CESifo Working Papers
www.cesifo.org/wp

Channel Systems: Why is there a Positive Spread?

Aleksander Berentsen
Alessandro Marchesiani
Christopher J. Waller

CESifo Working Paper No. 3251
Category 7: Monetary Policy and International Finance
November 2010

CESifo *Working Paper No. 3251*

Channel Systems: Why is there a Positive Spread?

Abstract

An increasing number of central banks implement monetary policy via two standing facilities: a lending facility and a deposit facility. In this paper we show that it is socially optimal to implement a non-zero interest rate spread. We prove this result in a dynamic general equilibrium model where market participants have heterogeneous liquidity needs and where the central bank requires government bonds as collateral. We also calibrate the model and discuss the behavior of the money market rate and the volumes traded at the ECB's deposit and lending facilities in response to the recent financial crisis.

JEL-Code: E52, E58, E59.

Keywords: monetary policy, open market, operations, standing facilities.

Aleksander Berentsen
University of Basel
Basel / Switzerlandl
aleksander.berentsen@unibas.ch

Alessandro Marchesiani
University of Minho
Braga / Portugal
marchesiani@gmail.com

Christopher J. Waller
University of Notre Dame
Notre Dame / USA
christopher.j.waller.6@nd.edu

1. Introduction

In a channel system, a central bank offers two facilities: a lending facility whereby it is ready to supply money overnight at a given lending rate against collateral and a deposit facility whereby banks can make overnight deposits to earn a deposit rate. The interest-rate corridor is chosen to keep the interest rate in the money market close to its target. A change in policy is implemented by simply changing the interest-rate corridor. In theory, there is no need for direct central bank intervention to control the market rate of interest, since money market participants will never mutually agree to trade at an interest rate that lies outside the interest rate corridor.[1] Furthermore, since market participants prefer to trade amongst themselves rather than access the standing facilities, it allows the central bank to control the money market rate while incurring very little operating costs (it reduces the interest paid on deposits and monitoring costs associated with lending).

In practice, all central banks that operate a channel system choose a non-zero interest-rate corridor and have a target rate above the deposit rate. From a theoretical perspective, this behavior is puzzling. First, if controlling the market interest rate is the objective, then why not set the spread to zero? Doing so allows the central bank to perfectly control the money market rate. Second, even with a positive spread, why set the target rate above the deposit rate? Cúrdia and Woodford (2010) argue that setting the target rate at the deposit rate is a way to run the Friedman rule which eliminates any inefficiencies owing to holding 'idle' reserves in the banking system.[2]

The fact that no central bank chooses a zero corridor and no bank keeps the target rate above the deposit rate suggests that there are significant frictions that induce the central bank to behave

[1]In practice, central banks still conduct open market operations to adjust the quantity of central bank money in circulation. In "normal" times, for technical reasons they do so to accommodate, for example, seasonal fluctuations in the demand for central bank money. In "exceptional" times, in response to severe aggregate shocks they do so to restore the functioning of money markets. Moreover, in practice, there are cases where the money market rate lies outside the interest-rate corridor which can often be explained by particular operational details. For example, in the US Fannie Mae and Freddy Mac have no access to the Federal Reserve's deposit facility. This institutional particularity has been brought forward to explain why the money market rate was often below the FED's deposit rate during the financial crisis.

[2]Our paper differs from Cúrdia and Woodford in two important ways. First, we do not have sticky prices. Second, we do not have inefficiencies in the financial intermediation process that gives rise to a need for reserves. It is this latter inefficiency that is eliminated by having sufficiently high bank reserves, i.e., when banks are satiated with reserves. Our framework works via a different mechanism – a combination of risk sharing and collateral requirements.

this way. One potential friction may be the fiscal ramifications associated with paying interest on deposits. If the central bank sets its target rate at the deposit rate, the central bank would actually need to make substantial interest payments. Without a portfolio of assets that generates a flow of income, the central bank must be able to either levy taxes or have the Treasury do so in order to make the interest payments.[3] This power to finance interest payments on reserves may be limited for several reasons. First, the central bank may not have the power to levy taxes or for political reasons, is loath to ask for tax revenues from the Treasury. Second, if lump-sum taxation is not available, then raising taxes to pay interest on reserves requires distortionary taxation.

In this paper, we study the optimality of having a non-zero interest-rate corridor in a general equilibrium model where market participants face idiosyncratic liquidity shocks, taking into account potential restrictions on the central bank's ability to extract tax revenue to pay interest on reserves. First, without restrictions on the central bank's ability to extract tax revenue, we prove that the optimal policy involves setting the deposit rate equal to the target rate. In line with Cúrdia and Woodford, doing so implements the Friedman rule. We then show that the optimal policy requires that the central bank is able to raise revenue to finance interest on reserves.[4] Second, with restrictions on the central bank's ability to extract tax revenue, the optimal policy necessitates setting the deposit rate strictly below the target rate. Moreover, it always involves a strictly positive interest-rate spread. The optimality of a non-zero corridor arises because it improves risk sharing and hence welfare by shifting central bank money to those market participants who need it most urgently.

[3]Fiscal considerations clearly factored in the Federal Reserve's ability to pay interest on reserves as pointed out in the following quote: "The Fed got the authority to start paying interest in October 2011 under the Financial Services Regulatory Relief Act of 2006, signed into law on Oct. 13, 2006. The reason for the late implementation was budgetary. Paying interest on reserves will reduce the amount of income the Fed earns on its securities portfolio and remits to Treasury each year. Congress pushed back the date of implementation to minimize the near-term impact on the deficit." Source: Real Time Economics Blog, Wall Street Journal, April 29, 2008.

[4]Moreover, it is optimal to have a positive interest-rate spread (the difference between the lending rate and the deposit rate). The optimality of a strictly positive interest- rate follows from the fact that under the optimal policy, the borrowing rate does not matter since market participants never borrow at the lending facility. The Friedman rule allows them to perfectly self-insure against any idiosyncratic liquidity shocks.

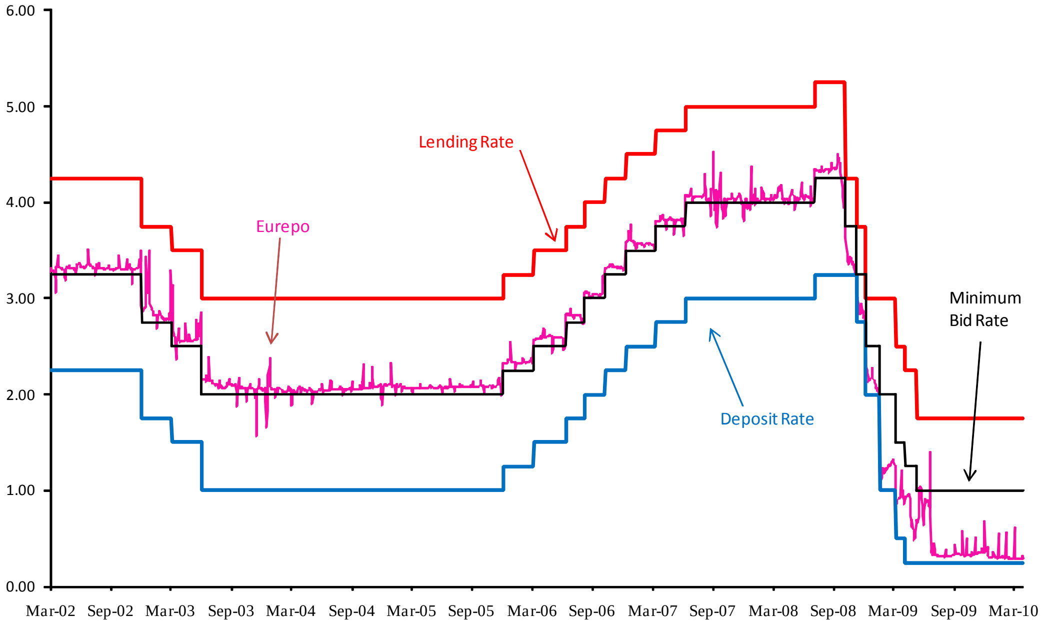

Figure 1: Standing Facilities of the ECB

In addition to studying the theoretical properties of our model, we also calibrate it to the channel system operated by the European Central Bank (ECB). We wish to discover whether our model can replicate the behavior of the money market interest rate (the Eurepo) and the volumes of trade at the ECB's standing facilities. Figure 1 displays the ECB's channel system.

As can be seen from Figure 1, the Eurepo fluctuates in the middle of the ECB's interest-rate corridor (around the minimum bid rate) until the fall of 2008. Thereafter, it fluctuates closely above the deposit rate set by the ECB. Moreover, the volume of deposits at the deposit facility and the volume of borrowing at the marginal lending facility of the ECB changed dramatically around the same time. This can be seen in Figure 2 where we display the daily volume at the two facilities from January 2007 to July 2010.

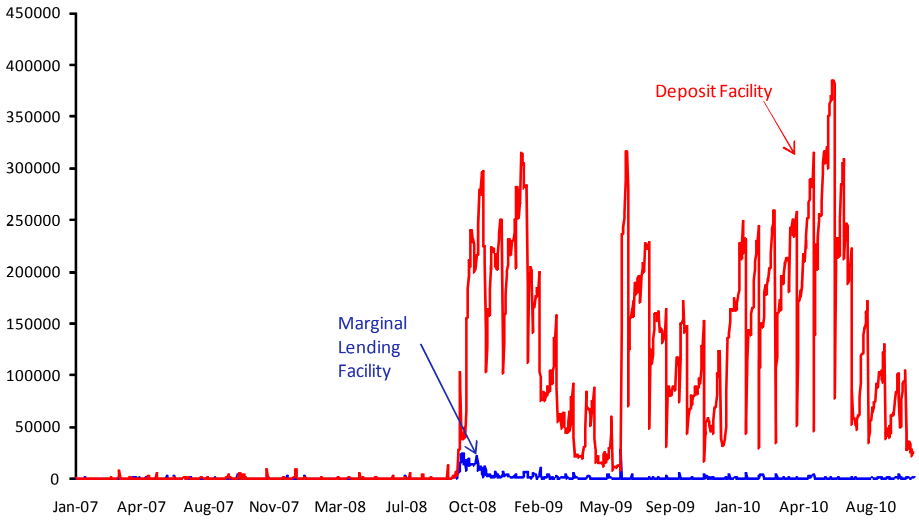

Figure 2: Deposit Facility and Marginal Lending Facility (Millions Euros) 2007-present.

Table 1 presents a summary statistics of the volumes traded at the standing facilities of the ECB. It displays the average daily volumes of trades at the deposit and lending facilities of the ECB before and after the Lehman bankruptcy. The average daily volume at the deposit facility of the ECB prior to the Lehman bankruptcy was 260 million euros. After the bankruptcy, the average daily volume increased to 150092 million euros; i.e., it is now 578 times higher than before the Lehman bankruptcy.

Table 1: Volumes at the ECB[a,b]

	pre Lehman	post Lehman	ratio post/pre
Deposit Facility	260	150092	578
Marginal Lending Facility	195	1818	9.3
Ratio	1.33	82.6	

[a]Average daily volume in million euros at the deposit and lending facilities of the ECB.

[b]Pre-Lehman data from 31.3.2002 until 14.09.2008; Post-Lehman data from 14.09.2008 until 31.03.2010.

