated to a less volatile diffusion $\tilde{X}_t$ characterized by the volatility coefficient $\tilde{\sigma}(x)$ satisfying the inequality $\tilde{\sigma}(x) \leq \sigma(x)$ for all $x \in \mathbb{R}_+$. Since $J \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+\backslash\{y^*\})$, $J''(y^*\pm) < \infty$, $J(x) \geq (1-t)x - (c+T)$ for all $x \in \mathbb{R}_+$, and

$$\frac{1}{2}\tilde{\sigma}^2(x)J''(x) + \mu(x)J'(x) - rJ(x) \leq (\mathcal{A}J)(x) - rJ(x) \leq 0$$

for all $x \in \mathbb{R}_+\backslash\{y^*\}$, we find that $J(x)$ constitutes a r-excessive majorant of the payoff $(1-t)x - (c+T)$ for the less volatile diffusion $\tilde{X}_t$. Since $\tilde{J}(x)$ is the smallest of these majorants, we find that $\tilde{J}(x) \geq J(x)$ proving that increased volatility increases the value of the single harvesting opportunity. Finally, if $x \in C_{\tilde{\sigma}} = \{x \in \mathbb{R}_+ : \tilde{J}(x) > (1-t)x - (c+T)\}$ then $x \in C_\sigma = \{x \in \mathbb{R}_+ : J(x) > (1-t)x - (c+T)\}$ as well and, therefore, we observe that $C_{\tilde{\sigma}} \subseteq C_\sigma$ which proves that increased volatility increases the optimal harvesting threshold. $\square$

C Proof of Corollary 2.3

Proof. As was established in the proof of Lemma 2.1 the optimal harvesting threshold x^* satisfies the condition

$$(1-t)\psi(x^*) = \psi'(x^*)((1-t)x^* - (c+T))$$

implying that

$$\psi'(x^*)x^* - \psi(x^*) = \frac{c+T}{1-t}\psi'(x^*) > 0.$$

Define now the mapping $F : (0,1) \times \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$F(t,x) = (1-t)\frac{\psi(x)}{S'(x)} - \frac{\psi'(x)}{S'(x)}((1-t)x - (c+T)).$$

Standard differentiation then yields that

$$\begin{aligned}
F_t(t,x) &= \frac{\psi'(x)x - \psi(x)}{S'(x)} \\
F_x(t,x) &= \psi(x)\theta(x)m'(x).
\end{aligned}$$

However, since $F(t,x^*) = 0$ we find by ordinary implicit differentiation that

$$\frac{\partial x^*}{\partial t} = -\frac{F_t(t,x^*)}{F_x(t,x^*)} = -\frac{1}{2}\sigma^2(x^*)\frac{\psi'(x^*)x^* - \psi(x^*)}{\psi(x^*)\theta(x^*)} > 0$$

since x^* is attained on the set where $\theta(x) < 0$. Inequality $\partial J(x)/\partial t < 0$ follows directly from the definition of the exercise payoff. Proving part (B) of the corollary is completely analogous with the previous analysis.

In order to consider the impact of increased taxation of interest income, denote as $J_{s_i}(x)$ the value of the harvesting opportunity and as x_i^* the optimal rotation threshold in the presence of the tax rate s_i, $i = 1, 2$. Since $J_{s_i}(x)$ satisfies the variational inequality $(\mathcal{A}J_{s_i})(x) \leq (1-s_i)rJ_{s_i}(x)$ for all $x \in \mathbb{R}_+\backslash\{x_i^*\}$ we find that if $s_1 \geq s_2$ then

$$(\mathcal{A}J_{s_1})(x) - (1-s_2)rJ_{s_1}(x) \leq -(s_1 - s_2)rJ_{s_1}(x) \leq 0$$

