



Munich Personal RePEc Archive

## **Harvesting natural resources: management and conflicts**

Halkos, George

University of Thessaly, Department of Economics

2010

Online at <http://mpra.ub.uni-muenchen.de/24119/>  
MPRA Paper No. 24119, posted 27. July 2010 / 08:28

# Harvesting Natural Resources: Management and Conflicts<sup>[1]</sup>

By

**George Emm. Halkos and George Papageorgiou**

University of Thessaly, Department of Economics,  
Korai 43, 38333, Volos, Greece

## **Abstract**

It is reasonable to consider the stock of any renewable resource as a capital stock and treat the exploitation of that resource in much the same way as one would treat accumulation of a capital stock. This has been done to some extent in earlier papers containing a discussion of this point of view. However, the analysis is much simpler than it appears in the literature especially since the interaction between markets and the natural biology dynamics has not been made clear. Moreover renewable resources are commonly analyzed in the context of models where the growth of the renewable resource under consideration is affected by two factors: the size of the resource itself and the rate of harvesting. This specification does not take into account that human activities other than harvesting can have an impact on the growth of the natural resource. Furthermore, natural resource harvesting are not productive factories. Fishery economic literature (based on the foundations of Gordon, 1954; Scott, 1955; and Smith, 1963) suggests particular properties of the ocean fishery which requires tools of analysis beyond those supplied by elementary economic theory. An analysis of the fishery must take into account the biological nature of fundamental capital, the fish and it must recognize the common property feature of the open sea fishery, so it must allow that the fundamental capital is the subject of exploitation. The purpose of this paper is the presentation of renewable resources dynamic models in the form of differential games aiming to extract the optimal equilibrium trajectories of the state and control variables for the optimal control economic problem. We show how methods of infinite horizon optimal control theory may be developed for renewable resources models.

**Keywords:** Renewable resources; exploitation of natural resources; dynamic optimization; optimal control.

**JEL classifications:** C61, C62, Q32

---

<sup>[1]</sup> Paper presented in the 11<sup>th</sup> Annual International Conference organized by the Association for Public Economic Theory (PET10) at Boğaziçi University, Istanbul Turkey, June 25 2010.

## 1. Introduction

Differential equations in dynamical systems (either in a continuous or a discrete framework) are of common use in most models that explain the optimal management of natural resources extraction. These systems depend on more than one parameter that measure different economic and biological characteristics of the exploited resource, so the structural stability is a key point to study, i.e. if the qualitative dynamical properties of the system persist when its structure is perturbed. In this context, the study of the structural stability is the first step to follow in the analysis of the system.

Moreover, many economic problems can be formulated as dynamic games in which strategically interacting agents choose actions that determine the current and future levels of a single capital stock. Consider, for example a single capital stock of an exhaustible or reproductive resource that is simultaneously exploited by several agents that do not cooperate. Each agent chooses an extraction strategy to maximize the discounted stream of future utility. The actions taken by agents not only determine their levels of utility but also the level of the capital stock. There are several implications of the above formulation. First, the actions taken by agents determine the size of a single capital stock that fully describes the current state of the economic system. Second, if there is no mechanism that forces players to coordinate their actions, they will act strategically and play a non-cooperative game. Third, the equilibrium outcome will critically depend on the strategy spaces available to the agents.

There is a wide choice of possible actions (strategies) taken by the agents. They may choose a simple time profile of actions and pre-commit themselves to these fixed actions over the entire planning horizon: the players then use open-loop strategies. Alternatively players might choose feedback or closed-loop or Markov strategies, they condition their actions on the current state of the system and react immediately every time the state variable changes, hence they are not required to pre-commit. We expose an example of several agents strategically exploiting the same renewable resource, like a stock of fish in order to expose the difference between open-loop and closed-loop strategies. If the fisheries use open-loop strategies they specify a time path of fishing effort in the beginning of the game and commit themselves to stick to these pre-announced actions over the entire planning horizon. Alternatively, if they use feedback strategies they choose decision rules that determine

current actions as a function of current stock of the resource. Feedback decision rules capture the strategic interactions present in a dynamic game. If a rival fishery makes a catch today that necessarily results in a lower level of the fish stock, the opponents react with actions that take this change in the stock into account. In that sense closed – loop strategies capture all the features of strategic interactions.

Within a differential game framework the feedback strategies have the property that players choose a state dependent decision rule that for every subgame assigns an equilibrium action to the current state of the economic system. This is in short the time consistent property in the subgame perfectness sense. The issue of time consistency is central in modern economic theory. Since the influential work by Kydland and Prescott (1977) economists have attempted different approaches to resolve the inconsistency problem. One possible strategy is to consider the interaction between the policy maker and the agent in a dynamic game setup. Many researchers such as Cohen and Michel (1988) found that a time consistent outcome corresponds to a feedback Nash equilibrium.

On the other hand, it is natural to consider the stock of any renewable resource as a capital stock and treat the exploitation of that resource in much the same way as one would treat accumulation of a capital stock. This has been done to some extent by Clark (1976), Clark and Munro (1975), whose papers contain a discussion of this point of view. However, the analysis is much simpler than it appears in the literature especially since the interaction between markets and the natural biology dynamics has not been made clear. Moreover renewable resources are commonly analyzed in the context of models where the growth of the renewable resource under study is affected by two factors: the size of the resource itself and the rate of harvesting. This specification does not take into account that human activities other than harvesting can have an impact on the growth of the natural resource (Levhari and Withagen, 1992).

Some externalities may arise in maximum sustained yield programs of replenishable natural resource exploitation followed by the two fundamental problems. The first is that the existence of a social discount factor (or interest rate) may cause the maximum sustained yield program to be nonoptimal (Plourde, 1970). The second problem relates to the many externalities which may be present in harvesting resources. The most significant of these externalities is the stock externality in production. That is, there is a potential misallocation of inputs in the

production of natural resource product due to the fact that one input, the natural resource, contributes to production but may not receive payment, because no one owns the resource.

Finally, natural resources harvesting differs from production. Renewable resources economic literature (based on the foundations of Gordon, 1954; Scott, 1955 and Smith, 1963), suggests particular properties of the ocean fishery which requires tools of analysis beyond those supplied by elementary economic theory. An analysis of the fishery must take into account the biological nature of fundamental capital, the renewable resource, and it must recognize the common property feature of land or sea, so it must allow that the fundamental capital is the subject of exploitation.