The average daily volume at the marginal borrowing facility displays a similar pattern, although the increase is less pronounced. The average daily volume prior to the Lehman bankruptcy was 195 million euros. After the bankruptcy, the average daily volume increased to 1818 million euros; i.e., it is now 9.3 times higher. In Table 1, we also display the ratio of deposits to lending prior to and after the Lehman bankruptcy. Before it, the ratio was 1.33, while it now stands at 82.6.

One would be hard pressed to say that the behavior of the money market rate and the volumes at the ECB's facilities are not linked to the severe financial crisis that erupted after the bankruptcy of Lehman brothers on September 15 2008. With therefore want to address two observations using our calibration: firstly, what explains the drop in the money market rate below the target rate during the fall of 2008; and secondly, what explains the extreme increase in volumes at the deposit facility during the fall of 2008.

We explore two possible explanations. The first hypothesis is that the Lehman bankruptcy triggered an aggregate variance shock that caused the drop in the Eurepo rate and the increase of deposits and loans at the ECB's standing facilities. By aggregate variance shock we mean that the dispersion in liquidity needs increased dramatically. The second hypothesis is that the Lehman bankruptcy triggered an aggregate collateral shock. We find that if we choose these two aggregate shocks appropriately and simultaneously, our model is able to replicate the large drop of the money market rate and the large increase in volumes at the standing facilities as observed in the data.

1.1. **Related Literature.** An increasing number of central banks are using channel systems or at least some features of the channel system for implementing monetary policy. Versions of a channel system are operated by the Bank of Canada, the Bank of England, the European Central Bank (see Figure 1), the Reserve Bank of Australia, and the Reserve Bank of New Zealand. The U.S. Federal Reserve System recently modified the operating procedures of its discount window facility in such a way that it now shares elements of a standing facility.[5]

Despite the growing use of channel systems to implement monetary policy, only a few theoretical studies on its use exist. The earlier literature on channel systems or aspects of channel

[5]Prior to 2003, the discount window rate was set below the target federal funds rate, but banks faced penalties when accessing the discount window. In 2003, the Federal Reserve decided to set the discount window rate 100 basis points above the target federal funds rate and eased access conditions to the discount window. The resulting framework was similar to a channel system, where the deposit rate is zero and the lending rate 100 basis points above the target rate. Furthermore, since October 2008, the Federal Reserve pays interest on required and excess reserve balances.

systems were conducted in partial equilibrium models. Berentsen and Monnet (2008) contains a discussion of these models. Berentsen and Monnet (2008) were the first to study monetary policy in a channel system within a general equilibrium framework. In their framework, the central bank requires a real asset as a collateral at its borrowing facility. Due to its liquidity premium, the social return of this real asset is lower than the private return to a market participant. From a social point of view, this results in an overaccumulation of the real collateral if the central bank implements a zero interest-rate spread. Consequently, by implementing a positive spread, the central bank can discourage the use of its borrowing facility and, thereby, improve the allocation. Their key result is therefore that it is socially optimal to implement a strictly positive interest-rate spread. However, this result hinges on the fact that collateral is a real asset that has a poor rate of return and would never be accumulated if collateral constraints did not bind. Hence they are socially 'useless' assets from a planner's perspective.

In this paper, we also find that it is optimal to implement a non-zero spread. The reason for this result, however, is very different. In our model, the central bank's borrowing facility accepts government bonds as collateral. Government bonds are essentially pieces of paper that are costless to produce and so there is no social waste in its use. Nevertheless, we also find that it is optimal to implement a strictly positive interest-rate spread. Without taxing frictions, the spread of the interest rate corridor is irrelevant. With taxing frictions, a non-zero corridor affects the distribution of central bank money in a welfare improving way, consequently there is an optimal spread.

Martin and Monnet (2010), which also builds on Berentsen and Monnet (2008), is a further general equilibrium model of a channel system. The purpose of their paper is to compare the feasible allocations that one can obtain when a central bank implements monetary policy either with a channel system or via plain vanilla open market operations in the Lagos-Wright framework. The focus in our paper is very different. We only consider channel systems and derive the optimal policy with and without taxing frictions. Furthermore, we calibrate the model to study the behavior of the money market and the volumes traded at the standing facilities in response to aggregate shocks. Moreover, we have a more complex structure of liquidity shocks than they have which allows us to study how policy affects the distribution of overnight-liquidity in a general equilibrium model.

The structure of the paper is the following. Section 2 describes the environment. Optimal decisions by market participants are characterized in Section 3. Section 4 studies symmetric stationary equilibria. Section 5 identifies the optimal policy if the central bank has no restrictions in its ability to extract tax revenue to pay interest on reserves. Section 6 does the same if the central bank faces restrictions in its ability to extract tax revenue. In Section 7 we calibrate the model to the ECB's channel system and derive the optimal interest-rate spread with taxing frictions. Furthermore, we study how aggregate shocks affect the behavior of the money market and the volumes traded at the standing facilities of the ECB. Section 8 concludes. All proofs are in the Appendix.

2. Environment

Our framework is motivated by the functioning of existing channel systems. For example, as discussed in Berentsen and Monnet (2008), the key features of the ECB's implementation framework and of the euro money market are the following. First, at the beginning of the day, any outstanding overnight loans at the ECB are settled. Second, the euro money market operates between 7 a.m. and 5 p.m. Third, after the money market has closed, market participants can access the ECB's facilities for an additional 30 minutes. This means that after the close of the money market, the ECB's lending facility is the only possibility for obtaining overnight liquidity. Also, any late payments received can still be deposited at the deposit facility of the ECB.

To capture the above sequence of trading in the money market and at the central bank's standing facilities, we assume that in each period two markets open sequentially. The first market is a money market, where market participants can trade money for bonds and where all claims from the previous day are settled. The second market is a goods market where market participants trade goods for central bank money. Its purpose is to generate a well defined demand for central bank money.[6] At the beginning of the goods market, agents receive an idiosyncratic liquidity shock which generates a role for the central bank's standing facility as explained below.[7]

In practice, only qualified financial intermediaries have access to the money market and the central bank's standing facilities. Nevertheless, these intermediaries act on the behalf of their customers: households and firms. We simplify the analysis by assuming that the economy is populated by infinitely-lived households who have direct access to the money market and the central bank's standing facilities. This simplifies the analysis and focuses and the varying liquidity needs of agents in the economy rather than the process of intermediation.

There is a generic good in the economy that is nonstorable and perfectly divisible in each market. Nonstorable means that they cannot be carried from one market to the next. There are two types of households: buyers and sellers. Each type has measure 1. Buyers consume in market 2 and consume and produce in market 1. Sellers produce in market 2 and can consume and produce in market 1.

In market 2, a buyer gets utility $\varepsilon u(q)$ from consuming q units of the good, where $u(q) = \log(q)$ and ε is a preference shock which affects the liquidity needs of buyers. The preference shock ε has a continuous distribution $F(\varepsilon)$ with support $(0, \infty]$, is iid across buyers and serially uncorrelated. Sellers incur a utility cost $c(q_s) = q_s$ from producing q_s units of market 2 goods. The discount factor across periods is $\beta = (1+r)^{-1} < 1$ where r is the time rate of discount.

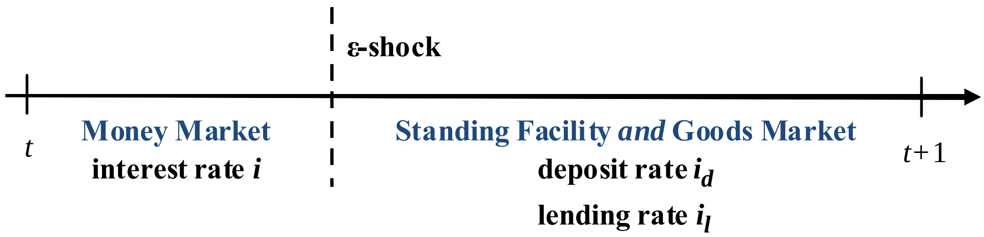

Figure 3: Sequence of events

[6] In Section 2.2 we discuss the necessary assumption imposed on the exchange process that makes the use of central bank money as a medium of exchange essential in market 2.

[7] This particular sequence of markets has been developed by Berentsen and Monnet (2008). Their model in turn, is based on the divisible money model of Lagos and Wright (2005). The framework by Lagos and Wright (2005) is useful because it allows us to introduce heterogeneous preferences while still keeping the distribution of money balances analytically tractable.

The preference shock creates random liquidity needs among buyers. The buyers learn the realization of the preference shock ε after the money market has closed but before market 2 opens. This generates a role for the central bank's standing facilities, since the money market is already closed.[8]

As in Lagos and Wright (2005), for tractability, we impose assumptions that yield a degenerate distribution of portfolios at the beginning of the goods market.[9] That is, we assume that in the money market agents can produce and consume goods. Goods are produced solely from inputs of labor according to a constant returns to scale production technology where one unit of the good is produced with one unit of labor generating one unit of disutility. Thus, producing h units of goods implies disutility $-h$, while consuming h units gives utility h. The utility of consuming x units of goods is $U(x) = x$.

2.1. **First-best allocation.** We assume without loss in generality that the planner treats all sellers symmetrically. He also treats all buyers experiencing the same preference shock symmetrically. Given this assumption, the weighted average of expected steady state lifetime utility of households and firms at the beginning of the money market can be written as follows

$$(1-\beta)\,\mathcal{W} = \int_0^\infty \left[\varepsilon u\left(q_\varepsilon\right) - h_\varepsilon\right] dF\left(\varepsilon\right) + x - q_s. \tag{2.1}$$

where h_ε is hours worked by an $\varepsilon-$buyer in the money market, q_ε is consumption of an $\varepsilon-$buyer in the goods market, and q_s is production of a seller in the goods market. The planner maximizes (2.1) subject to the feasibility constraints

$$\int_0^\infty q_\varepsilon dF\left(\varepsilon\right) - q_s \;\leq\; 0. \tag{2.2}$$

$$\int_0^\infty h_\varepsilon dF\left(\varepsilon\right) - x \;\leq\; 0 \tag{2.3}$$

[8]The interpretation for these shocks is that the $\varepsilon-$buyers are banks that receive liquidity shocks so late in the day that they have to rely on the ECB's standing facilities to readjust their portfolio.

[9]The idiosyncratic preference shocks play a similar role to that of random matching and bargaining in Lagos and Wright (2005). Due to these shocks, households spend different amounts of money in the goods market. Then, without quasilinear preferences and unbounded hours in market 1, the preference shocks would generate a nondegenerate distribution of money holdings, since the money holdings of individual households would depend on their history of shocks.

where x is consumption by a seller in the money market. The first-best allocation satisfies

$$\varepsilon u'\left(q_{\varepsilon}^{*}\right)=1 \text{ for all } \varepsilon. \tag{2.4}$$

These are the quantities chosen by a social planner who could dictate production and consumption in the goods market.

2.2. **Information frictions, money and bonds.** There are two perfectly divisible financial assets: money and one-period, nominal discount bonds. Both are intrinsically useless, since they are neither arguments of any utility function nor are they arguments of any production function. Both assets are issued by the central bank as described below.[10] New bonds are issued in the money market. They are payable to the bearer and default free. One bond pays off one unit of currency in the money market of the following period. The central bank is assumed to have a record-keeping technology over bond trades, and bonds are book-keeping entries – no physical object exists. This implies that households are not anonymous to the central bank. Nevertheless, despite having a record-keeping technology over bond trades, the central bank has no record-keeping technology over goods trades.

