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WICKSELLIAN THEORY OF FOREST ROTATION UNDER INTEREST RATE VARIABILITY

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Abstract

The current literature on optimal forest rotation makes the assumption of constant interest rate. However, the irreversible harvesting decisions of forest stands are typically subject to relatively long time horizons over which interest rates do fluctuate considerably. In this paper we apply the Wicksellian single rotation framework to extend the existing studies to cover the unexplored case of variable interest rate. Given the technical generality of the considered valuation problem, we provide a thorough mathematical characterization of the optimal timing problem and develop new results. We show that even in the deterministic case if the current interest rate deviates from its long-run steady state, interest rate variability changes the rotation age significantly when compared with the constant discounting case. Further, and importantly, allowing for interest rate uncertainty is shown to increase the optimal rotation period when the value of the optimal policy is a convex function of the current interest rate. In line with this finding, we also establish that increased interest rate volatility has a positive impact on the optimal rotation period.

JEL Classification: Q23, G31, C61.

Keywords: Wicksellian rotation, variable interest rates, linear diffusions, optimal stopping, free boundary problems.

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1 Introduction

In forest economics the well-known model by Faustmann 1849 has been the most often used starting point in studies considering the optimal rotation period of forest stands. Under the assumption of constant timber prices, constant total cost of clear-cutting and replanting as well as constant interest rate, perfect capital markets and perfect foresight the model leads to a constant rotation period for an even aged stand, which maximizes the present value of forest stand over an infinite time horizon (see e.g. Clark 1976, Johannsson and Löfgren 1985 and Samuelson 1976). The representative rotation age depends on timber price, total cost of clear-cutting and replanting, nature of forest growth as well as the interest rate.

The basic assumptions and predictions of the Faustmann model do not seem to lie in conformity with empirical evidence (see e.g. Kuuluvainen and Tahvonen 1999). This has led to ongoing research, which has extended the basic Faustmann model under perfect foresight to allow for amenity valuation of timber (see e.g. Hartman 1976), the potential interdependence of forest stands as producers of amenity services (see e.g. Swallow and Wear 1993 and Koskela and Ollikainen, 2000, 2001) as well as imperfect capital markets (see e.g. Tahvonen and Salo and Kuuluvainen 2001). The resulting rotation age has been shown to depend on the properties of amenity valuation function, the nature of stand interdependencies and potential borrowing constraints in the capital markets. In particular, in the latter case all the basic properties of optimal forest harvesting become different than the ones in the classical Faustmann model.

Finally, the perfect foresight assumption has been relaxed in studies focusing on the implications of stochastic timber prices (see e.g. Brazee and Mendelsohn 1988, Thomson 1992 and Plantinga 1998), risk of forest fire (see e.g. Reed 1984) and/or stochastic forest growth on optimal rotation age (see e.g. Reed 1993, Miller and Voltaire 1983, Morck and Schwartz and Stangeland 1989, Clarke and Reed 1989, 1990, Willassen 1998 and Alvarez 2001 b). The effect of uncertainties on the optimal rotation period depends on the type of uncertainty. In the case of forest fire risk modelled as a Poisson process the rotation age will become shorter due to the higher effective discount rate (see Reed 1984) while in the presence of timber price and/or forest growth risk usually the reverse happens; higher risk in price or in age-dependent growth will tend to lengthen the rotation period by lowering the effective discount rate (see e.g. Clarke and Reed 1989, Willassen 1998 and Alvarez 2001 b).

This rotation literature has covered several interesting cases and provided useful insights. There is, however, a very important issue, which has not yet been analyzed. To our knowledge in all the research associated with optimal rotation periods of forest stands the assumption of constant interest rate has been sticked to. As we know from empirical research, interest rates fluctuate a lot over time and the implications of this empirical finding for the term structure of interest rates, asset pricing etc. have been one of the major research areas in financial economics (for an up-to-date theoretical and empirical survey in the field Cochrane 2001; see also Björk 1998 for an extensive treatment of interest rate modelling). If the investment projects would be very liquid ones, then interest rate fluctuations would not necessarily matter very much. In the case of forestry, however, the situation is different. Given the relatively slow growth rate of forests, investing in replanting is a long-term investment project, over which the expected behavior of the interest rate as the opportunity cost will matter a lot. Similarly, since many real investments are productive over a considerably long time period, we are tempted to argue that the variability of interest rates should play a key role in the rational valuation and exercise policies of real irreversible investment opportunities as well. In an accompanying paper we study this issue (Alvarez and Koskela 2001). In this paper we study the unexplored issue of what is the impact of variable interest rate on optimal forest rotation and compare the results with those obtained by using the standard, though somewhat unrealistic, assumption of constant interest rate. Ingersoll and Ross 1992 have analyzed the effect of interest rate uncertainty on the timing of investment but under the assumption of zero-expected change in the interest rate. They show that the higher is the interest rate uncertainty, the longer the economy is willing to wait to invest before investing in a given project and, ceteris paribus, the less investment there will be.

We proceed as follows: In section 2 we present a framework to study the Wicksellian single rotation problem in the thus far unexplored situation of interest rate variability. Since the problem is more general than the constant discounting case, we first provide a thorough mathematical characterization of the optimal rotation policy and its value. More precisely, we state a set of sufficient conditions under which the considered optimal rotation problem admits a unique solution and under which the value of the optimal policy can be obtained from an associated boundary value problem expressed as a first order linear partial differential equation subject to the standard value matching and smooth fit (or smooth pasting) conditions. From an economic point of view we show that interest rate variability will change the rotation age compared with the constant discounting case in a way, which depends on the relationship between the current and the long run steady state interest rate. Section 3 provides an extension of the basic model, where the flow of returns consists not only of the revenues accrued from harvesting, but also of the flow of returns accrued from leaving the harvesting opportunity unexercised (recreational value of a forest a la' Hartman 1976). In section 4 we illustrate our qualitative findings by using numerical computations with logistic functions in the situations where the current interest rate is below or above the long-run steady state interest rate. Having considered the deterministic case, we generalize our analysis in section 5 to study the impact of interest rate uncertainty on optimal forest rotation and establish among others that allowing for interest rate uncertainty will increase the optimal rotation period under the natural condition when the value of the optimal policy is convex in terms of the current interest rate in the absence of uncertainty. Finally, we also establish that under a set of very general assumptions increased interest rate volatility will increase the value of the optimal policy and move the exercise date further, i.e. prolongs the rotation period. Section 6 provides some concluding remarks.

2 The Wicksellian Rotation Problem with Variable Interest Rate

In this section we formulate the Wicksellian rotation problem in more general terms than usually by allowing interest rate variability. We proceed as follows. First we provide a set of sufficient conditions under which the optimal rotation problem admits a unique solution and under which the value of optimal policy can be obtained from an associated boundary value problem expressed as a first order linear partial differential equation subject to the standard value matching and smooth fit (or smooth pasting) conditions. Then we study the relationship between the rotation periods under variable and constant discounting and finally we provide conditions under which the value of the optimal rotation policy is a decreasing and convex function of the current interest rate.

The underlying dynamics for the forest value X_t and interest rate r_t are described as

$$X_t' = \mu(X_t), \quad X_0 = x \tag{2.1}$$

and

$$r_t' = \alpha(r_t), \quad r_0 = r \tag{2.2}$$

where the mappings $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}$ are assumed to be continuously differentiable with Lipschitz-continuous derivative on $\mathbb{R}_+$. In order to capture the economically sensible models of the optimal rotation problem, we assume that there is a $\hat{r} > 0$ such that $\alpha(r) \gtreqless 0$, when $r \lesseqgtr \hat{r}$, that $\lim_{r\downarrow 0} \alpha(r) = 0$, that $\lim_{x\downarrow 0} \mu(x) = 0$ and that there is a $\hat{x} > 0$ such that $\mu(x) \gtreqless 0$, when $x \lesseqgtr \hat{x}$. In other words, we assume that the origin is an unstable equilibrium point for the two dimensional process (X_t, r_t) and that (X_t, r_t) tends towards the asymptotically stable long run steady state $(\hat{x}, \hat{r})$ for any possible interior initial state $(x, r) \in \mathbb{R}_+^2$. It is worth pointing out that although the interest rate dynamics are now defined on $\mathbb{R}_+$ our subsequent results are valid for interest rate processes defined on the entire $\mathbb{R}$ as well (for example, *flexible accelerator dynamics*). However, from an economical point of view only the non-negative interest rates are of interest and, therefore, we shall concentrate solely on that case. It is also worth observing that our analysis includes the purely compensatory case (i.e. dynamics subject to a decreasing percentage growth rate) appearing frequently in models considering the rational management of renewable resources. As usually, we denote as

$$\mathcal{A} = \mu(x)\frac{\partial}{\partial x} + \alpha(r)\frac{\partial}{\partial r} \tag{2.3}$$

the differential operator associated with the intertemporally time-homogeneous process (X_t, r_t).

