CONNECTED SUBSTITUTES AND INVERTIBILITY OF DEMAND

By

Steven Berry, Amit Gandhi, and Philip Haile

June 2011

COWLES FOUNDATION DISCUSSION PAPER NO. 1806

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

Connected Substitutes and Invertibility of Demand*

Steven Berry
Yale University

Amit Gandhi
University of Wisconsin-Madison

Philip Haile
Yale University

June 27, 2011

Abstract

We consider the invertibility of a nonparametric nonseparable demand system. Invertibility of demand is important in several contexts, including identification of demand, estimation of demand, testing of revealed preference, and economic theory requiring uniqueness of market clearing prices. We introduce the notion of "connected substitutes" and show that this structure is sufficient for invertibility. The connected substitutes conditions require weak substitution between all goods and sufficient strict substitution to necessitate treating them in a single demand system. These conditions are satisfied in many standard models, have transparent economic interpretation, and allow us to show invertibility without functional form restrictions, smoothness assumptions, or strong domain restrictions.

*This paper combines and expands on selected material first explored in Gandhi (2008) and Berry and Haile (2009a). We thank Jean-Marc Robin for helpful discussions. We received useful comments from participants in the 2011 "Econometrics of Demand" conference at MIT. Adam Kapor provided capable research assistance. Financial support from the National Science Foundation is gratefully acknowledged.

1 Introduction

We consider the invertibility of a nonparametric nonseparable demand system. Invertibility of demand is important in several theoretical and applied contexts, including identification of demand, estimation of demand systems, testing of revealed preference, and economic theory exploiting the uniqueness of market clearing prices. We introduce the notion of "connected substitutes" and show that this structure is sufficient for invertibility.

We consider a general setting in which demand for goods $1, \ldots, J$ is characterized by

$$\sigma(x) = (\sigma_1(x), \ldots, \sigma_J(x)) : \mathcal{X} \subseteq \mathbb{R}^J \to \mathcal{S} \tag{1}$$

where $x = (x_1, \ldots, x_J)$ is a vector of demand shifters associated with each good. All other arguments of the demand system are held fixed. This setup nests a number of special cases of interest. Points in $\mathcal{S}$ may be vectors of quantities demanded, choice probabilities, market shares, or expenditure shares. The demand shifters x might be prices, unobserved characteristics of the goods, or latent preference shocks. Several examples below illustrate.

The connected substitutes structure requires two conditions. First, goods must be weak substitutes in x in the sense that when x_j increases (e.g., price falls) only for a subset of goods j, demand for the remaining goods (taken as a whole) does not increase. Second, there must be sufficient *strict* substitution among the goods to require treating them all in one demand system. These conditions are weaker than other notions of substitution (e.g., strict gross substitutes) and allow us to show invertibility without functional form restrictions, smoothness assumptions, or domain restrictions relied on previously.

Important to our approach is the explicit treatment of a "good 0" whose "demand" is defined by the identity

$$\sigma_0(x) = 1 - \sum_{j=1}^{J} \sigma_j(x). \tag{2}$$

The interpretation will vary with the application. When demand is expressed in shares (e.g., choice probabilities or market shares), good 0 can be a "real" good—e.g., a numeraire good, an "outside good," or a good relative to which utilities are normalized. The identity (2) will then follow from the fact that shares sum to one. In other applications, good 0 will be a purely artificial notion introduced only as a technical device (see the examples below).

It is clear from (2) that even when good 0 is a real good, (1) characterizes the full demand system. Nonetheless, explicitly accounting for the demand for good 0 simplifies imposition of the connected substitutes structure on all goods. When good 0 is an artificial good, including it in the connected substitutes conditions proves useful as well. As will be clear below, it strengthens the weak substitution requirement in a natural way while weakening the requirement of minimal strict substitution.

Invertibility is naturally considered only on the interior of $\mathcal{S}$.[1] In some cases, one may wish to restrict the domain of interest further. For example, one might restrict attention to values of x observed in a particular data set, or consider only positive prices even when σ is defined on $\mathbb{R}^J$ (e.g., multinomial logit or probit). Let $\mathcal{X}^*$ denote the domain of interest, where

$$\mathcal{X}^* \subseteq \tilde{\mathcal{X}} \equiv \{x \in \mathcal{X} : \sigma(x) \in \operatorname{int}(\mathcal{S})\}.$$

Let

$$\sigma^* : \mathcal{X}^* \to \sigma(\mathcal{X}^*) \tag{3}$$

denote the restriction of σ to $\mathcal{X}^*$. Our goal is to provide sufficient conditions for invertibility of σ^*, i.e., such that for every $s \in \operatorname{int}(\mathcal{S})$ there is at most one $x \in \mathcal{X}^*$ satisfying $\sigma(x) = s$.[2]

Prior results on invertibility of demand have often relied on conditions that can be difficult to motivate and rule out important models of demand for differentiated products. A central result is the "univalence" theorem of Gale and Nikaido (1965), which showed global invertibility of a differentiable mapping when the domain $\mathcal{X}^*$ is a rectangle (a product of intervals) and the Jacobian is everywhere a $P-$matrix (i.e., all principal minors are strictly

---

[1] In our examples, $\mathcal{S}$ is either the unit J-simplex or $\mathbb{R}^J_+$, so $\operatorname{int}(\mathcal{S})$ excludes only points in $\mathcal{S}$ with zero demand for some real good. When demand is on a boundary, invertibility will generally fail. For example, if a good j has zero demand, an increase in its price typically will have no effect. When σ is differentiable, a good j with zero demand will typically have $\frac{\partial \sigma_j(x)}{\partial x_k} = 0 \forall k$, yielding a singular Jacobian matrix.

[2] Equivalently, such that for every $s \in \sigma(\mathcal{X}^*)$ there is a unique $x \in \mathcal{X}^*$ satisfying $\sigma(x) = s$.

positive). These sufficient conditions (see also the variations in, e.g., Garcia and Zangwill (1979) and Mas-Colell (1979)) can be problematic in applications to demand. Differentiability is essential, but fails in some important models. Examples include those defined on a discrete domain (e.g., a grid of prices) or a random utility model with discrete distributions. Even when differentiability is assumed, the Jacobian conditions can be difficult to interpret or to derive from widely applicable primitive conditions (see the examples below). Moreover, the Jacobian conditions can be problematic when combined with the requirement of a rectangular domain.[3] The Jacobian will generally be singular (and, thus, certainly not a P-matrix) outside the set $\tilde{\mathcal{X}}$ (recall footnote 1). Unless $\tilde{\mathcal{X}}$ is a rectangle, this limits the applicability of the Gale-Nikaido result.[4] Our connected substitutes conditions avoid these limitations. Although they rule out some models as well—most important, they require either indivisible goods or the absence of strict gross complements—they are easily interpreted, hold in wide range of models studied in practice, and avoid any smoothness requirement or restriction on the shape of the set $\mathcal{X}^*$.

The plan of the paper is as follows. In section 2 we present several examples that motivate our interest, tie our general formulation to more familiar special cases, and provide further connections to prior work. In section 3 we set up the model and discuss the connected substitutes conditions. Our invertibility result is given in section 4. In section 5 we examine a link between the connected substitutes conditions and the Jacobian condition of Gale and Nikaido (1965) when demand is differentiable.

---

[3] It is well known that the Gale and Nikaido (1965) result does not generally extend to non-rectangular domains (e.g., Parthasarathy and Ravindran (2003), Aleksandrov (1994)).

[4] Consider, for example, a market with vertically differentiated goods (e.g., Mussa and Rosen (1978)), where a lower quality good has no demand unless its price is strictly lower than that of all higher quality goods. If $-x$ is the price vector, the domain of interest $\mathcal{X}^*$ (e.g., $\tilde{\mathcal{X}}$ or the set of observed prices) generally will not be a rectangle. Other examples in which the natural domain of interest $\tilde{\mathcal{X}}$ is nonrectangular include models of spatial differentiation (e.g., Salop (1979)) or the "pure characteristics" model of Berry and Pakes (2007).