## 2. The optimal control of the recreational model

Consider an infinite horizon economy harvesting a natural resource. We denote by  $\nu(t)$  the resource stock at time  $t$ , by  $h(t)$  the harvesting function of the resource caused by overall human activities and by  $g(\nu(t))$  the regeneration of the natural resource. The function  $g(\nu(t))$  is set to zero in the trivial case which the natural resource is a non renewable one. With these functions in the model we obtain the system dynamics.

$$\dot{\nu}(t) = g(\nu(t)) - h(t) \quad (2.1)$$

This is an accounting identity stating that the natural resource accumulation  $\dot{\nu}(t)$  must be equal to the regeneration of the natural resource minus harvesting by human activities. It is assumed that the production regeneration function is  $g: [0, \infty) \rightarrow \mathbb{R}$  is continuous, twice continuously differentiable on  $(0, \infty)$  and strictly concave. In addition, we assume that  $g(0) = 0, \lim_{\nu \rightarrow 0} g'(\nu) = \infty$ , and that there exists a unique resource stock  $\bar{\nu} > 0$  such that  $g(\bar{\nu}) = 0$ . This implies that  $g(\nu) > 0$  for all  $\nu \in (0, \bar{\nu})$  and  $g(\nu) < 0$  for all  $\nu > \bar{\nu}$ .

The goal of the decision maker is to maximize the discounted utility derived over the infinite planning interval  $[0, \infty)$ . That is, the objective functional is

$$\int_0^{\infty} e^{-\rho t} F(h(t)) dt \quad (2.2)$$

where  $F : [0, \infty) \rightarrow \mathbb{R}$  is the utility function. Although the analysis can be carried out for a very general class of utility functions, we restrict ourselves to functions of the form

$$F_{\beta}(h) = \begin{cases} \frac{h^{\beta} - 1}{\beta} & \text{if } \beta \in (0, 1) \\ \ln h & \text{if } \beta = 0 \end{cases}$$

These utility functions have a constant elasticity of intertemporal substitution  $1/(1-\beta)$  often used in studies of economic growth. The optimal control consists in maximizing (2.2) subject to the system dynamics (2.1) and the non negativity constraints  $\nu(t) \geq 0, h(t) \geq 0$ .

### 2.1. Equilibrium analysis

The Hamiltonian of the problem under consideration is  $H(\nu, h, \lambda, t) = F_{\beta}(h) + \lambda[g(\nu) - h]$ . A necessary and sufficient condition that  $h(t)$  is an interior maximum of  $H(\nu(t), h, \lambda, t)$  is

$$F'_{\beta}(h(t)) = h(t)^{\beta-1} = \lambda(t) \quad (2.3)$$

Equation (2.3) has several implications for the costate  $\lambda(t)$  and the control function  $h(t)$ .

First, for an interior maximum of the Hamiltonian the costate has a positive value and second the maximizing value of the control is independent of the state variable  $\nu(t)$ . Solving (2.3) w.r.t the control the latter can be written as  $h(t) = \lambda(t)^{1/(\beta-1)}$ . Substituting the control found as a function of the costate into the Hamiltonian, the maximized Hamiltonian can be written as a function of the costate and state variables only, that is  $H^*(\nu, \lambda, t) = F_{\beta}(\lambda^{1/(1-\beta)}) - \lambda^{\beta/(1-\beta)} + \lambda g(\nu)$ , which implies that the maximized Hamiltonian is strictly concave w.r.t to the state variable  $\nu(t)$  whenever  $\lambda(t) > 0$ .

Consider now the time derivative of the costate variable. Sufficient conditions for the maximization problem (see for example Grass *et al*, 2008) yield the equation

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial \nu} + \rho \lambda(t) \Rightarrow \dot{\lambda}(t) = \lambda(t)[\rho - g'(\nu)] \quad (2.4)$$

Equation (2.4) implies that the stock converges to the steady state level  $\nu^*$  that is the unique solution of the equation  $g'(\nu) = \rho$ , which means, in the steady states, the biological rate of growth must be equal to the discount factor thought as the interest rate.

Further differentiation of (2.3) w.r.t. time yields  $\dot{\lambda}(t) = \frac{(\beta-1)h(t)^{\beta-1}\dot{h}(t)}{h(t)}$

and substitution of (2.4) in to the latter and making use of  $h(t) = \lambda(t)^{1/(\beta-1)}$  the final equation of time derivative of the harvesting function will be the following

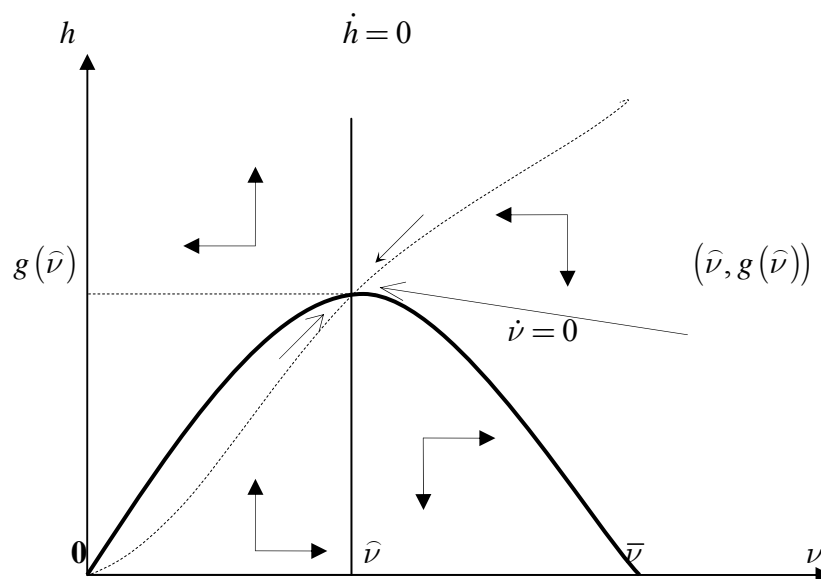
$$\dot{h}(t) = h(t) \left[ \rho - g'(\nu(t)) \right] / (\beta - 1) \quad (2.5)$$

Equations (2.1) and (2.5) is the system of differential equations that constitutes the main tool of economic analysis for the optimal control harvesting model. But first of all we can observe from equation (2.5) a kind of modified Hotelling rule. That is the harvesting rate  $\dot{h}(t)/h(t)$  must be equal to the biological rate of growth of the population  $g'(\nu(t))$  minus the discount rate  $\rho$  multiplied by the elasticity of intertemporal substitution  $1/(1-\beta)$ . Equation (2.5) verifies the claim of the Hotelling rule in the case of non-renewable resources, that is in the absence of the regeneration function, which becomes  $\dot{h}(t)/h(t) = \rho/(\beta-1)$  (2.6) and the latter says the utility grows at the interest rate, since  $\dot{u}(t)/u(t) = (1-\beta)\dot{h}(t)/h(t) \Rightarrow \dot{u}(t)/u(t) = -\rho$ .

