NECESSARY AND SUFFICIENT RESTRICTIONS FOR EXISTENCE OF A UNIQUE FOURTH MOMENT OF A UNIVARIATE GARCH(P,Q) PROCESS

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Abstract

A univariate GARCH(p,q) process is quickly transformed to a univariate autoregressive moving-average process in squares of an underlying variable. For positive integer m, eigenvalue restrictions have been proposed as necessary and sufficient restrictions for existence of a unique mth moment of the output of a univariate GARCH process or, equivalently, the 2mth moment of the underlying variable. However, proofs in the literature that an eigenvalue restriction is necessary and sufficient for existence of unique 4th or higher even moments of the underlying variable, are either incorrect, incomplete, or unecessarily long. Thus, the paper contains a short and general proof that an eigenvalue restriction is necessary and sufficient for existence of a unique 4th moment of the underlying variable of a univariate GARCH process. The paper also derives an expression for computing the 4th moment in terms of the GARCH parameters, which immediately implies a necessary and sufficient inequality restriction for existence of the 4th moment. Because the inequality restriction is easily computed in a finite number of basic arithmetic operations on the GARCH parameters and does not require computing eigenvalues, it provides an easy means for computing "by hand" the 4th moment and for checking its existence for low-dimensional GARCH processes. Finally, the paper illustrates the computations with some GARCH(1,1) processes reported in the literature.

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1. Introduction.

A univariate GARCH(p,q) process (Engle, 1982; Bollerslev, 1986) is quickly transformed to a univariate autoregressive moving-average (ARMA) process in squares of an underlying variable. Henceforth, for brevity, unless otherwise qualified, we indicate "GARCH(p,q)" by "GARCH." GARCH processes and their generalizations (Mittnik, Paolella, and Rachev, 2002) have been used to model volatilities or time-varying variances of underlying variables, usually financial-asset returns or residuals from estimated time-series models. GARCH processes are usually considered to have Gaussian or normally distributed disturbances, however, because asset returns can have large volatilities, such as the unexpected large drop in the U.S. stock market in October 1987, their distributions are often ascribed "heavier" tails than implied by the Gaussian distribution. Thus, Mandelbrot (1963a,b) studied asset-return distributions using stable-Paretian distributions. More recently, McCulloch (1997), Rachev and Mittnik (2000), and others studied time series of asset returns using GARCH and other processes driven by stable-Paretian and other heavy-tailed disturbances. Thus, it is useful to have a method for easily computing the 4th moment of an underlying variable of a univariate GARCH process, to check whether it exists and, if so, whether it indicates heavier than Gaussian tails. All the moments considered here are unconditional.

For positive integer m, eigenvalue restrictions have been proposed as necessary and sufficient conditions for existence of a unique 2mth moment of the underlying variable of a univariate GARCH process. Proofs in the literature that an eigenvalue restriction is necessary and sufficient for existence of unique 4th or higher even moments of the underlying variable of a GARCH process are either incorrect, incomplete, or unecessarily long. Before detailing the present paper's contribution to this literature, we clarify our use of the term "4th moment." A GARCH process is a type of ARMA process which linearly transforms squared disturbances, ε_t^2, to squared variables, y_t^2, so that the process nonlinearly transforms unsquared or underlying disturbances, ε_t, to unsquared or underlying variables, y_t. Thus, the "2mth moment of the underlying variable of a GARCH process" is the 2mth moment of y_t and is equivalent to the mth moment of y_t^2. In this regard, a GARCH process is (covariance) stationary if and only if the underlying variable, y_t, has a 2nd moment. Thus, unless otherwise qualified, "4th moment" means the 4th moment of y_t.

The paper contains a short and general proof that an eigenvalue restriction is necessary and sufficient for existence of the unique 4th moment of the underlying variable of a univariate GARCH process. The paper derives an expression for computing the 4th moment in terms of the GARCH parameters, which immediately implies a necessary and sufficient inequality restriction for checking the moment's existence. Because the eigenvalue and inequality restrictions are separately necessary and sufficient for the moment's existence, they are equivalent. Because the inequality restriction is easily computed in a finite number of basic arithmetic operations on the GARCH parameters and does not require computing eigenvalues, it provides an easy means for computing "by hand" the 4th moment and for checking its existence for low-dimensional GARCH processes.

The recent literature contains the following related results. In the following, all statements of "necessity" and "sufficiency" refer to the existence of the 4th moment of the underlying variable of a general univariate GARCH process.