2 Examples

Estimation of Discrete Choice Demand Models. A large empirical literature uses random utility discrete choice models of demand to study differentiated products markets, building on pioneering work of McFadden (1974, 1981), Bresnahan (1981, 1987) and others. Conditional indirect utilities are normalized relative to that of a good 0, often an outside good representing purchase of goods not explicitly under study.

Much of the recent literature follows Berry (1994) in modeling price endogeneity through a vector of product-specific unobservables x, with each x_j shifting tastes for good j monotonically.[5] Holding observables fixed, $\sigma(x)$ gives the vector of choice probabilities (or market shares). Typically each σ_j is a nonlinear function of the the entire vector of unobservables x. Invertibility is therefore nontrivial, and it is critical to estimation approaches, including those of Berry, Levinsohn, and Pakes (1995) and Dube, Fox, and Su (2009).[6] Berry (1994) provided sufficient conditions for invertibility that include linearity (the conditional indirect utility for good j is linear in x_j), differentiability of $\sigma_j(x)$, and strict gross substitutes.[7] Our result relaxes all three conditions, avoiding any functional form restriction or differentiability requirement, and imposing the weaker connected substitutes structure.

Note that a discrete choice framework can allow consumers to demand multiple indivisible goods (including complements), since every bundle can be redefined as a distinct good (e.g.,

[5] Examples include studies of the US automobile industry (e.g., Berry, Levinsohn and Pakes (1995, 1999, 2004), Petrin (2002)), the European automobile industry (e.g., Verboven (1996), Goldberg and Verboven (2001)), the breakfast cereal industry (e.g., Nevo (2000), Nevo (2001)), newspapers (e.g., Fan (2008)), movies (e.g., Davis (2001), Einav (2007)), radio (Berry and Waldfogel (1999)), airlines (e.g., Berry, Carnall, and Spiller (1996), Berry and Jia (2010)), pharmaceuticals (e.g., Azoulay (2002)), and banking (e.g., Dick (2008)).

[6] The Berry, Levinsohn, and Pakes (1995) estimation algorithm also exploits the fact that, in the models they consider, $\sigma(\mathcal{X}) = \text{int}(\mathcal{S})$. Given invertibility, this ensures that even at wrong values (i.e., trial values) of the model parameters, the observed choice probabilities can be inverted. This property is not necessary for all estimation methods or for other purposes motivating interest in the inverse. However, Gandhi (2010) provides sufficient conditions for a nonparametric model and also discusses a solution algorithm.

[7] Although Berry (1994) assumes strict gross substitutes, his proof only requires that each inside good strictly substitute to the outside good. Hotz and Miller (1993) provide an invertibility theorem for a similar class of models, although they provide a complete proof only for local, not global, invertibility. Berry and Pakes (2007) state an invertibility result for a discrete choice model relaxing some assumptions in Berry (1994), while still assuming the linearity of utility in x_j. Their proof is incomplete, although adding the second of our two connected substitutes conditions would correct this deficiency.

Gentzkow (2004)). Thus, the key restriction of a discrete choice setting is the indivisibility of goods.[8] Further, anything that raises the choice probability for good (or bundle) j must lower the aggregated choice probabilities over all other goods. Thus the restriction to weak substitutes (Assumption 2 below) is particularly mild in a discrete choice setting.

Nonparametric Identification of Discrete Choice Demand. Separate from practical issues of estimation, there has been growing interest in the question of whether discrete choice demand models in the spirit of Berry, Levinsohn, and Pakes (1995) are identified without the strong functional form and distributional assumptions typically used in applications. Berry and Haile (2009b, 2010) have recently provided affirmative answers for nonparametric random utility models in which each consumer's conditional indirect utilities for the "inside goods" have joint distribution

$$F_v\left(v_{i1t,}\ldots,v_{iJt}|c_{it}\right)$$

given characteristics c_{it} of the consumer i, the goods j, and the market t. Included in c_{it} is a vector $x_t = (x_{1t},\ldots,x_{Jt})$ of unobservables reflecting latent tastes in market t and/or unobserved characteristics of the goods. By conditioning out all other components of c_{it}, one obtains nonparametric choice probabilities of the form (1). Berry and Haile (2009b, 2010), relying on the invertibility result below, show that identification can be obtained using standard instrumental variables conditions or extensions of classical arguments for identification of supply and demand to a system of nonparametric simultaneous equations.

Inverting for Preference Shocks in Continuous Demand Systems. A recent paper considering invertibility of a nonparametric continuous demand system is Beckert and Blundell (2008). In their model, utility from a bundle of consumption quantities $q = (q_0,\ldots,q_J)$

[8] In a recent working paper, Azevedo, White, and Wyl (2011) consider a setting with indivisible goods, demand for bundles, and a continuum of consumers. They focus on existence of market clearing prices but also consider uniqueness. They require quasilinear utility and a strong notion of "large support" for preferences. Their uniqueness result can be interpreted as demonstrating invertibility of demand (in prices) under linear pricing of the elementary goods—a restriction on $\mathcal{X}^*$ in our notation. Each of those conditions would have analogs when x is the vector of product-specific unobservables. Our result requires neither quasilinearity nor large support, and does not limit attention to a set $\mathcal{X}^*$ consistent with linear pricing (or, analogously, linearity of bundle preferences in the product-specific unobservables).

is given by a strictly increasing C^2 function $u(q, \epsilon)$, with $\epsilon \in \mathbb{R}^J$ denoting latent demand shocks. The price of good 0 is normalized to 1 for simplicity. Given total expenditure m and prices $p = (p_1, \ldots, p_J)$ for the remaining goods, quantities demanded are given by

$$q_j = h_j(p, m, \epsilon) \quad j = 1, \ldots, J$$

with $q_0 = m - \sum_{j>0} p_j q_j$.

Beckert and Blundell (2008) consider invertibility of this demand system in the latent vector ϵ, pointing out that this is a necessary step toward identification of demand or testing of stochastic revealed preference restrictions (e.g., Block and Marschak (1960), McFadden and Richter (1971, 1990), Falmagne (1978), McFadden (2004)). They provide several invertibility results. One requires requires marginal rates of substitution between good 0 and goods $j > 0$ to be multiplicatively separable in ϵ, with an invertible matrix of coefficients. Alternatively, they provide conditions (on functional form and/or on derivative matrices of marginal rates of substitution) that imply the Gale-Nikaido Jacobian conditions.

When gross complementarities between goods can be ruled out, we provide alternative sufficient conditions for invertibility that may be more widely applicable. A natural (but hardly necessary) way to translate their model to ours is through expenditure shares. To do this, fix p and m, let $\sigma_j(\epsilon) = p_j h(p, m, \epsilon)/m$ for $j > 0$, and relabel $x = \epsilon$. Expenditure shares sum to one, implying the identity (2).

In the Beckert and Blundell (2008) model, the goods $j = 0, 1, \ldots, J$ under study represent all goods in the economy. A common alternative is to consider demand for a more limited set of goods—for example, goods in a particular product category. In that case, there will no longer be a good whose demand is determined from the others' through the budget constraint. Further, here it is natural to have a demand shock ϵ_j for every good j. This situation is also easily accommodated in our framework. Again let x denote the vector demand shock $(\epsilon_1, \ldots, \epsilon_J)$. Holding prices and all other demand shifters fixed, let $\sigma_j(x)$ give the quantity demand of good j, for $j = 1, \ldots, J$. To complete the mapping to our model, we let (2) define the object $\sigma_0(x)$. A hint at the role this artificial good 0 plays below can be seen by observing that a rise in $\sigma_0(x)$ represents a fall in the demand for goods $j > 0$ as a whole.