Into the two-dimensional system of differential equations (2.1) and (2.5) we observe that there is only one initial condition  $\nu(0) = \nu_0$ , which implies that there exist infinitely many solutions of the system. Fortunately, the transversality condition  $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) \nu(t) = 0$  will help us to reduce the number of candidates for optimality to a small number, even in a unique candidate. We have drawn the phase diagram of the system (2.1), (2.5) in order to expose the analysis that follows.

The solid concave curve  $h = g(\nu)$ , the locus of all points at which the right hand side of (2.1) becomes zero. The vertical line is the locus  $\nu = \hat{\nu}$ , where  $\hat{\nu}$  is the unique resource stock satisfying the steady state condition  $g'(\nu) = \rho$ . This line

together with the horizontal coordinate axis is the set of all points where the right hand side of (2.5) becomes zero. The phase space is partitioned in four regions by the two isoclines and for each region the flow is determined by a unique direction as the arrows shows. The three points of intersection of the two isoclines are the origin  $(0,0)$  the point  $(\hat{\nu}, g(\hat{\nu}))$  and the point  $(\bar{\nu}, 0)$ . The non trivial equilibrium point  $(\hat{\nu}, g(\hat{\nu}))$  has the property of saddle point, as can see by computing the Jacobian matrix,  $J$ , and its eigenvalues.



**Figure 1.** Phase diagram of the system (2.1), (2.5)

Simple calculations yield the Jacobian

$$J = \begin{pmatrix} \rho & -1 \\ A = g(\hat{\nu})g''(\hat{\nu})/(1-\beta) & 0 \end{pmatrix} \text{ and } A < 0 \text{ caused by assumptions.}$$

Consequently the two eigenvalues are  $r_{1,2} = \rho/2 \pm \sqrt{(\rho/2)^2 - A}$  and with  $A < 0$  we know that  $r_1$  is negative while  $r_2$  is positive. The last result for the two eigenvalues with different signs proves that  $(\hat{\nu}, g(\hat{\nu}))$  is indeed a saddle point. It follows that for every initial state  $\nu_0 \in (0, \infty)$  there exists a unique solution of the system (2.1), (2.5) that converges to the saddle point. The saddle point path is depicted by the dotted line in figure 1. Along this solution we have  $h(t) > 0$  and  $\nu(t) > 0$  and the corresponding



costate trajectory, which determined by (2.3) is also positive. We record the optimal natural resources management problem into the following proposition.

**Proposition 2.1.**

*The optimal management of the renewable resources model for which the benefits enjoyed by harvesting exhibit a constant elasticity of intertemporal substitution, admits a unique equilibrium path which converges to the saddle point. Along to the equilibrium path all the relevant variables even the costate have positive real values.*

**3. Extension in two state variables**

Intensive commercial extraction of natural resources requires sometimes improvements on the harvesting equipment in order to extract efficiently. But better equipment is subject to adjustment costs, e.g. electronic machines, vessels, boats and workmen hiring are some of these adjustment costs. The supposition of quadratic adjustment costs simplifies the arithmetic but is not essential. With these additional assumptions one can treat the harvesting effort not as an instantaneous control but rather as a stock variable, and integrating over past adjustments the new control variable  $e(t)$  enters into the model, which describes the evolution of the harvesting effort. Moreover, another modification is made in the objective functional, introducing the adjustment costs,  $C(e(t))$ . In this section we stress the analysis in concave natural resources regeneration function  $g(\nu)$ . The concavity of the function  $g(\nu)$  states that the law of diminishing returns applies here too. With these modifications the optimal management problem becomes,

$$\max_{e(t)} \int_0^{\infty} e^{-\rho t} [U(h(t), \nu(t)) - C(e(t))] dt \quad (3.1)$$

subject to 
$$\dot{\nu}(t) = g(\nu(t)) - h(t), \quad \nu(0) = \nu_0 \quad (3.2)$$

$$\dot{h}(t) = e(t), \quad h(0) = h_0 \quad (3.3)$$

Model (3.1)–(3.3) is an optimal control with two state and one control variable, and with a quadratic cost function. The necessary conditions required by the maximum principle, provide the following four dimensional system of equations:

$$\dot{\nu}(t) = g(\nu(t)) - h(t), \quad (3.4) \quad \dot{h}(t) = e(t) \quad (3.5)$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial \nu} + \rho\lambda_1, \quad (3.6) \quad \dot{\lambda}_2 = -\frac{\partial H}{\partial h} + \rho\lambda_2 \quad (3.7)$$

$$\text{together with the optimality } \frac{\partial H}{\partial e} = -C' + \lambda_1 = 0 \quad (3.8).$$

The function,  $H = U(h(t), \nu(t)) - C(e(t)) + \lambda_1 [g(\nu(t)) - h(t)] + \lambda_2 e(t)$  is the Hamiltonian current value of the problem (3.1)–(3.3) and  $\lambda_1, \lambda_2$  are the costate variables. According to Hartman (1963), the behavior of the trajectories of system (3.4)–(3.7) around certain equilibrium points can be deduced from the qualitative study of the linear system  $\dot{y} = Jy$ , where  $J$  is the Jacobian matrix given by the partial derivatives of the functions of the right hand side of system (3.4)–(3.7) w.r.t. each variable. The possibility of limit cycles appearance, in models with two state variables, was established by Dockner and Feichtinger (1991). Now, we can use an explicit quadratic formula for the adjustment cost function that helps the qualitative analysis of the system (3.4)–(3.7). Using the cost function  $C(e) = 1/2\beta e^2$   $\beta > 0$ , (3.8) becomes  $e = \lambda_1/\beta$  and finally the conditions that determine the optimal plan of a central decision maker, after the appropriate substitutions, are (time is neglected to avoid notation overburden) :