He and Terasvirta (1999) stated an inequality restriction (theorem 1, p. 827) on a univariate GARCH process, which they claim (p. 833-840) is necessary and sufficient. Ling and McAleer (2002) question (p. 724 and p. 728, note 1) whether He and Terasvirta's necessity proof is complete. In any case, because they do not use a state-space representation, He and Terasvirta's discussion is unnecessarily long. Karanasos (1999) stated (theorem 3.1, p. 66) an inequality restriction and proved (pp. 73-74) its necessity, but, as Ling and McAleer noted (pp. 723-724), he did not prove its sufficiency. Ling (1999) stated and proved (theorem 6.2, p. 702) an eigenvalue restriction's sufficiency.

Ling and McAleer (2002) stated (theorem 2.1, pp. 723-724) an eigenvalue restriction and purportedly proved (pp. 726-727) its necessity. However, they actually proved necessity only for the special case $\alpha_1 > 0$. Their proof works if and only if (iff) $\alpha_1 > 0$ or $\beta_1 > 0$, but not if $\alpha_1 = \beta_1 = 0$. Ling and McAleer claim (p. 727) their proof still works if $\alpha_1 = \beta_1 = 0$, but as written it does not because it requires a vector R to have all positive elements after a certain number of repeated steps and this is not the case if $\alpha_1 = \beta_1 = 0$. By contrast, the proof given here holds for any univariate GARCH process.

Eigenvalue restrictions require computing eigenvalues, which can generally be done analytically "by hand" only for matrices no larger than three dimensional, hence, for GARCH processes with no more than three lags.

Thus, generally, eigenvalue restrictions can be checked only numerically on a computer. By contrast, as illustrated in section 4, the present necessary and sufficient inequality restriction for the 4th moment's existence is easily checked "by hand" in a finite number of basic arithmetic operations.

The remainder of the paper is organized as follows. Section 2 states a univariate GARCH process in state-space form in order to derive the 4th-moment inequality restriction. Section 3 proves that the inequality restriction is necessary and sufficient for existence of a unique 4th moment of the underlying variable of a univariate GARCH process of any degree. Section 3 also proves that the 4th-moment inequality restriction is equivalent to an eigenvalue restriction. Section 4 numerically illustrates the 4th-moment inequality restriction with six GARCH(1,1) processes from the literature. Section 5 contains concluding remarks.

2. State-Space Form of a GARCH(p,q) Process.

For discrete-time periods t, let $y_t^2 = \varepsilon_t^2 h_t$ denote the square of the underlying variable in the process, where h_t is generated by the univariate GARCH process

$$(2.1) \qquad h_t = \alpha_0 + \sum_{i=1}^{n} \alpha_i \varepsilon_{t-i}^2 h_{t-i} + \sum_{i=1}^{n} \beta_i h_{t-i} ,$$

α_i and β_i are constant parameters, $n = \max(p,q)$, $\alpha_i = 0$ for $i > p$ if $p < n$, $\beta_i = 0$ for $i > q$ if $q < n$, and ε_t is a disturbance. Although the aim is to verify necessary and sufficient restrictions on the GARCH(p,q) parameters for existence of the 4th moment of the underlying variable, $Ey_t^4 = E(\varepsilon_t^4 h_t^2)$, because we assume the disturbance's 4th moment, $E\varepsilon_t^4$, exists, we concentrate on restrictions for existence of Eh_t^2.

Notation is simplified but no generality is lost when process (2.1) is written as a GARCH(n,n) process. The proofs in section 3 do not depend on the presence of the second summation in (2.1), $\sum_{i=1}^{n} \beta_i h_{t-i}$, hence, on whether $q \geq 1$ or $q \geq 0$. The following GARCH assumptions allow $q \geq 0$, but require $p \geq 1$.

We assume the following for GARCH process (2.1): (i) $n = \max(p,q)$, for $p \geq 1$ and $q \geq 0$; (ii) $\alpha_0 > 0$, (iii) $\alpha_i \geq 0$ and $\beta_i \geq 0$, for $i = 1, \ldots, n$; (iv) $\alpha_i > 0$, for one or more $i = 1, \ldots, n$; (v) $\alpha_n > 0$ or $\beta_n > 0$, or both; and, (vi)

ε_t is distributed identically, independently, with zero mean, $E\varepsilon_t = 0$, finite positive variance, $\sigma_\varepsilon^2 = E\varepsilon_t^2 > 0$, and finite positive 4th moment, $\sigma_\varepsilon^4 = E\varepsilon_t^4 > 0$, which is the central and noncentral 4th moment of ε_t because $E\varepsilon_t = 0$. As usual, without loss of generality, we set $\sigma_\varepsilon^2 = 1$ and, thus, effectively merge σ_ε^2 into α_i.