Inverting for Prices in Continuous Demand Systems. One could instead investigate invertibility in prices. Let $x_j = -p_j$, where p_j is the price of good j. Conditional on all other demand shifters, let $\sigma_j(x)$ give the quantity of good j demanded for the goods $j = 1, \ldots, J$ under consideration. Invertibility of the demand system then implies uniqueness of market clearing prices, an important property of demand in several contexts. For example, it is required for Cournot competition to be well defined and, typically, for the derivation of equilibrium comparative statics. Once again, the result of Gale and Nikaido (1965) has often been employed to show uniqueness. Cheng (1985) provided more easily interpretable conditions (and a connection to earlier results on uniqueness of Walrasian equilibrium prices, e.g., Arrow, Block, and Hurwicz (1959)) by showing that the Gale and Nikaido (1965) Jacobian condition holds under a standard dominant diagonal condition and a restriction to goods that are strict gross substitutes. The limitations of requiring differentiability and a rectangular domain are again a concern.[9] Further, the requirement of strict gross substitutes (here and in several other results cited above) rules out many standard models of differentiated products, which feature substitution that is only "local" — i.e., between goods that are adjacent in the product space (see, e.g., Figure 1 and Appendix A below). Our formulation avoids these limitations, once again using the identity (2) to introduce an artificial good 0 as a technical device capturing aggregate demand responses.

[9] The requirement of a rectangular domain is unstated in Cheng (1985), but required by the results relied on in the proof (in particular, Theorem 20.4 of Nikaido (1968)).

3 Model

3.1 Demand

Let $\mathcal{J} = \{0, 1, \ldots, J\}$ denote the set of all goods. Recall that $x \in \mathcal{X} \subseteq \mathbb{R}^J$ is a vector of demand shifters and that all other determinants of demand are held fixed.[10] Demand for each good is given by $\sigma_j(x)$, $j \in \mathcal{J}$, where we impose (2). Although we postpone assumptions on σ to section 3.2, one should think of x_j as a monotonic shifter of demand for good j. In the examples above, x_j is either (minus) the price of good j, the unobserved quality of good j, or a shock to taste for good j. In all of these examples, monotonicity is a standard property.

Given (2), the demand system can be characterized by $\sigma = (\sigma_1, \ldots, \sigma_J) : \mathcal{X} \to \mathcal{S}$. For $\mathcal{X}^* \subseteq \tilde{\mathcal{X}} \equiv \{x \in \mathcal{X} : \sigma(x) \in \text{int}(\mathcal{S})\}$, we seek conditions such that for every $s \in$ int$(\mathcal{S})$ there is at most one $x \in \mathcal{X}^*$ satisfying $\sigma(x) = s$.

Our first assumption is a condition on $\mathcal{X}$, the set on which σ is defined. This assumption is required only to ensure that $\sigma(x)$ is defined at all x used in our argument. Thus, the requirement is that $\mathcal{X}$ include a sufficiently rich set of points. Sufficient conditions include $\mathcal{X}$ being open (in the Euclidean topology) or a Cartesian product, the latter allowing a discrete domain.

Assumption 1. For all distinct $x, x' \in \mathcal{X}^*$ and all j such that $x_j \neq x'_j$, there exists $\lambda_j \in (0, 1]$ such that $\mathcal{X}$ contains either (a) $\tilde{x}$, where $\forall k$, $\tilde{x}_k = x_k + \lambda_j \left(x'_j - x_j\right) 1\{k = j\}$, or (b) $\tilde{x}'$, where $\forall k$, $\tilde{x}'_k = x'_k + \lambda_j \left(x_j - x'_j\right) 1\{k = j\}$.

Leading cases satisfying Assumption 1 are when either $\mathcal{X}$ or $\mathcal{X}^*$ is open (take λ_j sufficiently small) or when either $\mathcal{X}$ or $\mathcal{X}^*$ is a Cartesian product (take $\lambda_j = 1$). These cases may capture most environments of interest.[11] Even if this condition fails, it may often hold

[10] When good 0 is an real good relative to which prices or utilities are normalized, this includes all characteristics of this good. For example, we do not rule out the possibility that good 0 has a price x_0, but are holding it fixed (e.g., at 1).

[11] A more general sufficient condition, nesting these and others, is that $\mathcal{X}^* \subseteq \mathcal{Y} \subseteq \mathcal{X}$, where $\mathcal{Y}$ is either open or a Cartesian product. Examples of other sets satisfying Assumption 1 include the convex cone $\left\{x \in [0, \infty)^J : x_j > x_{j-1}\ \forall j\right\}$, the "quarter-disk" $\{x : x_j \geq 0\ \forall j,\ \|x\| < 1\}$, and a rectangle with an open hole, such as $\left\{x \in [0, 1]^J : \|x\| > \frac{1}{2}\right\}$.

under an extension of the domain $\mathcal{X}$ (e.g., to an open cover) or with a slightly smaller $\mathcal{X}^*$.

We contrast this assumption with Gale and Nikaido's (1965) requirement of a rectangular $\mathcal{X}^*$. Our assumption is much weaker: rectangular $\mathcal{X}^*$ is a special case of Cartesian $\mathcal{X}^*$. Further, our assumption need not place any restriction on the shape or other properties of the set $\mathcal{X}^*$ on which invertibility of σ is considered. Finally, Assumption 1 plays the role of a regularity condition for our result, whereas a rectangular domain is integral to the proof strategy in Gale and Nikaido (1965).

3.2 Connected Substitutes

Our main requirement for invertibility is a pair of conditions characterizing the notion of connected substitutes. The first is that the goods are weak substitutes in x in the sense that when x_j increases (e.g., price falls) for only a subset of goods j, demand for the remaining goods, taken as a whole, does not increase.

Assumption 2. For any $\mathcal{I} \subset \mathcal{J}$ and any $x, x' \in \mathcal{X}$ such that $x'_j \geq x_j$ $\forall j \in \mathcal{I}$ and $x'_j \leq x_j$ $\forall j \notin \mathcal{I}$, $\sum_{j \notin \mathcal{I}} \sigma_j(x') \leq \sum_{j \notin \mathcal{I}} \sigma_j(x)$.

Observe than when good 0 is an artificial good, its presence in $\mathcal{J}$ adds to the requirements of Assumption 2 in a natural way. Taking the case where x is (minus) price, all else equal, a fall in the price of some/all goods $j > 0$ cannot cause the total demand (over all goods) to fall.

An alternative notion of weak substitution (that of weak gross substitutes) is that $\sigma_k(x)$ is nonincreasing in x_j for all $k, j \neq k$. It is easy to see that this is implied by Assumption 2 (take $x'_j \geq x_j$, $x'_i = x_i$ $\forall i \neq j$, and $\mathcal{I} = \mathcal{J} \setminus \{k\}$). In many models these conditions are equivalent—for example, when $\mathcal{X}$ is a Cartesian product (see Appendix C). In a discrete choice model, it is easily verified that both notions of weak substitution are implied by the standard assumptions that x_j is excluded from the conditional indirect utilities of goods $k \neq j$ and that the conditional indirect utility for good j is increasing in x_j.

To state the second condition characterizing connected substitutes, we first define a directional notion of (strict) substitution.

Definition 1. *Good j substitutes to good k at x* if $\sigma_k(x)$ is strictly decreasing in x_j.

Consider a decline in x_j with all else held fixed. Assumption 2 implies that this weakly raises $\sigma_k(x)$ for all $k \neq j$. The goods to which j substitutes are those whose demands $\sigma_k(x)$ strictly rise. If x_j is (minus) the price of good j, this is a standard notion of strict gross substitution from j to k. Definition 1 merely extends this notion to other demand shifters that may play the role of x. Although this is a directional notion, substitution (as defined here) is typically *symmetric*; i.e., j substitutes to k iff k substitutes to j. An exception is substitution to good 0: since any demand shifters for good 0 are held fixed, good 0 does not substitute to other goods.[12]

It will be useful to represent substitution among the goods with the directed graph of the matrix $\Sigma(x)$ whose elements are

$$\Sigma_{j+1,k+1} = \begin{cases} 1\{\text{good } j \text{ substitutes to good } k \text{ at } x\} & j > 0 \\ 0 & j = 0. \end{cases}$$

The directed graph of $\Sigma(x)$ has nodes (vertices) representing each good and a directed edge from node k to node ℓ whenever good k substitutes to good ℓ at x.