Given the underlying dynamics, we plan to consider in this study the *Wicksellian single rotation problem*

$$V(x, r) = \sup_{t\geq 0} \left[e^{-\int_0^t r_s ds} g(X_t) \right], \tag{2.4}$$

where $g : \mathbb{R}_+ \mapsto \mathbb{R}$ is a twice continuously differentiable, non-decreasing, and concave mapping (i.e. $g \in C^2(\mathbb{R}_+)$, $g'(x) \geq 0$, and $g''(x) \leq 0$ for all $x \in \mathbb{R}_+$) denoting the *payoff accrued from exercising the irreversible harvesting opportunity* and satisfying the boundary condition $g(\hat{x}) > 0$ (implying that $g(x) \geq g(\hat{x}) > 0$ for all $x > \hat{x}$). It is now a simple exercise in ordinary analysis to demonstrate that given our smoothness assumptions, the optimal rotation problem (2.4) can be restated as (see, for example, Øksendal 1998, p. 199 and Protter 1990, p. 71)

$$V(x, r) = g(x) + \sup_{t\geq 0} \int_0^t e^{-\int_0^s r_y dy} \left[g'(X_s)\mu(X_s) - r_s g(X_s) \right] ds, \tag{2.5}$$

where the integral term

$$F(x,r)=\sup_{t\geq 0}\int_0^t e^{-\int_0^s r_y dy}\left[g'(X_s)\mu(X_s)-r_s g(X_s)\right]ds$$

constitutes the *early exercise premium accrued from undertaking optimally the irreversible policy prior expiration.* Before proceeding in our analysis, we now present the following auxiliary verification lemma which can be applied for solving either the optimal stopping problem (2.4) or its equivalence (2.5) (cf. Øksendal 1998, pp. 214–217 and Øksendal and Reikvam 1998).

Lemma 2.1. *Assume that there is a continuously differentiable mapping $J:\mathbb{R}_+^2\mapsto\mathbb{R}_+$ (i.e. $J\in C^1(\mathbb{R}_+^2)$) satisfying the variational inequalities*

$$\min\{J(x,r)-g(x), rJ(x,r)-(\mathcal{A}J)(x,r)\}=0$$

for all $(x,r)\in\mathbb{R}_+^2$. Then, $J(x,r)\geq V(x,r)$ for all $(x,r)\in\mathbb{R}_+^2$. Moreover, if there is a continuously differentiable mapping $W:\mathbb{R}_+^2\mapsto\mathbb{R}_+$ (i.e. $W\in C^1(\mathbb{R}_+^2)$) satisfying for all $(x,r)\in\mathbb{R}_+^2$ the variational inequalities

$$\min\{W(x,r), rW(x,r)-(\mathcal{A}W)(x,r)-(g'(x)\mu(x)-rg(x))\}=0,$$

then $W(x,r)\geq F(x,r)$ and, therefore, $W(x,r)+g(x)\geq V(x,r)$ for all $(x,r)\in\mathbb{R}_+^2$.

Proof. See Appendix A. □

Lemma 2.1 states a set of sufficient conditions (in terms of variational inequalities) which can be applied for deriving a majorant for the value of the optimal rotation problem (2.4). Lemma 2.1 also establishes a set of sufficient conditions which can be applied for deriving the explicit form of the early exercise premium $F(x,r)$. As usually in optimal stopping theory, we denote the *continuation region (i.e. the waiting or do-nothing region) where exercising the harvesting opportunity is suboptimal* as $C=\{(x,r)\in\mathbb{R}_+^2 : V(x,r)>g(x)\}$ and the *stopping region (i.e. the immediate exercise region, cutting region)* as $\Gamma=\{(x,r)\in\mathbb{R}_+^2 : V(x,r)=g(x)\}$. It is clear that the set $g^{-1}(\mathbb{R}_-)=\{x\in\mathbb{R}_+ : g(x)<0\}$ is a subset of the continuation region C since the decision maker can always attain at least a non-negative payoff by waiting up to the first moment when X_t arrives to the set where the payoff $g(x)$ is positive. Assume now that $(x,r)\in\bar{C}=\{(x,r)\in\mathbb{R}_+^2 : g'(x)\mu(x)>rg(x)\}$ and define the stopping date $\tau=\inf\{t\geq 0 : (X_t,r_t)\notin\bar{C}\}$. Then,

$$V(x,r)\ \geq\ e^{-\int_0^\tau r_s ds}g(X_\tau)=g(x)+\int_0^\tau e^{-\int_0^s r_y dy}\left[g'(X_s)\mu(X_s)-r_s g(X_s)\right]ds>g(x)$$

implying that $\bar{C}\subseteq C$. It is clear from (2.5) that if there is a finite date $t^*\in\mathbb{R}_+$ at which the opportunity is exercised, then we necessarily have that $g'(X_{t^*})\mu(X_{t^*})=r_{t^*}g(X_{t^*})$. On the other hand, t^* can be a maximum only if also the second order local sufficiency condition $\mu(X_{t^*})[g''(X_{t^*})\mu(X_{t^*})+g'(X_{t^*})(\mu'(X_{t^*})-r_{t^*})]<\alpha(r_{t^*})g(X_{t^*})$ is met. Consequently, we define the boundary curve of the continuation region (implicitly) as $\{(x,r) \in \mathbb{R}^2_+ : g'(x)\mu(x) = rg(x)\}$. Implicit differentiation then yields that along the boundary we have

$$\frac{dx}{dr} = \frac{g''(x)\mu(x) - g'(x)(r - \mu'(x))}{g(x)}$$

provided that $g(x) > 0$. Therefore, if the payoff $g(x)$ is non-decreasing and concave and $r > \mu'(x)$, then the boundary at which rotation is optimal is a decreasing mapping of the current rate of interest. Our main result characterizing the optimal rotation policy and its value for a broad class of problems is now summarized in the following.

Theorem 2.2. *Assume that* $D = (\hat{x}, \infty) \times (\hat{r}, \infty) = \{(x,r) \in \mathbb{R}^2_+ : \mu(x) < 0, \alpha(r) < 0\} \subset \{(x,r) \in \mathbb{R}^2_+ : g'(x)\mu(x) < rg(x)\}$ *and that* $\mu(x)[g''(x)\mu(x) - g'(x)(r - \mu'(x))] - \alpha(r)g(x) < 0$ *for all* $(x,r) \in \mathbb{R}^2_+ \backslash D$. *Then, the optimal rotation date* $t^* = \inf\{t \geq 0 : (X_t, r_t) \notin C\} < \infty$ *is for any* $(x,r) \in C$ *the root of the equation*

$$g'(X_{t^*})\mu(X_{t^*}) = r_{t^*}g(X_{t^*}).$$

Moreover, the value satisfies the boundary value problem

$$\begin{aligned} &\mu(x)\frac{\partial V}{\partial x}(x,r) + \alpha(r)\frac{\partial V}{\partial r}(x,r) - rV(x,r) = 0, \quad (x,r) \in C \\ &V(x,r) = g(x), \quad \frac{\partial V}{\partial x}(x,r) = g'(x), \quad \frac{\partial V}{\partial r}(x,r) = 0, (x,r) \in \partial C. \end{aligned} \tag{2.6}$$

Proof. See Appendix B. □

Theorem 2.2 provides a set of sufficient conditions under which the optimal rotation problem admits a unique solution and under which the value of optimal policy can be determined from an associated boundary value problem. Figure 1 describes the phase