for all $x \in \mathbb{R}_+\backslash\{x_1^*\} \cup \{x_2^*\}$ since $J_{s_1}(x) \geq 0$. Moreover, since $J_{s_1}(x) \geq (1-t)x - (c+T)$ for all $x \in \mathbb{R}_+$ we observe that $J_{s_1}(x)$ constitutes a $(1-s_2)r$-excessive majorant (cf. Borodin and Salminen 2002, pp. 32–35) of the underlying exercise payoff. Since $J_{s_2}(x)$ is the least of these majorants, we find that $J_{s_1}(x) \geq J_{s_2}(x)$ demonstrating that increased taxation of interest income increases the value of the harvesting opportunity. In order to establish that increased taxation

of interest income speeds up rational exercise of the harvesting opportunity, denote now as $C_i = \{x \in \mathbb{R}_+ : J_{s_i}(x) > (1-t)x - (c+T)\}$ the continuation region where exercising the opportunity is suboptimal in the presence of the tax rate s_i. If $x \in C_2$ then $J_{s_2}(x) > (1-t)x - (c+T)$. However, since $J_{s_1}(x) \geq J_{s_2}(x)$ we find that $x \in C_1$ as well and, therefore, that $C_2 \subseteq C_1$ from which the alleged result follows.

$\square$

D   Proof of Proposition 3.1

Proof. We prove the alleged result in two parts. First, we demonstrate that our assumptions imply that there is a unique harvesting threshold satisfying the optimality condition (3.7) and then we demonstrate that given this threshold the proposed value function indeed constitutes the value of the optimal rotation policy. Define now the continuously differentiable mapping $K : (x_0, \infty) \mapsto \mathbb{R}$ as

$$K(x) = (1-t)(\psi(x) - \psi(x_0)) - \psi'(x)((1-t)x - (c+T)).$$

We immediately observe that

$$\lim_{x \downarrow x_0} K(x) = -\psi'(x_0)((1-t)x_0 - (c+T)) > 0,$$

$$K((1-t)/(c+T)) = (1-t)(\psi((1-t)/(c+T)) - \psi(x_0)) > 0,$$

$$K(x^*) = -(1-t)\psi(x_0) < 0,$$

and $K(x) \leq -(1-t)\psi(x_0)$ on (x^*, ∞) since $(1-t)\psi(x) \leq \psi'(x)((1-t)x - (c+T))$ on (x^*, ∞) by the proof of Lemma 2.1. Consequently, equation $K(x) = 0$ has at least one root on (x_0, x^*). In order to establish uniqueness, we first observe that standard differentiation of $K(x)$ yields

$$K'(x) = -\psi''(x)((1-t)x - (c+T)) \gtreqless 0 \quad x \lesseqgtr \frac{c+T}{1-t}$$

since the increasing fundamental solution $\psi(x)$ is strictly convex. The uniqueness of the root of equation $K(x) = 0$ now follows from the continuity and monotonicity of $K(x)$ on $((c+T)/(1-t), \infty)$. It now remains to establish that the proposed value function satisfies the sufficient quasi-variational inequalities

$$\min\{rV(x) - (\mathcal{A}V)(x), V(x) - (1-t)x + (c+T) - V(x_0)\} = 0.$$

To accomplish this task, we first observe that $(\mathcal{A}V)(x) = rV(x)$ on $(0, y^*)$ and that

$$
\begin{aligned}
(\mathcal{A}V)(x) - rV(x) &= (1-t)\mu(x) - r((1-t)x - (c+T)) - r(1-t)\frac{\psi(x_0)}{\psi'(y^*)} \\
&\leq r\left((1-t)\left[\frac{\psi(x)}{\psi'(x)} - \frac{\psi(x_0)}{\psi'(y^*)}\right] - ((1-t)x - (c+T))\right),
\end{aligned}
$$

since $r\psi(x) > \mu(x)\psi'(x)$ by the strict convexity of $\psi(x)$. Consequently, letting $x \downarrow y^*$ yields

$$\lim_{x \downarrow y^*}[(\mathcal{A}V)(x) - rV(x)] \leq r\left((1-t)\frac{\psi(y^*) - \psi(x_0)}{\psi'(y^*)} - ((1-t)y^* - (c+T))\right) = 0.$$

