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CESifo Working Papers

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Luis H. R. Alvarez
Erkki Koskela

CESifo Working Paper No. 640 (8)

January 2002

Category 8: Resources and Environment

CESifo
Center for Economic Studies & Ifo Institute for Economic Research
Poschingerstr. 5, 81679 Munich, Germany
Phone: +49 (89) 9224-1410 - Fax: +49 (89) 9224-1409
e-mail: office@CESifo.de
ISSN 1617-9595

CESifo *Working Paper No. 640*
January 2002

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Abstract

The current extensive literature on irreversible investment decisions makes the assumption of constant interest rate. In this paper we study the impact of interest rate and revenue variability on the decision to carry out an irreversible investment project. Given the generality of the considered valuation problem, we first provide a thorough mathematical characterization of the problem and develop some new results. Contrary to what previous literature has suggested we establish that interest rate variability may have a profound decelerating or accelerating impact on investment demand depending on whether the current interest rate is below or above the long run steady state interest rate. Moreover, and importantly, allowing for interest rate uncertainty is shown to decelerate rational investment demand by raising both the required exercise premium of the irreversible investment opportunity and the value of waiting. Finally, we demonstrate that increased revenue volatility strengthens the negative impact of interest rate uncertainty and vice versa.

JEL Classification: Q23, G31, C61.

Keywords: irreversible investment, variable interest rates, free boundary problems.

Luis H. R. Alvarez
Department of Economics
Quantitative Methods in Management
Turku School of Economics and Business Administration
FIN-20500Turku
Finland

Erkki Koskela
Department of Economics
University of Helsinki
FIN-00014 University of Helsinki
Finland
erkki.koskela@helsinki.fi

1 Introduction

Most major investments are at least partly irreversible in the sense that firms cannot disinvest so that expenditures are sunk costs. This is because most capital is industry- or firm-specific so that it cannot be used in a different industry or by a different firm. Even though investment would not be firm or industry-specific, they still could be partly irreversible because of the "lemons" problem, i.e. their resale value is often below their purchase cost (cf. Dixit and Pindyck 1994, pp. 8–9). Since the seminal work by Arrow 1968 and Nickell 1974, 1978 who analyzed irreversible investments under certainty, decisions about irreversible investments in the presence of various types of uncertainties have been studied extensively (see e.g. Arrow and Fisher 1974, Baldursson and Karatzas 1997, Baldwin 1982, Bertola and Caballero 1994, Demers 1991, Henry 1974, Hu and Øksendal 1998, Kobila 1993, McDonald and Siegel 1986, Øksendal 2001, and Pindyck, 1988, 1991, 2000). In these studies option pricing techniques are used to show that in the presence of irreversibility the investment is undertaken when the net present value is "sufficiently high" compared with the opportunity cost. The various approaches and applications are excellently reviewed and extended in the seminal book by Dixit and Pindyck 1994. Finally, Bentolila and Bertola 1990, Brennan and Schwarz 1985, Dixit 1989, and Abel and Eberly 1996 have studied the implications of costly reversibility in the case of labor, decision to open or close a mine, costly entry and exit, and investments, respectively. In various contexts they show how in the presence of uncertainty even small sunk costs may produce a wide range of inaction. For a further analysis of the relationship between investment and uncertainty, see Caballero 1991. Bernanke 1983 and Cukierman 1980 have developed related models where firms have an incentive to postpone irreversible investment so that they can wait for new information to arrive.

In these studies dealing with the impact of irreversibility in a variety of problems and different types of frameworks the constancy of the discount rates has been one of the most predominant assumptions. The basic motivation of this argument is that interest rates are typically more stable (and consequently, less significant) than the revenue dynamics. As Dixit and Pindyck 1994 state:

> "Once we understand why and how firms should be cautious when deciding whether to exercise their investment options, we can also understand why interest rates seem to have so little effect on investment. (p. 13)"
> "Second, if an objective of public policy is to stimulate investment, the stability of interest rates may be more important than the level of interest rates. (p. 50)"

Although this argumentation is undoubtedly correct for short-lived investment projects, many real investment opportunities have considerably long planning and exercise periods. If the exercise of such investment opportunities takes a long time, the assumed constancy of the interest rate is quite questionable. This observation raises several questions: Does interest rate variability matter and, if it does, in what way and how much? What is the role of stochastic interest rate volatility from the point of view of exercising irreversible investment opportunities? Most studies considering this problem emphasize the role of uncertainty in general, not the role of variability. As Ingersoll and Ross 1992 state:

> "..., even for the simplest projects with deterministic cash flows, interest rate uncertainty has a significant effect on investment. (p. 3)"
> "..., the effect of interest rate uncertainty is ubiquitous and critical to understanding investment at the macroeconomic level. (p. 3)"

However, not all instability is necessarily caused by uncertainty but one could also argue that it is a natural process during the evolution towards the long run steady state of the economy. As Pindyck (1991) argues:

> "..., a major cost of political and economic instability may be its depressing effect on investment. (p. 112)"

It is known from empirical research that interest rates fluctuate a lot over time and that in the long run they follow mean reverting processes (for an up-to-date theoretical and empirical survey in the field see e.g. Cochrane 2001). Since variability may be deterministic and/or stochastic, we immediately observe that interest rate variability in general can be important from the point of view of exercising real investment opportunities.

Motivated by the argumentation that interest rates are neither constant nor deterministic for long-lived investments, we explore in this paper the impact of interest rate variability on the value and the optimal exercise policy of irreversible real investment opportunities. The impact of interest rate variability on optimal forest rotation has been analyzed in an accompanying paper by Alvarez and Koskela 2001. More precisely, we proceed as follows. We start our analysis in section 2 by considering the case where both the revenue and interest rate dynamics are variable but deterministic. We demonstrate, among others, that when the current interest rate is above (below) the long run steady state interest rate, then investment strategies based on the usual assumption of constant discounting will underestimate (overestimate) both the value of waiting and the required exercise premium of the irreversible investment policy. We also show a natural though new result according to which differences tend to become smaller as the growth rate of the interest rate process diminishes. In section 3 we extend our model to cover the situation, where the underlying interest rate dynamics is stochastic and demonstrate that interest rate uncertainty strengthens the effect of the interest rate variability on the value of waiting and optimal exercise policy. Section 4 further extends the analysis in section 3 by allowing the revenue dynamics to follow a geometric Brownian motion. We demonstrate that revenue uncertainty strengthens the negative impact of interest rate uncertainty and vice versa. Finally, there is a brief concluding section.

2 Irreversible Investment under Deterministic Interest Rate Variability

In this section we consider the determination of an optimal irreversible investment policy in the presence of deterministic interest rate variability. We proceed as follows: First, we provide a set of sufficient conditions under which the optimal exercise date of investment opportunity can be solved generally and in an interesting special case even explicitly. Second, we demonstrate the relationship between the optimal exercise dates with variable and constant discounting when the interest rate can be below or above the long-run steady state interest rate. Third, and finally, we show among others that the value of investment opportunity is a decreasing and convex function of the current interest rate.