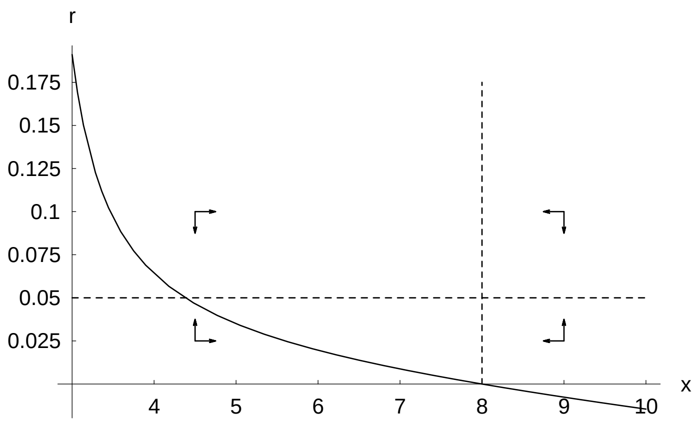

Figure 1: The Phase Diagram of the Controlled System

diagram of the controlled system under the assumptions of our Theorem 2.2. As was required in Theorem 2.2 the equilibrium state $(\hat{x}, \hat{r})$ of the controlled system has to be in the set where present value of the forest is decreasing over time, that is, above the boundary of the continuation region C. If that condition is met, then the two-dimensional system (X_t, r_t) will tend towards the stopping region from any initial state in the do-nothing region C and hit its boundary in finite time.

Given that important characterization we are next interested in the relationship between the exercise dates (i.e. the rotation periods) with variable and constant discounting under the plausible assumption that there exists the long-run steady state interest rate. In the deterministic case the answer is summarized in the following

Theorem 2.3. *Assume that the conditions of Theorem 2.2 are met. Then, the rotation period in the presence of variable discounting is shorter (longer) than the rotation period in the presence of constant discounting if* $r < \hat{r}$ $(r > \hat{r})$.

Proof. It is clear that in the absence of interest rate variability (i.e. when $r_t' = 0$ for all $t \geq 0$ and $r_0 = r$) the optimal rotation period $\tilde{t}$ satisfies the optimality condition $g'(X_{\tilde{t}})\mu(X_{\tilde{t}}) = rg(X_{\tilde{t}})$. Consequently, we find that

$$f(\tilde{t}) = g'(X_{\tilde{t}})\mu(X_{\tilde{t}}) - r_{\tilde{t}}g(X_{\tilde{t}}) = (r - r_{\tilde{t}})g(X_{\tilde{t}}) \gtreqqless 0 \quad r \gtreqqless \hat{r},$$

since

$$r_t \gtreqqless r \quad \forall t \geq 0 \quad \text{whenever} \quad r \lesseqqgtr \hat{r}.$$

Therefore, we find that $\tilde{t} \gtreqqless t^*$ whenever $r \lesseqqgtr \hat{r}$. □

According to Theorem 2.3 the rotation period with variable discounting falls short of the one with constant discounting when the current interest rate is known with certainty to increase over time. This is natural because in that case the opportunity costs of not harvesting increases over time. Naturally, the reverse happens in the case of falling interest rate when the opportunity cost of not harvesting goes down.

Later on in section 5 we analyze the determination of optimal rotation period in the presence of interest rate uncertainty and volatility. In this context the properties of the value of the optimal rotation policy in terms of the current interest rate turns out to be crucial. The next theorem provides a useful characterization from this point of view.

Theorem 2.4. *If* $\alpha(r)$ *is concave then the value* $V(x, r)$ *of the optimal rotation policy is decreasing and convex as a function of the current interest rate* r.

Proof. As was shown in the proof of Theorem 2.2 (cf. Alvarez 2001)

$$\frac{\partial r_t}{\partial r} = e^{\int_0^t \alpha'(r_s(r))ds} > 0,$$

proving the intuitively clear result that the interest rate process is an increasing function of the current state r. Moreover, if $r > \rho$ then $\alpha'(r_s(r)) < \alpha'(r_s(\rho))$ for all $s \in \mathbb{R}_+$ due to the monotonicity and uniqueness of the path $r_s(r)$ and the concavity of $\alpha(r)$. Therefore, we find that $r_t(r)$ is concave as a function of the current interest rate r. This, in turn, implies that the discount factor $e^{-\int_0^t r_s(r)ds}$ is a decreasing and convex function of the current interest rate r. Moreover, since the opportunity is exercised only when $g(x) > 0$, we may assume without loss of generality that $g(x) \geq 0$ and, therefore, that $e^{-\int_0^t r_s(r)ds}g(X_t) \geq 0$ (cf. Alvarez 2001 a, c). Therefore, since the maximum of a decreasing and convex mapping is convex and decreasing, we immediately find that $V(x, r)$ is convex and decreasing as a function of the current interest rate r, thus proving the alleged claim. □

3 Amenity Valuation and Forest Rotation

In the model considered in the previous section the forest owner was assumed to accrue revenue only when the irreversible harvesting opportunity was exercised. But it has been argued (see e.g. Hartman 1976) that forests provide not only harvest revenues, but also various types of amenity services under the circumstances when the harvesting opportunity is left unexercised. The purpose of this section is threefold: first, we state the auxiliary verification lemma, which can be applied for solving the corresponding optimal stopping problem with amenity valuation. Second, we present a set of sufficient conditions for the existence and uniqueness of the optimal policy and finally, we compare the optimal rotation periods with variable and constant discounting.

We consider now the following optimal rotation problem with amenity valuation

$$V(x,r) = \sup_{t\geq 0}\left[\int_0^t e^{-\int_0^s r_y dy}\pi(X_s)ds + e^{-\int_0^t r_s ds}g(X_t)\right], \tag{3.1}$$

where the mapping $\pi : \mathbb{R}_+ \mapsto \mathbb{R}$ measuring the *flow of returns accrued from leaving the harvesting opportunity unexercised* is assumed to be non-increasing, continuously differentiable and to satisfy the condition $\pi(\hat{x}) < 0$. It is clear in light of the analysis of the previous section that the optimal rotation problem (3.1) can be rewritten as

$$V(x,r) = g(x) + \sup_{t\geq 0}\int_0^t e^{-\int_0^s r_y dy}[\pi(X_s) + g'(X_s)\mu(X_s) - r_s g(X_s)]ds, \tag{3.2}$$

where the integral term

$$\tilde{F}(x,r) = \sup_{t\geq 0}\int_0^t e^{-\int_0^s r_y dy}[\pi(X_s) + g'(X_s)\mu(X_s) - r_s g(X_s)]ds$$

constitutes the early exercise premium accrued from undertaking optimally the irreversible policy prior expiration. In line with the results of our Lemma 2.1, we now can prove the following verification lemma (cf. Øksendal 1998, pp. 214–217 and Øksendal and Reikvam 1998).

Lemma 3.1. *Assume that there is a continuously differentiable mapping $\hat{J} : \mathbb{R}^2_+ \mapsto \mathbb{R}_+$ (i.e. $\hat{J} \in C^1(\mathbb{R}^2_+)$) satisfying the variational inequalities*

$$\min\{\hat{J}(x,r) - g(x), r\hat{J}(x,r) - (\mathcal{A}\hat{J})(x,r) - \pi(x)\} = 0$$

for all $(x,r) \in \mathbb{R}^2_+$. Then, $\hat{J}(x,r) \geq V(x,r)$ for all $(x,r) \in \mathbb{R}^2_+$. Moreover, if there is a continuously differentiable mapping $\hat{W} : \mathbb{R}^2_+ \mapsto \mathbb{R}_+$ (i.e. $\hat{W} \in C^1(\mathbb{R}^2_+)$) satisfying for all $(x,r) \in \mathbb{R}^2_+$ the variational inequalities

$$\min\{\hat{W}(x,r), r\hat{W}(x,r) - (\mathcal{A}\hat{W})(x,r) - (\pi(x) + g'(x)\mu(x) - rg(x))\} = 0,$$

then $\hat{W}(x,r) \geq F(x,r)$ and, therefore, $\hat{W}(x,r) + g(x) \geq V(x,r)$ for all $(x,r) \in \mathbb{R}^2_+$.