Assumption (v) is convenient but unnecessary, because if $\alpha_n = \beta_n = 0$, then, assumption (iv) guarantees that we can reduce n until $\alpha_i > 0$ for some i = 1, ..., n-1. We take "ε_t is distributed independently" to mean that ε_t is distributed independently not just of past values of itself but also of past values of h_t. We do not need any particular distributional assumption such as Gaussianity.

Finally, we assume GARCH process (2.1) is stationary. Throughout, by "stationarity" we mean weak or covariance stationarity. By constrast, for example, Nelson (1990) considers strong stationarity of GARCH(1,1) processes. Following Milhoj (1985), Bollerslev (1986) proved that, for $\alpha_i \geq 0$ and $\beta_i \geq 0$, process (2.1) is stationary iff (vii) $\sum_{i=1}^{n} f_i < 1$, where $f_i = \alpha_i + \beta_i$. Although the final theorem 4 here proves that GARCH process (2.1) is stationary if Eh_t^2 exists, we assume stationarity in order to simplify the discussion.

In sum, we assume (i) to (vii) for GARCH process (2.1) and call these "the GARCH assumptions."

If GARCH process (2.1) is stationary, then, its mean, $Eh_t = \mu$, exists, is positive, and is given by

$$(2.2) \qquad \mu = \alpha_o/(1 - \sum_{i=1}^{n} f_i).$$

Let a tilde denote a mean-adjusted variable, so that $\tilde{h}_t = h_t - \mu$ and $\tilde{\varepsilon}_t^2 = \varepsilon_t^2 - \sigma_\varepsilon^2 = \varepsilon_t^2 - 1$. Then, we write the mean-adjusted form of GARCH process (2.1) in the ARMA form

$$(2.3) \qquad \tilde{h}_t = \sum_{i=1}^{n} f_i \tilde{h}_{t-i} + \sum_{i=1}^{n} g_i \xi_{t-i},$$

where $\xi_t = \tilde{\varepsilon}_t^2 h_t$, $f_i = \alpha_i + \beta_i$, and $g_i = \alpha_i$.

A state-space form of an ARMA process comprises an observation equation and a state equation in terms of a state vector. Let $x_t = (x_{1,t}, \ldots, x_{n,t})^T$ denote an n×1 state vector, where superscript T denotes transposition. Then, following Ansley and Kohn (1983), we can write ARMA equation (2.3) in state-space form, with observation equation $\tilde{h}_t = e_1^T x_t$, where $e_1 = (1, 0, \ldots, 0)^T$ is the n×1 vector with first element 1 and all other elements 0, and state equation

$$(2.4) \qquad x_t = Fx_{t-1} + g\xi_{t-1}, \quad F = \begin{bmatrix} f_1 & 1 & 0 & . & . & . & 0 \\ f_2 & 0 & 1 & . & & & . \\ . & . & . & . & . & & . \\ . & . & & . & . & . & . \\ . & . & & & . & . & 0 \\ . & . & & & & . & 1 \\ f_n & 0 & . & . & . & . & 0 \end{bmatrix}, \quad g = \begin{bmatrix} g_1 \\ . \\ . \\ . \\ . \\ . \\ g_n \end{bmatrix}.$$

Observation equation $\tilde{h}_t = e_1^T x_t$ implies $\tilde{h}_t$ is the first element of x_t. Because F is a companion matrix, its eigenvalues are identical to the roots of the characteristic equation $\lambda^n - f_1\lambda^{n-1} - \ldots - f_{n-1}\lambda - f_n = 0$ (Anderson, 1958, p. 177), so that stationarity condition $\sum_{i=1}^{n} f_i < 1$ is equivalent to $\rho(F) < 1$, where $\rho(F)$ denotes the spectral radius or maximal absolute eigenvalue of F. We say that the state equation (2.4) is stationary and the state-transition matrix F is stable iff $\rho(F) < 1$.

3. 4th-Moment Inequality and Eigenvalue Restrictions.

We are interested in proving that derived restrictions for existence of $Ey_t^4 = E(\varepsilon_t^4 h_t^2)$ are necessary and sufficient. However, we assume that $E\varepsilon_t^4$ exists; Eh_t^2 and $E\tilde{h}_t^2$ are linked by $Eh_t^2 = E\tilde{h}_t^2 + \mu^2$; $E\tilde{h}_t^2$ and V are linked by $E\tilde{h}_t^2 = v_{11} = e_1^T V e_1$; and, theorem 2 proves that v_{11} exists iff V exists, where v_{11} and V are defined two paragraphs below. Thus, proving that restrictions for existence of Ey_t^4 are necessary and sufficient reduces to proving such restrictions for V.