At time t, the central bank sells one-period, nominal discount bonds in market 1 and redeems bonds that were sold in $t-1$. Private households are anonymous to each other and cannot commit to honor inter-temporal promises. Since bonds are intangible objects, they are incapable of being used as media of exchange in market 2, hence they are illiquid.[11] Since households are anonymous and cannot commit, a household's promise in market 2 to deliver bonds to a seller in market 1 of the following period is not credible.

To motivate a role for fiat money, search models of money typically impose three assumptions on the exchange process (Shi 2008): a double coincidence problem, anonymity, and costly communication. First, our preference structure creates a single-coincidence problem in market

[10]Strictly speaking, these bonds are not government bonds. Rather, these are nominal claims on the central bank that are redeemable in central bank money. They resemble, for example, the SNB-bills issued by the Swiss National Bank or term deposits issued to banks by the Federal Reserve.

[11]The beneficial role of illiquid bonds has been studied by Kocherlakota (2003), Shi (2008) and Berentsen and Waller (2010). More recent models with illiquid assets include, Lagos and Rocheteau (2008), Lagos (2010), Lester, Postlewaite and Wright (2010), and many others.

2 since households do not have a good desired by sellers. Second, agents in market 2 are anonymous, which rules out trade credit between individual buyers and sellers. Third, there is no public communication of individual trading outcomes (public memory), which, in turn, eliminates the use of social punishments in support of gift-giving equilibria. The combination of these frictions implies that sellers require immediate compensation from buyers. In short, there must be immediate settlement with some durable asset, and money is the only durable asset. These are the micro-founded frictions that make money essential for trade in market 2. Araujo (2004), Kocherlakota (1998), Wallace (2001), and Aliprantis, Camera and Puzzello (2007) provide a more detailed discussion of the features that generate an essential role for money. In contrast, in the money market all agents can produce for their own consumption or use money balances acquired earlier. In this market, money is not essential for trade.[12]

2.3. **Central bank policy and the money supply process.** At the beginning of market 1, after all preference shocks are observed, the central bank offers a borrowing and a deposit facility. The central bank operates at zero cost and offers nominal loans ℓ at an interest rate i_ℓ and promises to pay interest rate i_d on nominal deposits d with $i_\ell \geq i_d$. Let $\rho_d = 1/(1+i_d)$ and $\rho_\ell = 1/(1+i_\ell)$. Since we focus on facilities provided by the central bank, we restrict financial contracts to overnight contracts. An agent who borrows ℓ units of money from the central bank in market 2 repays $(1+i_\ell)\,\ell$ units of money in market 1 of the following period. Also, an agent who deposits d units of money at the central bank in market 2 receives $(1+i_d)\,d$ units of money in market 1 of the following period.

In a channel system, then the money stock evolves as follows

$$M^+ = M - i_\ell L + i_d D - \rho B^+ + B + T \tag{2.5}$$

where M and B are the stock of money, respectively the stock of bonds, at the beginning of the current-period money market, M^+ and B^+ the stock of money, respectively the stock of bonds, at the beginning of the next-period money market, T the lump sum transfer, ρ the price of bonds in the money market. In the money market, total loans L are repaid and total deposits D are

[12] One can think of agents as being able to barter perfectly in this market. Obviously in such an environment, money is not needed.

redeemed. Since interest-rate payments by the agents are $i_\ell L$, the stock of money shrinks by this amount. Interest payments by the central bank on total deposits are $i_d D$. The central bank simply prints additional money to make these interest payments, causing the stock of money to increase by this amount. The central bank also issues new one-period bonds which it sells at discount ρ. This shrinks the amount of money by ρB^+. In the money market, it also redeems the stock of bonds it issued in the previous period B, which increases the stock of money by B. Finally, the central bank can also change the stock of money via lump-sum transfers $T = \tau M$ in the money market.

3. Household decisions

The money price of goods in the money market is P, implying that the goods price of money in the money market is $\phi = 1/P$. Let p be the money price of goods in the goods market.

3.1. **Money market.** $V_M(m, b, \ell, d)$ denotes the expected value of entering the money market with m units of money, b bonds, ℓ loans, and d deposits. $V_G(m, b)$ denotes the expected value from entering the goods market with m units of money and b collateral. For notational simplicity, we suppress the dependence of the value function on the time index t.

In the money market, the problem of the representative buyer is:

$$V_M(m, b, \ell, d) = \max_{h, m\prime, b\prime} -h + V_G\left(m', b'\right)$$

$$s.t. \quad \phi m' + \phi\rho b' = h + \phi m + \phi b + \phi\left(1 + i_d\right) d - \phi\left(1 + i_\ell\right) \ell + \phi T.$$

where h is hours worked in market 1, m' is the amount of money brought into the goods market, and b' is the amount of bonds brought into the goods market. Using the budget constraint to eliminate h in the objective function, one obtains the first-order conditions

$$V_G^{m'} \leq \phi \; (= \text{ if } m' > 0 \;) \tag{3.1}$$

$$V_G^{b'} \leq \phi\rho \; (= \text{ if } b' > 0 \;) \tag{3.2}$$

$V_G^{m'} \equiv \frac{\partial V_G(m', b')}{\partial m'}$ is the marginal value of taking an additional unit of money into the goods market. Since the marginal disutility of working is one, $-\phi$ is the utility cost of acquiring one unit of money in the money market. $V_G^{b'} \equiv \frac{\partial V_G(m',b')}{\partial b'}$ is the marginal value of taking additional bonds into the goods market. Since the marginal disutility of working is 1, $-\phi\rho$ is the utility cost of acquiring one unit of bonds in the money market. The implication of (3.1) and (3.2) is that all agents enter the goods market with the same amount of money and the same quantity of bonds (which can be zero).

The envelope conditions are

$$V_M^m = V_M^b = \phi; V_M^d = \phi\left(1 + i_d\right); V_M^\ell = -\phi\left(1 + i_\ell\right) \tag{3.3}$$

where V_M^j is the partial derivative of $V_M(m, b, \ell, d)$ with respect to $j = m, b, \ell, d$.

3.2. **Goods Market.** We first consider the problem solved by sellers and then the one solved by buyers. During the goods market, the central bank operates a borrowing facility and a deposit facility which allows households to borrow at rate i_ℓ and deposit unspent money at rate i_d.

3.2.1. *Decisions by sellers.* Sellers produce goods in the goods market with linear cost $c\left(q\right) = q$ and consume in the money market obtaining linear utility $U(x) = x$. It is straightforward to show that that sellers are indifferent as to how much they sell in the goods market if

$$p\beta\phi^+\left(1 + i_d\right) = 1. \tag{3.4}$$

Since we focus on a symmetric equilibrium, we assume that all sellers produce the same amount. With regard to bond holdings, it is straightforward to show that, in equilibrium, sellers are indifferent to holding any bonds if the Fisher equation holds and will hold no bonds if the yield on the bonds does not compensate them for inflation or time discounting. Thus, for brevity of analysis, we assume sellers carry no bonds across periods.

It is also clear that sellers always deposit their proceeds from sales at the deposit facility, since they can earn the interest rate i_d.

3.2.2. *Decisions by buyer.* The indirect utility function of an $\varepsilon-$buyer in the goods market is

$$V_G(m,b|\,\varepsilon) = \max_{q_\varepsilon,d_\varepsilon,\ell_\varepsilon} \varepsilon u\left(q_\varepsilon\right) + \beta V_M\left(m+\ell_\varepsilon - pq_\varepsilon - d_\varepsilon, b, \ell_\varepsilon, d_\varepsilon|\,\varepsilon\right)$$
$$\text{s.t.}\quad m+\ell_\varepsilon - pq_\varepsilon - d_\varepsilon \geq 0, \text{ and } \frac{b}{1+i_\ell} - \ell_\varepsilon \geq 0$$

where d_ε is the amount of money an ε-buyer deposits at the central bank, and ℓ_ε is the loan received from the central bank. The first inequality is the buyer's budget constraint. The second inequality is the collateral constraint. Let $\beta\phi^+\lambda_\varepsilon$ denote the Lagrange multiplier for the first inequality and denote $\beta\phi^+\lambda_\ell$ the Lagrange multiplier of the second inequality. Then, using (3.3) to replace V_M^m, V_M^ℓ and V_M^d, the first-order conditions for q_ε, d_ε, and ℓ_ε can be written as follows:

$$\begin{aligned} \varepsilon u'\left(q_\varepsilon\right) - \beta p\phi^+\left(1+\lambda_\varepsilon\right) &= 0 \\ i_d - \lambda_\varepsilon &\leq 0 \qquad (= 0 \text{ if } d_\varepsilon > 0) \\ -i_\ell + \lambda_\varepsilon - \lambda_\ell &\leq 0 \qquad (= 0 \text{ if } \ell_\varepsilon > 0) \end{aligned} \tag{3.5}$$

Lemma 1 below characterizes the optimal borrowing and lending decisions by an $\varepsilon-$buyer and the quantity of goods obtained by the $\varepsilon-$buyer:

LEMMA 1. *There exist critical values ε_d, ε_ℓ, $\varepsilon_{\bar\ell}$, with $0 \leq \varepsilon_d \leq \varepsilon_\ell \leq \varepsilon_{\bar\ell}$, such that the following is true: if $0 \leq \varepsilon < \varepsilon_d$, a buyer deposits money at the central bank; if $\varepsilon_\ell < \varepsilon \leq \varepsilon_{\bar\ell}$, he borrows money and the collateral constraint is nonbinding; if $\varepsilon_{\bar\ell} \leq \varepsilon$, he borrows money and the collateral constraint is binding; and if $\varepsilon_d \leq \varepsilon \leq \varepsilon_\ell$, he neither borrows nor deposits money. The critical values solve:*

$$\varepsilon_d = \left(1+i_d\right)\beta\phi^+ m,\ \varepsilon_\ell = \left(1+i_\ell\right)\beta\phi^+ m, \text{ and } \varepsilon_{\bar\ell} = \left(1+i_\ell\right)\beta\phi^+ m + \beta\phi^+ b. \tag{3.6}$$

In any equilibrium, the amount of borrowing and depositing by a buyer with a taste shock ε and the amount of goods purchased by the buyer satisfy:

$$\begin{array}{llll} q_\varepsilon = \varepsilon, & d_\varepsilon = p\left(\varepsilon_d - \varepsilon\right), & \ell_\varepsilon = 0, & \text{if } 0 \leq \varepsilon \leq \varepsilon_d \\ q_\varepsilon = \varepsilon_d, & d_\varepsilon = 0, & \ell_\varepsilon = 0, & \text{if } \varepsilon_d \leq \varepsilon \leq \varepsilon_\ell \\ q_\varepsilon = \varepsilon\rho_\ell/\rho_d, & d_\varepsilon = 0, & \ell_\varepsilon = p\left(\varepsilon\rho_\ell/\rho_d - \varepsilon_d\right), & \text{if } \varepsilon_\ell \leq \varepsilon \leq \varepsilon_{\bar\ell}, \\ q_\varepsilon = \varepsilon_{\bar\ell}\rho_\ell/\rho_d, & d_\varepsilon = 0, & \ell_\varepsilon = \rho_\ell b, & \text{if } \varepsilon_{\bar\ell} \leq \varepsilon. \end{array} \tag{3.7}$$

The optimal borrowing and lending decisions follow the cut-off rules according to the realization of the taste shock. The cut-off levels, ε_d, ε_ℓ, and $\varepsilon_{\bar{\ell}}$ partition the set of taste shocks into four regions. For shocks lower than ε_d, a buyer deposits money at the standing facility; for shocks higher than ε_ℓ, the buyer borrows at the standing facility. For values between ε_d and ε_ℓ, the buyer does not use the central bank's standing facility. Finally, the cut-off value $\varepsilon_{\bar{\ell}}$ determines whether a buyer's collateral constraint is binding or not.