$$\dot{\nu} = g(\nu) - h, \quad \nu(0) = \nu_0 \quad (3.9)$$

$$\dot{h} = \lambda_1/\beta, \quad h(0) = h_0 \quad (3.10)$$

$$\dot{\lambda}_1 = (\rho - g')\lambda_1 - U_\nu \quad (3.11)$$

$$\dot{\lambda}_2 = \rho\lambda_2 - U_h + \lambda_1 \quad (3.12)$$

### 3.1. Theoretic results

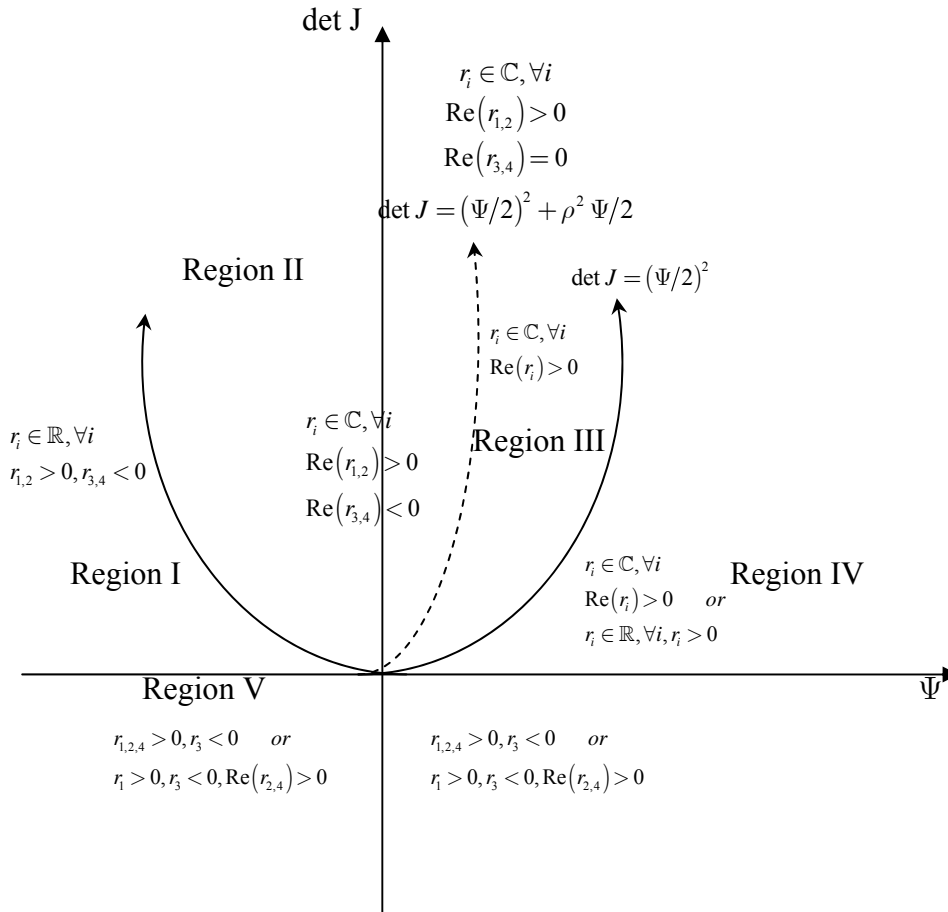
We study the dynamic properties of the system (3.9)–(3.12). Stability of this system is restricted to saddlepoint stability, i.e., to a two dimensional manifold in the four dimensional space of state and costates. According to Dockner's explicit formula (Dockner, 1985) the four eigenvalues  $r_i, i=1, \dots, 4$  of the linearized dynamics of the canonical equations given by:

$$r_{1,2,3,4} = \rho/2 \pm \sqrt{\rho^2/4 - \Psi/2} \pm \sqrt{\Psi^2 - 4 \det J} \quad (3.13)$$

and the magnitude  $\Psi$  is the sum of determinants of submatrices of the Jacobian  $J$  expressed as:

$$\Psi = \left\| \begin{array}{cc} \frac{\partial \dot{\nu}}{\partial \nu} & \frac{\partial \dot{\nu}}{\partial \lambda_1} \\ \frac{\partial \dot{\lambda}_1}{\partial \nu} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_1} \end{array} \right\| + \left\| \begin{array}{cc} \frac{\partial \dot{h}}{\partial h} & \frac{\partial \dot{h}}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_2}{\partial h} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_2} \end{array} \right\| + 2 \left\| \begin{array}{cc} \frac{\partial \dot{\nu}}{\partial h} & \frac{\partial \dot{\nu}}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_1}{\partial h} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_2} \end{array} \right\| \quad (3.14)$$

From Dockner's formula (3.13), it is well known that sufficient conditions for the saddle point are: first the positive determinant of the Jacobian matrix and second the negativity of the coefficient  $\Psi$  given by (3.14). A positive determinant of the Jacobian is crucial for stability, because a negative determinant restricts the stability to a one dimensional manifold of initial conditions (with one negative eigenvalue, the other three are positive or have positive real parts) and the generic solution is unstable. The following figure classifies the eigenvalues depending on  $\det J$  and  $\Psi$ .



**Figure 2.** Classification of the eigenvalues depending on  $\det J$  and  $\Psi$ . The solid curve is given by  $(\Psi/2)^2$  and the dashed curve by  $(\Psi/2)^2 + \rho^2(\Psi/2)$ .

As we can see in figure 2 there not exist at least one case for which all eigenvalues are negative numbers, the latter means that complete stability is

impossible. Dockner and Feichtinger (1991) show that a necessary and sufficient condition for the eigenvalues to be pure imaginary numbers is  $\det J > (\frac{1}{2}\Psi)^2$ ,  $\det J = (\frac{1}{2}\Psi)^2 + \frac{1}{2}\rho^2\Psi$ . In figure 2 the necessary and sufficient conditions correspond to the eigenvalues  $r_{3,4}$  that cross the imaginary axis when they go from one side of the dashed curve to the other. Considering the discount rate  $\rho$  as a parameter, the values of  $\rho$  for which the conditions are met, are possible Hopf bifurcations<sup>[2]</sup> (Kuznetsov, 1997) and a limit cycle will emerge if the complex eigenvalues  $r_{3,4}$  crosses the imaginary axis with non zero velocity at  $\rho = \rho_0$ , i.e.  $\left. \frac{d}{d\rho}(\text{Re}(\rho_{3,4})) \right|_{\rho=\rho_0} \neq 0$

**Table 1.** Classification of the equilibrium according to  $\Psi$  and  $\det J$ , where the regions depicted in figure 2.

Region/curve	Type of equilibrium	Local behavior
I	Saddle node	Monotonic
II	Saddle focus	Transient oscillation
III	Focus	Unstable
IV	Focus/node	Unstable
V	Saddle	Diverging except stable path

Following Dockner's formula (3.13) we compute the Jacobian  $J$  of equations (3.9)–(3.12) at the equilibrium:

$$J = \begin{bmatrix} g' & -1 & 0 & 0 \\ 0 & 0 & 0 & 1/\beta \\ -U_{vv} - g''U_h & -U_{vh} & \rho - g' & 0 \\ -U_{vh} & -U_{hh} & 1 & \rho \end{bmatrix}$$

and the determinant of  $J$  is:

$$|J| = \frac{(r - 2g')U_{vv} + g'(r - g')U_{hh} - U_{vv} - g''U_h}{\beta} \quad (3.15)$$

<sup>[2]</sup> Hopf bifurcations occurs when there exist two pure imaginary eigenvalues of the Jacobian matrix. Hopf bifurcations, so called bifurcations of co-dimension one, are related to the existence of a simple real eigenvalue of Jacobian matrix equal to zero. The dynamic change produced by values of the parameter higher than the bifurcation value has the result of closed trajectories (limit cycles). The equilibrium point for which there exist any of these two types of eigenvalues is known as non hyperbolic equilibrium point.