We shall generally maintain two terminological conventions. First, when we write an inequality such as "x ≥ 0" we shall mean not only that the inequality holds but that the variable in it exists and, hence, is finite.

Second, as noted before, we refer to assumptions (i) to (vii) on GARCH process (2.1) as "the GARCH assumptions."

Post-multiplying equation (2.4) by its transpose, taking unconditional expectation, E, of the result, and apply independence assumption (vi), so that $Ex_t\xi_t^T = 0_{n\times 1}$, the n×1 zero vector, implies the Lyapunov equation $V = FVF^T + (E\xi_t^2)gg^T$, where $V = Ex_tx_t^T$. Then, $E\xi_t^2 = E(\tilde{\varepsilon}_t^2)^2(E\tilde{h}_t^2 + 2E\tilde{h}_t\mu + \mu^2) = E(\tilde{\varepsilon}_t^2)^2(E\tilde{h}_t^2 + \mu^2)$, because $\tilde{\varepsilon}_t$ and $\tilde{h}_t$ are stochastically independent and $E\tilde{h}_t = 0$. In particular, $\tilde{h}_t^2$ depends only on constant parameters and variables dated before period t. Thus, the Lyapunov equation is equivalent to

$$(3.1) \qquad V = FVF^T + \theta g e_1^T V e_1 g^T + \mu^2\theta gg^T,$$

where $\theta = E(\tilde{\varepsilon}_t^2)^2 = E\varepsilon_t^4 - (E\varepsilon_t^2)^2 = \sigma_\varepsilon^4 - (\sigma_\varepsilon^2)^2 = \sigma_\varepsilon^4 - 1 > 0$ (e.g., by Jensen's inequality). When ε_t is Gaussian, $\sigma_\varepsilon^4 = 3$, so that $\theta = 2$.

Using $vec(ABC) = [C^T \otimes A]vec(B)$, for matrices A, B, and C conformable to the product ABC (Magnus and Neudecker, 1988, p. 30), where $vec(\cdot)$ denotes the column vectorization of a matrix (column 1 on top of column 2, etc.) and $\otimes$ denotes the Kronecker product. We state equation (3.1) equivalently as

$$(3.2) \qquad w = Aw + \mu^2\theta(g \otimes g),$$

where $w = vec(V)$ and $A = F \otimes F + \theta(g \otimes g)(e_1 \otimes e_1)^T$.

Gantmacher (1959, pp. 50-57) discusses the following implications of irreducibility. A matrix or vector, M, is nonnegative (M ≥ 0) or positive (M > 0) iff all of its elements are nonnegative or positive. A real, n×n, and nonnegative matrix, M, is irreducible iff it has no invariant coordinate subspace with dimension less than n. A theorem by Frobenius says that if M is a real, n×n, nonnegative, and irreducible matrix, then, M has an eigenvalue, λ, and associated left or right eigenvector, z, such that λ is real, positive, and equal to the maximal absolute eigenvalue of M or $\lambda = \rho(M)$, and z is real and positive. We need the following lemma.

LEMMA 1: Assume that GARCH assumptions (i) to (vii) hold. Then, matrices F ≥ 0 and $A = F \otimes F + \theta(g \otimes g)(e_1 \otimes e_1)^T \geq 0$ are irreducible.

PROOF OF LEMMA 1: Let e_i denote the n×1 elementary vector i, with one in position i and zeroes elsewhere. The n-dimensional real vector space is spanned by e_1, ..., e_n. Consider $f_1 = 0$, ..., $f_{n-1} = 0$ and n successive mappings with F starting from e_n: $Fe_n = e_{n-1}$, $F^2e_n = e_{n-2}$, ..., $F^{n-1}e_n = e_1$, and $F^ne_n = f_ne_n$. The first n-1 mapped vectors are equal to the first n-1 elementary vectors, e_1, ..., e_{n-1}; because $f_n > 0$, the last mapped vector lies in the space spanned by the last elementary vector, e_n. Thus, starting from e_n and successively mapping n times with F, the mapped vectors span the n-dimensional real vector space. The same conclusion holds if we start the mappings from any other elementary vector and if $f_1 \geq 0$, ..., $f_{n-1} \geq 0$. Thus, F is irreducible. $F \otimes F \geq 0$ and, by a similar argument in n×n-block form, is irreducible. Because $F \otimes F \geq 0$ and $\neq 0$, $\theta > 0$, $(g \otimes g)(e_1 \otimes e_1)^T \geq 0$ and $\neq 0$, adding $(g \otimes g)(e_1 \otimes e_1)^T \geq 0$ and $\neq 0$ to $F \otimes F$ only increases the absolute values of the nonnegative elements of the first block-column of $F \otimes F$ and does not change its irreducibility. Thus, $A = F \otimes F + \theta(g \otimes g)(e_1 \otimes e_1)^T$ is irreducible and lemma 1 is proved.