In order to accomplish these tasks, we describe the underlying dynamics for the value of investment X_t and the interest rate r_t as

$$X_t' = \mu X_t, \quad X_0 = x \tag{2.1}$$

and

$$r_t' = \alpha r_t(1 - \beta r_t), \quad r_0 = r, \tag{2.2}$$

where μ, γ, α, and β are exogenously determined positive constants. That is, we assume that the revenues accrued from exercising the irreversible investment opportunity increase at an exponential rate and that the interest rate dynamics follow a logistic dynamical system which is consistent with the notion that the interest rate is a mean reverting process. As usually, we denote as

$$\mathcal{A} = \mu x \frac{\partial}{\partial x} + \alpha r(1 - \beta r)\frac{\partial}{\partial r}$$

the differential operator associated with the inter-temporally time homogeneous process (X_t, r_t).

Given these assumptions, we now plan to consider the optimal irreversible investment problem

$$V(x, r) = \sup_{t \geq 0} \left[e^{-\int_0^t r_s ds}(X_t - c) \right]. \tag{2.3}$$

As usually in the literature on real options, the determination of the optimal exercise date of the irreversible investment policy can be viewed as the valuation of a perpetual American forward contract on a dividend paying asset. However, *in contrast to previous models relying on constant interest rates, the valuation is now subject to a variable interest rate and, therefore, constitutes a two-dimensional optimal timing (i.e. two-dimensional optimal stopping) problem.* The continuous differentiability of the exercise payoff implies that (2.3) can also be restated as (cf. Øksendal 1998, p. 199, and Protter 1990, p. 71)

$$V(x, r) = (x - c) + F(x, r), \tag{2.4}$$

where

$$F(x, r) = \sup_{t \geq 0} \int_0^t e^{-\int_0^s r_y dy}[\mu X_s - r_s(X_s - c)]ds \tag{2.5}$$

is known as *the early exercise premium of the considered irreversible investment opportunity.* We can now prove the following.

Theorem 2.1. *Assume that $1 > \beta\mu$, so that the percentage growth rate μ of the revenues X_t is below the long run steady state β^{-1} of the interest rate r_t. Then, for all $(x, r) \in$*

$C = \{(x,r) \in \mathbb{R}^2_+ : rc > (r-\mu)x\}$ *the optimal exercise date of the investment opportunity* $t^*(x,r) = \inf\{t \geq 0 : r_t c - (r_t - \mu)X_t \leq 0\}$ *is finite and the value*

$$V(x,r) = e^{-\int_0^{t^*(x,r)}}(X_{t^*(x,r)} - c)$$

constitutes the solution of the boundary value problem

$$\begin{aligned} &(\mathcal{A}V)(x,r) - rV(x,r) = 0 \quad (x,r) \in C \\ &V(x,r) = x - c \quad \frac{\partial V}{\partial x}(x,r) = 1 \quad \frac{\partial V}{\partial r}(x,r) = 0 \quad (x,r) \in \partial C. \end{aligned}$$

In line with this finding, the early exercise premium $F(x,r)$ *satisfies the boundary value problem*

$$\begin{aligned} &(\mathcal{A}F)(x,r) - rF(x,r) + \mu x - r(x-c) = 0 \quad (x,r) \in C \\ &F(x,r) = 0 \quad \frac{\partial F}{\partial x}(x,r) = 0 \quad \frac{\partial F}{\partial r}(x,r) = 0 \quad (x,r) \in \partial C. \end{aligned}$$

Proof. See Appendix A. □

Theorem 2.1 states a set of sufficient conditions under which the optimal investment problem (2.3) can be solved in terms of the initial states (x,r) and the exogenous variables of the problem. The non-linearity of the optimality condition implies that it is typically very difficult, if possible at all, to solve explicitly the optimal exercise date of the investment opportunity in the general case. Fortunately, there is an interesting special case under which the investment problem can be solved explicitly. This case is treated in the following.

Corollary 2.2. *Assume that* $\beta^{-1} > \mu$ *and that* $\mu = \alpha$. *Then, for all* $(x,r) \in C = \{(x,r) \in \mathbb{R}^2_+ : rc > (r-\mu)x\}$ *the optimal exercise date of the investment opportunity is*

$$t^*(x,r) = \frac{1}{\mu}\ln\left(1 + \frac{rc - (r-\mu)x}{rx(1-\mu\beta)}\right).$$

In this case, the value reads as

$$V(x,r) = \begin{cases} x - c & (x,r) \in \mathbb{R}^2_+ \setminus C \\ \frac{\mu x}{r}\left(\frac{x - \beta r(x-c)}{x(1-\mu\beta)}\right)^{1-1/(\mu\beta)} & (x,r) \in C, \end{cases} \tag{2.6}$$

and the early exercise premium reads as

$$F(x,r) = \begin{cases} 0 & (x,r) \in \mathbb{R}^2_+ \setminus C \\ \frac{\mu x}{r}\left(\frac{x - \beta r(x-c)}{x(1-\mu\beta)}\right)^{1-1/(\mu\beta)} - (x-c) & (x,r) \in C. \end{cases} \tag{2.7}$$

Moreover,

$$\frac{\partial t^*}{\partial x}(x,r) = -\frac{rc}{\mu x(rx(1-\mu\beta) + rc - (r-\mu)x)} < 0$$

and

$$\frac{\partial t^*}{\partial r}(x,r) = -\frac{x}{r(rx(1-\mu\beta) + rc - (r-\mu)x)} < 0.$$

Proof. See Appendix B. □

Corollary 2.2 shows that whenever the percentage growth rates at low values (i.e. as $(X_t, r_t) \to (0,0)$) of the revenue and interest rate process coincide, that is, whenever $\mu = \alpha$, both the value and the optimal exercise date of the irreversible investment policy can be solved explicitly in terms of the current states (x, r) and the exogenous variables of the problem. Another important implication of our Theorem 2.1 demonstrates how the value and early exercise premium of our problem are related to their counterparts in the absence of interest rate variability. This relationship is summarized in the following.

Corollary 2.3. *Assume that the conditions $\beta^{-1} > \mu$ and $r > \mu$ are satisfied. Then,*

$$\lim_{\alpha \downarrow 0} V(x,r) = x^{r/\mu} \sup_{y \geq x} \left[\frac{y-c}{y^{r/\mu}} \right] = \tilde{V}(x,r), \tag{2.8}$$

$$\lim_{\alpha \downarrow 0} F(x,r) = x^{r/\mu} \left[\sup_{y \geq x} \left[\frac{y-c}{y^{r/\mu}} \right] - \frac{x-c}{x^{r/\mu}} \right] = \tilde{F}(x,r), \tag{2.9}$$

and

$$\lim_{\alpha \downarrow 0} t^*(x,r) = \frac{1}{\mu} \ln \left(\frac{rc}{(r-\mu)x} \right) = \tilde{t}(x,r), \tag{2.10}$$

where $\tilde{V}(x,r) = \sup_{t \geq 0}[e^{-rt}(X_t - c)]$ denotes the value,

$$\tilde{F}(x,r) = \sup_{t \geq 0} \int_0^t e^{-rs}[rc - (r-\mu)X_s]ds$$

the early exercise premium, and $\tilde{t}(x,r)$ the optimal exercise date in the absence of interest rate variability, respectively.