Assumption 3. For all $x \in \mathcal{X}^*$, the directed graph of $\Sigma(x)$ has, from every node $k \neq 0$, a directed path to node 0.

Figure 1 illustrates the directed graphs of $\Sigma(x)$ at generic $x \in \mathcal{X}^*$ for some standard models of differentiated products, letting $x = (-p_1, \ldots, -p_J)$, where p_j is the price of good j, and assuming (as usual) that conditional indirect utilities are strictly decreasing in price. In all of these models, Assumptions 2 and 3 hold. As panel f illustrates, they hold even when

[12] If good 0 is a real good designated to normalize utilities or prices, one can imagine expanding x to include x_0 and defining substitution from good 0 to other goods prior to the normalization that fixes x_0. If Assumption 3 holds under the original designation of good 0, it will hold for all designations of good 0 as long as substitution (using the expanded vector $x = (x_0, \ldots x_J)$) is symmetric at all $x \in \chi^*$, a property that would hold in all the examples below.

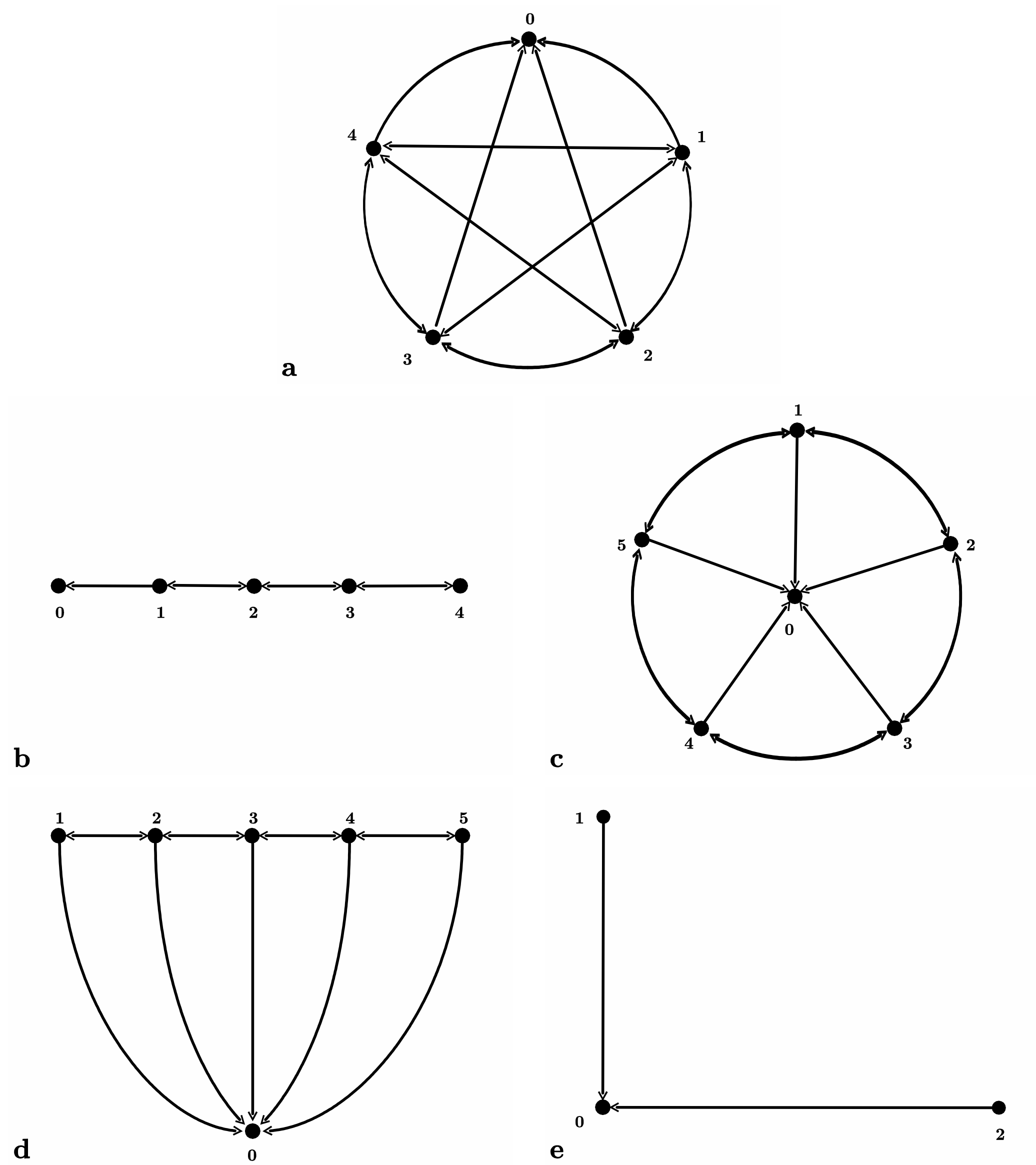

Figure 1: Directed graphs of $\Sigma(x)$ for $x \in \tilde{X}^*$ (x equals minus price) in some standard models of differentiated products. Panel a: multinomial logit, multinomial probit, mixed logit, etc.; Panel b: models of pure vertical differentiation, (e.g., Mussa and Rosen (1978), Bresnahan (1981b), etc.); Panel c: Salop (1979) with random utility for the outside good; Panel d: Rochet and Stole (2002); Panel e: independent goods with either an outside good or an artificial good 0.

$\mathcal{J}$ is comprised of independent goods and either an outside good or an artificial good 0. Each of these examples has an extension to models of discrete/continuous demand (e.g., Novshek and Sonnenschein (1979), Hanemann (1984), Dubin and McFadden (1984)), multiple discrete choice (e.g., Hendel (1999), Dube (2004)), and models of differentiated products demand (e.g., Deneckere and Rothschild (1992), Perloff and Salop (1985)) that provide a foundation for representative consumer models of monopolistic competition (e.g., Spence (1976), Dixit and Stiglitz (1977)).[13]

It may be useful to compare the connected substitutes conditions to a strict gross substitutes assumption, where $\sigma_j(x)$ is strictly decreasing in x_k for all $k \neq j$. The latter obviously implies Assumption 2, and it further implies that every good $j > 0$ substitutes to every other good at all $x \in \mathcal{X}^*$—a strong sufficient condition for Assumption 3. In Figure 1, however, only the models represented in panel *a* satisfy the strict gross substitutes condition (see also Appendix A).

The following lemma provides a helpful interpretation of Assumption 3 and is useful below.