Now we consider the one dimensional control problem without adjustment costs, studied by Berck (1981). The Hamiltonian current value of one dimensional problem is:  $H_1 = U(h, \nu) + \lambda(g(\nu) - h)$  and the optimality conditions

$$H_h = U_h - \lambda = 0 \quad (3.16)$$

$$\dot{\lambda} = (\rho - g')\lambda - U_\nu \quad (3.17)$$

$$\dot{\nu} = g(\nu) - h \quad (3.18)$$

Setting the optimal control  $h = \phi(\nu, \lambda)$ , the derivatives w.r.t.  $\nu, \lambda$   $h_\nu = \phi_\nu = -U_{\nu h}/U_{hh}$  and  $h_\lambda = \phi_\lambda = 1/U_{hh}$ . The Jacobian matrix  $\hat{J}$  of the one dimensional model without adjustment costs, after these calculations, becomes

$$\hat{J} = \begin{bmatrix} \frac{\partial \dot{\nu}}{\partial \nu} & \frac{\partial \dot{\nu}}{\partial \lambda} \\ \frac{\partial \dot{\lambda}}{\partial \nu} & \frac{\partial \dot{\lambda}}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} g' + U_{\nu h}/U_{hh} & -1/U_{hh} \\ -U_{\nu\nu} - g''U_h + U_{\nu\nu}^2/U_{hh} & (\rho - g') - U_{\nu h}/U_{hh} \end{bmatrix}$$

Calculating the determinant of the Jacobian  $\hat{J}$  one can see that the determinant is

$$|\hat{J}| = \frac{(\rho - 2g')U_{\nu h} + g'(\rho - g')U_{hh} - g''U_{\nu\nu}}{U_{hh}} \quad (3.19)$$

Comparing (3.15) and (3.19) one can see that the relation between these

determinants is 
$$|J| = \frac{|\hat{J}|U_{hh}}{\beta} \quad (3.20)$$

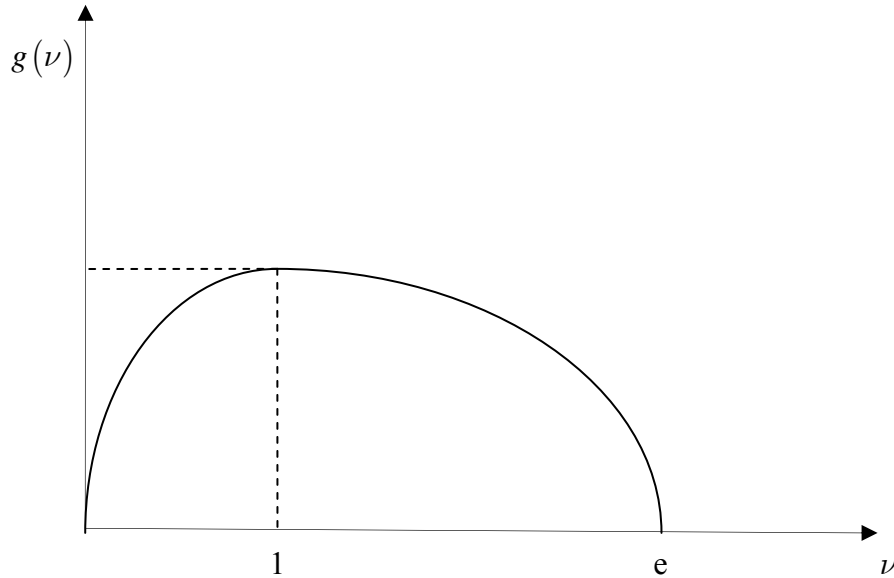
A simple observation of (3.14) reveals that the three terms that summed are essentially, the first and second are the determinants of the one dimensional without adjustment costs problems, while the third measures the interaction between these one dimensional problems. Application of (3.14) yields

$$\Psi = g'(\rho - g') + U_{hh}/\beta \quad (3.21)$$

Hence, the negative marginal growth of the natural resource  $g' \leq 0$ , and  $U_{hh} = U_{\nu h} = 0$ , are assumptions that ensure saddle point stability. These assumptions fulfilled for example for logistic growth resource functions.

#### 4. The Gompertz law of population growth

We consider the basic renewable resource model with a Gompertz growth function of the resource. The above growth function is given by the expression (see for instance Schafer, 1994)  $g(\nu) = \nu(t)[1 - \ln(\nu(t))]$



**Figure 3.** The shape of the Gompertz growth function  $g(\nu) = \nu(t)[1 - \ln(\nu(t))]$

The Gompertz growth function fulfills the conditions

$$g'(\nu) = -\ln(\nu) \quad g''(\nu) = -\frac{1}{\nu} < 0 \quad g(0) = 0$$

it is right-skew and has the same properties as the logistic growth function, e.g., it has the ‘pure compensation property’ in the sense of Clark (1976), and the upper stationary solution of  $\dot{\nu} = g(\nu)$ , i.e.,  $\nu = e$ , is asymptotically stable.

With the Gompertz growth function the stock of the resource evolves according to the differential equation:

$$\dot{\nu}(t) = \nu(t)[1 - \ln(\nu(t))] - u_1 - u_2$$

where  $u_i, i = 1, 2$  is the harvesting function for the two players of the model.

We define the fishing effort for the  $i$  player as  $a_i(t) = u_i(t)/\nu(t)$ , then the game is a non cooperative one for which every agent choosing a time path of his own fishing effort  $a_i(t)$  that maximizes the discounted utility. We transform the utility in the form

of an additive separable function, i.e. dependent on the fish stock  $\nu(t)$  and on utility that every player enjoys from harvesting  $u_i(t)$  as well.