Proof. The alleged results are direct consequences of the proof of our Theorem 2.1. □

Remark: It is worth observing that the value of the optimal investment policy in the absence of interest rate variability can also be expressed as

$$\tilde{V}(x,r) = \begin{cases} x - c & x \geq rc/(r-\mu) \\ \frac{\mu x}{r} \left(\frac{rc}{(r-\mu)x} \right)^{1-r/\mu} & x < rc/(r-\mu) \end{cases}.$$

Corollary 2.3 proves that the value, the early exercise premium, and the optimal exercise date of the investment policy in the presence of interest rate variability tend towards their counterparts in the presence of constant discounting as the growth rate of the interest rate process tends to zero. This means interestingly that *if the interest rate process evolves towards its long run steady state β^{-1} at a very slow rate, then the conclusions obtained in models neglecting interest rate variability will not be grossly in error when compared with the predictions obtained in models taking into account the variability of interest rates.* In order to illustrate the potential quantitative role of these qualitative differences we next provide some numerical computations. In Table 1 we have used the assumption that $c = 1$, $\mu = 1\%$, $\beta^{-1} = 3\%$, $r = 5\%$ and $x = 0.1$ (implying that $\tilde{t}(0.1, 0.05) = 91.6291$). Hence, in this case the long-run steady state of interest is below the current interest rate. As Table 1 and Figure 1 illustrate, interest rate variability affects both the exercise date and the value of waiting.

α	$t^*(0.1, 0.05)$	$X(t^*(0.1, 0.05)) - c$
5%	109.779	0.498761
1%	102.962	0.4
0.5%	98.3206	0.336506
10^{-6}	91.6306	0.250019

Table 1: The Optimal Exercise Date and Required Exercise Premium.

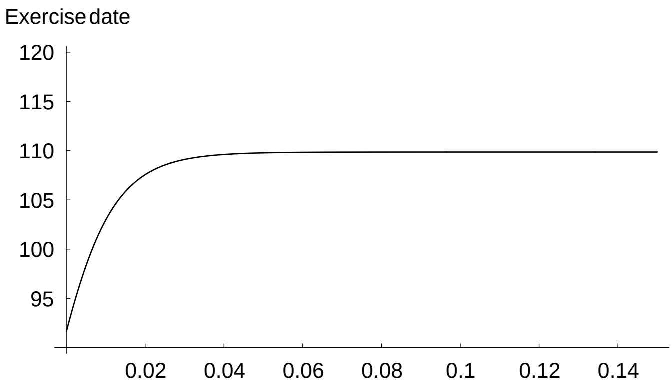


Figure 1: The Optimal Exercise Date $\tilde{t}(0.1, 0.05)$ as a function of α

In Table 2 we illustrate our results under the assumption that the long-run steady state interest rate is above the current interest rate. More precisely, we assume that $c = 1$, $\mu = 1\%$, $\beta^{-1} = 3\%$, $r = 1.5\%$ and $x = 0.1$ (implying that $\tilde{t}(0.1, 0.015) = 179.176$). In this case interest rate variability has the reverse effect on the exercise date and the value of waiting.

α	$t^*(0.1, 0.015)$	$X(t^*(0.1, 0.015)) - c$
5%	110.065	0.503061
1%	125.276	0.75
0.5%	138.629	1
10^{-6}	179.158	1.99946

Table 2: The Optimal Exercise Date and Required Exercise Premium.

After having characterized a set of conditions under which the optimal investment problem with variable discounting can be solved in terms of the initial states of the system and exogenous variables and having provided explicit solutions in an interesting special case, we now ask the following important, but thus far unexplored question:

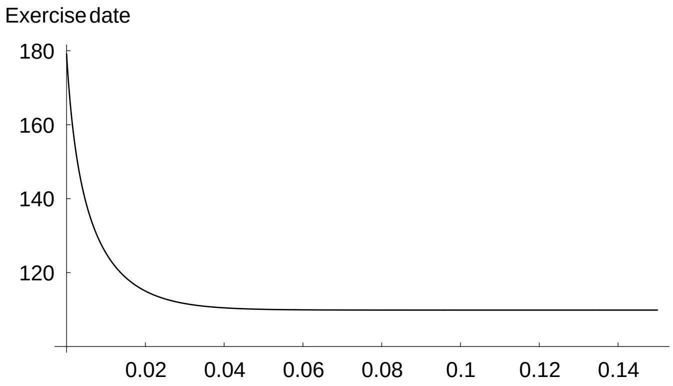


Figure 2: The Optimal Exercise Date $\tilde{t}(0.1, 0.015)$ as a function of α

What is the relationship between the optimal exercise policy and the value of the investment opportunity with variable and constant discounting. Given the definitions of the optimal policy and its value under future evolution of the interest rate, we are now in the position to establish the following new set of results summarized in

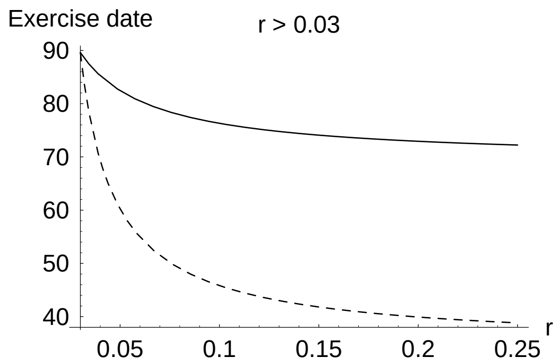
Theorem 2.4. *Assume that $\beta^{-1} > \mu$ and that $r > \mu$. Then,*

$$t^*(x,r) \gtreqqless \tilde{t}(x,r), \quad V(x,r) \gtreqqless \tilde{V}(x,r) \quad \textit{and} \quad F(x,r) \gtreqqless \tilde{F}(x,r) \quad \textit{when} \quad r \gtreqqless \beta^{-1}.$$

Proof. See Appendix C. □

Theorem 2.4 presents two qualitatively new important results, which characterize the differences of the optimal exercise policy and the value of the investment opportunity with constant and variable discounting. First, the required exercise premium and the value of the investment opportunity is higher in the presence of variable discounting than in the presence of constant discounting when the current interest rate is above the long-run steady state interest rate. Second, the reverse happens when the current interest rate is below the long-run steady state interest rate. More specifically, these findings imply the following: *When the current interest rate is above (below) the long run steady state value, then the investment strategies based on the usual approach neglecting the interest rate variability will underestimate (overestimate) both the value of waiting and the required exercise premium of the irreversible investment policy.*