4. Equilibrium

We focus on symmetric stationary equilibria where money is used as a medium of exchange and there is a positive demand for nominal government bonds. Such equilibria meet the following requirements: (i) Households' decisions are optimal, given prices; (ii) The decisions are symmetric across all sellers and symmetric across all buyers with the same preference shock; (iii) The goods and bond markets clear; (iv) All real quantities are constant across time; (v) The law of motion for the stock of money (2.5) holds in each period.

Point (iv) requires that the real stock of money is constant; i.e.,

$$\phi M = \phi^+ M^+. \tag{4.1}$$

This implies that $\phi/\phi^+ = M^+/M \equiv \gamma$ where γ is the gross steady-state money growth rate. Symmetry requires $m = M^+$ and $b = B^+$. Let $\mathcal{B} = B/M$ denote the bonds-to-money ratio.

In any stationary equilibrium, ρ has to be constant. A constant bond price then implies that the bond-to-money ratio has to be constant, and this can be only achieved when the growth rates of money and bonds are equal. We assume there are positive initial stocks of money M_0 and government bonds B_0.[13]

Market clearing in the goods market requires

$$q_s - \int_0^\infty q_\varepsilon dF(\varepsilon) = 0, \tag{4.2}$$

where q_s is aggregate production by sellers in the goods market.

In equilibrium, the critical values solve:

[13]Since the assets are nominal objects, the central bank can start the economy off by one-time injections of cash M_0 and bonds B_0.

$$\varepsilon_\ell = \varepsilon_d \frac{\rho_d}{\rho_\ell} \text{ and } \varepsilon_{\bar{\ell}} = \varepsilon_d \frac{\rho_d}{\rho_\ell} \left(1 + \rho_\ell \mathcal{B}\right) \tag{4.3}$$

The critical values ε_ℓ and $\varepsilon_{\bar{\ell}}$ are functions of the interest rates i_d and i_ℓ, the bonds-to-money ratio $\mathcal{B}$, the critical value ε_d and the price of bonds ρ. In the Appendix, we show that ε_d and ρ solve equations (4.4) and (4.5) below.

Proposition 1. *An equilibrium is a policy (i_d, i_ℓ, γ) and endogenous variables (ρ, ε_d) satisfying*

$$\frac{\rho_d \gamma}{\beta} = \int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon) \tag{4.4}$$

$$\frac{\rho \gamma}{\beta} = \int_0^{\varepsilon_{\bar{\ell}}} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}} dF(\varepsilon) \tag{4.5}$$

A detailed derivation of (4.4)-(4.5) can be found in the Appendix. Equation (4.4) is obtained from the choice of money holdings (3.1). Equation (4.5) is obtained from (3.1) and (3.2); in any equilibrium with a strictly positive demand for money and bonds, we must have $\rho V_G^m(m, b) = V_G^b(m, b)$. We then use this arbitrage equation to derive (4.5).

One can easily solve these two equations numerically for ρ and ε_d. All remaining endogenous variables can then be calculated as follows: The critical values are obtained from (4.3). The amount of borrowing and depositing by a buyer and the amount of goods purchased by the buyer are obtained from (3.7); from (3.6), the real stock of money is $\phi M = \rho_d \varepsilon_d / \beta$ and the real stock of bonds is $\phi B = (\varepsilon_{\bar{\ell}} - \varepsilon_\ell)/\beta$. Finally, from the law of motion of money holdings (2.5) one obtains the value of τ that is consistent with the policy choice (i_d, i_ℓ, γ).

5. Optimal interest rate policy

The central bank chooses (i_d, i_ℓ, γ) to maximize (2.1) subject to (4.4) and (4.5).

Proposition 2. *The optimal policy is to set $\rho_d = \rho = \beta/\gamma$. This policy implements the first-best allocation.*

As in Cúrdia and Woodford (2010), it is optimal to set the deposit rate equal to the operating target rate for the policy rate (the money market rate i in our model) in each period. The optimal policy makes holding money costless and therefore satiates money demand as described by the Friedman rule.[14] In Cúrdia and Woodford (2010) the same policy satiates the demand for central bank reserves. Note that such a policy means that the money market rate and the central bank's deposit rate exactly compensate market participants for their impatience and for inflation. To see this, define $r = (1-\beta)/\beta$ and $\pi = \gamma - 1$ and rewrite $\gamma = \beta/\rho_d$ to get $1 + i_d = (1+\pi)(1+r)$, which is the Fisher equation. Under this policy, the rate of return of money is the same as the rate of return on government bonds. Hence, they have the same marginal liquidity value, which is zero.[15] Finally, note that under the optimal policy the lending rate is irrelevant, since buyers never borrow.

COROLLARY 1. *The optimal policy requires that the central bank is able to raise tax revenue.*

In the proof of Proposition 2, we show that the optimal policy requires that the central bank is able to receive tax revenue to finance the interest payments on deposits. From the law of motion of money holding, the lump-sum transfers that are necessary under the optimal policy are:

$$\tau^{opt} \equiv (\beta - 1)(\mathcal{B} + \gamma/\beta) < 0.$$

Thus, the optimal policy requires that the central bank is able to tax households directly or receives funds from the treasury.[16] Cúrdia and Woodford (2010) never discuss how the central bank finances interest on reserves. Implicitly, they must assume that the central bank can directly tax households or receives tax revenue from the treasury. It does not matter for the argument whether the central bank holds government bonds that finances interest on reserves. In this case, the government has to levy taxes to finance interest payments on the government bonds which it then hands over to the central bank.

In practice, central banks have no fiscal power to levy taxes and therefore need to rely on the treasury to provide the funds necessary to run the Friedman rule. Therefore, in the following section we study the case where the central bank ability to tax is limited or, for political reasons,

[14]See Andolfatto (2010) and Lagos (2010) who derive results on the optimality and implementation of the Friedman rule in the Search Theory of Money.

[15]Since the first-best quantities are $q_\varepsilon = \varepsilon$ with the support of ε being unbounded, the real value of money approaches infinity; i.e., the price level approaches zero. Any finite upperbound would yield a finite strictly positive price level.

[16]Recall that $\tau < 0$ is a tax, and $\tau > 0$ a subsidy.

the treasury is unwilling to transfer sufficient resources to run the Friedman rule. Under this friction, we study the optimal policy.

6. Optimal interest rate policy with taxing frictions

In this section, we assume that the central bank faces taxing frictions and derive the optimal interest rate policy for this case. In the previous section, we have shown that implementing the first-best allocation requires the central bank to levy lump-sum taxes equal to $\tau = \tau^{opt} < 0$. In Definition 1 below, we define the term *taxing friction.* Define $\tau^{\max}$ as the largest feasible tax.

Definition 1. *A central bank faces taxing friction if* $\tau^{\max} > \tau^{opt}$.

Note that it might well be that the central bank can raise taxes; i.e., $\tau^{\max} < 0$, but, the largest tax satisfies $0 > \tau^{\max} > \tau^{opt}$, preventing it from implementing the first-best allocation. A special case would be $\tau^{\max} = 0$; i.e., the central bank receives no tax revenue. In an economy with taxing frictions, the central bank cannot set $\tau = \tau^{opt}$. That might be for technical reasons since raising tax revenue is costly or it is politically costly to ask for tax revenues.

Lemma 2 below shows that if taxing frictions exist for the central bank, then in equilibrium $\gamma > \beta(1 + i_d)$.

Lemma 2. *In any equilibrium with taxing frictions* $\gamma > \beta(1 + i_d)$.

In the proof of Lemma 2, we show that taxing frictions imply $\gamma > \beta(1 + i_d)$. We use this latter condition to characterize the optimal policy in an economy with taxing frictions in Proposition 3 below.

Proposition 3. *If* $\gamma > \beta(1 + i_d)$, *it is optimal to choose a strictly positive interest rate spread. Furthermore, under the optimal policy the money market rate must satisfy* $i > i_d$.

Proposition 3 deviates from the optimal policy when there are no taxing frictions along two dimensions. First, it is optimal to choose a strictly positive interest-rate spread. Recall that without taxing frictions, the interest-rate spread is irrelevant. Second, with taxing frictions the optimal policy is to set the deposit rate strictly below the money market rate. Thus, with taxing frictions the optimal policy does not confirm the optimal policy in Cúrdia and Woodford (2010), who recommend setting the deposit rate equal to the operating target rate for the policy rate (the money market rate i in our model) in each period.

What is the economic rational for the optimality of a strictly positive spread? In the proof of Proposition 3, we show that if $\gamma > \beta(1 + i_d)$ and if the central bank sets $i_d = i_\ell$, then increasing the loan rate marginally is strictly welfare improving. Thus, it is never optimal to have a zero spread with taxing frictions. This result is driven by the reallocation of consumption that occurs from increasing the loan rate as depicted in Figure 4.

Figure 4 graphically illustrates of why a strictly positive interest-rate spread is welfare improving. The black dotted linear curve (the 45-line) plots the first-best consumption quantities. The red curve (labelled zero band) plots the quantities for some given policy $\gamma > \beta(1 + i_d)$ and a zero spread; i.e., $i_d = i_\ell$. Up to some critical value for ε, $\tilde{\varepsilon}$, the buyer receives the first-best consumption quantities after which the collateral constraint is binding, as indicated by the consumption quantities that are independent of ε. The blue curve (labelled positive band) plots the quantities for the same policy $\gamma > \beta(1 + i_d)$ and a strictly positive spread; i.e., $i_\ell > i_d$. Up to the critical value ε_d, the buyer consumes the first-best quantity; i.e., $q_\varepsilon = \varepsilon$. For $\varepsilon \in [0, \varepsilon_d]$ he deposits any excess money at the deposit facility. For $\varepsilon \in [\varepsilon_d, \varepsilon_\ell]$ he neither deposits nor borrows money. He simply spends all the money brought into the period and consumes $q_\varepsilon = \varepsilon_d$. For $\varepsilon \in [\varepsilon_\ell, \varepsilon_{\bar{\ell}}]$, the buyer borrows but his collateral constraint is non-binding. Finally, for $\varepsilon > \varepsilon_{\bar{\ell}}$ the collateral constraint is binding.