The proof of lemma 1 depends on $f_n > 0$, so that if $f_n = 0$ because $\alpha_n = \beta_n = 0$, maintaining the lemma requires reducing n until $f_n > 0$.

3.1. Necessary and Sufficient Inequality Restriction.

Stationarity or $\rho(F) < 1$ implies that $B = \sum_{i=0}^{\infty} F^i gg^T (F^T)^i \geq 0$ exists (Wilkinson, 1965, p. 59), so that equation (3.1) can be stated equivalently as

$$(3.3) \qquad V = (v_{11} + \mu^2)\theta B,$$

where $v_{11} = e_1^T V e_1$, the (1,1) element of V. Pre-multiplying equation (3.3) by e_1^T, postmultiplying the result by e_1, and rearranging, leads to equation (3.4), where $b_{11} = e_1^T B e_1$, the (1,1) element of B. Thus, we have theorem 1, which states the necessary and sufficient restriction $0 < \theta b_{11} < 1$ for existence of $E\widetilde{h}_t^2 = v_{11} > 0$.

THEOREM 1: Assume that GARCH assumptions (i) to (vii) hold. Then,

$$(3.4) \qquad v_{11} = \mu^2 \left(\frac{\theta b_{11}}{1 - \theta b_{11}} \right) > 0$$

exists iff, in addition, $0 < \theta b_{11} < 1$.

PROOF OF THEOREM 1: The GARCH assumptions imply that $\mu > 0$ and $\theta > 0$ exist. In particular, stationarity implies that $B \geq 0$ exists. Because $F \geq 0$ and is irreducible (lemma 1) and $g \geq 0$ and $\neq 0$, $b_{11} = e_1^T B e_1 \geq e_1^T F^j g g^T (F^T)^j e_1 > 0$, for some positive integer j. Thus, $\mu^2 \theta b_{11} > 0$ and equation (3.4) implies that $v_{11} > 0$ exists iff, in addition, $\theta b_{11} < 1$ and theorem 1 is proved.

Next, equation (3.3) implies theorem 2, which links the existence of V with the existence of v_{11}.

THEOREM 2: Assume that GARCH assumptions (i) to (vii) hold. Then, $V \geq 0$ exists iff $v_{11} > 0$ exists.

PROOF OF THEOREM 2: Suppose that $V \geq 0$ exists. Then, $v_{11} = e_1^T V e_1 \geq 0$ exists. The GARCH assumptions imply that $\mu > 0$ and $\theta > 0$ exist, that F is stable, nonnegative, and irreducible, and that $g \geq 0$ and $\neq 0$. These properties imply that $B \geq 0$ and $b_{11} > 0$ exist. Thus, equation (3.3) implies that $v_{11} = (v_{11} + \mu^2)\theta b_{11} \geq \mu^2 \theta b_{11} > 0$ and necessity is proved. Suppose that $v_{11} > 0$ exists. The GARCH assumptions imply that $\mu > 0$, $\theta > 0$, and $B \geq 0$ exist. Thus, equation (3.3) implies that $V \geq 0$ exists and sufficiency and theorem 2 are proved.

3.2. Necessary and Sufficient Eigenvalue Restriction.

Equation (3.2) has the unique solution $w = \mu^2 \theta (I_{n^2} - A)^{-1}(g \otimes g)$ iff $I_{n^2} - A$ is nonsingular, where I_{n^2} denotes the $n^2 \times n^2$ identity matrix, and this occurs iff A has no eigenvalue equal to one, because each eigenvalue of $I_{n^2} - A$ is one minus an eigenvalue of A. Theorem 3 tells us that $v_{11} > 0$ only if $\rho(A) < 1$, because otherwise w might have unacceptable negative values.

THEOREM 3: Assume that GARCH assumptions (i) to (vii) hold. Then, w = vec(V) $\geq$ 0 and $v_{11} = w_1 > 0$ exist, where w_1 denotes the first element of w, iff, in addition, $\rho(A) < 1$.