Proof. The proof is analogous with the proof of Lemma 2.1. □

In line with our results in the previous section, we are now in a position to state a set of sufficient conditions under which the value of optimal policy can be determined from an associated boundary value problem and under which the optimal problem admits a unique solution in the presence of amenity valuation. More precisely, we can now establish the following.

Theorem 3.2. *Assume that* $D = (\hat{x}, \infty) \times (\hat{r}, \infty) = \{(x, r) \in \mathbb{R}^2_+ : \mu(x) < 0, \alpha(r) < 0\} \subset \{(x, r) \in \mathbb{R}^2_+ : \pi(x) + g'(x)\mu(x) < rg(x)\}$ *and that* $\mu(x)[\pi'(x) + g''(x)\mu(x) - g'(x)(r - \mu'(x))] - \alpha(r)g(x) < 0$ *for all* $(x, r) \in \mathbb{R}^2_+ \backslash D$*. Then, for any* $(x, r) \in C$ *the optimal rotation date* $t^* = \inf\{t \geq 0 : (X_t, r_t) \notin C\} < \infty$ *is the root of the equation*

$$g'(X_{t^*})\mu(X_{t^*}) + \pi(X_{t^*}) = r_{t^*}g(X_{t^*}).$$

Moreover, the value of the policy satisfies the boundary value problem

$$\mu(x)\frac{\partial V}{\partial x}(x, r) + \alpha(r)\frac{\partial V}{\partial r}(x, r) - rV(x, r) + \pi(x) = 0 \quad (x, r) \in C$$
$$V(x, r) = g(x), \quad \frac{\partial V}{\partial x}(x, r) = g'(x), \quad \frac{\partial V}{\partial r}(x, r) = 0, \quad (x, r) \in \partial C.$$

Proof. The proof is analogous with the proof of Theorem 2.2. □

Theorem 3.2 extends the results of Theorem 2.2 to the Wicksellian single rotation problem in the presence of amenity valuation. The optimal rotation period is longer with than without amenity valuation Again, we observe that essentially sufficiency is guaranteed provided that the steady states of the forest value and the interest rate process belongs to the set where the present value of the payoff of the rotation strategy is negative and a growth rate condition is met. As intuitively is clear, we are now in position to extend the conclusions of our Theorem 2.3 also to the present case as is demonstrated in the following.

Theorem 3.3. *Assume that the conditions of Theorem 3.2 are met. Then, the rotation period in the presence of variable discounting is shorter (longer) than the rotation period in the presence of constant discounting if* $r < \hat{r}$ $(r > \hat{r})$*.*

Proof. The proof is analogous with the proof of Theorem 2.3. □

The economic interpretation of Theorem 3.3. is obvious and similar to the one for Theorem 2.3. Allowing for variable discounting will shorten the rotation period when the current interest rate is known to increase over time. This is because the opportunity cost of not harvesting will become higher and vice versa for the case of falling interest rate when the opportunity costs of not harvesting decreases over time.

4 A Numerical Illustration in the Case of Logistic Growth

In section 2 we proved Theorem 2.3, according to which the rotation period with variable interest rate is shorter (longer) that the one in the presence with constant discounting when the current interest rate is smaller (higher) than its long run steady state value. Now we illustrate this finding quantitatively by using a model based on logistic dynamics of our two dimensional process (X_t, r_t). In other words we now assume that

$$X_t' = \mu X_t(1-\gamma X_t), \quad X_0 = x \tag{4.1}$$

and

$$r_t' = \alpha r_t(1-\beta r_t), \quad r_0 = r, \tag{4.2}$$

where μ, γ, α, and β are exogenously determined non-negative constants. It is now a simple exercise in ordinary analysis to demonstrate that

$$X_t = \frac{xe^{\mu t}}{1+\gamma x(e^{\mu t}-1)}, \quad r_t = \frac{re^{\alpha t}}{1+\beta r(e^{\alpha t}-1)},$$

and

$$e^{-\int_0^t r_s ds} = \left(1+\beta r(e^{\alpha t}-1)\right)^{-1/(\alpha\beta)}.$$

Consequently, if $g(x) = x - c$, then (2.4) reads as

$$V(x,r) = \sup_{t\geq 0}\left[\left(1+\beta r(e^{\alpha t}-1)\right)^{-1/(\alpha\beta)}\left(\frac{xe^{\mu t}}{1+\gamma x(e^{\mu t}-1)} - c\right)\right]$$

Therefore if an optimal rotation date t^* exists, it is implicitly given by the equation

$$\frac{\mu x(1-\gamma x)e^{\mu t^*}}{(1+\gamma x(e^{\mu t^*}-1))^2} = \left(\frac{xe^{\mu t^*}}{1+\gamma x(e^{\mu t^*}-1)} - c\right)\frac{re^{\alpha t^*}}{1+\beta r(e^{\alpha t^*}-1)}.$$

The optimal exercise date t^* is illustrated in Figure 2 under the assumptions that $g(x) = x - c$, $c = 5, \mu = 1\%, \gamma = 2\%, \alpha = 1\%, x = 5$, and $\beta = 20$ (implying that the long run steady state can be defined as $(\hat{x}, \hat{r}) = (50, 5\%)$). Figure 2 illustrates the case when the current interest rate r is below its long-run steady state value $\hat{r}$. The solid line describes the optimal exercise date (the optimal rotation period) in the presence of a variable interest rate while the dotted line the optimal exercise date with constant discounting. The exercise date with variable discounting falls short of the exercise date with constant discounting because the interest rate is known with certainty to increase over time. One can see that the difference between the rotation periods becomes larger with lower interest rates. This is obvious because in the presence of lower current interest rate it will increase much more over time with variable discounting. Therefore, the optimal exercise dates will differ more in that case.

Analogously, Figure 3 illustrates the alternative case where the current interest rate r is above its long-rung steady sate value $\hat{r}$. Again, the solid line describes the optimal exercise date (the optimal rotation period) in the presence of variable interest rate while the dotted line the optimal exercise date with constant discounting. Now contrary to the findings in Figure 2, the exercise date with variable discounting exceeds the one with constant discounting because the interest rate is known to decrease over time. Compared with Figure 2 now the reverse happens also in the sense that the difference between the optimal exercise dates becomes larger with higher interest rates. The reason for that is that in the presence of higher current interest rate it will decrease much more over time with variable discounting.

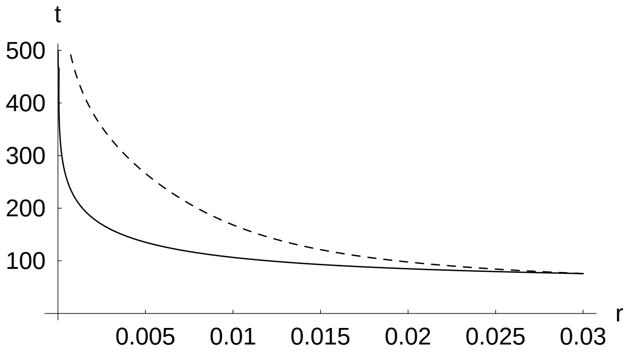

Figure 2: The optimal exercise dates as function of the current interest rate r when $r < \hat{r}$

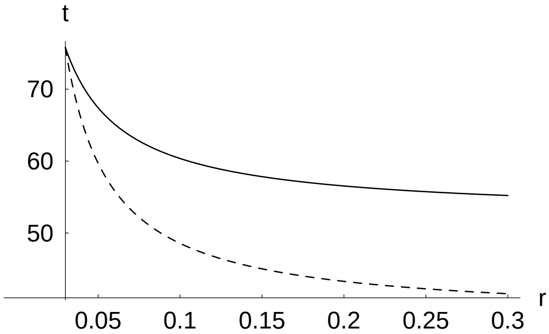

Figure 3: The optimal exercise dates as function of the current interest rate r when $r > \hat{r}$

5 Interest Rate Uncertainty and Forest Rotation

In the analyzes we have carried out thus far, the underlying dynamics for the forest value X_t and the interest rate r_t has been postulated to be deterministic. The reason for this was that in order to present a thorough characterization of the optimal rotation problem we first wanted to consider the impact of variable discounting on the optimal rotation period in the simpler case without uncertainty. Of course, in light of the length of the standard forest rotation decisions the assumption of completely deterministic dynamics is difficult to defend, to say the least. To mention a specific example: do we know the behavior of the interest rates over the next five decades? Certainly the right answer is: We do not know, but we still might have a good knowledge about the stochastic process generating the interest rate fluctuations.