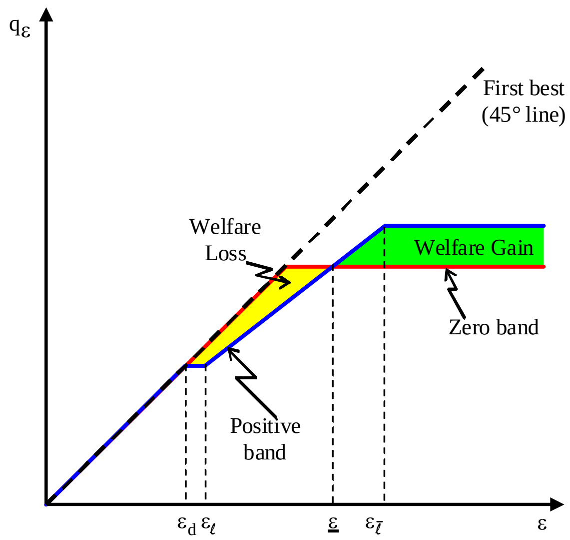

Figure 4: Welfare effects

As indicated by Figure 4, the welfare gain from increasing the borrowing rate i_ℓ marginally rises because it lowers the consumption of medium ε-buyers and increases the consumption of high-ε buyers. The first effect lowers welfare, while the second increases welfare. In the proof of Proposition 3, we show that, starting from $i_d = i_\ell$, the net gain is always positive.

The mechanism works as follows. By marginally increasing i_ℓ, the central bank makes it relatively more costly to turn bonds into money and hence consumption. This affects the portfolio choice of agents in the money market. The demand for money and hence its value increases. For those who are not borrowing-constrained, i.e., for buyers with $\varepsilon \in [\varepsilon_d, \varepsilon_{\bar{\ell}}]$ the higher marginal borrowing cost lowers their consumption at the margin. However, starting from $i_d = i_\ell$, this welfare loss is of second order. For those who are borrowing-constrained, i.e., for buyers with $\varepsilon > \varepsilon_{\bar{\ell}}$, the marginal higher borrowing cost has no effect on their consumption yet their higher real balances allow them to consume more. Again, starting from $i_d = i_\ell$, this welfare gain is of first order.

We have shown that, with taxing frictions, increasing the spread by increasing the loan rate is always welfare improving starting from $i_d = i_\ell$. Alternatively, one could consider lowering i_d starting from $i_d = i_\ell$. However, lowering i_d lowers the demand for money and hence its value, which reduces consumption for all constrained buyers and does not increase the consumption of unconstrained buyers. This is clearly welfare reducing (see the proof at the end of the Appendix).

7. Money market rate

In the previous section we established that a strictly positive interest-rate spread is optimal if the central bank cannot raise sufficient tax revenue to run the Friedman rule. A natural question to ask is what the optimal size of the spread is. We can not answer this question analytically because it depends on parameter values. Therefore, to get answers to this question, we now calibrate the model to the ECB's standing facilities.

Furthermore, we also use the calibration to study the behavior of the money market rate in the euro area. As can be seen from Figure 1, the Eurepo money market rate fluctuates around the middle of the ECB's interest-rate corridor until the fall of 2008. Thereafter, it fluctuates closely above the deposit rate set by the ECB. Moreover, the volume of deposits at the deposit facility and the volume of borrowing at the marginal lending facility of the ECB changed dramatically in response to the financial crisis that erupted after the Lehman bankruptcy. This can be seen in Figure 2 in the Introduction.

For what follows, we distinguish between two periods: the pre-Lehman period and the post-Lehman period. The former starts on March 4 2002 and ends on September 14 2008 and the latter starts on September 15 2008 and ends on March 31 2010.[17] Table 1, which we replicate here for easier reference, displays the average daily volumes of trades at the deposit and lending facilities of the ECB for the two periods. The average daily volume at the deposit facility of the ECB prior to the Lehman bankruptcy was 260 million euros. After the bankruptcy the average daily volume increased to 150092 million euros; i.e., it is now 578 times higher than before the Lehman bankruptcy.

Table 1: Volumes at the ECB[a,b]

	pre Lehman	post Lehman	ratio post/pre
Deposit Facility	260	150092	578
Marginal Lending Facility	195	1818	9.3
Ratio	1.33	82.6	

[a]Average daily volume in million euros at the deposit and lending facilities of the ECB.

[b]Pre-Lehman data from 31.3.2002 until 14.09.2008; Post-Lehman data from 14.09.2008 until 31.03.2010.

The average daily volume at the marginal borrowing facility displays a similar pattern, although the increase is less pronounced. The average daily volume prior to the Lehman bankruptcy was 195 million euros. After the bankruptcy, the average daily volume increased to 1818 million euros; i.e., it is now 9.3 times higher. In Table 1, we also display the ratio of deposits to lending prior to and after the Lehman bankruptcy. Beforehand, the ratio was 1.33 and now it is 82.6. We will use the former ratio as a target for our calibration, as explained below.

To explain the break in the behavior of the money market rate and the dramatic increase in the volumes at the deposit and borrowing facilities of the ECB after the Lehman bankruptcy we

[17]We start the pre-Lehman period March 31 2002 because this is the first day the ECB reports the EUREPO rate.

now calibrate the model. We then explore two possible explanations. The first hypothesis is that the Lehman bankruptcy triggered an aggregate variance shock that caused the Eurepo rate to change its behavior and the increase of deposits and loans at the ECB's standing facilities. The second hypothesis is that the Lehman bankruptcy triggered a collateral shock.

7.1. **Calibration and the optimal interest rate spread.** For the calibration, we assume that the model period is one day, and we only use data prior to the Lehman bankruptcy.[18] Throughout this paper, we report annualized interest rates. For the calibration we use the six targets listed in Table 2. We use the average of the annualized daily Eurepo rate for the pre-Lehman period. The average is 2.88%.[19] For the same period, we calculate the annualized interest rates at the lending facility and the deposit facility. The average values are 1.81% and 3.81%, respectively. We assume that the annualized real interest rate is 2%, and we calculate the inflation rate for this period. It is 2.32%. Finally, we use the ratio of deposits to lending at the ECB's standing facilities as explained in Table 1.

Table 2: Targets[a,b]

EUREPO rate	=	0.0288
Lending Facility: interest rate	=	0.0381
Deposit Facility: interest rate	=	0.0181
Real interest rate	=	0.02
Inflation	=	0.0232
Ratio of deposits to lending	=	1.33

[a] Interest rate targets are the average of the daily annualized rates.

[b] Pre-Lehman data from 31.3.2002 until 14.09.2008.

Given these targets, the calibration is straightforward. The policy variables of the model are i_d, i_ℓ and γ. We can directly use the deposit rate to set $i_d = 0.181$ and the lending rate to set $i_\ell = 0.381$. We use the inflation target to set $\gamma = 1.0232$ and the real interest rate target to

[18] We assume that a year has 300 days.

[19] We chose the Eurepo rate with the shorter maturity (less than one week), called the Eurepo TN rate, downloadable from www.eurepo.org.

set $\beta = 0.98$. The utility function in the goods market is assumed to be $u(q) = \frac{q^{1-\alpha}}{1-\alpha}$, and the preference shocks are assumed to be random draws from a uniform distribution with support $[0, \varepsilon_H]$.[20] We set $\varepsilon_H = 1$. We have experimented with other values for ε_H, but it did not affect the calibration and simulation results at all. We are left with the two parameters α and $\mathcal{B}$. We simultaneously use the Eurepo rate target and deposit-to-lending target to pin down α and $\mathcal{B}$. We find $\mathcal{B}$ =15.33 and $\alpha = 0.278$. The Mathematica file with the calculations is available on request.

Given this parametrization we can calculate the optimal interest-rate spread. Our calibration suggests that the ECB's spread was too narrow in the pre-Lehman area. Given the deposit rate of $i_d = 0.181$, our calibration suggests that the optimal lending rate is 0.067 instead of $i_\ell = 0.038$ as set by the ECB. Although this appears to be a large difference, our calibration suggests that the welfare implications are negligible. We only find that welfare would have been higher by 0.00065 percentage points if the ECB's had chosen $i_\ell = 0.067$ instead of $i_\ell = 0.038$.

The optimal interest-rate spread is independent of the value of ε_H. Nevertheless, it does depend on the choice of the distribution for the liquidity shocks ε. If we do the same experiment for a log-normal distribution, we find a much larger optimal interest-rate spread. But again, the welfare loss is modest. The message that can be taken from our calibration is that the welfare loss of not choosing an optimal spread is likely to be small. In the next section, we consider how well our model describes the behavior of the money market rate.

7.2. **Money market rate after Lehman.** In what follows we simulate the model to see how well the model is able to track the money market rate and the volumes traded at the standing facilities of the ECB prior and after the bankruptcy of Lehman brothers on September 15 2008. To simulate the money market rate we use the daily values of the ECB's deposit and borrowing rates and solve (4.4) and (4.5) for $\rho = 1/(1+i_t)$ and $\varepsilon_{d,t}$ for each t. In Figure 5 below, we then plot the model-generated money market rate i_t. Figure 5 shows that our model tracks the Eurepo very well up to the bankruptcy of Lehman brothers on September 15 2008. After this event, the Eurepo fluctuates very close to the deposit rate set by the ECB. Our simulated money

[20] The theoretical model presented earlier assumes $u(q) = \ln q$ and $\varepsilon_H \to \infty$. These two assumptions simplify the presentation of the model. However, it is straightforward to rewrite the model for the more general utility function $u(q) = \frac{q^{1-\alpha}}{1-\alpha}$, and a distribution of the idiosyncratic $\varepsilon-$shocks with a finite upper bound ε_H.

market rate does not follow this path, which suggest, that an aggregate shock affected the money markets around this time.

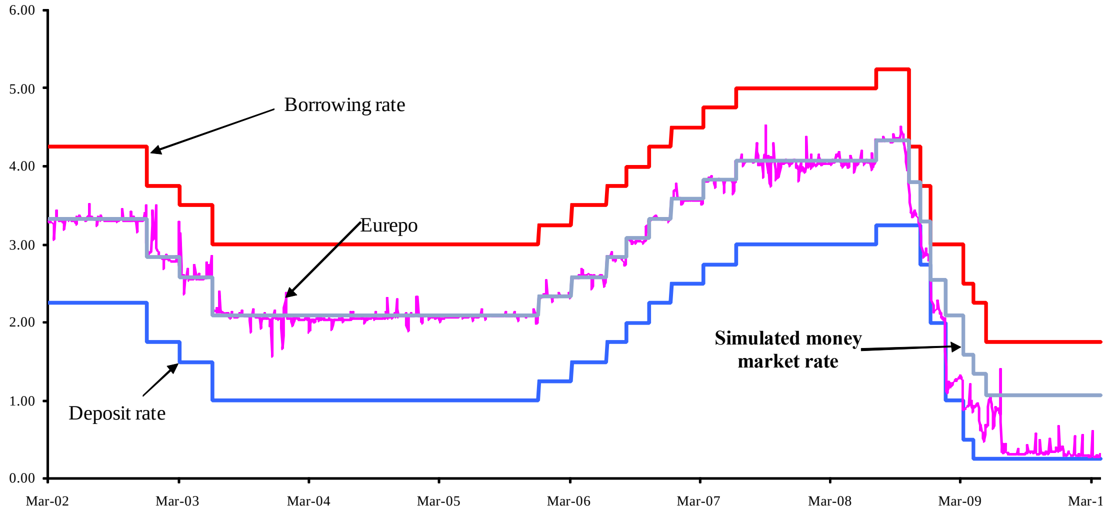

Figure 5: Simulated money market rate.

Table 3 also shows that the model fails to account for the large increase in volumes at the deposit and borrowing facilities of the ECB after the Lehman bankruptcy. In fact, in the simulation the volumes stay almost the same in the pre-Lehman and post-Lehman periods, while in the data they increase dramatically after the bankruptcy of Lehman brothers on September 15 2008.