Lemma 1. *Assumption 3 holds iff for all $x \in \mathcal{X}^*$ and any nonempty $\mathcal{K} \subseteq \mathcal{J}\backslash 0$, there exist $k \in \mathcal{K}$ and $\ell \notin \mathcal{K}$ such that $\sigma_\ell(x)$ is strictly decreasing in x_k.*

Proof. (necessity of Assumption 3) Let $\mathcal{I}_0(x) \subseteq \mathcal{J}$ be comprised of 0 and the indexes of all other goods whose nodes have a directed path to node 0 in the directed graph of $\Sigma(x)$. If Assumption 3 fails, then for some $x \in \mathcal{X}^*$ the set $\mathcal{K} = \mathcal{J}\backslash\mathcal{I}_0(x)$ is nonempty. Further, by construction there is no directed path from any node in $\mathcal{K}$ to any node in $\mathcal{I}_0(x)$. Thus, there do not exist $k \in \mathcal{K}$ and $\ell \notin \mathcal{K}$ such that $\sigma_\ell(x)$ is strictly decreasing in x_k.
(*sufficiency*) Assumption 3 implies that for all $x \in \mathcal{X}^*$ and any nonempty $\mathcal{K} \subseteq \mathcal{J}\backslash 0$, every

[13] Mosenson and Dror (1972) used a graphical representation to characterize the possible patterns of substitution for Hicksian demand. Suppose x is minus the price vector, expanded to include the price of good zero (see footnote 12). Suppose further that σ is differentiable and represents the Hicksian (compensated) demand of an individual consumer. Let $\Sigma^+(x)$ be the expanded subsitution matrix, with elements $\Sigma^+_{j+1,k+1} = 1\{\text{good } j \text{ substitutes to good } k \text{ at } x\}$. Mosenson and Dror (1972) show that the directed graph of $\Sigma^+(x)$ must be strongly connected. This is a sufficient condition for Assumption 3.

node $k \in \mathcal{K}$ has a directed path in $\Sigma(x)$ to node $0 \notin \mathcal{K}$. This is impossible unless there exist $k \in \mathcal{K}$ and $\ell \notin \mathcal{K}$ such that good k substitutes to good ℓ. $\square$

Thus, Assumption 3 requires that there be no way to partition $\mathcal{J}$ into subsets of goods that substitute only among themselves. Note that when good 0 is an artificial good, its presence in $\mathcal{J}$ weakens the requirements of Assumption 3. In particular, taking the case where x is (minus) price, when the price of some good $j > 0$ falls, it may be only the demand for good zero that strictly declines.

Finally, when introducing the model we suggested that x_j should be thought of as a monotonic shifter of demand for good j. The following remark shows that we have implicitly imposed this monotonicity with the connected substitutes conditions.

Remark 1. *Suppose Assumptions 2 and 3 hold. Then for all $x \in \mathcal{X}^*$ and $j > 0$, $\sigma_j(x)$ is strictly increasing in x_j.*

Proof. Take $x \in \mathcal{X}^*$ and $x' \in \mathcal{X}$ such that $x'_j > x_j$, $x'_k = x_k\ \forall k \neq j$. Assumption 2 implies $\sigma_k(x') \leq \sigma_k(x)\ \forall k \neq j$. Further, by Lemma 1, $\sigma_\ell(x') < \sigma_\ell(x)$ for some $\ell \neq j$. Thus, $\sum_{k \neq j} \sigma_\ell(x') < \sum_{k \neq j} \sigma_\ell(x)$. The claim then follows from (2). $\square$

4 Invertibility of Demand

We now establish our main result. We begin with a key lemma.[14]

Lemma 2. *Let Assumptions 1–3 hold. If $x, x' \in \mathcal{X}^*$ are such that $\mathcal{I}^+ \equiv \left\{j : x'_j > x_j\right\}$ is nonempty, then $\sum_{j \in \mathcal{I}^+} \sigma_j(x') > \sum_{j \in \mathcal{I}^+} \sigma_j(x)$.*

Proof. Since $0 \notin \mathcal{I}^+$, Lemma 1 ensures that for some $k \in \mathcal{I}^+$ and some $\ell \notin \mathcal{I}^+$, $\sigma_\ell(x)$ is strictly decreasing in x_k. Take one such pair (k, ℓ). Taking case (a) of Assumption 1, suppose

[14]If $\mathcal{X}^*$ is open then, given Assumptions 1 and 2, Assumption 3 is necessary for the conclusion of this lemma. Suppose Assumption 3 fails. Then by Assumption 2 and Lemma 1 there is some $x \in \chi^*$ and some nonempty $\mathcal{K} \subseteq \mathcal{J} \backslash 0$, such that $\sigma_\ell(x)$ is constant in x_k for all $k \in \mathcal{K}$ and all $\ell \notin \mathcal{K}$. For some $\epsilon > 0$ and each $k \in \mathcal{K}$ let $x'_k = x_k + \epsilon$, while $x'_j = x_j$ for $j \notin \mathcal{K}$. Now $\mathcal{I}^+ = \mathcal{K}$. For sufficiently small ϵ we have $x' \in \mathcal{X}^*$ and $\sum_{j \notin \mathcal{I}^+} \sigma_j(x') = \sum_{j \notin \mathcal{I}^+} \sigma_j(x)$, which implies $\sum_{j \in \mathcal{I}^+} \sigma_j(x') = \sum_{j \in \mathcal{I}^+} \sigma_j(x)$.

that for some $\lambda \in (0,1]$ the point $\tilde{x}$ lies in $\mathcal{X}$, where $\tilde{x}_j = x_j + \lambda\left(x'_k - x_k\right) 1\left\{j = k\right\}\ \forall j$. By Assumption 2, $\sigma_j\left(\tilde{x}\right) \leq \sigma_j\left(x\right)$ for all $j \neq k$. Further, $\sigma_\ell\left(\tilde{x}\right) < \sigma_\ell\left(x\right)$ by our choice of (k, ℓ). So, since $\ell \notin \mathcal{I}^+$,

$$\sum_{j \notin \mathcal{I}^+} \sigma_j\left(\tilde{x}\right) < \sum_{j \notin \mathcal{I}^+} \sigma_j\left(x\right).$$

Assumption 2 also implies $\sum_{j \notin \mathcal{I}^+} \sigma_j\left(x'\right) \leq \sum_{j \notin \mathcal{I}^+} \sigma_j\left(\tilde{x}\right)$, so we obtain

$$\sum_{j \notin \mathcal{I}^+} \sigma_j\left(x'\right) \leq \sum_{j \notin \mathcal{I}^+} \sigma_j\left(\tilde{x}\right) < \sum_{j \notin \mathcal{I}^+} \sigma_j\left(x\right)$$

and the result follows from (2). If $\tilde{x} \notin \mathcal{X}$, so that case (b) of Assumption 1 applies, instead take $k \in \mathcal{I}^+$ and $\ell \notin \mathcal{I}^+$ for which $\sigma_\ell\left(x'\right)$ is strictly decreasing in x_k (such a pair being sure to exist by Lemma 1). For some $\lambda \in (0,1]$, the point $\tilde{x}'$ must lie in $\mathcal{X}$, where $\tilde{x}'_j = x'_j + \lambda\left(x_k - x'_k\right) 1\left\{j = k\right\}\ \forall j$. By a symmetric argument we obtain

$$\sum_{j \notin \mathcal{I}} \sigma_j\left(x\right) \geq \sum_{j \notin \mathcal{I}} \sigma_j\left(\tilde{x}\right) > \sum_{j \notin \mathcal{I}} \sigma_j\left(x'\right)$$

and the result follows from (2). □

To demonstrate invertibility of demand under the connected substitutes conditions, we will first show that σ^* (recall 3) is *inverse isotone*. Below we use $\leq$ to denote the component-wise weak partial order on $\mathbb{R}^n$. Thus for $y, y' \in \mathbb{R}^n$, $y \leq y'$ iff $y_i \leq y'_i$ for all $i = 1, \ldots, n$.

Definition 2. A mapping $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is *inverse isotone* if for any $y, y' \in D$, $F\left(y'\right) \leq F\left(y\right)$ implies $y' \leq y$.

Theorem 1. *Under Assumptions 1–3, $\sigma^* : \mathcal{X}^* \to \mathcal{S}$ is inverse isotone.*

Proof. Take any $x, x' \in \mathcal{X}^*$ such that

$$\sigma^*\left(x'\right) \leq \sigma^*\left(x\right) \tag{4}$$

and suppose, contrary to the claim, that the set $\mathcal{I}^+ = \left\{j : x'_j > x_j\right\}$ is non-empty. By

Lemma 2 this requires

$$\sum_{j \in \mathcal{I}^+} \sigma_j^* \left(x'\right) > \sum_{j \in \mathcal{I}^+} \sigma_j^* \left(x\right)$$

which contradicts (4). □

The following remark documents a well known connection between inverse isotone mappings and invertible mappings (see, e.g., Rheinboldt (1970)).[15]

Remark 2. *If* $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is* inverse isotone, it is injective.