In Theorem 2.4 we characterized qualitatively the differences of the optimal exercise policy and the value of investment opportunities with constant and variable discounting. In Figure 3, we illustrate these findings quantitatively in an example where the steady state interest rate $\hat{r}$ is 3% and the current interest rate is either above the steady state interest rate (the l.h.s. of Figure 3) or below the steady state interest rate (the r.h.s. of Figure 3). The solid lines describe the exercise dates in the presence of variable interest rate while dotted lines the optimal exercise dates with constant discounting. One can see from Figure 3 that when the current interest rate is above the steady state interest rate, the difference between the exercise dates becomes larger the higher is the current interest rate. Naturally, the reverse happens when the current interest rate is below the



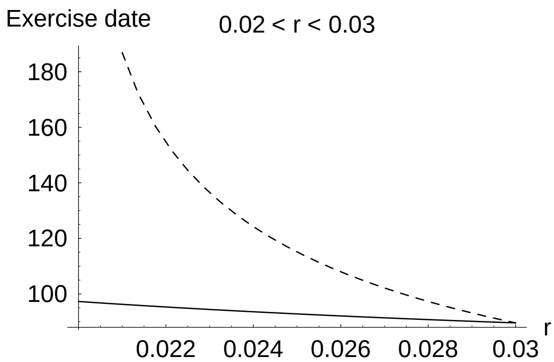


Figure 3: The Optimal Exercise Date $t^*(x, r)$

steady state interest rate. The differences between the exercise dates can be very large if the variability of interest rate is big enough.

It is worth observing that if $\alpha = \mu$, then the *required exercise premiums* read as

$$P(x, r) = X_{t^*(x,r)} - c = \frac{\mu c}{\beta^{-1} - \mu}\left[1 + \frac{x(1 - \beta r)}{\beta r c}\right] \tag{2.11}$$

and as

$$\tilde{P}(x, r) = X_{\tilde{t}(x,r)} - c = \frac{\mu c}{r - \mu}. \tag{2.12}$$

Moreover, as intuitively is clear, $P(x, \beta^{-1}) = \tilde{P}(x, \beta^{-1})$. That is, *the required exercise premiums coincide at the long run asymptotically stable steady state of the interest rate.* As we can observe from (2.11)

$$\frac{\partial P}{\partial x}(x, r) = \frac{\mu c}{\beta^{-1} - \mu}\left[\frac{1 - \beta r}{\beta r c}\right] \gtreqqless 0, \quad r \lesseqqgtr \beta^{-1},$$

and

$$\frac{\partial P}{\partial r}(x, r) = -\frac{\mu c}{\beta^{-1} - \mu}\left[\frac{x}{r^2 \beta c}\right] < 0.$$

Thus, the *required exercise premium is a decreasing function of the current interest rate r at all states.* However, *the sign of the sensitivity of the required exercise premium is positive (negative) provided that the current interest rate r is below (above) the long run steady state β^{-1}*. Before proceeding further in our analysis, we prove the following result characterizing the monotonicity and curvature properties of the value of the investment opportunity.

Lemma 2.5. *Assume that the conditions of Theorem 2.1 are satisfied. Then, the value of the investment opportunity is an increasing and convex function of the current revenues x and a decreasing and convex function of the current interest rate r.*

Proof. See Appendix D. □

Later on we generalize these properties to cover the case of stochastic interest rate and stochastic revenue.

3 Interest Rate Uncertainty and Irreversible Investment

In the analyzes we have carried out thus far, the underlying dynamics for the revenue X_t and the interest rate r_t has been postulated to be deterministic. The reason for this was that we first wanted to show the impact of variable discounting on the investment decisions in the simpler case without uncertainty. In this section we generalize our earlier analysis by exploring the optimal investment decision in the presence of interest rate uncertainty, i.e. when the interest rate process has certain properties, but fluctuates stochastically. We proceed as follows. First, we characterize a set of sufficient conditions for the optimality of investment strategy and second, we show how under certain conditions the interest rate uncertainty has the impact of postponing the rational exercise of investment opportunity.

We assume that the interest rate process $\{r_t; t \geq 0\}$ is defined on a complete filtered probability space $(\Omega, P, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ satisfying the usual conditions and that r_t is described on $\mathbb{R}_+$ by the (Itô-) stochastic differential equation

$$dr_t = \alpha r_t(1-\beta r_t)dt + \sigma(r_t)dW_t, \quad r_0 = r, \tag{3.1}$$

where $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a sufficiently smooth mapping for guaranteeing the existence of a solution for (3.1) (at least continuous; cf. Borodin and Salminen 1996, pp. 46–47). We also assume that $\sigma(r) > 0$ for all $r \in (0,\infty)$, that ∞ is an unattainable boundary for the diffusion r_t, and that 0 is either unattainable or exit for r_t (cf. Borodin and Salminen 1996, pp. 14–19). It is now clear that in the present example given our assumptions on the underlying dynamics the differential operator associated with the two-dimensional process (X_t, r_t) now reads as

$$\hat{\mathcal{A}} = \frac{1}{2}\sigma^2(r)\frac{\partial^2}{\partial r^2} + \mu x\frac{\partial}{\partial x} + \alpha r(1-\beta r)\frac{\partial}{\partial r}.$$

If both boundaries are unattainable and

$$\int_0^\infty m'(y)dy < \infty,$$

where $m'(r) = \frac{2}{\sigma^2(r)S'(r)}$ denotes the density of the speed measure m of the diffusion r_t and

$$S'(r) = \exp\left(-\int \frac{2\alpha r(1-\beta r)}{\sigma^2(r)}dr\right)$$

denotes the density of the scale function of the diffusion r_t, then we know that the diffusion r_t will tend in the long run towards a random variable distributed according to the stationary distribution with density (cf. Borodin and Salminen 1996, pp. 35–36)

$$p(r) = \frac{m'(r)}{\int_0^\infty m'(y)dy}.$$

For example, if in the present example we have that $\sigma(r) = \eta r$, where $\eta > 0$ is an exogenous constant, then a stationary distribution exists provided that $\alpha > \eta^2/2$. In that case, the density of the stationary distribution reads as

$$p(r) = \left(\frac{2\alpha\beta}{\eta^2}\right)^{\frac{\theta}{2}} \frac{r^{\frac{(\theta-2)}{2}} e^{-\frac{2\alpha\beta r}{\eta^2}}}{\Gamma(\frac{\theta}{2})},$$

where $\theta/2 = \frac{2\alpha}{\eta^2} - 1$.