PROOF OF THEOREM 3: Suppose that $\rho(A) < 1$. Then, $\rho(I_{n^2}-A) < 1$, so that w = $\mu^2\theta(I_{n^2}-A)^{-1}(g\otimes g)$ uniquely solves equation (3.2). The GARCH assumptions imply that $\theta > 0$ and $\mu > 0$ exist. The solution w = $\mu^2\theta(I_{n^2}-A)^{-1}(g\otimes g)$ implies that w = $\mu^2\theta(I_{n^2} + A + A^2 + ...)(g\otimes g)$. The irreducibility of A implies that $(e_1\otimes e_1)^T A^j(g\otimes g) > 0$, for some positive integer j. See the proof of lemma 1. Thus, $v_{11} = w_1 > 0$ and sufficiency is proved. Suppose that $\delta x^T = x^T A$, where δ is a maximal eigenvalue of A, so that $|\delta| = \rho(A)$, and x is an associated left eigenvector of A. Because A is real, nonnegative, and irreducible, Frobenius's theorem (Gantmacher, 1959, pp. 50-57) implies that δ and x are real and positive. Pre-multiplying equation (3.2) by x^T implies that

(3.5) $(1 - \delta)x^T w = \mu^2\theta x^T(g\otimes g)$.

However, x > 0, w $\geq$ 0, $w_1 > 0$, and $g\otimes g \geq 0$ and $\neq 0$ imply that $x^T w > 0$ and $x^T(g\otimes g) > 0$. Thus, $\mu^2\theta > 0$ implies that $1-\delta = \mu^2\theta x^T(g\otimes g)/x^T w > 0$, so that equation (3.5) implies that $\delta = \rho(A) < 1$ and necessity and theorem 3 are proved.

Putting together theorems 1 to 3 implies that $\theta b_{11} < 1$ and $\rho(A) < 1$ are equivalent necessary and sufficient restrictions for existence of the 4th moment of the underlying variable of GARCH process (2.1). We state this conclusion formally as corollary 1.

COROLLARY 1: Assume that GARCH assumptions (i) to (viii) hold. Then, $\theta b_{11} < 1$ iff $\rho(A) < 1$.

Statistics tells us that a 4th moment exists only if the corresponding 2nd moment exists. Here, this means that $\rho(A) < 1$ only if $\rho(F) < 1$. Theorem 4 generalizes this result and shows that 4th-moment existence implies stronger restrictions than stationarity or 2nd-moment existence. For example, under

the GARCH assumptions, stationarity occurs iff $B \geq 0$ and $b_{11} > 0$ exist, but 4th-moment existence occurs iff, in addition, $\theta b_{11} < 1$.

THEOREM 4: $\rho(F)^2 < \rho(A)$, where $A = F \otimes F + \theta(g \otimes g)(e_1 \otimes e_1)^T$.

PROOF of THEOREM 4: Let $\lambda^2(z \otimes z)^T = (z \otimes z)^T(F \otimes F)$ and $\delta x = Ax$, where λ and δ are maximal eigenvalues of F and A, so that $|\lambda| = \rho(F)$ and $|\delta| = \rho(A)$, and z and x are associated left and right eigenvalues of F and A. The GARCH assumptions and Frobenius's theorem imply that λ, δ, z, and x are real and positive. Pre-multiplying $\delta x = Ax$ by $(z \otimes z)^T$ and rearranging implies that

$$(3.6) \qquad (\delta - \lambda^2)(z \otimes z)^T x = \theta(z^T g)^2 x_1,$$

where x_1 denotes the first element of x. However, $\theta > 0$, $z > 0$, $x > 0$, $g \geq 0$, and $g \neq 0$, imply that $(z \otimes z)^T x > 0$, $z^T g > 0$, and $x_1 > 0$, so that equation (3.6) implies that $\delta - \lambda^2 = \theta(z^T g)^2 x_1/(z \otimes z)^T x > 0$ and theorem 4 is proved.

4. Illustration of the 4th-Moment Inequality Restriction.

We now illustrate equation (3.4) in table 1 with six GARCH(1,1) processes from the literature. To do this, we first write b_{11} in terms of a finite number of basic arithmetic operations on f_i and g_i. Pre-multiplying $B = \sum_{i=0}^{\infty} F^i g g^T (F^T)^i$ by F, post-multiplying the result by F^T, subtracting the result from the initial equation for B, and rearranging, implies that

$$(3.7) \qquad B = FBF^T + gg^T,$$

which is a Lyapunov equation, linear in B.

The companion form of F suggests that equation (3.7) can be solved for b_{11} by eliminating elements of B from "back to front" until only b_{11} remains. Although such an approach might generally be overly complex and impractical, it is easily applied when n = 2. We consider n = 2 in part because it includes nearly all the GARCH processes we have seen in the empirical literature. For n = 2, equation (3.7) implies that

$$(3.8) \qquad b_{11} = \frac{(1-f_2)(g_1^2 + g_2^2) + 2f_1 g_1 g_2}{(1-f_2)(1-f_1^2 - f_2^2) - 2f_1^2 f_2} .$$

The GARCH nonegativity assumptions $f_i \geq 0$ and $g_i \geq 0$ and the stationarity assumption $\sum_{i=1}^{n} f_i < 1$ imply that the numerator and denominator in equation (3.8) are both positive.