Table 3: Simulated Eurepo and volumes at the ECB[a,b,c]

	pre-Lehman	post-Lehman	ratio post/pre
Eurepo	0.0289 (0.0288)	0.0171 (0.0112)	
Deposit Facility	0.2384	0.2379	0.9981 (578)
Marginal Lending Facility	0.1792	0.1795	1.0015 (9.3)
Ratio	1.3305 (1.32)	1.3259 (82.6)	

[a]Average daily volume in million euros at the deposit and lending facilities of the ECB.

[b]Pre-Lehman data from 31.3.2002 until 14.09.2008; Post-Lehman data from 14.09.2008 until 31.03.2010.

[c]Data from Table 1 in parenthesis.

7.3. **Aggregate shocks.** We now propose two aggregate shocks that might explain the behavior of the money market rate the large increase in volumes at the deposit and borrowing facilities of the ECB after the Lehman brother bankruptcy: an aggregate variance shock and an aggregate collateral shock.

7.3.1. *Aggregate variance shock.* Here we assume that the Lehman bankruptcy triggered an aggregate variance shock; i.e., it increased the variance of the idiosyncratic liquidity shocks. That is, we assume that on September 15 2008 the upper bound of the uniform distribution jumps from $\varepsilon_H = 1$ for the pre-Lehman area to $\varepsilon_H = 4.5$ for the post-Lehman area.

Table 4: Aggregate variance shock[a]

	pre-Lehman	post-Lehman	ratio post/pre
Eurepo	0.0289 (0.0288)	0.017 (0.0112)	
Deposit Facility	0.2384	53.24	223 (578)
Marginal Lending Facility	0.179	40.152	224 (9.3)
Ratio	1.33 (1.32)	1.326 (82.6)	

[a] Data from Table 1 in parenthesis.

Table 4 summarizes the effects of an aggregate variance shock. First, it does not affect the money market rate since the simulated money market rate is still 0.017 as for the case without such a shock (see Table 3). Second, the volume at the deposit facility increases by a factor of 223. This is not enough to explain the increase by a factor of 578 in the data. Third, the volume at the lending facility increases by the same factor. This is not consistent with the data where it only increases by a factor of 9.3. Finally, the ratio of deposits to loans does not increase. The reason is that this shock affects the volumes at the deposit and lending facilities proportionately.

7.3.2. *Aggregate collateral shock.* Here we assume that Lehman triggered an aggragate collateral shock in the money market. That is, we assume that the ratio of collateral to money dropped after the Lehman bankruptcy. This seems to be a reasonable aggregate shock since the Lehman bankruptcy dramatically increased the uncertainty about the validity of certain collateral. Moreover, the ECB and many other central banks increased the stock of central bank money, which also reduced the ratio of collateral to money. To construct Table 5, we have decreased the bonds-to-money ratio by 95%. That is, we assume that the bonds-to-money ratio dropped by 95% on September 15 2008 where it then stays until the end of the sample. As can be seen, this drop can explain to a large extent the behavior of the money market rate after the Lehman bankruptcy.

Table 5: Aggregate collateral shock[a]

	pre Lehman	post Lehman	ratio post/pre
Eurepo	0.0289 (0.0288)	0.0112 (0.0112)	
Deposit Facility	0.2384	0.5919	2.48 (578)
Marginal Lending Facility	0.1792	0.0298	0.166 (9.3)
Ratio	1.33 (1.32)	19.866 (82.6)	

[a] Data from Table 1 in parenthesis.

Table 5 summarizes the effects of an aggregate collateral shock. First, it decreases the money market rate substantially. In fact, the model now matches the Eurepo for the pre-Lehman period *and* the post-Lehman period. Second, the volume at the deposit facility increases by a factor of 2.48. This is not enough to explain the increase in the data by a factor of 578. Third, the volume at the lending facility decreases. This is not consistent with the data where it increases by a factor of 9.3. Finally, the ratio of deposits to loans increases by a factor of 19.866 which is again not enough to explain the increase in the data by a factor of 82.6.

7.3.3. *Combined aggregate shocks.* In what follows, we show that a combination of the two aggregate shocks studied above can explain to a large extent the behavior of the money market rate and the dramatic increase in volumes at the standing facilities of the ECB after the Lehman bankruptcy.

Table 6 summarizes the effects of a combined aggregate collateral and variance shock. To obtain these numbers, we have simply combined the two shocks described above. We find the following. First, the model simulation now matches the Eurepo for the pre-Lehman period *and* the post-Lehman period. Second, the volume at the deposit ratio increases by a factor of 555.

This is consistent with the data where the increase is by a factor of 578. Third, the volume at the lending facility increases by a factor of 33, which is too large since in the data it is only 9.3. Finally, the ratio of deposits to loans increases by a factor of 19.86, which is still not enough to explain the increase in the data by a factor of 82.6.

Table 6: Combined aggregate shocks[a]

	pre-Lehman	post-Lehman	ratio post/pre
Eurepo	0.0288 (0.0288)	0.0112 (0.0112)	
Deposit Facility	0.2384	132.4	555 (578)
Marginal Lending Facility	0.1792	6.66	37 (9.3)
Ratio	1.33 (1.32)	19.86 (82.6)	

[a] Data from Table 1 in parenthesis.

We believe that our model can successfully explain the behavior of the Eurepo and the volumes at the standing facilities of the ECB after the Lehman bankruptcy if we simultaneously take into account an aggregate variance and an aggregate collateral shock. This is not to say that other factors did also affect their behavior. Further research that identifies these factors could explain some of the remaining discrepancies between our model simulation and the data.

8. Conclusions

The focus of this paper is to study optimal policy in a general equilibrium model where market participants face idiosyncratic liquidity shocks, taking into account potential restrictions on the central bank's ability to extract tax revenue to pay interest on reserves. First, without restrictions on the central bank's ability to extract tax revenue, we prove that the optimal policy involves setting the deposit rate equal to the target rate. Second, with restrictions on the central bank's ability to extract tax revenue, the optimal policy necessitates setting the deposit rate strictly below the target rate, which involves a strictly positive interest-rate spread (the difference between the lending rate and deposit rate). We then calibrate the model to the ECB's channel system to explain the behavior of the Eurepo and the volumes at the standing facilities of the ECB. We find that if we simultaneously take into account an aggregate variance and an aggregate collateral shock that hit the economy at the time of the Lehman bankruptcy, the model can successfully replicate the behavior of the Eurepo and the volumes at the standing facilities of the ECB before and after the Lehman bankruptcy.

Appendix

Proof of Lemma 1. We first derive the cut-off values ε_d and ε_ℓ. For this proof, to the notation of the consumption level of a buyer, we add a subscript d if the buyer deposits money at the central bank, a subscript ℓ if the buyer takes out a loan and the collateral constraint is nonbinding, a subscript $\bar{\ell}$ if the buyer takes out a loan and the collateral constraint is binding, and a subscript 0 if the buyer does neither.

From (3.5), the consumption level of a buyer who enters the loan market satisfies:

$$q_d(\varepsilon) = \frac{\varepsilon}{p\beta\phi^+ (1+i_d)}, \quad q_\ell(\varepsilon) = \frac{\varepsilon}{p\beta\phi^+ (1+i_\ell)}, \tag{8.1}$$

where we have used the functional form $u(q) = \ln(q)$.

A buyer who does not use the deposit facilities will spend all his money on goods, since, if he anticipated that he would have idle cash after the goods trade, it would be optimal to deposit the idle cash in the intermediary, provided $i_d > 0$. Thus, consumption of such a buyer is:

$$q_0(\varepsilon) = \frac{m}{p}. \tag{8.2}$$

At $\varepsilon = \varepsilon_d$, the household is indifferent between depositing and not depositing. We can write this indifference condition as:

$$\varepsilon_d u(q_d) - \beta\phi^+ (pq_d - i_d d) = \varepsilon_d u(q_0) - \beta\phi^+ pq_0.$$

By using (8.1), (8.2), and $d = m - pq_d$, we can write the equation further as

$$\varepsilon_d \ln\left[\frac{\varepsilon_d}{\beta\phi^+ (1+i_d) m}\right] = \varepsilon_d - (1+i_d)\beta\phi^+ m.$$

The unique solution to this equation is $\varepsilon_d = (1+i_d)\beta\phi^+ m$, which implies that $\beta\phi^+ m < \varepsilon_d$.

At $\varepsilon = \varepsilon_\ell$, the household is indifferent between borrowing and not borrowing. We can write this indifference condition as

$$\varepsilon_\ell u\left(q_\ell\right) - \beta\phi^+\left(pq_\ell + i_\ell \ell\right) = \varepsilon_\ell u\left(q_0\right) - \beta\phi^+ pq_0.$$

Using (8.1), (8.2) and $\ell = pq_\ell - m$, we can write this equation further as

$$\varepsilon_\ell \ln\left[\frac{\varepsilon_\ell}{\left(1+i_\ell\right)\beta\phi^+ m}\right] = \varepsilon_\ell - \left(1+i_\ell\right)\beta\phi^+ m.$$

The unique solution to this equation is $\varepsilon_\ell = \left(1+i_\ell\right)\beta\phi^+ m$. Using the expression for ε_d we get

$$\varepsilon_\ell = \varepsilon_d\left(\rho_d/\rho_\ell\right). \tag{8.3}$$

We now calculate $\varepsilon_{\bar\ell}$. There is a critical buyer who enters the goods market and wants to take out a loan, whose collateral constraint is just binding. From (3.5), for this buyer we have the following equilibrium conditions: $q_{\bar\ell} = \frac{\rho_\ell \varepsilon_{\bar\ell}}{\beta\phi^+ p}$ and $pq_{\bar\ell} = m + \rho_\ell b$. Eliminating $q_{\bar\ell}$ we get

$$\varepsilon_{\bar\ell} = \left(1+i_\ell\right)\beta\phi^+ m + \beta\phi^+ b.$$

Using (8.3) we get

$$\varepsilon_{\bar\ell} = \varepsilon_d \frac{\rho_d}{\rho_\ell}\left(1 + \rho_\ell \frac{b}{m}\right).$$

It is then evident that

$$0 \leq \varepsilon_d \leq \varepsilon_\ell \leq \varepsilon_{\bar\ell}.$$

□

Proof of Proposition 1. In this proof we derive equations (4.4) and (4.5). We first derive equation (4.4). Differentiate $V_G\left(m,b\right) = \int_0^\infty V_G\left(m,b|\,\varepsilon\right) dF\left(\varepsilon\right)$ with respect to m to get

$$V_G^m\left(m,b\right) = \int_0^\infty \left[\beta V_M^m\left(m + \ell_\varepsilon - pq_\varepsilon - d_\varepsilon, b, \ell_\varepsilon, d_\varepsilon|\,\varepsilon\right) + \beta\phi^+\lambda_\varepsilon\right] dF\left(\varepsilon\right).$$

Then, use (3.3) to replace V_M^m and (3.5) to replace $\beta\phi^+\lambda_\varepsilon$ to obtain

$$V_G^m(m,b) = \int_0^\infty \frac{\varepsilon u'(q_\varepsilon)}{p} dF(\varepsilon). \tag{8.4}$$

Use the first-order condition for q_s in (3.4) to get

$$V_G^m(m,b) = \beta\phi^+(1+i_d)\int_0^\infty \varepsilon u'(q_\varepsilon)\,dF(\varepsilon).$$

Using (3.1) to replace $V_G^m(m,b)$ and (4.1) to replace the ϕ^+/ϕ yields

$$\frac{\gamma}{\beta} = (1+i_d)\int_0^\infty \varepsilon u'(q_\varepsilon)\,dF(\varepsilon).$$

Finally, note that $u'(q) = 1/q$ and replace q_ε using Lemma 1 to get

$$\frac{\rho_d\gamma}{\beta} = \int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^\infty \frac{\varepsilon}{\varepsilon_{\bar{\ell}}}\frac{\rho_d}{\rho_\ell} dF(\varepsilon) \tag{8.5}$$

Note that since $\varepsilon_\ell = \varepsilon_d\frac{\rho_d}{\rho_\ell}$ and $\varepsilon_{\bar{\ell}} = \left(\varepsilon_d\frac{\rho_d}{\rho_\ell}\right)(1+\rho_\ell\mathcal{B})$, (8.5) yields ε_d.