Given these technical assumptions, we next consider the valuation of the irreversible investment opportunity in the presence of interest rate uncertainty. More precisely, we consider the optimal stopping problem

$$\hat{V}(x,r) = \sup_{\tau} E_{(x,r)} \left[e^{-\int_0^\tau r_s ds}(X_\tau - c) \right], \tag{3.2}$$

where τ is an arbitrary $\mathcal{F}_t$-stopping time. In line with our results of the previous section, Dynkin's theorem (cf. Karlin and Taylor 1981, pp. 297–313 and Øksendal 1998, pp. 118-120) implies that the optimal stopping problem (3.2) can also be rewritten as in (2.4) with the exception that the early exercise premium now reads as

$$\hat{F}(x,r) = \sup_{\tau} E_{(x,r)} \int_0^\tau e^{-\int_0^s r_y dy}(\mu X_s - r_s(X_s - c))ds. \tag{3.3}$$

Before proceeding in our analysis, we first present the following auxiliary result.

Lemma 3.1. *Assume that there is a mapping $J : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ satisfying the following conditions*

$$\text{(i) } J \in C^{1,2}(\mathbb{R}_+^2)$$
$$\text{(ii) } \min\{J(x,r) - (x-c), rJ(x,r) - (\hat{\mathcal{A}}J)(x,r)\} = 0 \text{ for all } (x,r) \in \mathbb{R}_+^2.$$

Then, $J(x,r) \geq \hat{V}(x,r)$ for all $(x,r) \in \mathbb{R}_+^2$. Consequently, if there is a mapping $L : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ satisfying the following conditions

$$\text{(i) } L \in C^{1,2}(\mathbb{R}_+^2)$$
$$\text{(ii) } \min\{L(x,r), rL(x,r) - (\hat{\mathcal{A}}L)(x,r) - \mu x + r(x-c)\} = 0 \text{ for all } (x,r) \in \mathbb{R}_+^2.$$

Then, $L(x,r) \geq \hat{F}(x,r)$ and, therefore, $L(x,r) + (x-c) \geq \hat{V}(x,r)$ for all $(x,r) \in \mathbb{R}_+^2$.

Proof. The result is a straightforward consequence of Theorem 2.1 in Øksendal and Reikvam 1998. □

Lemma 3.1 states a set of sufficient conditions which can be applied for the verification of the optimality of a proposed investment strategy. Unfortunately, multi-dimensional optimal stopping problems of the type (3.2) are extremely difficult, if possible at all, to be solved explicitly in terms of the current states and the parameters of the problem.

We can now also establish a qualitative connection between the deterministic stopping problem (2.3) and the present stochastic problem (3.2). This is summarized in the following theorem which could be called *the fundamental qualitative characterization of the value of an irreversible investment opportunity in the presence of interest rate uncertainty.*

Theorem 3.2. *Assume that $1 > \beta\mu$, so that the expected percentage growth rate μ of the revenues X_t is below the long run steady state β^{-1} of the interest rate r_t. Then,*

$$\hat{V}(x,r) \geq V(x,r) \quad \text{and} \quad \hat{F}(x,r) \geq F(x,r) \tag{3.4}$$

for all $(x,r) \in \mathbb{R}^2_+$ and, therefore, $C \subset \hat{C} = \{(x,r) \in \mathbb{R}^2_+ : \hat{V}(x,r) > x - c\}$. Hence, interest rate uncertainty increases both the required exercise premium and the value of the irreversible investment opportunity and, consequently, postpones the optimal exercise of investment opportunities.

Proof. See Appendix E. □

Theorem 3.2 shows that given the conditions of our paper, both the value and the rational exercise boundary of the investment opportunity is higher in the presence of interest rate uncertainty than in the absence of it. It would be of interest to characterize more precisely the difference between the optimal policy in the absence of uncertainty with the optimal policy in the presence of uncertainty. Unfortunately, stopping problems of the type (3.2) are seldom solvable and, consequently, the difference between the optimal policies can typically be illustrated only numerically. In any case, this is an area for further research.

Although Theorem 3.2 demonstrates that uncertainty decelerates rational investment when compared with the certain situation, it does not characterize the impact of increased interest rate volatility on investment. In order to be able to present an unambiguous characterization of the sign of the relationship between increased volatility and the rational exercise of investment opportunities, we assume that the interest rate process $\{\hat{r}_t; t \geq 0\}$ is described on $\mathbb{R}_+$ by the (Itô-) stochastic differential equation

$$d\hat{r}_t = \alpha(\hat{r}_t)dt + \hat{\sigma}(\hat{r}_t)dW_t, \quad \hat{r}_0 = r, \tag{3.5}$$

where $\hat{\sigma} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a sufficiently smooth mapping satisfying the inequality $\hat{\sigma}(r) \geq \sigma(r)$. That is, $\hat{r}_t$ can be interpreted as a diffusion evolving at the same rate as r_t but subject to greater stochastic fluctuations than r_t (more volatile dynamics). Given this definition, we define the value $\tilde{V} : \mathbb{R}^2_+ \mapsto \mathbb{R}_+$ as

$$\tilde{V}(x,r) = \sup_{\tau} E_{(x,r)}\left[e^{-\int_0^\tau \hat{r}_s ds}(X_\tau - c)\right], \tag{3.6}$$

where τ is an arbitrary $\mathcal{F}_t$-stopping time. Before establishing the sign of the relationship between interest rate volatility and investment, we first present an important result characterizing the form of the value function $\hat{V}(x,r)$ as a function of the current revenues x and the current interest rate r. This is accomplished in the following.

Lemma 3.3. *(A) The value function $\hat{V}(x,r)$ is increasing and convex as a function of the current revenues x. That is, $\hat{V}_x(x,r) > 0$ and $\hat{V}_{xx}(x,r) \geq 0$ for all $(x,r) \in \mathbb{R}^2_+$.*
(B) Assume that $\sigma(r)$ is continuously differentiable with Lipschitz-continuous derivative, and that the standard Novikov-condition

$$E_r\left[\exp\left(\frac{1}{2}\int_0^t \sigma'^2(r_s)ds\right)\right] < \infty$$

is satisfied for all $(t,r) \in \mathbb{R}^2_+$. Then, the value function $\hat{V}(x,r)$ is decreasing and convex as a function of the current interest rate r. That is, then $\hat{V}_r(x,r) < 0$, and $\hat{V}_{rr}(x,r) \geq 0$ for all $(x,r) \in \mathbb{R}^2_+$.

Proof. See Appendix F. □

Lemma 3.3 shows that typically the value function is an increasing and convex function of the current revenues x and an decreasing and convex function of the current interest rate r. This result is very important since it implies that the sign of the relationship between interest rate volatility and investment in unambiguously negative. More precisely,

Theorem 3.4. *Assume that the conditions of part (B) of Lemma 3.3 are satisfied. Then, for all $(x,r) \in \mathbb{R}^2_+$ we have*

$$\tilde{V}(x,r) \geq \hat{V}(x,r) \quad \text{and} \quad \tilde{F}(x,r) \geq \hat{F}(x,r),$$

where

$$\tilde{F}(x,r) = \sup_{\tau} E_{(x,r)} \int_0^{\tau} e^{-\int_0^t r_s ds}[\mu X_t - \hat{r}_t(X_t - c)]dt$$

denotes the early exercise premium in the presence of the more volatile interest rate process $\hat{r}_t$. Moreover, $\{(x,r) \in \mathbb{R}^2_+ : \hat{V}(x,r) > (x-c)\} \subset \{(x,r) \in \mathbb{R}^2_+ : \tilde{V}(x,r) > (x-c)\}$, that is, increased interest rate volatility postpones the optimal exercise of investment opportunities.