Since we have focused in our paper on the potential role of variable discounting, in this section we generalize our earlier analysis by exploring the optimal rotation problem in the presence of interest rate uncertainty, i.e. when the interest rate process fluctuates stochastically. More precisely, we assume that the interest rate process $\{r_t; t \geq 0\}$ is defined on a complete filtered probability space $(\Omega, P, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ satisfying the usual conditions and that r_t is described on $\mathbb{R}_+$ by the (Itô-) stochastic differential equation

$$dr_t = \alpha(r_t)dt + \sigma(r_t)dW_t, \quad r_0 = r, \tag{5.1}$$

where W_t denotes standard Brownian motion and $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a sufficiently smooth mapping for guaranteeing the existence of a solution for (5.1) (at least continuous; cf. Borodin and Salminen 1996, pp. 46–47). In order to avoid interior singularities, we also assume that $\sigma(r) > 0$ for all $r \in (0, \infty)$, that ∞ is an unattainable boundary for the diffusion r_t (non-explosive paths), and that 0 is either unattainable or exit for r_t (cf. Borodin and Salminen 1996, pp. 14–19). It is now clear that given our assumptions on the underlying dynamics the differential operator associated with the two-dimensional process (X_t, r_t) now reads as

$$\hat{\mathcal{A}} = \frac{1}{2}\sigma^2(r)\frac{\partial^2}{\partial r^2} + \mu(x)\frac{\partial}{\partial x} + \alpha(r)\frac{\partial}{\partial r}.$$

It is worth observing that if both boundaries are unattainable and

$$\int_0^\infty m'(y)dy < \infty,$$

where $m'(r) = \frac{2}{\sigma^2(r)S'(r)}$ denotes the density of the speed measure m of the diffusion r_t and

$$S'(r) = \exp\left(-\int \frac{2\alpha(r)}{\sigma^2(r)}dr\right)$$

denotes the density of the scale function of the diffusion r_t, then we know that the diffusion r_t will tend in the long run towards a random variable distributed according to the stationary distribution with density (cf. Borodin and Salminen 1996, pp. 35–36)

$$p(r) = \frac{m'(r)}{\int_0^\infty m'(y)dy}.$$

Given the stochastic interest rate dynamics (5.1) we next consider in this section the following stochastic rotation problem (an optimal stopping problem)

$$\hat{V}(x, r) = \sup_\tau E_{(x,r)}\left[\int_0^\tau e^{-\int_0^s r_t dt}\pi(X_s)ds + e^{-\int_0^\tau r_s ds}g(X_\tau)\right], \tag{5.2}$$

where τ is an arbitrary $\mathcal{F}_t$-stopping time. As in the deterministic case, we can again restate (5.2) by decomposing the value of the optimal rotation policy into the immediate exercise payoff and the early exercise premium as is indicated by the observation

$$\hat{V}(x, r) = g(x) + \hat{F}(x, r),$$

where

$$\hat{F}(x, r) = \sup_\tau E_{(x,r)}\int_0^\tau e^{-\int_0^t r_s ds}\left[\mu(X_t)g'(X_t) - r_t g(X_t) + \pi(X_t)\right]dt \tag{5.3}$$

denotes the early exercise premium in the presence of interest rate uncertainty. Our first result characterizing the value of the optimal policy and extending the results of Lemma 2.1 to the stochastic interest rate case is summarized in the following.

Lemma 5.1. *Assume that there is a mapping* $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ *satisfying the condition* $J \in C^{1,2}(\mathbb{R}_+^2)$ *and the variational inequalities*

$$\min\{J(x,r) - g(x), rJ(x,r) - (\hat{\mathcal{A}}J)(x,r) - \pi(x)\} = 0$$

for all $(x,r) \in \mathbb{R}_+^2$*. Then,* $J(x,r) \geq \hat{V}(x,r)$ *for all* $(x,r) \in \mathbb{R}_+^2$*. Moreover, if there is a mapping* $W : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ *satisfying the condition* $J \in C^{1,2}(\mathbb{R}_+^2)$ *and the variational inequalities*

$$\min\{W(x,r), rW(x,r) - (\hat{\mathcal{A}}W)(x,r) - (g'(x)\mu(x) - rg(x) + \pi(x))\} = 0,$$

then $W(x,r) \geq \hat{F}(x,r)$ *and, therefore,* $W(x,r) + g(x) \geq \hat{V}(x,r)$ *for all* $(x,r) \in \mathbb{R}_+^2$*.*

Proof. See Appendix C. □

Lemma 5.1 states a set of variational inequalities from which the value of optimal policy can be determined in most of the cases provided that a set of regularity and smoothness conditions are met (cf. Øksendal and Reikvam 1998). It is worth observing that the results of Lemma 5.1 do not characterize the value or the required exercise premium of the irreversible rotation opportunity. Hence, these results do not define the relationship between the values of the optimal policies in the deterministic and stochastic interest rate cases, i.e. they do not characterize the relationship between $V(x,r)$ and $\hat{V}(x,r)$. This relationship is summarized without and with amenity valuation, respectively, in

Theorem 5.2. *(A) Assume that the value* $V(x,r)$ *is convex as a function of the current interest rate* r *in the presence of amenity valuation. Then,*

$$\hat{V}(x,r) \geq V(x,r) \quad \hat{F}(x,r) \geq F(x,r) \quad (x,r) \in \mathbb{R}_+^2.$$

Especially, $C \subset \{(x,r) \in \mathbb{R}_+^2 : \hat{V}(x,r) > g(x)\}$*. That is, uncertainty increases both the required exercise premium and the value of waiting and, therefore, prolongs the optimal rotation period.*
(B) Assume that $\pi(x) \equiv 0$ *and that* $\alpha(r)$ *is concave. Then,*

$$\hat{V}(x,r) \geq V(x,r) \quad \hat{F}(x,r) \geq F(x,r) \quad (x,r) \in \mathbb{R}_+^2.$$

Especially, $C \subset \{(x,r) \in \mathbb{R}_+^2 : \hat{V}(x,r) > g(x)\}$*. That is, uncertainty increases both the required exercise premium and the value of waiting and, therefore, prolongs the optimal rotation period.*

Proof. See Appendix D. □

According to Theorem 5.2 under plausible assumptions uncertainty will increase both the required exercise premium and the value of waiting. Allowing for interest rate uncertainty means that the opportunity cost of not harvesting becomes more uncertain, which will move the exercise date further into the future and, therefore, prolongs the rotation period.

Although this result demonstrates that uncertainty slows down the rational harvesting policy when compared with the certain situation, it does not characterize the impact of increased volatility. To this end, we assume that the interest rate process $\{\hat{r}_t; t \geq 0\}$ is described on $\mathbb{R}_+$ by the (Itô-) stochastic differential equation

$$d\hat{r}_t = \alpha(\hat{r}_t)dt + \hat{\sigma}(\hat{r}_t)dW_t, \quad \hat{r}_0 = r, \tag{5.4}$$

where $\hat{\sigma} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a sufficiently smooth mapping satisfying the inequality $\hat{\sigma}(r) \geq \sigma(r)$. That is, $\hat{r}_t$ can be interpreted as a diffusion evolving at the same rate as r_t but subject to greater stochastic fluctuations than r_t. Given this definition, we define the value $\tilde{V} : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ as

$$\tilde{V}(x,r) = \sup_{\tau} E_{(x,r)} \left[\int_0^{\tau} e^{-\int_0^s \hat{r}_t dt} \pi(X_s) ds + e^{-\int_0^{\tau} \hat{r}_s ds} g(X_{\tau}) \right], \tag{5.5}$$

where τ is an arbitrary $\mathcal{F}_t$-stopping time. Again, we observe that the value can also be expressed as $\tilde{V}(x,r) = g(x) + \tilde{F}(x,r)$ where the early exercise premium $\tilde{F}(x,r)$ reads as

$$\tilde{F}(x,r) = \sup_{\tau} E_{(x,r)} \int_0^{\tau} e^{-\int_0^t \hat{r}_s ds} \left[\mu(X_t)g'(X_t) - \hat{r}_t g(X_t) + \pi(X_t) \right] dt$$

An interesting result characterizing the relationship between the values $V(x,r)$ and $\tilde{V}(x,r)$ is now summarized in the following.