Although a GARCH process linearly transforms squared disturbances, ε_t^2, into squared variables, y_t^2, it nonlinearly transforms ε_t to y_t, so that y_t is non-Gaussian even when ε_t is Gaussian. We may consider this nonpreservation of Gaussianity in terms of kurtosis, by considering equation (3.4) as

$$(3.9) \qquad \frac{k_y}{k_\varepsilon} = \frac{\theta b_{11}}{1 - \theta b_{11}} + 1,$$

where $k_y = Ey_t^4 / (Ey_t^2)^2$ and $k_\varepsilon = E\varepsilon_t^4 / (E\varepsilon_t^2)^2$ are the kurtoses of y_t and ε_t. Equation (3.9) implies that a GARCH transformation always increases kurtosis.

Table 1 considers only the case of n = 1 or GARCH(1,1) processes, which covers most of the empirical literature. Some exceptions are Engle (1982) and Geweke (1988) who consider nonstationary GARCH(0,4) and GARCH(0,2) processes which we exclude because they do not have 4th moments. In table 1, we assume that $\varepsilon_t \sim N(0,1)$ (Gaussian, zero mean, unit variance) even though Bollerslev, Drost and Klaassen, and Hsieh assume that ε_t is non-Gaussian. The cases in table 1 are ordered by increasing $f_1 = \alpha_1 + \beta_1$ and, for n = 1, $\sigma_\varepsilon^2 = 1$, and $\theta = 2$, equations (3.8) and (3.9) reduce to

$$(3.10) \qquad b_{11} = \frac{g_1^2}{1 - f_1^2} = \frac{\alpha_1^2}{1 - (\alpha_1 + \beta_1)^2} ,$$

$$(3.11) \qquad \frac{k_y}{k_\varepsilon} = \frac{1 - f_1^2}{1 - f_1^2 - 2g_1^2} = \frac{1 - (\alpha_1 + \beta_1)^2}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2} .$$

Table 1 contains values of θb_{11} and k_y / k_ε, for $\theta = 2$, according to equations (3.10) and (3.11) for six GARCH(1,1) processes from the literature.

Table 1: Examples of θb_{11} and k_y/k_ε for GARCH(1,1) processes and $\theta = 2$.

Case	Date, Authors	α_1	β_1	f_1	θb_{11}	k_y/k_ε
1	1982, Engle	.955	.000	.955	20.6	∞
2	1986, Bollerslev	.135	.829	.964	.516	2.07
3	1989, Baillie-Bollerslev	.061	.910	.971	.130	1.15
4	1987, Bollerslev	.057	.921	.978	.149	1.18
5	1997, Drost-Klaassen	.052	.932	.984	.170	1.21
6	1989, Hsieh	.191	.806	.997	12.2	∞

Table 1 depicts small α_1 (cases 2 to 5) associated with small $\theta b_{11} < 1$, for $\theta = 2$, and finite k_y/k_ε, larger and large α_1 (cases 6 and 1) associated with large $\theta b_{11} > 1$ and infinite k_y/k_ε, and, this pattern appears to be independent of the value of β_1 and to depend more on the value of α_1 (compare cases 1 and 6 with cases 2 to 5). The pattern is confirmed in figure 1, which depicts the feasible area of α_1 and β_1 in which the underlying variable in a scalar GARCH(1,1) process with $\varepsilon_t \sim N(0,1)$ has 2nd and 4th moments. The feasible area for stationarity or 2nd-moment existence is between the horizontal-α_1 axis, the vertical-β_1 axis, and the straight line $\beta_1 = 1 - \alpha_1$; the feasible subarea for 4th-moment existence is between the axes and the curved line $\beta_1 = -\alpha_1 + \sqrt{1 - 2\alpha_1^2}$. Evidently, 4th-moment existence restricts both α_1 and β_1, but restricts α_1 more: if a univariate GARCH(1,1) process is stationary and its underlying variable has a 4th moment, then, a minimum $\alpha_1 = 0$ implies a maximum $\beta_1 = 1$, but a minimum $\beta_1 = 0$ implies a maximum $\alpha_1 = 1/\sqrt{3} \cong .5773$, because β_1 is either real and negative or complex with a negative real part when $1/\sqrt{3} < \alpha_1 < 1$. Adding 4th-moment existence to 2nd-moment existence, reduces the feasible area of α_1 and β_1 by about one third. Adding more GARCH lags (n > 1) or higher even-moment existence further reduces the feasible area of α_1 and β_1.