Proof. The proof is analogous with the proof of Theorem 5.4. in Alvarez and Koskela 2001. □

4 Interest Rate and Revenue Uncertainty

In the earlier section we characterized the value and optimal exercise of investment opportunities when the underlying interest rate dynamics was assumed to be stochastic and the revenue dynamics were deterministic. In order to further extend the analysis of the previous section, we now assume that the interest rate dynamics follow the diffusion described by the stochastic differential equation (3.1) and that the revenue dynamics are described on $\mathbb{R}_+$ by the stochastic differential equation

$$dX_t = \mu X_t dt + \gamma X_t d\bar{W}_t \quad X_0 = x, \tag{4.1}$$

where $\bar{W}_t$ is a Brownian motion independent of W_t and $\mu > 0$, $\gamma > 0$ are exogenously given constants. It is clear that given the stochasticity of the revenue dynamics, the differential operator associated with the process (X_t, r_t) now reads as

$$\bar{\mathcal{A}}_\gamma = \frac{1}{2}\sigma^2(r)\frac{\partial^2}{\partial r^2} + \frac{1}{2}\gamma^2 x^2 \frac{\partial^2}{\partial x^2} + \mu x \frac{\partial}{\partial x} + \alpha r(1-\beta r)\frac{\partial}{\partial r}.$$

Given the dynamics of the process (X_t, r_t) we now plan to consider the following valuation problem

$$\bar{V}(x,r) = \sup_{\tau} E_{(x,r)}\left[e^{-\int_0^{\tau} \hat{r}_s ds}(X_\tau - c)\right], \tag{4.2}$$

where τ is an arbitrary stopping time. In line with our previous findings, we can now establish the following.

Lemma 4.1. *The value $\bar{V}(x,r)$ is increasing and convex as a function of the current revenues x. That is, $\bar{V}_x(x,r) > 0$ and $\bar{V}_{xx}(x,r) \geq 0$ for all $(x,r) \in \mathbb{R}_+^2$. Moreover, if the conditions of part (B) of Lemma 3.3 are satisfied, then the value $\bar{V}(x,r)$ is increasing and convex as a function of the current interest rate r. That is, then $\bar{V}_r(x,r) < 0$ and $\bar{V}_{rr}(x,r) \geq 0$ for all $(x,r) \in \mathbb{R}_+^2$.*

Proof. It is now clear that the solution of the stochastic differential equation (4.1) is $X_t = xe^{\mu t}M_t$, where $M_t = e^{\gamma \bar{W}(t) - \gamma^2 t/2}$ is a positive martingale. Consequently, all the elements in the sequence of value functions $V_n(x,r)$ presented in the proof of part (A) of Lemma 3.3 are increasing and convex as functions of the current revenues x (cf. El Karoui, Jeanblanc-Picqué, and Shreve 1998). This implies that the value function is increasing and convex as a function of the current revenues x. The rest of the proof is analogous with the proof of part (B) of Lemma 3.3. □

The key implication of Lemma 4.1 is now presented in

Theorem 4.2. *Assume that the conditions of Lemma 4.1 are satisfied, and that $1 > \beta\mu$, so that the expected percentage growth rate μ of the revenues X_t is below the long run steady state β^{-1} of the interest rate. Then, for all $(x,r) \in \mathbb{R}_+^2$ we have that*

$$\bar{V}(x,r) \geq \hat{V}(x,r) \geq V(x,r) \quad \text{and} \quad \bar{F}(x,r) \geq \hat{F}(x,r) \geq F(x,r),$$

where

$$\bar{F}(x,r) = \sup_{\tau} E_{(x,r)} \int_0^{\tau} e^{-\int_0^t r_s ds}[r_t c - (r_t - \mu)X_t]dt$$

denotes the early exercise premium in the presence of revenue and interest rate uncertainty. Moreover, increased volatility γ increases the value of the investment opportunity and postpones rational exercise.

Proof. See Appendix G. □

Theorem 4.2 shows that revenue uncertainty strengthens the negative effect of interest rate uncertainty and vice versa. Consequently, our results clearly verify the intuitively clear result that uncertainty, independently of its source, slows down rational investment demand by increasing the required exercise premium of a rational investor.

5 Conclusions

In this paper we have considered the determination of an optimal irreversible investment policy with variable discounting and demonstrated several new results. We started our analysis by considering the case of deterministic interest rate variability. First, we provided a set of sufficient conditions under which the optimal exercise date of investment opportunity can be solved generally and in an interesting special case explicitly. Second, we demonstrated the relationship between the optimal exercise dates with variable and constant discounting when the interest rate can be below or above the long-run steady state interest rate. Third, we showed that the value of the investment opportunity is a decreasing and convex function of the current interest rate.

We have also generalized our deterministic analysis in two respects. First, we have explored the optimal investment decision in the presence of interest rate uncertainty, i.e. when the interest rate process has certain properties, but fluctuates stochastically, and second, we have allowed for revenue dynamics to follow geometric Brownian motion. In this setting we characterized a set of sufficient conditions which can be applied for the verification of the optimality of an investment strategy. Moreover, we have showed how under certain plausible conditions the interest rate uncertainty postpones the rational exercise of investment opportunity. Finally, and importantly, we demonstrated that revenue uncertainty strengthens the negative impact of interest rate uncertainty and vice versa.

Acknowledgements: The research of Luis H. R. Alvarez has been supported by the *Foundation for the Promotion of the Actuarial Profession* and the *Yrjö Jahnsson Foundation.* Erkki Koskela thanks the *Research Unit of Economic Structures and Growth (RUESG)* in the University of Helsinki for financial support and *CES* in the University of München for its great hospitality. The authors are grateful to *Heikki Ruskeepää* for his assistance in the *MATHEMATICA*©-calculations.