Proof. Suppose $F(y) = F(y')$ for $y, y' \in D$. Since F is inverse isotone this implies both $y \leq y'$ and $y' \leq y$; hence $y' = y$. □

Given Remark 2, our invertibility result follows as a corollary to Theorem 1.

Corollary 1. *Under Assumptions 1–3, for any* $s \in$ int$(\mathcal{S})$ *there is at most one* $x \in \mathcal{X}^*$ *such that* $\sigma(x) = s$*.*

5 Connected Substitutes and the Jacobian Matrix

It is possible to provide a connection between our connected substitutes conditions and the global Jacobian condition required by the univalence theorem of Gale and Nikaido (1965). Suppose σ is differentiable on $\mathcal{X}^*$ and let $J_\sigma(x)$ denote the Jacobian matrix

$$\begin{bmatrix} \frac{\partial \sigma_1(x)}{\partial x_1} & \cdots & \frac{\partial \sigma_1(x)}{\partial x_J} \\ \vdots & \ddots & \vdots \\ \frac{\partial \sigma_J(x)}{\partial x_1} & \cdots & \frac{\partial \sigma_J(x)}{\partial x_J} \end{bmatrix}.$$

Recall that a square matrix is said to be a P-matrix if all its principal minors are positive.

[15] Another application of Theorem 1 appears in a recent paper by Gandhi, Lu, and Shi (2011). They exploit the inverse isotone property shown here in studying identification and estimation of multinomial choice demand models under mismeasurement of market shares.

Theorem 2. *Suppose that Assumptions 1–3 hold. Suppose further that for all $x \in \mathcal{X}^*$, σ is differentiable and substitution is symmetric between all goods $j, k > 0$. Then $J_\sigma(x)$ is a P-matrix for all $x \in \mathcal{X}^*$.*

Proof. See Appendix B. □

One would never use this result to establish invertibility—if the connected substitutes conditions hold, Corollary 1 establishes invertibility without the additional differentiability and domain restrictions Gale and Nikaido (1965) require.[16] However, Theorem 2 is useful for understanding the relationship between the two results. Ours avoids the smoothness and domain restrictions of Gale and Nikaido (1965) while imposing restrictions on substitution that, given differentiability and a weak symmetry condition, are sufficient for the Gale-Nikaido Jacobian requirement.

Conditions ensuring that the Jacobian of σ^* is a $P-$matrix are of independent interest as well. Berry and Haile (2010) use this result in establishing the identifiability of supply in differentiated products markets—i.e., identification of marginal costs and of the model of oligopoly competition. In particular, the P-matrix property ensures invertibility of the derivative matrix of market shares with respect to prices for goods produced by the same firm—a matrix appearing in the first-order conditions for each firm. Their identification results generalize immediately to oligopoly supply models with continuous demand. The P-matrix property can also be applied instead to the matrix of derivatives of market shares with respect to the latent demand shocks, which is sufficient to ensure a key condition used in Berry, Linton, and Pakes (2004) (and confirmed there for special cases) to provide the asymptotic distribution theory for a class of estimators for discrete choice demand models. In particular, it ensures the Jacobian of demand is always full rank over $\mathcal{X}^*$.

[16] Note that Assumption 2 implies that the matrix $J_\sigma(x)$ is of the "Leontieff type." Thus, given the connected substitutes conditions we used to show that σ was inverse isotone in Theorem 1, one could use Theorem 5 in Gale and Nikado (1965) to show this same property under the additional requirements of differentiability and rectangular domain.

6 Conclusion

We have introduced the notion of "connected substitutes" and shown that this structure is sufficient for invertibility in a large class of nonparametric nonseparable demand systems. The connected substitutes conditions are satisfied in a large class of models used in practice, have transparent economic interpretation, and allow us to show invertibility without functional form restrictions, smoothness assumptions, or strong domain restrictions. We have also provided a link between the connected substitutes conditions and the Jacobian condition required by the classical univalence result of Gale and Nikaido (1965).

Appendix A. An Example

Here we present a simple variation of Lancaster's (1966) "diet example," illustrating a continuous demand system with only local substitution, with a non-rectangular domain of interest , and where the introduction of an artificial good 0 is useful even though an outside good is already modeled.

A representative consumer has a budget of y and chooses consumption quantities (q_1, q_2, q_3) of three goods: wine, bread, and cheese, respectively. Her preferences are given by a utility function

$$u(q_1, q_2, q_3) = \ln(z_1) + \ln(z_2) + \ln(z_3) + m$$

where (z_1, z_2, z_3) are consumption of calories, protein, and calcium, and m is money left to spend on other goods. The mapping of goods consumed to characteristics consumed is given by[17]

$$\begin{aligned} z_1 &= q_1 + q_2 + q_3 \\ z_2 &= q_2 + q_3 \\ z_3 &= q_3. \end{aligned}$$

We assume $y > 3$ and that prices (p_1, p_2, p_3) are such that all goods are purchased, i.e.,

$$0 < p_1 < p_2 - p_1 < p_3 - p_2. \tag{5}$$

Since p plays the role of x here, (5) also defines $\tilde{\mathcal{X}}$. $\tilde{\mathcal{X}}$ is not a rectangle; however, because $\tilde{\mathcal{X}}$ is open Assumption 1 holds for any $\mathcal{X}^* \subseteq \tilde{\mathcal{X}}$ (see footnote 11).

[17] Unlike Lancaster (1966), we sacrifice accuracy of nutritional information for the sake of simplicity.

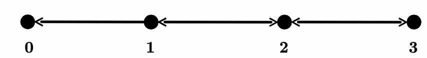

Figure 2: The directed graph of $\Sigma(x)$ for the "diet example."

It is then easily verified that demand for each inside good is given by

$$\begin{aligned} \sigma_1(p) &= \frac{1}{p_1} - \frac{1}{p_2 - p_1} \\ \sigma_2(p) &= \frac{1}{p_2 - p_1} - \frac{1}{p_3 - p_2} \\ \sigma_3(p) &= \frac{1}{p_3 - p_2} \end{aligned} \tag{6}$$

for $p \in \mathcal{X}^*$. These equations fully characterize demand for all goods. We introduce the artificial quantity of "good 0", defined by

$$q_0 \equiv 1 - \sum_{j=1}^{3} q_j. \tag{7}$$

Observe that this artificial good is not the outside good m. Further, the connected substitutes conditions would not hold if the outside good were treated as good 0.

With (6), (7) implies

$$\sigma_0(p) = 1 - \frac{1}{p_1}.$$

From these equations, it is now easily confirmed that Assumption 2 holds Further, goods 2 and 3 substitute to each other, goods 1 and 2 substitute to each other, and good 1 substitutes to good 0. Thus, Assumption 3 also holds. Figure 2 illustrates.

Appendix B. Proof of Theorem 2

We prove Theorem 2 by showing that, under its hypotheses, every principal submatrix of $J_\sigma(x)$ is invertible at all $x \in \mathcal{X}^*$. We first review some definitions.

A square matrix is *reducible* if it can be placed in block upper triangular form by permutations of rows and columns. A square matrix that is not reducible is *irreducible.* A square matrix A with elements a_{ij} is *diagonally dominant* if for all i

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|.$$

An *irreducibly diagonally dominant* matrix is a square matrix that is irreducible and diagonally dominant, with at least one diagonal being strictly dominant, i.e., with at least one row such that

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|. \tag{8}$$

A directed graph G is *strongly connected* if for every pair of distinct nodes (i, j) in G there exists a directed path from i to j.

We will rely on the following well known result (see, e.g., Taussky (1949) or Horn and Johnson (1990), p. 363).