We obtain the second equation that determines ρ as follows. In any equilibrium with a strictly positive demand for money and bonds, we must have $\rho V_G^m(m,b) = V_G^b(m,b)$. We now use this arbitrage equation to derive (4.5). We have already derived $V_G^m(m,b)$ above. To get $V_G^b(m,b)$ differentiate $V_G(m,b)$ by b to get

$$V_G^b(m,b) = \int_0^\infty \left[\beta V_M^b(m+\ell_\varepsilon - pq_\varepsilon - d_\varepsilon, b, \ell_\varepsilon, d_\varepsilon|\,\varepsilon) + \rho_\ell\beta\phi^+\lambda_\ell\right] dF(\varepsilon).$$

Use (3.3) to replace V_M^b to get

$$V_G^b(m,b) = \beta\phi^+\int_0^\infty (1+\rho_\ell\lambda_\ell)\,dF(\varepsilon).$$

Use (3.5) to replace λ_ℓ and rearrange to get

$$V_G^b(m,b) = \int_0^{\varepsilon_{\bar{\ell}}} \beta\phi^+ dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \rho_\ell \frac{\varepsilon u'(q_\varepsilon)}{p} dF(\varepsilon).$$

Use the first-order condition for q_s in (3.4) to get:

$$V_G^b(m,b) = \int_0^{\varepsilon_{\bar{\ell}}} \beta\phi^+ dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \beta\phi^+ (\rho_\ell/\rho_d)\, \varepsilon u'(q_\varepsilon)\, dF(\varepsilon).$$

Equate $\rho V_G^m(m,b) = V_G^b(m,b)$ and simplify to get

$$\int_0^{\infty} \varepsilon u'(q_\varepsilon)\, dF(\varepsilon) = \int_0^{\varepsilon_{\bar{\ell}}} (\rho_d/\rho)\, dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} (\rho_\ell/\rho)\, \varepsilon u'(q_\varepsilon)\, dF(\varepsilon).$$

Note that $\int_0^\infty \varepsilon u'(q_\varepsilon)\, dF(\varepsilon) = \rho_d\gamma/\beta$ and use Lemma 1

$$\frac{\rho\gamma}{\beta} = \int_0^{\varepsilon_{\bar{\ell}}} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}} dF(\varepsilon). \tag{8.6}$$

Note that since $\varepsilon_\ell = \varepsilon_d \frac{\rho_d}{\rho_\ell}$ and $\varepsilon_{\bar{\ell}} = \left(\varepsilon_d \frac{\rho_d}{\rho_\ell}\right)(1+\rho_\ell \mathcal{B})$, ρ depends on ε_d only. □

Proof of Proposition 2. Setting $\rho_d = \rho$ reduces (4.4) and (4.5) as follows

$$\frac{\rho\gamma}{\beta} = \int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\rho}{\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}}\frac{\rho}{\rho_\ell} dF(\varepsilon)$$

$$\frac{\rho\gamma}{\beta} = \int_0^{\varepsilon_{\bar{\ell}}} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}} dF(\varepsilon).$$

Then, equilibrium requires that

$$\int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\rho}{\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}}\frac{\rho}{\rho_\ell} dF(\varepsilon) = \int_0^{\varepsilon_{\bar{\ell}}} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{\varepsilon_{\bar{\ell}}} dF(\varepsilon)$$

This equation holds if and only if $\varepsilon_d \longrightarrow \infty$. Then, from (4.3), $\varepsilon_\ell, \varepsilon_{\bar{\ell}} \longrightarrow \infty$. Thus, from Lemma 1, the first-best allocation $q_\varepsilon = \varepsilon$ for all ε is attained. Moreover, from (4.5), it is clear that the money market rate must satisfies $\rho = \beta/\gamma$.

We next show that the optimal policy requires that the government is able to tax agents. We first derive the law of motion of money holdings in equilibrium. In any equilibrium, the sellers' money holdings satisfy

$$pq_s = M + L - \int_0^{\varepsilon_d} (M - pq_\varepsilon)\, dF(\varepsilon).$$

The left-hand side is the aggregate money receipts of sellers. The right-hand side is the beginning of period quantity of money, M; plus aggregate lending of money by the central bank, L; minus deposits by late-buyers at the central bank. These buyers simply deposit any "idle" money to receive interest on it. Furthermore, in any equilibrium aggregate deposits satisfy

$$D = pq_s + \int_0^{\varepsilon_d} (M - pq_\varepsilon)\, dF(\varepsilon),$$

where pq_s is deposits by sellers. These two equations imply that in any equilibrium, total deposits satisfy

$$D = M + L. \tag{8.7}$$

From Lemma 1, we know that only buyers with a shock $\varepsilon \geq \varepsilon_\ell$ borrow. Thus aggregate lending is $L = \int_{\varepsilon_\ell}^{\infty} \ell_\varepsilon dF(\varepsilon)$. From Lemma 1, we also know that $\ell_\varepsilon = p\left[(\rho_\ell/\rho_d)\,\varepsilon - \varepsilon_d\right]$ if $\varepsilon_\ell \leq \varepsilon \leq \varepsilon_{\bar{\ell}}$, and $\ell_\varepsilon = b/(1+i_\ell) = p\left[(\rho_\ell/\rho_d)\,\varepsilon_{\bar{\ell}} - \varepsilon_d\right]$ if $\varepsilon \geq \varepsilon_{\bar{\ell}}$. Thus, real aggregate lending is

$$L/p = \Psi, \tag{8.8}$$

where

$$\Psi \equiv \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \left[(\rho_\ell/\rho_d)\,\varepsilon - \varepsilon_d\right] dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[(\rho_\ell/\rho_d)\,\varepsilon_{\bar{\ell}} - \varepsilon_d\right] dF(\varepsilon).$$

Divide both sides of (2.5) by M to get

$$\gamma = 1 + \frac{i_d D - i_\ell L}{M} + \mathcal{B}(1 - \rho\gamma) + \tau.$$

Eliminating D and L using (8.7), respectively (8.8), and noting that $M/p = \varepsilon_d$, the last expression can be rewritten as follows

$$\gamma = 1 + i_d + \frac{i_d - i_\ell}{\varepsilon_d}\Psi + \mathcal{B}\left(1 - \rho\gamma\right) + \tau. \tag{8.9}$$

Finally, under the optimal policy $\rho_d = \rho$ we have $\rho = \beta/\gamma$. Replacing ρ_d and ρ by β/γ and noting that $\Psi = 0$ under the optimal policy yields

$$\tau = \tau^{opt} \equiv \left(\gamma/\beta + \mathcal{B}\right)\left(\beta - 1\right) \leq 0.$$

Thus, the optimal policy requires that the central bank is able to raise tax revenue. □

Proof of Lemma 2. In an economy with taxing frictions, $\tau > \tau^{opt} = \left(\gamma/\beta + \mathcal{B}\right)\left(\beta - 1\right)$. Use (8.9) to substitute τ to get

$$\gamma - \left(1 + i_d\right) - \frac{i_d - i_\ell}{\varepsilon_d}\Psi - \mathcal{B}\left(1 - \rho\gamma\right) > \left(\gamma/\beta + \mathcal{B}\right)\left(\beta - 1\right).$$

Rewrite the previous expression to get

$$\gamma/\beta - \left(1 + i_d\right) + \frac{i_\ell - i_d}{\varepsilon_d}\Psi + \mathcal{B}\beta\left(\rho\gamma/\beta - 1\right) > 0.$$

In any equilibrium, $\frac{i_\ell - i_d}{\varepsilon_d}\Psi \geq 0$ and $\rho\gamma/\beta \geq 1$. Hence, in any equilibrium with taxing frictions,

$$\gamma/\beta - \left(1 + i_d\right) > 0.$$

□

Proof of Proposition 3. Equations (4.4) and (4.5) are block recursive. We can first solve (4.4) for ε_d.[21] Once we know ε_d, we get ρ from (4.5).

[21]Such a value exists and is unique, since the right-hand side of (4.4) is decreasing in ε_d. Furthermore, the right-hand side is approaching infinity for $\varepsilon_d \to 0$ and is approaching 1 for $\varepsilon_d \to \infty$. Accordingly, there exists a unique value for $\varepsilon_d \in (0, \infty)$ that solves (4.4) if $\gamma/\beta > 1/\rho_d$.

The welfare function is

$$\begin{aligned}\mathcal{W} &= \int_0^{\varepsilon_d} [\varepsilon u(q_\varepsilon) - q_\varepsilon]\, dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} [\varepsilon u(q_\varepsilon) - q_\varepsilon]\, dF(\varepsilon) \\ &\quad + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} [\varepsilon u(q_\varepsilon) - q_\varepsilon]\, dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} [\varepsilon u(q_\varepsilon) - q_\varepsilon]\, dF(\varepsilon).\end{aligned} \tag{8.10}$$

To show that it is never optimal to choose a zero band, we calculate $d\mathcal{W}/d\rho_\ell$, evaluate it $\rho_\ell = \rho_d = \rho$, and then show that $d\mathcal{W}/d\rho_\ell|_{\rho_\ell=\rho_d=\rho} < 0$.

Note that ρ_ℓ affects $\mathcal{W}$ directly and indirectly via ε_d; that is

$$\frac{d\mathcal{W}}{d\rho_\ell} = \frac{\partial \mathcal{W}}{\partial \varepsilon_d}\frac{d\varepsilon_d}{d\rho_\ell} + \frac{\partial \mathcal{W}}{\partial \rho_\ell}.$$

We get the term $\frac{d\varepsilon_d}{d\rho_\ell}$ by taking the total derivative of the equilibrium equation (4.4), which we replicate here for easier reference:

$$\frac{\rho_d \gamma}{\beta} = \int_0^{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon \rho_d}{\varepsilon_{\bar{\ell}} \rho_\ell} dF(\varepsilon).$$

From this equation, we get

$$\frac{d\varepsilon_d}{d\rho_\ell} = -\frac{\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \frac{\rho_d}{(\rho_\ell)^2} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon \mathcal{B}}{\varepsilon_d (1+\rho_\ell \mathcal{B})^2} dF(\varepsilon)}{\int_{\varepsilon_d}^{\varepsilon_\ell} \frac{\varepsilon}{(\varepsilon_d)^2} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\varepsilon}{(\varepsilon_d)^2 (1+\rho_\ell \mathcal{B})} dF(\varepsilon)} < 0,$$

since $\varepsilon_\ell = \varepsilon_d \frac{\rho_d}{\rho_\ell}$ and $\varepsilon_{\bar{\ell}} = \varepsilon_d \frac{\rho_d}{\rho_\ell}(1 + \rho_\ell \mathcal{B})$.