Lemma 5.3. *Assume that the value $\tilde{V}(x,r)$ is convex as function of the current interest rate r. Then, $\tilde{V}(x,r) \geq V(x,r)$ and $\tilde{F}(x,r) \geq F(x,r)$ for all $(x,r) \in \mathbb{R}_+^2$. Moreover, $\{(x,r) \in \mathbb{R}_+^2 : \hat{V}(x,r) > g(x)\} \subset \{(x,r) \in \mathbb{R}_+^2 : \tilde{V}(x,r) > g(x)\}$ That is, if the value $\tilde{V}(x,r)$ is convex as function of the current interest rate r, then increased volatility increases both the value and the early exercise premium of the irreversible policy and, therefore, prolongs the optimal rotation period.*

Proof. See Appendix E. □

An economic interpretation of Lemma 5.3 goes as follows. Increased interest rate volatility means that the opportunity cost of not harvesting (i.e. leaving the harvesting opportunity unexercised) becomes more uncertain which will move the exercise date further into the future if the value $\hat{V}(x,r)$ of the optimal policy is a convex function of the current interest rate.

Next we have to ask: under what conditions the value $\hat{V}(x,r)$ of the optimal policy under interest rate uncertainty is a convex function of the current interest rate. Given the results of Lemma 5.3 we next provide a set of sufficient conditions under which we can fix the relationship between the optimal rotation period and interest rate volatility. Our characterization is presented in

Theorem 5.4. *Assume that $\sigma(r)$ is continuously differentiable with Lipschitz-continuous derivative, that the standard Novikov-condition*

$$E_r \left[e^{\frac{1}{2}\int_0^t \sigma'^2(r_s)ds} \right] < \infty \quad (t,r) \in \mathbb{R}_+^2$$

is satisfied, that $\pi(x) \equiv 0$, and that $\alpha(r)$ is concave. Then, $\tilde{V}(x,r) \geq V(x,r)$ and $\tilde{F}(x,r) \geq F(x,r)$ for all $(x,r) \in \mathbb{R}_+^2$, and $\{(x,r) \in \mathbb{R}_+^2 : \hat{V}(x,r) > g(x)\} \subset \{(x,r) \in$

$\mathbb{R}^2_+ : \tilde{V}(x,r) > g(x)\}$. *That is, increased volatility increases both the value and the early exercise premium of the irreversible policy and, therefore, prolongs the optimal rotation period.*

Proof. We follow the proof of Theorem 2 in Alvarez 2001 d. Denote now as $r_t(i)$, $t \geq 0$, the solution of the stochastic differential equation (5.1) subject to the initial condition $r_0 = i \in \mathbb{R}_+$. Given our smoothness assumptions $r_t(i)$ can be expressed in the (Itô-) form

$$r_t(i) = i + \int_0^t \mu(r_s(i))ds + \int_0^t \sigma(r_s(i))dW_s. \tag{5.6}$$

Given our assumptions, $r_t(i)$ constitutes a continuously differentiable flow in i (cf. Protter 1990, Theorem V. 38 and 39). Define now the process $\{Y_t; t \geq 0\}$ as $Y_t = \partial r_t(i)/\partial i$. It is then well-known that (cf. Protter 1990, Theorem V. 39)

$$Y_t = 1 + \int_0^t \mu'(r_s(i))Y_s ds + \int_0^t \sigma'(r_s(i))Y_s dW_s. \tag{5.7}$$

Applying Itô's theorem to the mapping $y \mapsto \ln y$ then implies that the solution of the stochastic differential equation (5.7) can be expressed as

$$Y_t = \frac{\partial r_t(i)}{\partial i} = \exp\left(\int_0^t \mu'(r_s(i))ds\right) Z_t(1), \tag{5.8}$$

where, given our assumptions, the process $\{Z_t(1); t \geq 0\}$ defined as

$$Z_t(1) = \exp\left(\int_0^t \sigma'(r_s(i))dW_s - \frac{1}{2}\int_0^t \sigma'^2(r_s(i))ds\right)$$

is a positive martingale starting at date 0 from 1 for any possible $i \in \mathbb{R}_+$. The strong uniqueness of a solution for the stochastic differential equation

$$dZ_t = \sigma'(r_t(i))Z_t dW_t \quad Z_0 = 1$$

then, in turn, implies that $Z_t(1)$ is not affected by i. The concavity of the drift $\mu(r)$ then implies that $\mu'(r)$ is non-increasing in r and, therefore, that $\mu'(r_s(\rho)) \leq \mu'(r_s(i))$ for all $\rho \geq i$ and $s \in [0,t]$. Consequently, we find that $\partial r_t(i)/\partial i$ is non-increasing in i, proving the alleged concavity of the solution $r_t(i)$ as a function of the current short rate i.

Given the concavity of the solution as a function of the current short rate we observe as in Theorem 2.4 that the discount factor $e^{-\int_0^t r_s ds}$ is decreasing and convex as function of the current interest rate. Define now the increasing sequence $\{V_n(x,r)\}_{n\in\mathbb{N}}$ iteratively as

$$V_0(x,r) = g(x), \quad V_{n+1}(x,r) = \sup_{t\geq 0} E_{(x,r)}\left[e^{-\int_0^t r_s ds} V_n(X_t, r_t)\right].$$

It is now clear that $V_1(x,r)$ is convex as a function of r since the maximum of a convex function is convex. Consequently, all elements in the sequence $\{V_n(x,r)\}_{n\in\mathbb{N}}$ are convex as functions of r. Since $V_n(x, r) \uparrow \hat{V}(x, r)$ as $n \to \infty$ (cf. Øksendal 1998, p. 200) we find that for all $\lambda \in [0, 1]$ and $r, \rho \in \mathbb{R}_+$ we have that

$$\lambda\hat{V}(x, r) + (1-\lambda)\hat{V}(x, \rho) \geq \lambda V_n(x, r) + (1-\lambda)V_n(x, \rho) \geq V_n(x, \lambda r + (1-\lambda)\rho).$$

Letting $n \to \infty$ then implies that $\lambda\hat{V}(x, r)+(1-\lambda)\hat{V}(x, \rho) \geq \hat{V}(x, \lambda r+(1-\lambda)\rho)$ proving the convexity of $\hat{V}(x, r)$. The alleged results follow then from Lemma 5.3. □

According to Theorem 5.4 under quite plausible assumptions that the diffusion term is sufficiently smooth as a function of the interest rate and the drift term is a concave function of the interest rate, increasing interest rate volatility will lengthen the optimal rotation period in the absence of amenity valuation. Unfortunately, it is very difficult, if possible at all, to establish simple conditions under which the value of the optimal harvesting policy in the presence of amenity valuation would be convex as a function of the current interest rate. More precisely, reconsider the valuation problem (5.2). An application of the strong Markov property of diffusions then shows that (provided that the functionals exist)

$$\hat{V}(x, r) = (R\pi)(x, r) + \sup_{\tau} E_{(x,r)}\left[e^{-\int_0^{\tau} r_s ds}(g(X_\tau) - (R\pi)(X_\tau, r_\tau))\right],$$

where

$$(R\pi)(x, r) = E_{(x,r)}\int_0^{\infty} e^{-\int_0^s r_t dt}\pi(X_s)ds$$

denotes the expected cumulative present value of the flow of revenues accrued from the amenity services. As is clear from this expression, in the presence of amenity valuation there are three components depending on the current interest rate r, not just one as in the absence of amenity services. Consequently, it is difficult to present simple conditions leading to an unambiguously negative relationship between uncertainty and rotation. However, numerical experimentation seems to indicate that the sign of the relationship between volatility and harvesting is typically negative in this setting as well.