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A Proof of Theorem 2.1

Proof. It is a simple exercise in ordinary analysis to demonstrate that

$$X_t = xe^{\mu t}, \quad r_t = \frac{re^{\alpha t}}{1+\beta r(e^{\alpha t}-1)},$$

$$e^{-\int_0^t r_s ds} = \left(1+\beta r(e^{\alpha t}-1)\right)^{-1/(\alpha\beta)},$$

and that

$$\frac{d}{dt}\left[e^{-\int_0^t r_s ds}(X_t - c)\right] = e^{-\int_0^t r_s ds}(\mu X_t - r_t(X_t - c)). \tag{A.1}$$

Given the solutions of the ordinary differential equations (2.1) and (2.2), we observe that (A.1) can be rewritten as

$$(1+\beta r(e^{\alpha t}-1)))e^{\int_0^t r_s ds}\frac{d}{dt}\left[e^{-\int_0^t r_s ds}(X_t - c)\right] = \mu x(1-\beta r) + rce^{(\alpha-\mu)t} - rx(1-\beta\mu)e^{\alpha t}.$$

Consider now the mapping $f : \mathbb{R}_+ \mapsto \mathbb{R}$ defined as

$$f(t) = \mu x(1-\beta r) + rce^{(\alpha-\mu)t} - rx(1-\beta\mu)e^{\alpha t}.$$

It is now clear that $f(0) = rc - (r-\mu)x$ and that $\lim_{t\to\infty} f(t) = -\infty$. Moreover, since

$$f'(t) = (\alpha-\mu)rce^{(\alpha-\mu)t} - \alpha rx(1-\beta\mu)e^{\alpha t},$$

we find that $f'(t) < 0$ for all $t \geq 0$ whenever $\alpha \leq \mu$ and, therefore, that for any initial state on C, the optimal stopping date $t^*(x,r)$ satisfying the optimality condition $f(t^*(x,r)) = 0$ exists and is finite (because of the monotonicity and the boundary behavior of $f(t)$). Assume now that $\alpha > \mu$. Then, $f'(0) = (\alpha-\mu)rc - \alpha rx(1-\beta\mu)$ and $\lim_{t\to\infty} f'(t) = -\infty$. Moreover, since

$$f''(t) = (\alpha-\mu)^2 rce^{(\alpha-\mu)t} - \alpha^2 rx(1-\beta\mu)e^{\alpha t},$$

we find that $0 = \operatorname{argmax}\{f(t)\}$ provided that $(\alpha-\mu)rc \leq \alpha rx(1-\beta\mu)$ and that

$$\tilde{t} = \frac{1}{\mu}\ln\left(\frac{(\alpha-\mu)c}{\alpha x(1-\beta\mu)}\right)$$

provided that $(\alpha-\mu)rc > \alpha rx(1-\beta\mu)$. However, since

$$f''(\tilde{t}) = -\alpha rx(1-\mu\beta)\mu e^{\alpha\tilde{t}} < 0$$

we find that $f'(t) < 0$ for all $(x,r) \in \mathbb{R}_+^2$ in that case as well and, therefore, that for any initial state on C, the optimal stopping date $t^*(x,r)$ satisfying the optimality condition $f(t^*(x,r)) = 0$ exists and is finite.

Having established the existence and finiteness of the optimal exercise date $t^*(x,r)$ we now have to prove that the value satisfies the boundary value problem. Standard differentiation yields (after simplifications)

$$\frac{\partial V}{\partial x}(x,r) = \left(1+\beta r(e^{\alpha t^*(x,r)}-1)\right)^{-1/(\alpha\beta)}$$

and

$$\frac{\partial V}{\partial r}(x,r) = -\left(1+\beta r(e^{\alpha t^*(x,r)}-1)\right)^{-1/(\alpha\beta)}\frac{1}{\alpha}(X_{t^*(x,r)}-c)(e^{\alpha t^*(x,r)}-1).$$

Applying these equations then proves that $(\mathcal{A}V)(x,r)-rV(x,r)=0$ for all C. Moreover, since $t^*(x,r)=0$ whenever $(x,r)\in\partial C$, we find that $\frac{\partial V}{\partial x}(x,r)=1$ and $\frac{\partial V}{\partial r}(x,r)=0$ for all $(x,r)\in\partial C$. Our results on the early exercise premium $F(x,r)$ are direct implications of the definition (2.4). □

B Proof of Corollary 2.2

Proof. As was established in the proof of Theorem 2.1, the optimal exercise date $t^*(x,r)$ is the root of the equation $\mu X_{t^*(x,r)} = r_{t^*(x,r)}(X_{t^*(x,r)}-c)$, that is, the root of the equation

$$\mu x e^{\mu t^*(x,r)}(1+\beta r(e^{\mu t^*(x,r)}-1)) = r e^{\mu t^*(x,r)}(x e^{\mu t^*(x,r)}-c).$$

Multiplying this equation with $e^{-\mu t^*(x,r)}$ and reordering terms then yields

$$rx(\mu\beta-1)e^{\mu t^*(x,r)} = \mu x(\beta r-1)-rc$$

from which the alleged result follows by taking logarithms from both sides of the equation. Inserting the optimal exercise date $t^*(x,r)$ to the expression

$$V(x,r) = e^{-\int_0^{t^*(x,r)} r_s ds}(X_{t^*(x,r)}-c)$$

then yields the alleged value. Our conclusions on the early exercise premium $F(x,r)$ then follow directly from (2.4). Finally, the comparative static properties of the optimal exercise date $t^*(x,r)$ can then be established by ordinary differentiation. □

C Proof of Theorem 2.4

Proof. It is clear that since $\tilde{t}(x,r)$ satisfies the condition $\mu X_{\tilde{t}(x,r)} = r(X_{\tilde{t}(x,r)}-c)$. Define now the mapping $\hat{f}(t)=\mu X_t - r(X_t-c)$. We then find that

$$\hat{f}(\tilde{t}(x,r)) = \mu X_{\tilde{t}(x,r)} - r_{\tilde{t}(x,r)}(X_{\tilde{t}(x,r)}-c) = (r-r_{\tilde{t}})(X_{\tilde{t}(x,r)}-c) \gtreqless 0, \quad \text{if } r \gtreqless \beta^{-1},$$

since $r_t \gtreqless r$ for all $t\geq 0$ when $r \lesseqgtr \beta^{-1}$. However, since $\hat{f}(t^*(x,r))=0$ we find that $t^*(x,r) \gtreqless \tilde{t}(x,r)$ provided that $r \gtreqless \beta^{-1}$.

Assume that $r<\beta^{-1}$ and, therefore, that $r_t > r$ for all $t\geq 0$. Since $\mu x\frac{\partial \tilde{V}}{\partial x}(x,r) \leq r\tilde{V}(x,r)$ and $\tilde{V}(x,r)\geq g(x)$ for all $x\in\mathbb{R}_+$ we find by ordinary differentiation that

$$\begin{aligned}\frac{d}{dt}\left[e^{-\int_0^t r_s ds}\tilde{V}(X_t,r)\right] &= e^{-\int_0^t r_s ds}\left[\mu X_t\frac{\partial \tilde{V}}{\partial x}(X_t,r) - r_t\tilde{V}(X_t,r)\right] \\ &\leq e^{-\int_0^t r_s ds}\left[r-r_t\right]\tilde{V}(X_t,r)\leq 0\end{aligned}$$

for all $t \geq 0$. Therefore,

$$\tilde{V}(x,r) \geq e^{-\int_0^t r_s ds}\tilde{V}(X_t,r) \geq e^{-\int_0^t r_s ds}g(X_t)$$

implying that $\tilde{V}(x,r) \geq V(x,r)$ when $r < \beta^{-1}$. The proof in the case where $r > \beta^{-1}$ is completely analogous. The conclusions on the early exercise premiums $F(x,r)$ and $\tilde{F}(x,r)$ follow directly from their definitions. □