The partial derivative $\frac{\partial \mathcal{W}}{\partial \varepsilon_d}$ is

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial \varepsilon_d} &= \int_0^{\varepsilon_d} [\varepsilon u'(q_\varepsilon) - 1] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} [\varepsilon u'(q_\varepsilon) - 1] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon) \\ &\quad + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} [\varepsilon u'(q_\varepsilon) - 1] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} [\varepsilon u'(q_\varepsilon) - 1] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon).\end{aligned}$$

Using (3.7), we can write this partial derivative as follows:

$$\frac{\partial \mathcal{W}}{\partial \varepsilon_d} = \int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u'(q_\varepsilon) - 1\right] dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] (1 + \rho_\ell \mathcal{B})\, dF(\varepsilon). \tag{8.11}$$

Note that $\frac{\partial \mathcal{W}}{\partial \varepsilon_d}$ is strictly positive.

For $\frac{\partial \mathcal{W}}{\partial \rho_\ell}$ we get

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial \rho_\ell} &= \int_0^{\varepsilon_d} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_\ell} dF(\varepsilon) \\ &\quad + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_\ell} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_\ell} dF(\varepsilon),\end{aligned}$$

which using (3.7) can be written as

$$\frac{\partial \mathcal{W}}{\partial \rho_\ell} = \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{\varepsilon}{\rho_d} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] \varepsilon_d \mathcal{B} dF(\varepsilon),$$

which is strictly positive. This implies that $\frac{d\mathcal{W}}{d\rho_\ell}$ can be either positive or negative. Increasing ρ_ℓ (decreasing i_ℓ) has a positive effect on welfare through its direct effect $\frac{\partial \mathcal{W}}{\partial \rho_\ell}$, but an negative effect through its indirect effect $\frac{\partial \mathcal{W}}{\partial \varepsilon_d}\frac{d\varepsilon_d}{d\rho_\ell}$.

We now evaluate these derivatives at $\rho_\ell = \rho_d = \rho$. We get

$$\begin{aligned}\left.\frac{d\varepsilon_d}{d\rho_\ell}\right|_{\rho_\ell=\rho_d=\rho} &= -\frac{\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\rho\varepsilon\mathcal{B}}{\varepsilon_{\bar{\ell}}(1+\rho\mathcal{B})} dF(\varepsilon)}{\int_{\varepsilon_{\bar{\ell}}}^{\infty} \frac{\rho\varepsilon}{\varepsilon_d \varepsilon_{\bar{\ell}}} dF(\varepsilon)} \\ \left.\frac{\partial \mathcal{W}}{\partial \rho_\ell}\right|_{\rho_\ell=\rho_d=\rho} &= \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] \varepsilon_d \mathcal{B} dF(\varepsilon) \\ \left.\frac{\partial \mathcal{W}}{\partial \varepsilon_d}\right|_{\rho_\ell=\rho_d=\rho} &= \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] (1 + \rho\mathcal{B})\, dF(\varepsilon),\end{aligned}$$

since $\varepsilon_d = \varepsilon_\ell$ and $\varepsilon_{\bar{\ell}} = \varepsilon_d(1 + \rho\mathcal{B})$ at $\rho_\ell = \rho_d = \rho$. Use these expressions to write $\left.\frac{d\mathcal{W}}{d\rho_\ell}\right|_{\rho_\ell=\rho_d=\rho}$ as follows

$$\left.\frac{d\mathcal{W}}{d\rho_\ell}\right|_{\rho_\ell=\rho_d=\rho} = -\int_{\varepsilon_{\bar{\ell}}}^{\infty}\left[\varepsilon u'\left(q_\varepsilon\right)-1\right]dF\left(\varepsilon\right)\frac{\left(\varepsilon_{\bar{\ell}}\right)^2\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}}dF\left(\varepsilon\right)}{\rho\int_{\varepsilon_{\bar{\ell}}}^{\infty}\varepsilon dF\left(\varepsilon\right)}<0.$$

Hence, a marginal decrease of ρ_ℓ (marginal increase of i_ℓ) from $\rho_\ell=\rho_d=\rho$ is welfare improving. It follows that a zero band is not optimal policy if $\gamma/\beta>\rho_d$. □

Proof that $\frac{\partial\mathcal{W}}{\partial\rho_d}<0$. The welfare function is

$$\begin{aligned}\text{(8.12)}\qquad \mathcal{W} &= \int_0^{\varepsilon_d}\left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right]dF\left(\varepsilon\right)+\int_{\varepsilon_d}^{\varepsilon_\ell}\left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right]dF\left(\varepsilon\right)\\ &\quad+\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}}\left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right]dF\left(\varepsilon\right)+\int_{\varepsilon_{\bar{\ell}}}^{\infty}\left[\varepsilon u\left(q_\varepsilon\right)-q_\varepsilon\right]dF\left(\varepsilon\right).\end{aligned}$$

To show that it is never optimal to lower i_d if $\gamma/\beta>1+i_d$, we calculate $d\mathcal{W}/d\rho_d$. Note that ρ_d affects $\mathcal{W}$ directly and indirectly via ε_d; that is

$$\frac{d\mathcal{W}}{d\rho_d}=\frac{\partial\mathcal{W}}{\partial\varepsilon_d}\frac{d\varepsilon_d}{d\rho_d}+\frac{\partial\mathcal{W}}{\partial\rho_d}.$$

We get the term $\frac{d\varepsilon_d}{d\rho_d}$ by taking the total derivative of the equilibrium equation (4.4) which we replicate here for easier reference:

$$\frac{\rho_d\gamma}{\beta}=\int_0^{\varepsilon_d}dF\left(\varepsilon\right)+\int_{\varepsilon_d}^{\varepsilon_\ell}\frac{\varepsilon}{\varepsilon_d}dF\left(\varepsilon\right)+\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}}\frac{\rho_d}{\rho_\ell}dF\left(\varepsilon\right)+\int_{\varepsilon_{\bar{\ell}}}^{\infty}\frac{\varepsilon}{\varepsilon_d\left(1+\rho_\ell\mathcal{B}\right)}dF\left(\varepsilon\right).$$

From this equation, we get

$$\frac{d\varepsilon_d}{d\rho_d}=-\frac{\frac{\gamma}{\beta}-\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}}\frac{1}{\rho_\ell}dF\left(\varepsilon\right)}{\int_{\varepsilon_d}^{\varepsilon_\ell}\frac{\varepsilon}{\left(\varepsilon_d\right)^2}dF\left(\varepsilon\right)+\int_{\varepsilon_{\bar{\ell}}}^{\infty}\frac{\varepsilon}{\left(\varepsilon_d\right)^2\left(1+\rho_\ell\mathcal{B}\right)}dF\left(\varepsilon\right)}<0,$$

since $\varepsilon_\ell=\varepsilon_d\frac{\rho_d}{\rho_\ell}$ and $\varepsilon_{\bar{\ell}}=\varepsilon_d\frac{\rho_d}{\rho_\ell}\left(1+\rho_\ell\mathcal{B}\right)$.

The partial derivative $\frac{\partial \mathcal{W}}{\partial \varepsilon_d}$ is

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial \varepsilon_d} &= \int_0^{\varepsilon_d} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon) \\ &\quad + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\varepsilon_d} dF(\varepsilon).\end{aligned}$$

Using (3.7), we can write this partial derivative as follows:

$$\frac{\partial \mathcal{W}}{\partial \varepsilon_d} = \int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u'(q_\varepsilon) - 1\right] dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] (1 + \rho_\ell \mathcal{B})\, dF(\varepsilon).$$

Note that $\frac{\partial \mathcal{W}}{\partial \varepsilon_d}$ is strictly positive.

For $\frac{\partial \mathcal{W}}{\partial \rho_d}$ we get

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial \rho_d} &= \int_0^{\varepsilon_d} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_d} dF(\varepsilon) + \int_{\varepsilon_d}^{\varepsilon_\ell} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_d} dF(\varepsilon) \\ &\quad + \int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_d} dF(\varepsilon) + \int_{\varepsilon_{\bar{\ell}}}^{\infty} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{dq_\varepsilon}{d\rho_d} dF(\varepsilon),\end{aligned}$$

which using (3.7) can be written as

$$\frac{\partial \mathcal{W}}{\partial \rho_d} = -\int_{\varepsilon_\ell}^{\varepsilon_{\bar{\ell}}} \left[\varepsilon u'(q_\varepsilon) - 1\right] \frac{\rho_\ell}{(\rho_d)^2} dF(\varepsilon) < 0,$$

which is strictly negative. This implies that $\frac{d\mathcal{W}}{d\rho_d}$ is negative. Thus, increasing ρ_d (decreasing i_d) has a negative effect on welfare. It follows that increasing i_d is optimal if $\gamma/\beta > \rho_d$. □

References

[1] Andolfatto, D. 2010. Essential interest-bearing money, Journal of Economic Theory, 145, 1495-1507.

[2] Araujo, L. 2004. Social norms and money. Journal of Monetary Economics 51, 241-256.

[3] Berentsen, A. and Monnet, C. 2008. Monetary policy in a channel system. Journal of Monetary Economics, 55, 1067-1080.

[4] Berentsen, A. and Waller, C. 2010. Outside versus inside bonds: A Modigliani-Miller type result for liquidity constrained households, Journal of Economic Theory, forthcoming.

[5] Cúrdia, V. and Woodford, M. 2010. The central-bank balance sheet as an instrument of monetary policy. Journal of Monetary Economics, forthcoming.

[6] Kocherlakota, N. 1998. Money is memory. Journal of Economic Theory 81, 232-251.

[7] Kocherlakota, N. 2003. Societal benefits of illiquid bonds. Journal of Economic Theory, 108, 179-193.

[8] Lagos, R. 2010. Some results on the optimality and implementation of the Friedman rule in the Search Theory of Money. Journal of Economic Theory 145 (July): 1508–24.

[9] Lagos, R. 2010. Asset Prices, Liquidity, and Monetary Policy in the Search Theory of Money. Federal Reserve Bank of Minneapolis Quarterly Review, 33, 14–20.

[10] Lagos, R. and Rocheteau, G. 2008. Money and capital as competing media of exchange. Journal of Economic Theory 142, 247-258.

[11] Lagos, R. and Wright, R. 2005. A unified framework for monetary theory and policy analysis. Journal of Political Economy 113, 463-484.

[12] Lester, B., Postlewaite, A. and Wright, L. 2010. Information and liquidity. Journal of Money, Credit and Banking, forthcoming.

[13] Martin, A. and Monnet, C. 2010. Monetary policy implementation frameworks: A Comparative Analysis. Macroeconomic Dynamics, forthcoming.

[14] Shi, S. 2008. Efficiency improvement from restricting the liquidity of nominal bonds. Journal of Monetary Economics, 55, 1025-1037.

[15] Wallace, N. 2001. Whither monetary economics? International Economic Review 42, 847-869.

(Aleksander Berentsen) University of Basel and Federal Reserve Bank of St.Louis

(Alessandro Marchesiani) University of Minho

(Christopher J. Waller) Federal Reserve Bank of St. Louis and University of Notre Dame