6 Conclusions

There is currently an extensive literature about the determination of optimal forest rotation under various circumstances when amenity valuation of forest stands matters, when capital markets are imperfect so that landowners might be subject to credit rationing or when there is uncertainty about timber prices and/or forest growth due either to forest growth uncertainty or to risk of forest fire. Undoubtedly this literature has provided useful insights about the potential determinants of forest rotation. There is, however, an important issue, which has not yet been analyzed. To our knowledge all the literature makes a simplifying but in the forestry case an unrealistic assumption that the interest rate is constant. Clearly the irreversible harvesting decision of forest stands is a decision subject to a relatively long time horizon. Hence, given the relatively slow growth rate of forests, thinking about harvesting and investing in replanting is a long-term investment project over which the behavior of interest rates as the opportunity cost should matter a lot.

In this paper we have used the Wicksellian single rotation framework to extend the existing studies to cover the unexplored case of variable interest rate. Since the problem is more general than the constant discounting case, we first had to provide a new mathematical characterization of the optimal rotation policy. More precisely, we provided a set of sufficient conditions under which the optimal rotation problem admits a unique solution and under which the value of optimal policy can be obtained from an associated boundary value problem expressed as a first order linear partial differential equation subject to the standard value matching and smooth fit (or smooth pasting) conditions.

From an economic point of view we have established several new findings. First, the variability of interest rate will change the rotation age compared with the constant discounting in a way which depends on the relationship between the current and long run steady state interest rate. More specifically, if the current interest rate is lower than the asymptotically stable one, then the variable interest rate rotation age is lower than the one with constant discounting. The reverse happens in the case when current interest rate is above the long-run steady state interest rate. We illustrated this qualitative finding also quantitatively by using numerical computations in section 4 with logistic functions. Second, we have demonstrated that allowing for interest rate uncertainty will increase the optimal rotation period under the natural condition that the value of the optimal policy is convex in terms of interest rate in the absence of uncertainty. Finally, and importantly, under the plausible assumptions that the diffusion term in the (Itô-) stochastic differential equation for the interest rate is sufficiently smooth as a function of the interest rate and the drift term is concave function of the interest rate, higher interest rate volatility will increase the value of waiting and prolong the optimal rotation period in the absence of amenity valuation.

Whether our conclusions remain valid in the Faustmann's ongoing rotation problem is an open question beyond the scope of the present study. However, given the close connection of impulse control problems and optimal stopping theory (impulse control problems can be viewed as sequential stopping problems; cf. Alvarez 2001 b), we are tempted to conjecture that most probably our conclusions would remain valid with only minor modifications in the ongoing rotation case as well. Of course, the verification of this claim is still an open and challenging problem left for future research.

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A Proof of Lemma 2.1

Proof. Given the assumptions of our lemma, we know that

$$\begin{aligned} e^{-\int_0^{t_n} r_s ds} g(X_{t_n}) &\leq e^{-\int_0^{t_n} r_s ds} J(X_{t_n}, r_{t_n}) \\ &= J(x,r) + \int_0^{t_n} e^{-\int_0^s r_y dy} [(\mathcal{A}J)(X_s, r_s) - r_s J(X_s, r_s)] ds \leq J(x,r), \end{aligned}$$

where $t_n = t \wedge n \wedge \inf\{t \geq 0 : \sqrt{X_t^2 + r_t^2} > n\}$ is a truncation date. Since $\lim_{n\to\infty} t_n = t$, we find that

$$J(x,r) \geq e^{-\int_0^t r_s ds} g(X_t)$$

for all $(x,r) \in \mathbb{R}_+^2$ and $t \geq 0$. Since this inequality is valid for any date t, it must be valid for the optimal date as well and, therefore, $J(x,r) \geq V(x,r)$ for all $(x,r) \in \mathbb{R}_+^2$.

To prove the second claim of our lemma, we first observe that given our conditions

$$\begin{aligned} 0 &\leq e^{-\int_0^{t_n} r_s ds} W(X_{t_n}, r_{t_n}) = W(x,r) + \int_0^{t_n} e^{-\int_0^s r_y dy} [(\mathcal{A}W)(X_s, r_s) - r_s W(X_s, r_s)] ds \\ &\leq W(x,r) - \int_0^{t_n} e^{-\int_0^s r_y dy} \left[g'(X_s)\mu(X_s) - r_s g(X_s) \right] ds. \end{aligned}$$

Letting the $n \to \infty$ and invoking the same principle as above then shows that $W(x,r) \geq F(x,r)$ for all $(x,r) \in \mathbb{R}_+^2$. However, since $V(x,r) = g(x) + F(x,r)$ we find that $V(x,r) \leq g(x) + W(x,r)$ for all $(x,r) \in \mathbb{R}_+^2$, thus completing the proof of our claim. □

B Proof of Theorem 2.2

Proof. Standard differentiation yields that

$$\frac{d}{dt}\left[e^{-\int_0^t r_s ds} g(X_t)\right] = e^{-\int_0^t r_s ds}\left[g'(X_t)\mu(X_t) - r_t g(X_t)\right].$$

Define the mapping $f : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$f(t) = g'(X_t)\mu(X_t) - r_t g(X_t).$$

Assume now that $(x,r) \in D$. Since $D \subset \{(x,r) \in \mathbb{R}_+^2 : g'(x)\mu(x) < rg(x)\}$, we find that $f(t) < 0$ for all $t \geq 0$ implying that $D \subset \Gamma$ and, therefore, that $t^* = 0$. Assume now that $(x,r) \in \mathbb{R}_+^2 \setminus D$. If $(x,r) \in (\mathbb{R}_+^2 \setminus D) \cap \{(x,r) \in \mathbb{R}_+^2 : g'(x)\mu(x) > rg(x)\}$, then it is clear that our assumptions imply that $f(0) = g'(x)\mu(x) - rg(x) > 0$, $\lim_{t\to\infty} f(t) = -\hat{r}g(\hat{x}) < 0$, and

$$f'(t) = \mu(X_t)[g''(X_t)\mu(X_t) - (r_t - \mu'(X_t))g(X_t)] - \alpha(r_t)g(X_t) < 0. \tag{B.1}$$

Consequently, the continuity and monotonicity of the mapping $f(x)$ imply that for any $(x,r) \in (\mathbb{R}_+^2 \setminus D) \cap \{(x,r) \in \mathbb{R}_+^2 : g'(x)\mu(x) > rg(x)\}$ there is a unique $t^*(x,r) < 0$ satisfying the condition

$$f(t) = g'(X_t)\mu(X_t) - r_t g(X_t) \gtreqless 0, t \lesseqgtr t^*(x,r)$$

and, therefore, maximizing the present value $e^{-\int_0^t r_s ds} g(X_t)$. The same observation implies that if $(x,r) \in (\mathbb{R}_+^2 \backslash D) \cap \{(x,r) \in \mathbb{R}_+^2 : g'(x)\mu(x) \leq rg(x)\}$, then $f(t) \leq 0$ for all $t \geq 0$ so that $t^*(x,r) = 0$.