D Proof of Lemma 2.5

Proof. Consider first the discount factor $e^{-\int_0^t r_s ds}$. Since

$$e^{-\int_0^t r_s ds} = \left(1+\beta r(e^{\alpha t}-1)\right)^{-1/(\alpha\beta)},$$

we find by ordinary differentiation that

$$\frac{d}{dr}\left[e^{-\int_0^t r_s ds}\right] = -\frac{1}{\alpha}\left(1+\beta r(e^{\alpha t}-1)\right)^{-(1/(\alpha\beta)+1)}(e^{\alpha t}-1) < 0$$

and that

$$\frac{d^2}{dr^2}\left[e^{-\int_0^t r_s ds}\right] = \frac{1}{\alpha}\left(\frac{1}{\alpha\beta}+1\right)\left(1+\beta r(e^{\alpha t}-1)\right)^{-(1/(\alpha\beta)+2)}\beta(e^{\alpha t}-1)^2 > 0$$

implying that the discount factor is decreasing and convex as a function of the current interest rate. Since the maximum of a decreasing and convex mapping is decreasing and convex, we find that the value is a decreasing and convex function of the current interest rate r. Similarly, since the exercise payoff $X_t - c$ is increasing and linear as a function of the current state x, we find that the maximum, that is, the value of the opportunity is increasing and convex as a function of the initial revenues x (by classical duality arguments of nonlinear programming). □

E Proof of Theorem 3.2

Proof. As was established in Lemma 2.5, the value of the investment opportunity is convex in the absence of uncertainty. Consequently, we find that for all $(x,r) \in C$ we have that

$$(\hat{\mathcal{A}}V)(x,r) - rV(x,r) = \frac{1}{2}\sigma^2(r)\frac{\partial^2 V}{\partial r^2}(x,r) \geq 0,$$

since $(\mathcal{A}V)(x,r) - rV(x,r) = 0$ for all $(x,r) \in C$. Let τ_n be a sequence of stopping times converging towards the stopping time $\tau^* = \inf\{t \geq 0 : \mu X_t \leq r_t(X_t - c)\}$. Dynkin's theorem then yields that

$$E_{(x,r)}\left[e^{-\int_0^{\tau_n} r_s ds}V(X_{\tau_n}, r_{\tau_n})\right] \geq V(x,r).$$

Letting $n \to \infty$ and invoking the continuity of the value $V(x,r)$ across the boundary ∂C then yields that

$$V(x,r) \leq E_{(x,r)}\left[e^{-\int_0^{\tau_n} r_s ds}(X_{\tau_n} - c)\right] \leq \hat{V}(x,r)$$

for all $(x,r) \in C$. However, since $V(x,r) = x - c$ on $\mathbb{R}^2_+ \backslash C$ and $\hat{V}(x,r) \geq x - c$ for all $x \in \mathbb{R}^2_+$, we find that $\hat{V}(x,r) \geq V(x,r)$ for all $x \in \mathbb{R}^2_+$.

Assume that $(x,r) \in C$. Since $\hat{V}(x,r) \geq V(x,r) > (x-c)$, we find that $(x,r) \in \{(x,r) \in \mathbb{R}^2_+ : \hat{V}(x,r) > x - c\}$ as well and, therefore, that $C \subset \{(x,r) \in \mathbb{R}^2_+ : \hat{V}(x,r) > x - c\}$, thus completing the proof of our theorem. □

F Proof of Lemma 3.3

Proof. (A) To establish the monotonicity and convexity of the value function $\hat{V}(x,r)$ as a function of the current revenues x, we first define the increasing sequence $\{V_n(x,r)\}_{n\in\mathbb{N}}$ iteratively as

$$V_0(x,r) = (x-c), \quad V_{n+1}(x,r) = \sup_{t\geq 0} E_{(x,r)}\left[e^{-\int_0^t r_s ds} V_n(X_t, r_t)\right].$$

It is now clear that since $V_0(x,r)$ is increasing and linear as a function of x and $X_t = xe^{\mu t}$, the value $V_1(x,r)$ is increasing and convex as a function of x by standard duality arguments from nonlinear programming theory. Consequently, all elements in the sequence $\{V_n(x,r)\}_{n\in\mathbb{N}}$ are increasing and convex as functions of x. Since $V_n(x,r) \uparrow \hat{V}(x,r)$ as $n \to \infty$ (cf. Øksendal 1998, p. 200) we find that for all $\lambda \in [0,1]$ and $x, y \in \mathbb{R}_+$ we have that

$$\lambda\hat{V}(x,r) + (1-\lambda)\hat{V}(y,r) \geq \lambda V_n(x,r) + (1-\lambda)V_n(y,r) \geq V_n(\lambda x + (1-\lambda)y, r).$$

Letting $n \to \infty$ and invoking monotonic convergence then implies that $\lambda\hat{V}(x,r) + (1-\lambda)\hat{V}(y,r) \geq V(\lambda x + (1-\lambda)y, r)$ proving the convexity of $\hat{V}(x,r)$. Similarly, if $x \geq y$ then

$$\hat{V}(x,r) \geq V_n(x,r) \geq V_n(y,r) \uparrow \hat{V}(y,r), \quad \text{as } n \to \infty$$

proving the alleged monotonicity of $\hat{V}(x,r)$ as a function of x.
(B) As was established in Alvarez and Koskela 2001, the assumed smoothness of the diffusion coefficient $\sigma(r)$, the Novikov-condition, and the strict concavity of the drift $\alpha r(1-\beta r)$ imply that the discount factor $e^{-\int_0^t r_s ds}$ is an almost surely decreasing and strictly convex function of the current interest rate r and, consequently, that the value function is decreasing and strictly convex as a function of the current interest rate r. □

G Proof of Theorem 4.2

Proof. The proof of the first part of the theorem is analogous with the proof of Theorem 3.2. In order to establish that increased volatility γ increases the value and postpones rational exercise, we observe that the convexity of the value implies that if $\hat{\gamma} > \gamma$ and $(x,r) \in \{(x,r) \in \mathbb{R}^2_+ : \bar{V}(x,r) > x - c\}$, then

$$((\bar{\mathcal{A}}_{\hat{\gamma}} - r)\bar{V})(x,r) = ((\bar{\mathcal{A}}_{\hat{\gamma}} - \bar{\mathcal{A}}_{\gamma} + \bar{\mathcal{A}}_{\gamma} - r)\bar{V})(x,r) = \frac{1}{2}(\hat{\gamma} - \gamma^2)x^2\bar{V}_{xx}(x,r) \geq 0.$$

That is the value function $\bar{V}(x,r)$ is r-subharmonic for the more volatile revenue process on the continuation region $\{(x,r) \in \mathbb{R}^2_+ : \bar{V}(x,r) > x - c\}$. The rest of the proof is then analogous with the proof of Theorem 3.2. □