However, the monotonicity of $\rho(x)$ implies that $\theta(x)$ is non-increasing as well and, therefore, that $(\mathcal{A}V)(x) \leq rV(x)$ on (y^*, ∞). Consider now, in turn, the continuously differentiable mapping $\Delta(x) = V(x) - (1-t)x + (c+T) - V(x_0)$. It is now clear that

$$\Delta(x) = \begin{cases} 0 & x \geq y^* \\ (1-t)\frac{(\psi(x) - \psi(x_0))}{\psi'(y^*)} - (1-t)x + (c+T) & x < y^* \end{cases}$$

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and, therefore, that

$$\Delta'(x) = \begin{cases} 0 & x \geq y^* \\ (1-t)\frac{\psi'(x)}{\psi'(y^*)} - (1-t) & x < y^*. \end{cases}$$

The convexity of $\psi(x)$ implies that $\psi'(x)/\psi'(y^*) < 1$ for all $x < y^*$ and, therefore, that $\Delta(x) > 0$ for all $x < y^*$. This demonstrates that $V(x)$ indeed constitutes the value of the optimal ongoing rotation problem (2.2). The monotonicity and convexity of $V(x)$ follow directly from the monotonicity and convexity of $\psi(x)$. Finally, the representation (3.9) follows directly from the definition of the value and the convexity of $\psi(x)$. $\qquad\square$

E   Proof of Proposition 3.2

Proof. Denote as $\tilde{\psi}(x)$ the increasing fundamental solution of the differential equation

$$\frac{1}{2}\tilde{\sigma}^2(x)u''(x) + \mu(x)u'(x) - ru(x) = 0.$$

As was established in Proposition 3.8 of Alvarez 2004a, the fundamental solutions $\psi(x)$ and $\tilde{\psi}(x)$ satisfy for all $x \in (x_0, \infty)$ the inequality

$$\frac{\psi'(x)}{\psi(x) - \psi(x_0)} \leq \frac{\tilde{\psi}'(x)}{\tilde{\psi}(x) - \tilde{\psi}(x_0)}$$

which, in turn, implies that

$$(1-t) - \frac{\psi'(x)}{(\psi(x) - \psi(x_0))}((1-t)x - (c+T)) \geq (1-t) - \frac{\tilde{\psi}'(x)}{(\tilde{\psi}(x) - \tilde{\psi}(x_0))}((1-t)x - (c+T))$$

for all $x \geq \tilde{x}$, where $\tilde{x}$ denotes the break-even state satisfying the condition $(1-t)\tilde{x} = c+T$. Consequently, we observe that

$$(1-t) - \frac{\tilde{\psi}'(y^*)}{(\tilde{\psi}(y^*) - \tilde{\psi}(x_0))}((1-t)y^* - (c+T)) \leq 0$$

which proves that increased volatility increases the optimal harvesting threshold. Establishing that increased volatility increases the value of the harvesting opportunity is analogous with the proof of Corollary 2.2. $\qquad\square$

F   Proof of Proposition 3.3

Proof. As was established in the proof of Proposition 3.1 the optimal exercise threshold y^* satisfies the ordinary first order condition

$$(1-t)(\psi(y^*) - \psi(x_0)) = \psi'(y^*)((1-t)y^* - (c+T))$$

implying that

$$\psi'(y^*)y^* - (\psi(y^*) - \psi(x_0)) = \frac{c+T}{1-t}\psi'(y^*) > 0.$$

As in the proof of Corollary 2.3, define the continuously differentiable mapping $G : (0,1) \times \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$G(t,x) = (1-t)(\psi(x) - \psi(x_0)) - \psi'(x)((1-t)x - (c+T)).$$

Standard differentiation yields

$$
\begin{aligned}
G_t(t,x) &= \psi'(x)x - (\psi(x) - \psi(x_0)) \\
G_x(t,x) &= -\psi''(x)((1-t)x - (c+T)).
\end{aligned}
$$