We specify utility functions to be in logarithmic form given, arisen from the following utility function specification usually used in growth models.

$$F(\nu) = \begin{cases} \frac{\nu^\beta - 1}{\beta} & \beta \in (0,1) \\ \ln(\nu) & \beta = 0 \end{cases}$$

for which the elasticity of intertemporal substitution is given by  $1/(1-\beta)$ . We define  $y(t) = \ln \nu(t)$  in the case  $\beta = 0$ . Easy calculations made in order to set up the problem. These calculations are:

$y(t) = \ln \nu(t) \Rightarrow \nu(t) = e^{y(t)} \Rightarrow \frac{d\nu(t)}{dt} = \dot{\nu}(t) = e^{y(t)} \dot{y}(t) \Rightarrow \dot{\nu}(t) = \nu(t) \dot{y}(t)$  and the transformed evolution equation now becomes

$$\dot{\nu}(t) = \nu(t) [1 - \ln(\nu(t))] - u_1 - u_2 \Rightarrow \frac{\dot{\nu}(t)}{\nu(t)} = 1 - \ln(\nu(t)) - \frac{u_1}{\nu(t)} - \frac{u_2}{\nu(t)} \Rightarrow$$

$\Rightarrow \dot{y}(t) = 1 - y(t) - a_1(t) - a_2(t)$  which is the transformed stock evolution equation, that depends on the logarithm of the resource stock and on the players' fishing effort as well. Utility function that maximized is dependent on the resource stock and on the effort as follows. We have assume that original present value maximized utility is

dependent on the harvesting function, that is  $\max \int_0^\infty e^{-\rho t} \ln(u_i(t)) dt$  but the latter can

be transformed as follows:

$$\begin{aligned} \max_{u_i} \int_0^\infty e^{-\rho t} \ln(u_i(t)) dt &= \max_{u_i} \int_0^\infty e^{-\rho t} [\ln(u_i(t)) - \ln(\nu(t)) + \ln(\nu(t))] dt \\ &= \max_{u_i} \int_0^\infty e^{-\rho t} \left[ \ln\left(\frac{u_i(t)}{\nu(t)}\right) + \ln(\nu(t)) \right] dt = \max_{a_i} \int_0^\infty e^{-\rho t} [\ln(a_i(t)) + y(t)] dt \end{aligned}$$

The differential game now becomes

$$\begin{aligned} &\max_{a_i} \int_0^\infty e^{-\rho t} [\ln(a_i(t)) + y(t)] dt \\ \text{s.t.} &\quad \dot{y}(t) = 1 - y(t) - a_1(t) - a_2(t) \end{aligned}$$

In what follows we explore the Nash equilibria of the game which is may be a time consistent one, in the sense of subgame perfectness.

Time consistency could be seen as a minimal requirement for the credibility of an equilibrium strategy. If player  $i, i=1,2$  had an incentive to deviate from his strategy  $\phi_i$  during the time interval  $[0, T)$ , the other player  $j, j=1,2$  would not believe his announcement of  $\phi_i$  in the first place. Consequently, player  $j$  computes his own strategy taking into account the expected future deviation of player  $i$  which, in general, would lead to strategies different from  $\phi_j, j \neq i$ . Open loop informational structure strategies are not in general time consistent, while closed loop or Markovian strategies are certainly time consistent (Dockner *et al.*, 2000). On the other hand subgame perfectness is the concept for which an equilibrium strategy remains unchanged regardless the starting period the game begins. So, subgame perfectness is a sole requirement for the credibility of an equilibrium strategy that is time consistency for that strategy. We conclude if we can found an equilibrium strategy for the game, independently on the initial state and regardless the informational structure employed, the strategy has the subgame perfectness property and can be a time consistent strategy.

#### 4.1 Equilibrium analysis

##### Proposition 4.1

*The game with the Compertz growth function in the resource stock evolution admits an equilibrium strategy of the form  $a_i = \rho + 1$  that is time consistent.*

##### Proof

The Hamiltonian of the above problem for the  $i, i=1,2$  player is

$$H_i = y(t) + \ln a_i(t) + \lambda(t)[1 - y(t) - a_1(t) - a_2(t)]$$

and the conditions for an interior solution are  $\frac{\partial H_i}{\partial a_i} = \frac{1}{a_i(t)} - \lambda(t) = 0 \Rightarrow a_i(t) = \frac{1}{\lambda(t)}$

While the adjoint equation becomes  $\dot{\lambda}(t) = -\frac{\partial H_i}{\partial y} + \rho\lambda(t) \Rightarrow \dot{\lambda}(t) = -1 + (\rho + 1)\lambda(t)$

with solution  $\lambda(t) = \frac{1}{\rho + 1} + e^{(\rho+1)t}\Omega$  along the transversality condition

$\lim_{t \rightarrow \infty} \lambda(t)y(t) = 0$ , which must satisfied, is reasonable to set  $\Omega = 0$ , and the costate



variable becomes  $\lambda(t) = \frac{1}{\rho+1}$ . Substituting the value of the costate variable into the strategy, the resulting strategy becomes  $a_i = \rho+1$  independent on the initial state, consequently is time consistent.

**Proposition 4.2**

*In the case the players cooperate the joint cooperative time consistent equilibrium harvesting strategy is given by the expression  $a(t) = \frac{\rho+1}{2}$ .*

**Proof**

The evolution equation in the cooperative case becomes

$\dot{y}(t) = 1 - y(t) - 2a(t)$  where  $a(t) = a_1(t) + a_2(t)$  is the joint fishing effort of the two players. Hamiltonian for the cooperative case is,

$H_c = y(t) + \ln a(t) + \lambda(t)[1 - y(t) - 2a(t)]$  and the rest of algebraic manipulations for maximization reveals the cooperative equilibrium strategy  $a = \frac{\rho+1}{2}$  which is again

time consistent.

**4.2. The Value function**

**Proposition 4.3**

*In the case the players do not cooperate the value function for each player is*

$$V_i = \frac{y}{1+\rho} + \frac{1}{\rho} \left[ \ln(1+\rho) + \frac{1}{1+\rho} - 2 \right]$$

**Proof**

We verify that equilibrium strategies are by proposition 4.1 given. The satisfactory HJB becomes:

$$\rho V_i = \max \left\{ y(t) + \ln a_i(t) + \frac{\partial V_i}{\partial y} [1 - y(t) - a_i(t) - a_j(t)] \right\}$$

while maximization of the LHS of the HJB equation is:

$$\frac{\partial \left[ y(t) + \ln a_i(t) + \frac{\partial V_i}{\partial y} [1 - y(t) - a_i(t) - a_j(t)] \right]}{\partial a_i} = 0 \Rightarrow \frac{\partial V_i}{\partial y} = \frac{1}{a_i(t)} \quad (3.1).$$

Differentiation of the value function w.r.t the state variable  $y$  yields  $\frac{\partial V_i}{\partial y} = \frac{1}{1+\rho}$  and

with substitution in the above (3.1)  $a_i = \rho+1$ .

## 5. A Stackelberg Game of Natural Resources Extraction

One basic difference between the extraction of non renewable and renewable resources is the fact the renewable resource is subject to regeneration while the former isn't. In this way the regeneration function that appears in the right hand side of the resource accumulation equation is not present in the case of non renewable resources. Similarly, if we can distinguish the two phases of every biomass in the regeneration and living phases, in the last phase of its live (for which no recreation process exists) the regeneration function vanishes from the model.