Figure 1: Parameter Regions for 2nd and 4th Moment Existence
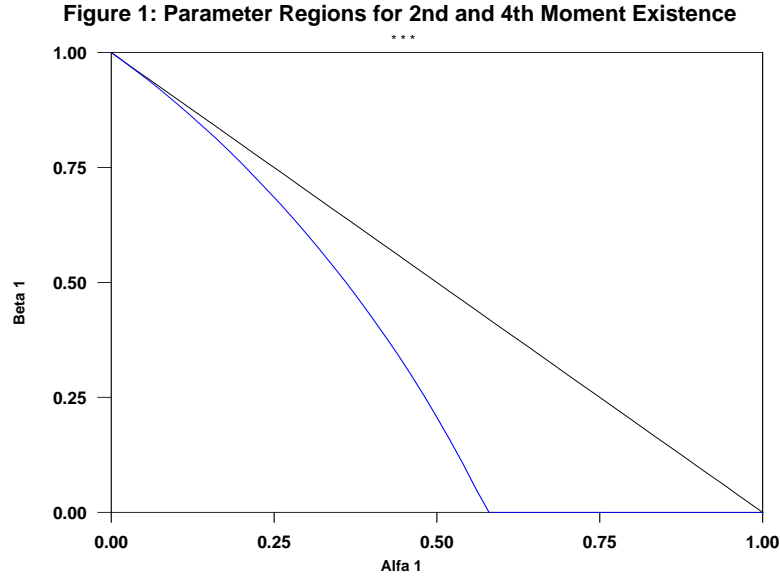
Equations (3.8) to (3.11), table 1, and figure 1, illustrate an important advantage of inequality restriction $\theta b_{11} < 1$ over eigenvalue restriction $\rho(A) < 1$. For n = 2, equation (3.8) states $\theta b_{11} < 1$ in terms of a finite number of basic arithmetic operations on the GARCH parameters, and, for n = 1, table 1 evaluates and figure 1 depicts this inequality. In principle, this could be done for any n, but, when n is large enough, algebraic detail becomes overwhelming. By contrast, generally, an eigenvalue restriction can be written out explicitly in terms of the GARCH parameters only when n ≤ 3. When n > 3, this can be done only in unlikely cases in which the characteristic polynomial of $A = F \otimes F + \theta(g \otimes g)(e_1 \otimes e_1)^T$ factors into polynomials of degree ≤ 3.

5. Conclusion.

Either equation (3.4) or (3.9) indicate that the underlying variable, y_t, of univariate GARCH process (2.1) has a 4th moment iff $0 < \theta b_{11} < 1$ exists. Non-Gaussian disturbances without 4th moments, in particular, stable Paretian disturbances, have been considered (McCulloch, 1997; Rachev and Mittnik, 2000). Whether or not a disturbance has a 4th moment, in practice we would like to know whether an estimated GARCH process preserves existence of a disturbance's 4th moment. This would involve developing a statistical test of

$\theta b_{11} < 1$ which accounts for the sampling variability of an estimated θb_{11}, based on the estimated GARCH parameters. Ideally, this would be an easily formed test, based on standard parameter estimates, along the lines of Dickey and Fuller's (1979) test of unit-root nonstationarity of an autoregressive linear time-series process.

Various papers in the literature claim to prove necessity and sufficiency of restrictions for 4th-moment existence in univariate GARCH processes, but most do not discuss any sort of nonnegativity (e.g., Hafner, 2003). Here, we see the crucial role of nonnegativity in the necessity proofs. Nonnegativity also figures crucially in existence proofs of stationarity of GARCH processes, but there nonnegativity occurs more simply, as is seen in equation (2.2). Bougerol and Picard (1992) and Ling and McAleer (2002) are exceptions in the literature for discussing nonnegativity in their respective existence proofs for 2nd and 4th moments of GARCH processes.

The state-space form of a GARCH process used here is simpler than the forms used by Bougerol and Picard (1992), Ling and McAleer (2002), and Mittnik, Paolella, and Rachev (2002), because the transition matrix F is nonstochastic. Thus, the proofs here are based on more elementary concepts and are simpler. For example, there is no need to consider a stochastic Lyapunov exponent and to verify its convergence, as in Bougerol and Picard (1992).

Finally, for any positive integer m, it would be interesting to generalize present restrictions to necessary and sufficient restrictions for existence of unique 2mth-moments of the underlying variable of a univariate GARCH process (cf., Ling and McAleer, 2002).

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