Therefore, implicit differentiation of the first order condition $G(t,y^*) = 0$ yields

$$
\frac{\partial y^*}{\partial t} = \frac{(c+T)\psi'(y^*)}{(1-t)\psi''(y^*)((1-t)y^* - (c+T))} > 0,
$$

since $\psi''(y^*)((1-t)y^* - (c+T)) > 0$ at the optimal boundary. Establishing that increased revenue taxation decreases the value of the ongoing harvesting opportunity follows directly from the definition of the auxiliary mapping $g(y)$ and the exercise payoff $(1-t)y - (c+T)$. This proves part (A) of our Proposition. Establishing part (B) is completely analogous. In order to establish part (C) we first observe that the value of the ongoing harvesting opportunity at the regeneration state x_0 can be re-expressed as

$$
V_y(x_0) = ((1-t)y - (c+T))\frac{\psi(x_0)/\psi(y)}{1 - \psi(x_0)/\psi(y)} = ((1-t)y - (c+T))\frac{\mathbf{E}_{x_0}\left[e^{-(1-s)r\tau_y}\right]}{1 - \mathbf{E}_{x_0}\left[e^{-(1-s)r\tau_y}\right]}.
$$

Define the mapping $\beta : (0,1) \mapsto \mathbb{R}_+$ as $\beta(s) = \mathbf{E}_{x_0}\left[e^{-(1-s)r\tau_y}\right]$, where $\tau_y = \inf\{t \geq 0 : X_t \geq y\}$. It is now a simple exercise in ordinary differentiation to establish that $\beta'(s) > 0$ and, therefore, that the fraction $\beta(s)/(1 - \beta(s))$ is non-decreasing as a function of the tax rate s. Consequently, we find that increased taxation of interest income increases the value of the ongoing harvesting opportunity at the generic initial state x_0. However, since

$$
V_y(x) = ((1-t)y - (c+T) + V_y(x_0))\mathbf{E}_x\left[e^{-(1-s)r\tau_y}\right]
$$

we find that increased taxation of interest income increases the value of the ongoing harvesting opportunity at all states. It remains to establish that increased taxation of interest income increases the optimal rotation threshold. To see that this is indeed the case, we first denote the increasing fundamental solution of the ordinary differential equation $(\mathcal{A}u)(x) = (1-s)ru(x)$ as $\psi_s(x)$. It is now clear from our analysis above that if $s_1 > s_2$ then $\psi_{s_1}(x)/\psi_{s_1}(y) \geq \psi_{s_2}(x)/\psi_{s_2}(y)$ for all $x \leq y$. Thus, we find that the mapping $U : (x_0, y) \mapsto \mathbb{R}_+$ defined as

$$
U(x) = \frac{\psi_{s_1}(x)}{\psi_{s_1}(y)} - \frac{\psi_{s_1}(x) - \psi_{s_1}(x_0)}{\psi_{s_1}(y) - \psi_{s_1}(x_0)}
$$

is non-negative, decreasing and satisfies the condition $\lim_{x \uparrow y} U(x) = 0$. Hence, we find that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $U(x) < \varepsilon$ for all $x \in (y - \delta, y)$. Therefore, we observe that for all $x \in (y - \delta, y)$ we have

$$
\frac{\psi_{s_1}(x) - \psi_{s_1}(x_0)}{\psi_{s_1}(y) - \psi_{s_1}(x_0)} > \frac{\psi_{s_1}(x)}{\psi_{s_1}(y)} - \varepsilon \geq \frac{\psi_{s_2}(x)}{\psi_{s_2}(y)} - \varepsilon \geq \frac{\psi_{s_2}(x) - \psi_{s_2}(x_0)}{\psi_{s_2}(y) - \psi_{s_2}(x_0)} - \varepsilon,
$$

which finally implies that

$$
\frac{\psi'_{s_1}(y)}{\psi_{s_1}(y) - \psi_{s_1}(x_0)} \leq \frac{\psi'_{s_2}(y)}{\psi_{s_2}(y) - \psi_{s_2}(x_0)}
$$

for all $y \in (x_0, \infty)$. Define the continuously differentiable mapping $\tilde{G} : (0,1) \times \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$
\tilde{G}(s,x) = (1-t)(\psi_s(x) - \psi_s(x_0)) - \psi'_s(x)((1-t)x - (c+T)).
$$