Lemma 3. *An irreducibly diagonally dominant matrix is invertible.*

For nonempty $\mathcal{K} \subseteq \{1, 2, \ldots, J\}$, let $D_{\mathcal{K}}(x)$ denote the principal submatrix of $J_\sigma(x)$ obtained by deleting rows $r \notin \mathcal{K}$ and columns $c \notin \mathcal{K}$.

Lemma 4. *Suppose σ is differentiable on $\mathcal{X}^*$ and that Assumptions 1–3 hold. Then for all $x \in \mathcal{X}^*$ and nonempty $\mathcal{K} \subseteq \{1, 2, \ldots, J\}$, $D_{\mathcal{K}}(x)$ is diagonally dominant, with at least one strictly dominant diagonal.*

Proof. Take $x \in \mathcal{X}^*$. Because $\sum_{k \in \mathcal{J}} \sigma_k(x) = 1$, $\sum_{k \in \mathcal{J}} \frac{\partial \sigma_k(x)}{\partial x_j} = 0$. By Assumption 2 and

Remark 1, $\frac{\partial \sigma_j(x)}{\partial x_j} > 0$ while $\frac{\partial \sigma_k(x)}{\partial x_j} \leq 0$ for all $k \neq j$. Thus, for any $j \in \mathcal{K}$,

$$\left| \frac{\partial \sigma_j (x)}{\partial x_j} \right| = \sum_{k \in \mathcal{K} - \{j\}} \left| \frac{\partial \sigma_k (x)}{\partial x_j} \right| + \sum_{\ell \notin \mathcal{K}} \left| \frac{\partial \sigma_\ell (x)}{\partial x_j} \right|. \tag{9}$$

which implies

$$\left| \frac{\partial \sigma_j (x)}{\partial x_j} \right| \geq \sum_{k \in \mathcal{K} - \{j\}} \left| \frac{\partial \sigma_k (x)}{\partial x_j} \right|. \tag{10}$$

Furthermore, since $0 \notin \mathcal{K}$, Lemma 1 implies that for some $j \in \mathcal{K}$ the second sum in (9) is strictly positive. For that j the inequality (10) must be strict. □

The following lemma states a useful elementary result in matrix theory (see, e.g., Horn and Johnson (1990), p. 362).

Lemma 5. *The directed graph of a square matrix A is strongly connected iff A is irreducible.*

We now complete the proof of Theorem 2. Take arbitrary $x \in \mathcal{X}^*$ and nonempty $\mathcal{K} \subseteq \{1, 2, \ldots, J\}$. We will show that $D_{\mathcal{K}}(x)$ is invertible. First suppose the directed graph of $D_{\mathcal{K}}(x)$ is strongly connected. By Lemmas 4 and 5 it is then an irreducibly diagonally dominant matrix and, therefore, invertible by Lemma 3. So suppose instead that the directed graph of $D_{\mathcal{K}}(x)$ is not strongly connected. Since substitution is symmetric on $\mathcal{X}^*$, all edges in the directed graph of $D_{\mathcal{K}}(x)$ must be bidirectional. Since this graph is not strongly connected, it must then be possible to partition it into isolated strongly connected subgraphs, each of which corresponds to a subset of goods that substitute only among themselves in $\mathcal{K}$. We can therefore rearrange the order of goods in $\mathcal{K}$, with those in one strongly connected subset coming first, another subset following, and so on. The resulting permutation of $D_{\mathcal{K}}(x)$ is block diagonal, with each block being irreducible by Lemma 5. Further, by Lemma 4, each block is diagonally dominant with at least one strictly dominant diagonal. Therefore, by Lemma 3, each block is invertible. This implies that the entire $D_{\mathcal{K}}(x)$ matrix is invertible. Since $\mathcal{K}$ and $x \in \mathcal{X}^*$ were arbitrary, every principal submatrix of $J_\sigma (x)$ is invertible for all $x \in \mathcal{X}^*$.

Appendix C

We pointed out in the text that our notion of weak substitution always implies the alternative notion, that $\sigma_j(x)$ is weakly decreasing in x_k for all $k \neq j$. The following result provides one sufficient condition for the two notions to be equivalent.

Proposition 1. *Suppose $\mathcal{X}$ is a Cartesian product and that for all $x \in \mathcal{X}$, $\sigma_j(x)$ is weakly decreasing in x_k for all $k \neq j$. Then Assumption 2 holds.*

Proof. Suppose $x'_j \geq x_j$ for all $j \in \mathcal{I}$, while $x'_j \leq x_j$ for all $j \notin \mathcal{I}$. Let $\tilde{x}$ be such that $\tilde{x}_j = x_j$ for $j \in \mathcal{I}$ and $\tilde{x}_j = x'_j$ for $j \notin \mathcal{I}$. Since $\mathcal{X}$ is a Cartesian product, $\tilde{x} \in \mathcal{X}$. Because $\sigma_j(x)$ is weakly decreasing in x_k for $k \neq j$

$$\sum_{j \in \mathcal{I}} \sigma_j(\tilde{x}) \geq \sum_{j \in \mathcal{I}} \sigma_j(x).$$

With (2), this implies

$$\sum_{j \notin \mathcal{I}} \sigma_j(\tilde{x}) \leq \sum_{j \notin \mathcal{I}} \sigma_j(x),$$

Further, because $\sigma_j(x)$ is weakly decreasing in x_k for $k \neq j$,

$$\sum_{j \notin \mathcal{I}} \sigma_i(x') \leq \sum_{j \notin \mathcal{I}} \sigma_i(\tilde{x})$$

and the result follows. $\square$

References

Aleksandrov, V. (1994): “On the Gale-Nikaido-Inada Fundamental Theorem on Univalence of Mappings,” *Siberian Mathematical Journal*, 35, 637–639.

Arrow, K. J., H. D. Block, and L. Hurwicz (1959): “On the Stability of the Competitive Equilibrium. II,” *Econometrica*, 27, 82–109.

Azevedo, E., A. White, and E. G. Wyl (2011): “Walrasian Equilibrium without Gross Substitutes,” Discussion paper, Harvard University.

Azoulay, P. (2002): “Do Pharmaceutical Sales Respond to Scientific Evidence?,” *Journal of Economics and Management Strategy*, 11, 551–594.

Beckert, W., and R. Blundell (2008): “Heterogeneity and the Non-Parametric Analysis of Consumer Choice: Conditions for Invertibility,” *Review of Economic Studies*, 75, 1069–1080.

Berry, S. (1994): “Estimating Discrete Choice Models of Product Differentiation,” *RAND Journal of Economics*, 23(2), 242–262.

Berry, S., M. Carnall, and P. Spiller (1996): “Airline Hubs: Costs, Markups and the Implications of Consumer Heterogeneity,” Discussion Paper 5561, NBER.

Berry, S., and P. Jia (2010): “Tracing the Woes: An Empirical Analysis of the Airline Industry,” *American Economic Journals: Microeconomics*.

Berry, S., J. Levinsohn, and A. Pakes (1995): “Automobile Prices in Market Equilibrium,” *Econometrica*, 60(4), 889–917.

——— (1999): “Voluntary Export Restraints on Automobiles: Evaluating a Strategic Trade Policy,” *American Economic Review*, 89(3), 189–211.

——— (2004): “Differentiated Products Demand Systems from a Combination of Micro and Macro Data: The New Vehicle Market,” *Journal of Political Economy*, 112(1), 68–105.

Berry, S., O. Linton, and A. Pakes (2004): "Limit Theorems for Differentiated Product Demand Systems," *Review of Economic Studies*, 71(3), 613–614.

Berry, S., and J. Waldfogel (1999): "Free Entry and Social Inefficiency in Radio Broadcasting," *RAND Journal of Economics*, 70(3), 397–420.

Berry, S. T., and P. A. Haile (2009a): "Identification in Differentiated Products Markets Using Market Level Data," Discussion paper, Yale University.