A second but with much more economic implication difference between the natural resources is that some renewable, like fish, can be thought as a migratory capital. For the recreational reasons the renewables are not stay hooked in a determined sea place but migrate from the place in that spawn eggs to another place to find food and so forth. For example, pacific salmon travels along the coastline of the Pacific Ocean that includes Canada and United States spatial borders, staying in the USA high seas for some years, depending on the species, and return back into the Canadian seas for breeding. An economic implication for the fishery economy of each involved country could be the fact of the sequential mood of play (countries thought as players of the fishery game), caused by the migration nature of the resource. Fishery takes place regardless the place in that natural resource temporarily lives but one of the two players of the game (the leader) plays first, while the other follows. In this section we propose a sequential dynamic game, well known as a Stackelberg game, of natural resource exploitation for which the resource migrates form a spatial harvesting place into another.

### 5.1. The model

We assume the linear growth function of the form  $g(\nu) = A\nu$  and with this function the evolution equation that describes the resource rate of growth becomes  $\dot{\nu}(t) = A\nu(t) - h_1(t) - h_2(t)$  where  $h_i, i = 1, 2$  is the harvesting rate for both players. Linearity implies, in the absence of human harvesting, an exponential growth of rate for the resource as the solution of equation  $\dot{\nu}(t) = A\nu(t)$  becomes  $\nu(t) = \nu(0)e^{At}$ . The latter may be in reality, but under ideal conditions, where the availability of space and other resources does not inhibit growth, many biological populations are observed to grow at an approximately exponential rate initially. As the growth of the fish is limited by neither the big spatial size of coastline in that travels, nor by the infinite

food supply along the same spatial coastline in that species lives. Finally it is true that species survival is endangered by mass exploitation rather than by natural forces.

The objective of player 1 is to maximize the discounted sum of its instantaneous net

benefits  $J_1 = \int_0^{\infty} e^{-\rho t} R_1(h_1(t)) dt$  subject to the evolution equation.

Similarly the objective of player 2 is to maximize

$J_2 = \int_0^{\infty} e^{-\rho t} R_2(h_2(t), h_1, \nu(t)) dt$  subject to the same constraint. Assuming that both

players use Markovian informational structures, to ensure time consistency, we seek to find in equilibrium the stationary strategies  $h_1 = h_1(\nu)$  and  $h_2 = h_2(h_1(\nu), \nu)$ . The above strategies follows the sequential (or Stackelberg) nature of the game, because the first player (the leader) conditions his actions only on the current resource stock while the second (the follower) is informed of the catch of the first player before taking its action. Since the harvesting rate of the follower depends on the rate of the leader, which it observes before taking its own rate, and since the leader knows this (and takes into account), the solution of the described game is a purely Stackelberg one. Normally a Stackelberg equilibrium is a Nash equilibrium for which the strategy space of the follower is restricted over the set of all feasible reaction functions<sup>[3]</sup>.

## 5.2. Equilibrium Analysis

The net benefits for every player of the game is assumed to depend on the harvesting rate  $h_i(t)$  which is dependent on the fishing effort  $E_i(t)$  and on the remaining stock of fish biomass  $\nu(t)$ , in a multiplicative form  $h_i(t) = E_i(t)\nu(t)$ . As fishing effort we may think for example the total number of vessel – days per unit time or in other cases more detailed information regarding the number of nets, lines, or traps hauled is available. On the other hand, the stock may be a good proxy for the recreational value of the resource (think of a sand beach), or it may have a direct effect on the agents' profit (for example, the cost of harvesting fish may depend on the stock of fish). With these considerations net benefits can take the functional form.

$v_i(\nu(t), E_i(t)) = [E_i(t)\nu(t)]^\beta$ ,  $0 < \beta < 1/2$ , which is a concave function in the harvest rate  $h_i$ . For example, if the harvest is sold in an international market at a constant

---

<sup>[3]</sup> See Dockner *et al* (2000) chapter 5 for more details about the hierarchical games

price, then the net benefit determined as the total revenues minus the total costs. If total revenue is linear and total cost is convex, then net benefit is concave. In the case where the harvest sold in the home country's market is segregated from the world market, the net benefit to the home country is taken to be the sum of consumers' surplus plus the producers' surplus, since the consumer's surplus is normally concave and the cost function of catching fish is usually concave the final result of the net benefit is a concave function. Furthermore, for the special nature of some species for which the recreation of the population occurs in some protected areas at which commercial fishing is prohibited by the law and in some cases the areas are not the open seas (e.g. the pacific salmon) but the rivers, we assume the regeneration function becomes zero and we treat the natural resource reduction caused only by human activities like harvesting.

With these assumptions, the dynamic problem is formulated as follows.

$$\begin{aligned} & \max_{E_i \geq 0} \int_0^{\infty} e^{-\rho t} [E_i(t)\nu(t)]^{\beta} dt \\ & \text{s.t. } \dot{\nu}(t) = f(\nu(t), E_1(t), E_2(t)) = -\sum_{j=1}^2 E_j(t) \end{aligned}$$

In the strongly time consistent Nash equilibria the following HJB equation must be

satisfied

$$\rho V_i = \max_{E_i} \left\{ (E_i \nu)^{\beta} - \frac{\partial V_i(\nu)}{\partial \nu} \left[ \sum_{j=1}^2 E_j(t) \right] \right\}$$

Maximization of the right hand side of the above equation yields the feedback equilibrium strategies

$$\begin{aligned} & \frac{\partial \left\{ (E_i \nu)^{\beta} - \frac{\partial V_i(\nu)}{\partial \nu} \left[ \sum_{j=1}^2 E_j(t) \right] \right\}}{\partial E_i} = 0 \Rightarrow \beta \nu^{\beta} E_i^{\beta-1} = \frac{\partial V_i(\nu)}{\partial \nu} \Rightarrow \\ & \Rightarrow E_i = \left[ \frac{\frac{\partial V_i(\nu)}{\partial \nu}}{\beta \nu^{\beta}} \right]^{\frac{1}{\beta-1}} \Rightarrow E_i^* = \left[ \frac{\beta \nu^{\beta}}{\frac{\partial V_i(\nu)}{\partial \nu}} \right]^{\frac{1}{1-\beta}} \end{aligned} \quad (5.1)$$

Substitution of the strategy into the satisfactory HJB equation yields

$$\begin{aligned} \rho V_i(\nu) &= \left[ \left( \frac{\beta \nu^\beta}{\left[ \frac{\partial V_i(\nu)}{\partial \nu} \right]} \right)^{\frac{1}{1-\beta}} \nu - 2 \frac{\partial V_i}{\partial \nu} \left[ \frac{\beta \nu^\beta}{\left[ \frac{\partial V_i(\nu)}{\partial \nu} \right]} \right]^{\frac{1}{\beta-1}} \right] \Rightarrow \\ &\Rightarrow \rho V_i(\nu) \left[ \frac{\partial V_i(\nu)}{\partial \nu} \right]^{\frac{\beta}{1-\beta}} = (\beta \nu)^{\frac{\beta}{\beta-1}} (1-2\beta) \end{aligned}$$