It is then clear that above the break-even level $(c+T)/(1-t)$ we have

$$
\tilde{G}(s_1, x) \geq \frac{(\psi_{s_1}(x) - \psi_{s_1}(x_0))}{(\psi_{s_2}(x) - \psi_{s_2}(x_0))}\tilde{G}(s_2, x)
$$

from which the alleged result follows. $\qquad\square$

G Proof of Proposition 4.1

Proof. Applying Dynkin's proposition to the mapping $x \mapsto U(\pi(x))$ yields

$$\mathbf{E}_x \left[e^{-r\tau^*} U(\pi(X_{\tau^*})) \right] = U(\pi(x)) + \mathbf{E}_x \int_0^{\tau^*} e^{-rs} \Lambda(X_s) ds, \qquad (G.1)$$

where $\tau^* = \inf\{t \geq 0 : X_t \notin (a,b)\}$ denotes the first exit time of the stand value diffusion X_t from the open set (a,b), where $0 < a < b < \infty$. As is known from the literature on linear diffusions, the expected value appearing on the left hand side of (G.1) satisfies the ordinary second order differential equation $(\mathcal{A}u)(x) = ru(x)$ subject to the boundary conditions $u(a) = U(\pi(a))$ and $u(b) = U(\pi(b))$ (cf. Karlin and Taylor 1981, pp. 191–204). Hence, we find that left hand side of (G.1) can be re-expressed as

$$\mathbf{E}_x \left[e^{-r\tau^*} U(\pi(X_{\tau^*})) \right] = U(\pi(a)) \frac{\hat{\varphi}(x)}{\hat{\varphi}(a)} + U(\pi(b)) \frac{\hat{\psi}(x)}{\hat{\psi}(b)},$$

where $\hat{\psi}(x) = \psi(x) - \psi(a)\varphi(x)/\varphi(a)$ denotes the increasing and $\hat{\varphi}(x) = \varphi(x) - \psi(x)\varphi(b)/\psi(b)$ the decreasing fundamental solutions of the ordinary second order differential equation $(\mathcal{A}u)(x) = ru(x)$ subject to the boundary conditions $\hat{\psi}(a) = 0$ and $\hat{\varphi}(b) = 0$ (given up to a multiplicative constant). Similarly, the expected cumulative term on the right-hand side of equation (G.1) satisfies the ordinary second order differential equation $(\mathcal{A}u)(x) - ru(x) + \Lambda(x) = 0$ subject to the boundary conditions $u(a) = u(b) = 0$. Consequently, we find that it can be re-expressed as

$$\mathbf{E}_x \int_0^{\tau^*} e^{-rs} \Lambda(X_s) ds = \hat{B}^{-1} \hat{\varphi}(x) \int_a^x \hat{\psi}(x) \Lambda(y) m'(y) dy + \hat{B}^{-1} \hat{\psi}(x) \int_x^b \hat{\varphi}(x) \Lambda(y) m'(y) dy,$$

where

$$\hat{B} = \left(1 - \frac{\psi(a)\varphi(b)}{\varphi(a)\psi(b)} \right) B$$

denotes the constant Wronskian of the solutions $\hat{\psi}(x)$ and $\hat{\varphi}(x)$. Combining these findings then imply that

$$\frac{d}{dx} \left[\frac{U(\pi(x))}{\hat{\psi}(x)} \right] = \frac{S'(x)}{\hat{\psi}^2(x)} \left[\int_a^x \hat{\psi}(y) \Lambda(y) m'(y) dy - \frac{B U(\pi(a))}{\varphi(a)} \right],$$

which can be re-expressed as

$$\frac{(1-t)U'(\pi(x))}{S'(x)} \hat{\psi}(x) - \frac{\hat{\psi}'(x)}{S'(x)} U(\pi(x)) = \int_a^x \hat{\psi}(y) \Lambda(y) m'(y) dy - \frac{B U(\pi(a))}{\varphi(a)}.$$