It remains to prove that the value function $V(x,r)$ satisfies the boundary value problem (2.6). In order to accomplish this, we first observe that since

$$X_t = x + \int_0^t \mu(X_s)ds \quad \text{and} \quad r_t = r + \int_0^t \alpha(r_s)ds$$

we find by ordinary differentiation that (cf. Protter 1990, pp. 245–255)

$$\frac{\partial X_t}{\partial x} = 1 + \int_0^t \mu'(X_s)\frac{\partial X_s}{\partial x}ds \quad \text{and} \quad \frac{\partial r_t}{\partial r} = 1 + \int_0^t \alpha'(r_s)\frac{\partial r_s}{\partial r}ds$$

implying that

$$\frac{\partial X_t}{\partial x} = e^{\int_0^t \mu'(X_s)ds} \quad \text{and} \quad \frac{\partial r_t}{\partial r} = e^{\int_0^t \alpha'(r_s)ds}.$$

However, since

$$V(x,r) = e^{-\int_0^{t^*(x,r)} r_s ds} g(X_{t^*(x,r)}),$$

we obtain by ordinary differentiation that

$$\frac{\partial V}{\partial x}(x,r) = e^{-\int_0^{t^*(x,r)} (r_s - \mu'(X_s))ds} g'(X_{t^*(x,r)})$$

and that

$$\frac{\partial V}{\partial r}(x,r) = -e^{-\int_0^{t^*(x,r)} r_s ds} g(X_{t^*(x,r)}) \int_0^{t^*(x,r)} e^{\int_0^s \alpha'(r_t)dt} ds.$$

Since $t^*(x,r) = 0$ whenever $(x,r) \in \partial C$, we find that the value function satisfies the alleged boundary conditions at ∂C. Moreover, we also find that

$$\begin{aligned}(\mathcal{A}V)(x,r) - rV(x,r) = {} & e^{-\int_0^{t^*(x,r)} r_s ds} \mu(x) e^{\int_0^{t^*(x,r)} \mu'(X_s)ds} g'(X_{t^*(x,r)}) \\ & - e^{-\int_0^{t^*(x,r)} r_s ds} \left(r + \int_0^{t^*(x,r)} \alpha(r) e^{\int_0^s \alpha'(r_t)dt} ds \right) g(X_{t^*(x,r)}).\end{aligned}$$

Finally, since

$$\mu(X_t) = \mu(x) e^{\int_0^t \mu'(X_s)ds} \quad \text{and} \quad \alpha(r_t) = \alpha(r) e^{\int_0^t \alpha'(r_s)ds}$$

we find that

$$(\mathcal{A}V)(x,r) - rV(x,r) = e^{-\int_0^{t^*(x,r)} r_s ds} \left[\mu(X_{t^*(x,r)}) g'(X_{t^*(x,r)}) - r_{t^*(x,r)} g(X_{t^*(x,r)}) \right] = 0$$

thus completing the proof of our theorem. □

C Proof of Lemma 5.1

Proof. As in Øksendal and Reikvam 1998 and in the proof of Lemma 2.1 we find by invoking Dynkin's theorem that

$$\begin{aligned} E_{(x,r)}\left[e^{-\int_0^{\tau_n} r_s ds} g(X_{\tau_n})\right] &\leq E_{(x,r)}\left[e^{-\int_0^{\tau_n} r_s ds} J(X_{\tau_n}, r_{\tau_n})\right] \\ &= J(x,r) + E_{(x,r)} \int_0^{\tau_n} e^{-\int_0^s r_y dy}[(\hat{\mathcal{A}}J)(X_s, r_s) - r_s J(X_s, r_s)]ds \\ &\leq J(x,r) - E_{(x,r)} \int_0^{\tau_n} e^{-\int_0^s r_y dy} \pi(X_s) ds, \end{aligned}$$

where $\tau_n = \tau \wedge n \wedge \inf\{t \geq 0 : \sqrt{X_t^2 + r_t^2} > n\}$ is an almost surely finite stopping time converging to the stopping time τ as $n \to \infty$. Reordering terms and invoking Fatou's theorem then yields that

$$J(x,r) \geq E_{(x,r)}\left[e^{-\int_0^{\tau} r_s ds} g(X_\tau) + \int_0^\tau e^{-\int_0^s r_y dy} \pi(X_s) ds\right].$$

Since this inequality is valid for any stopping time τ, it must be valid for the optimal stopping time as well and, therefore, $J(x,r) \geq \hat{V}(x,r)$ for all $(x,r) \in \mathbb{R}_+^2$. The second claim of our lemma follows the directly from the first after noticing that $\hat{V}(x,r) = g(x) + \hat{F}(x,r)$. □

D Proof of Theorem 5.2

Proof. (A) If the value $V(x,r)$ is convex as function of r, then for all $(x,c) \in C$ we have

$$E_{(x,r)}\left[e^{-\int_0^{\tau_n} r_s ds} V(X_{\tau_n}, r_{\tau_n})\right] = V(x,r) + E_{(x,r)} \int_0^{\tau_n} e^{-\int_0^t r_s ds}\left[(\hat{\mathcal{A}}V)(X_t, r_t) - r_t V(X_t, r_t)\right] dt,$$

where τ_n is a sequence of almost surely finite stopping times converging towards $\tau^* = \inf\{t \geq 0 : (X_t, r_t) \notin C\}$. Since

$$(\hat{\mathcal{A}}V)(x,r) - rV(x,r) = \frac{1}{2}\sigma^2(r) \frac{\partial^2 V}{\partial r^2}(x,r) \geq 0$$

whenever $(x,c) \in C$, we find that

$$V(x,r) \leq E_{(x,r)}\left[e^{-\int_0^{\tau_n} r_s ds} V(X_{\tau_n}, r_{\tau_n})\right].$$

Letting $n \to \infty$ and invoking continuity of the value across the boundary then yields that

$$V(x,r) \leq E_{(x,r)}\left[e^{-\int_0^{\tau^*} r_s ds} g(X_{\tau^*})\right] \leq \hat{V}(x,r),$$

proving that $V(x,r) \leq \hat{V}(x,r)$ on the continuation region C. However, since $V(x,r) = g(x)$ on $\mathbb{R}_+^2 \setminus C$ and $\hat{V}(x,r) \geq g(x)$ for all $(x,r) \in \mathbb{R}_+^2$, we find that $\hat{V}(x,r) \geq V(x,r)$ and $\hat{F}(x,r) \geq F(x,r)$ for all $(x,r) \in \mathbb{R}_+^2$. Moreover, if $(x,r) \in C$, then $\hat{V}(x,r) \geq V(x,r) > g(x)$ proving that $(x,r) \in \{(x,r) \in \mathbb{R}_+^2 : \hat{V}(x,r) > g(x)\}$ as well. Proving part (B) is analogous after using the results of Theorem 2.4. □

E Proof of Lemma 5.3

Proof. Given our assumptions, we find that for all $(x,r) \in \mathbb{R}^2_+$ we have that

$$(\hat{\mathcal{A}}\tilde{V})(x,r) - r\tilde{V}(x,r) + \pi(x) \leq \frac{1}{2}(\sigma^2(r) - \hat{\sigma}^2(r))\frac{\partial^2 \tilde{V}}{\partial r^2}(x,r) \leq 0$$

since

$$\frac{1}{2}\hat{\sigma}^2(r)\frac{\partial^2 \tilde{V}}{\partial r^2}(x,r) + \mu(x)\frac{\partial \tilde{V}}{\partial x}(x,r) + \alpha(r)\frac{\partial \tilde{V}}{\partial r}(x,r) - r\tilde{V}(x,r) + \pi(x) \leq 0$$

for all $(x,r) \in \mathbb{R}^2_+$ by the r-excessivity of $\tilde{V}(x,r)$. Consequently, we observe that

$$E_{(x,r)}\left[e^{-\int_0^{\tau_n} r_s ds}\tilde{V}(X_{\tau_n}, r_{\tau_n})\right] \leq \tilde{V}(x,r) - E_{(x,r)}\int_0^{\tau_n} e^{-\int_0^t r_s ds}\pi(X_t)dt$$

where $\tau_n = \tau \wedge n \wedge \inf\{t \geq 0 : \sqrt{X_t^2 + r_t^2} > n\}$. Reordering terms, invoking the condition $V(x,r) \geq g(x)$, letting $n \to \infty$, and applying Fatou's theorem then yields that

$$\tilde{V}(x,r) \geq E_{(x,r)}\left[e^{-\int_0^{\tau} r_s ds}g(X_\tau) + \int_0^{\tau} e^{-\int_0^t r_s ds}\pi(X_t)dt\right]$$

proving that $\tilde{V}(x,r) \geq \hat{V}(x,r)$ for all $(x,r) \in \mathbb{R}^2_+$. The inequality $\tilde{F}(x,r) \geq F(x,r)$ then follows from the definition of the early exercise premiums. Finally, if $(x,r) \in \{(x,r) \in \mathbb{R}^2_+ : \hat{V}(x,r) > g(x)\}$, then $(x,r) \in \{(x,r) \in \mathbb{R}^2_+ : \tilde{V}(x,r) > g(x)\}$ as well, since then $\tilde{V}(x,r) \geq \hat{V}(x,r) > g(x)$. □