——— (2009b): "Nonparametric Identification of Multinomial Choice Demand Models with Heterogeneous Consumers," Discussion paper, Yale University.

——— (2010): "Identification in Differentiated Products Markets Using Market Level Data," Discussion paper, Yale University.

Berry, S. T., and A. Pakes (2007): "The Pure Characteristics Demand Model," Discussion paper, Yale University.

Block, H., and J. Marschak (1960): "Random Orderings and Stochastic Theories of Responses," in *Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling*, ed. by I. Olkin, S. Ghurye, W. Hoeffding, W. G. Mado, and H. B. Mann. Stanford University Press.

Bresnahan, T. (1981a): "Departures from Marginal Cost Pricing in the American Automobile Industry," *Journal of Econometrics*, 17, 201–227.

——— (1981b): "Departures from Marginal Cost Pricing in the Auomobile Industry," *Journal of Econometrics*, 17, 457–482.

——— (1987): "Competition and Collusion in the American Automobile Oligopoly: The 1955 Price War," *Journal of Industrial Economics*, 35, 457–482.

Cheng, L. (1985): "Inverting Systems of Demand Functions," *Journal of Economic Theory*, 37, 202–210.

DAVIS, P. (2001): "Spatial Competition in Retail Markets: Motion Theaters," Discussion paper, London School of Economics.

DENECKERE, R., AND M. ROTHSCHILD (1992): "Monopolistic Competition and Preference Diversity," *Review of Economic Studies*, 59, 361–373.

DICK, A. (2008): "Demand Estimation and Consumer Welfare in the Banking Industry," *Journal of Banking and Finance*, 32, 1661–1676.

DIXIT, A., AND J. E. STIGLITZ (1977): "Monopolistic Competition and Optimum Product Diversity," *American Economic Review*, 67, 297–308.

DUBE, J.-P. (2004): "Multiple Discreteness and Product Differentiation: Demand for Carbonated Soft Drinks," *Marketing Science*, 23, 66–81.

DUBE, J.-P., J. FOX, AND C.-L. SU (2009): "Improving the Numerical Performance of BLP StaticCoefficients Demand Estimation," Discussion paper, University of Chicago.

DUBIN, J., AND D. MCFADDEN (1984): "An Econometric Analysis of Residential Electric Appliance Holdings and Consumption," *Econometrica*, 52(2), 345–62.

EINAV, L. (2007): "Seasonality in the U.S. Motion Picture Industry," *Rand Journal of Economics*, 38, 127–145.

FALMAGNE, J.-C. (1978): "A Representation Theorem for Finite Random Scale Systems," *Journal of Mathematical Psychology*, 18, 52–72.

FAN, Y. (2008): "Market Structure and Product Quality in the U.S. Daily Newspaper Market," Discussion paper, Yale University.

GALE, D., AND H. NIKAIDO (1965): "The Jacobian Matrix and Global Univalence of Mappings," *Mathematische Annalen*, 159, 81–93.

GANDHI, A. (2008): "On the Nonparametric Foundations of Discrete Choice Demand Estimation," Discussion paper, University of Wisconsin-Madison.

——— (2010): "Inverting Demand in Product Differentiated Markets," Discussion paper, University of Wisconsin-Madison.

Gandhi, A., Z. Lu, and X. Shi (2011): "Estimating Discrete Choice Models with Market Level Data: A Bounds Approach," Discussion paper, University of Wisconsin-Madison.

Garcia, C., and W. Zangwill (1979): "On Univalence and P-Matrices," *Linear Algebra and Its Applications*, 24, 239–250.

Gentzkow, M. (2004): "Valuing New Goods in a Model with Complementarity: Online Newspapers," Working paper, Chicago GSB.

Goldberg, P., and F. Verboven (2001): "The Evolution of Price Dispersion in the European Car Market," *The Review of Economics Studies*, 68(4), 811–848.

Hanemann, W. M. (1984): "Discrete/Continuous Models of Consumer Demand," *Econometrica*, 52, 541–561.

Hendel, I. (1999): "Estimating Multiple-Discrete Choice Models: An Application to Computerization Returns," *Review of Economic Studies*, 66(2), 423–46.

Horn, R. A., and C. R. Johnson (1990): *Matrix Analysis*. Cambrige University Press.

Hotz, J., and R. A. Miller (1993): "Conditional Choice Probabilites and the Estimation of Dynamic Models," *Review of Economic Studies*, 60, 497–529.

Lancaster, K. J. (1966): "A New Approach to Consumer Theory," *Journal of Political Economy*, 74, 132–157.

Mas-Colell, A. (1979): "Homeomorphisms of Compact, Convex Sets and the Jacobian Matrix," *Siam Journal of Mathematics*, 10, 1105–1109.

McFadden, D. (1974): "Conditional Logit Analysis of Qualitative Choice Behavior," in *Frontiers of Econometrics*, ed. by P. Zarembka. Academic Press, New York.

——— (1981): "Econometric Models of Probabilistic Choice," in *Structural Analysis of Discrete Data with Econometric Applications*, ed. by C. Manski, and D. McFadden. MIT Press, Cambridge, MA.

——— (2004): "Revealed Stochastic Preference: A Synthesis," Discussion paper, UC Berkeley.

McFadden, D., and M. K. Richter (1971): "On the Extension of a Set Function on a Set of Events to a Probabiliyt on the Generated Boolean σ-Algebra," Discussion paper, UC Berkeley.

——— (1990): "Stochastic Rationality and Revealed Stochastic Preference," in *Preferences, Uncertainty, and Rationality*, ed. by J. Chipman, D. McFadden, and M. Richter, pp. 161–186. Westview Press.

Mosenson, R., and E. Dror (1972): "A Solution to the Qualitative Substitution Problem in Demand Theory," *Review of Economic Studies*, 39, 433–441.

Mussa, M., and S. Rosen (1978): "Monopoly and Product Quality," *Journal of Economics Theory*, 18, 301–307.

Nevo, A. (2000): "Mergers with Differentiated Products: The Case of the Ready-to-Eat Cereal Industry," *RAND Journal ofEconomics*, 31(3), 395–421.

——— (2001): "Measuring Market Power in the Ready-to-Eat Cereal Industry," *Econometrica*, 69(2), 307–42.

Nikaido, H. (1968): *Convex Structures and Economic Theory*. Academic Press.

Novshek, W., and H. Sonnenschein (1979): "Marginal Consumers and Neoclassical Demand Theory," *Journal of Political Economy*, 87, 1368–1376.

Parthasarathy, T., and G. Ravindran (2003): "Global Univalence and the Jacobian Conjecture," in *Applicable Mathematics in the Golden Age*, ed. by J. Misra. Narosa Publishing House.

Perloff, J. M., and S. C. Salop (1985): "Equilibrium with Product Differentiation," *Review of Economic Studies*, 52, 107–120.

Petrin, A. (2002): "Quantifying the Benefits of New Products: The Case of the Minivan," *JPE*, 110(4), 705–729.

Rheinboldt, W. C. (1970): "On M-Functions and Their Application to Nonlinear Gauss-Seidel Iterations and to Network Flows," *Journal of Mathematical Analysis and Applications*, 32, 274–307.

Rochet, J.-C., and L. A. Stole (2002): "Nonlinear Pricing with Random Participation," *The Review of Economic Studies*, 69(1), 277–311.

Salop, S. (1979): "Monopolistic Competition with Outside Goods," *RAND Journal of Economics*, 10, 141–156.

Spence, M. (1976): "Product Selection, Fixed Costs and Monopolistic Competition," *Review of Economic Studies*, 43, 217–236.

Taussky, O. (1949): "A Recurring Theorem on Determinants," *The American Mathematical Monthly*, 56(10), 672–676.

Verboven, F. (1996): "International Price Discrimination in the European Car Market," *Rand Journal of Economics*, 27, 240–268.