Letting $a \downarrow 0$, invoking the unattainability of 0 for the underlying stand value diffusion, and noticing that $\lim_{a \downarrow 0} \hat{\psi}(x) = \psi(x)$ then yields

$$\frac{(1-t)U'(\pi(x))}{S'(x)} \psi(x) - \frac{\psi'(x)}{S'(x)} U(\pi(x)) = \int_0^x \psi(y) \Lambda(y) m'(y) dy,$$

since $\Lambda \in \mathcal{L}^1(\mathbb{R}_+)$. Establishing now that the value of the optimal rotation policy reads as in (4.3) and that the optimal stopping boundary

$$x^* = \operatorname{argmax} \left\{ \frac{U(\pi(x))}{\psi(x)} \right\}$$

exists and is unique is then completely analogous with the proof of Lemma 2.1. $\qquad \square$

H Proof of Proposition 4.2

Proof. Consider the mapping $F_\sigma : \mathbb{R}_+ \mapsto \mathbb{R}$ defined as

$$F_\sigma(x) = (1-t)U'(\pi(x))\frac{\psi(x)}{\psi'(x)} - U(\pi(x)).$$

We observe that $F_\sigma(x^*) = 0$ and that

$$F_\sigma'(x) = (1-t)^2 U''(\pi(x))\frac{\psi(x)}{\psi'(x)} - (1-t)U'(\pi(x))\frac{\psi(x)\psi''(x)}{\psi'^2(x)} < 0,$$

since the increasing fundamental solution $\psi(x)$ is convex by the monotonicity of the appreciation rate. Denote as $\tilde{\psi}(x)$ the increasing fundamental solution associated to the less volatile stand value dynamics characterized by the volatility coefficient $\tilde{\sigma}(x)$ satisfying the condition $\tilde{\sigma}(x) \leq \sigma(x)$ for all $x \in \mathbb{R}_+$. As in the proof of Proposition 3.2 we observe that $\psi'(x)/\psi(x) \leq \tilde{\psi}'(x)/\tilde{\psi}(x)$ implying that

$$F_\sigma(x) \geq (1-t)U'(\pi(x))\frac{\tilde{\psi}(x)}{\tilde{\psi}'(x)} - U(\pi(x)) = F_{\tilde{\sigma}}(x)$$

which proves that increased volatility increases the optimal harvesting threshold. $\square$

I Proof of Proposition 4.5

Proof. We first establish that given the conditions of our Proposition, the mapping

$$H(x) = (1-t)U'(\pi(x))(\psi(x) - \psi(x_0)) - U(\pi(x))\psi'(x)$$

has a unique root on $(\tilde{x}, x^*)$. It is now clear that our assumptions imply that

$$\lim_{x\downarrow\tilde{x}} H(x) = (1-t)U'(0+)(\psi(\tilde{x}+) - \psi(x_0)) - U(0+)\psi'(\tilde{x}+) > 0,$$

and that

$$H(x) < -(1-t)U'(\pi(x))\psi(x_0) < 0$$

for all $x \geq x^*$. Hence $H(x)$ has at least one root $y^* \in (\tilde{x}, x^*)$. To demonstrate that y^* is unique, we first observe by applying the proof of Proposition 4.1 that

$$\frac{d}{dx}\left[\frac{(1-t)U'(\pi(x))}{\psi'(x)}\right] = \frac{2rS'(x)}{\sigma^2(x)\psi'^2(x)}\int_0^x \psi(y)(\Lambda(x) - \Lambda(y))m'(y)dy \leq 0$$

proving that $(1-t)U'(\pi(x))/\psi'(x)$ is non-increasing on $\mathbb{R}_+$. Since

$$H(x) = \psi'(x)\left[\frac{(1-t)U'(\pi(x))}{\psi'(x)}(\psi(x) - \psi(x_0)) - U(\pi(x))\right]$$

we finally find that y^* is unique. $\square$

J Proof of Proposition 4.6

Proof. The proof is analogous with the proof of Proposition 3.2 after re-expressing the optimality condition as

$$(1-t)U'(\pi(y^*))\frac{(\psi(y^*) - \psi(x_0))}{\psi'(y^*)} = U(\pi(y^*)).$$

$\square$

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