We try with the value function of the form  $V_i(\nu) = \beta \nu^{2\beta} \Rightarrow \frac{\partial V_i(\nu)}{\partial \nu} = 2\beta^2 \nu^{2\beta-1}$  and further substitution into the satisfactory HJB yields the feedback Nash equilibrium strategies:

$$E_i^*(\nu) = \frac{\rho \nu}{2-4a} \quad (5.2)$$

We turn now into the Stackelberg game. We suppose player 1 is the leader. The leader takes into account the follower's reaction function(5.1)and the HJB equation for his problem now is described by

$$\rho V_1(x) = \max_{E_1 \geq 0} \left\{ (E_1 \nu)^\beta - \frac{\partial V_1(\nu)}{\partial \nu} \left[ E_1 + \left[ \frac{\beta \nu^\beta}{V_2(\nu)} \right]^{\frac{1}{1-\beta}} \right] \right\} \quad (5.3)$$

The maximization of the right hand side of equation (5.3) yields the feedback equilibrium strategy for the leader

$$\frac{\partial \left\{ (E_1 \nu)^\beta - \frac{\partial V_1(\nu)}{\partial \nu} \left[ E_1 + \left( \frac{\beta \nu^\beta}{V_2(\nu)} \right)^{\frac{1}{1-\beta}} \right] \right\}}{\partial E_1} = 0 \Rightarrow E_1 = \left[ \frac{\beta \nu^\beta}{\frac{\partial V_1(\nu)}{\partial \nu}} \right]^{\frac{1}{1-\beta}} \quad (5.4)$$

which is the same with the Nash equilibrium feedback strategy. We suppose now that player 2 is the leader. By the same argument the feedback equilibrium strategy is

given by

$$E_2 = \left[ \frac{\beta \nu^\beta}{\frac{\partial V_2(\nu)}{\partial \nu}} \right]^{\frac{1}{1-\beta}} \quad (5.5)$$

The coincidence of the two equilibria leads us to conclude the next proposition.

### **Proposition 5.1.**

*In the two players differential game of harvesting for which one is the leader and the other is the follower – a Stackelberg differential game – the first mover advantage disappears and the feedback Stackelberg equilibria coincides with the feedback Nash.*

### **6. Conclusions**

In this study we investigate the natural resources as capital resource and treat the exploitation of these resources as one would treat accumulation of a capital stock. The analysis taken place is concentrated on the two basic factors that affect the fishing industry, the size of the resource itself and the rate of human harvesting. The above specification does not take into account any other human activities that affects on biomass, for example like coastlines pollution. The analysis of the fishery takes into account the biological nature of fundamental capital, for which we have recognized the common property feature of the open sea fishery. We setup and solve the optimal control management for the recreational model in that under a well known growth function a kind of modified Hotelling rule emerges. The properties of the variables involved in a fishery game model are analyzed and some special interdependence for these variables recognized.

Moreover a differential game for two players is proposed for which the well known from biology population growth function (the Gompertz growth function) is used. In equilibrium of the model we found the time consistent strategies for every player in order to have an efficient economic outcome. The last model of the paper is a Stackelberg model for a special kind of species that travels along the coastlines in their living phase. In the last model we make the assumption that human harvesting takes place in unprotected places, while the protected are the habitats of the species in which they spawn eggs, e.g. rivers, therefore the regeneration function vanishes from the basic model of harvesting. Further analysis of the model reveals that is indifferent of which player plays first, having the first mover advantage, which disappears and the equilibrium feedback strategies coincide with the Nash equilibrium strategies.

### **References**

- Berck, P., (1981), Optimal management of renewable resources with growing demand and stock externalities, *Journal of Environmental Economics and Management*, 8, 105–117.
- Clark, C., (1973), Profit maximization and the extinction of animal species, *Journal of Political Economy*, 81, 950 – 961.

- Clark, C., (1990), *Mathematical Bioeconomics*, 2<sup>nd</sup>, Wiley Interscience.
- Clark, C.W., Munro, G.R., (1975), Economics of fishing and modern capital theory: a simplified approach, *Journal of Environmental Economics and Management*, 2, 92 – 106.
- Cohen, D., Michel, P., (1998) How should control theory be used to calculate a time-consistent government policy?. *Review of Economic Studies*, 55, 2, 263 – 274.
- Dockner, E., Jorgensen, S., Long, N.V., Sorger, G., (2000), *Differential games in economics and management science*, Cambridge University Press.
- Grass, D., Caulkins, J., Feichtinger, G., Trangler, G., Behrens, D., (2008), *Optimal Control of Nonlinear Processes*, Springer.
- Dockner, E.J., Feichtinger, G., (1991), On the optimality of limit cycles in dynamic economic systems. *Journal of Economics*, 53, 31 – 50.
- Gordon, H.S., (1954), The economic theory of a common property resource, *Journal of Political Economy*, 62, 124 – 142.
- Hartl, P.F., (1987), A simple proof of the monotonicity of the state trajectories in autonomous control problems. *Journal of Economic Theory*, 40, 211 – 215.
- Hartman, P., (1963), On the local linearization of differential equations. In: *Proceedings of the American Mathematical Society*, pp. 568 – 573.
- Kydland, F., Prescott, E., (1977), Rules rather discretion: The inconsistency of optimal plans. *Journal of Political Economy*, 85, 473 – 492.
- Kuznetsov, Y., (1997). *Elements of Applied Bifurcation Theory*. Springer, Berlin.
- Levhari, D., Withagen, C., (1992), Optimal Management of the Growth Potential of Renewable Resources, *Journal of Economics*, 3, 297 – 309.
- Lewis, T., Schmalensee, R., (1980), On Oligopolistic Markets of Nonrenewable Natural Resources, *Quarterly Journal of Economics*, 95, 475 – 491.
- Plourde, C.G., (1970), A simple model of replenishable natural resource exploitation, *American Economic Review*, 62, 518 – 521.
- Schafer, M., (1994), Exploitation of natural resources and pollution. Some differential game models, *Annals of Operations Research*, 54, 237 – 262.
- Scott, A., 1955, *Natural resources: The economics of conservation*, Univ. of Toronto Press, Toronto, Canada.
- Smith, V.L., (1969), On models of commercial fishing, *Journal of Political Economy*, 77, 181 – 198.