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ESTIMATION ERROR COVARIANCE IN REGRESSION WITH SEQUENTIALLY VARYING PARAMETERS*

BY BARR ROSENBERG

In regression with parameters that obey a first-order Markov Process, it is useful to estimate the parameters' trajectory over time. Formulas to compute the time series of "smoothed" estimators of the stochastic parameters, and the estimation error variance at any time, are well known. This article derives the intertemporal estimation error covariance and thence the sampling distribution of estimated parameter shifts. Contemporaneous and intertemporal covariances are linked by the "gain" matrices of "forward" and "backward filters."

I. INTRODUCTION

This note completes the results of the preceding paper (Cooley, Rosenberg, and Wall, 1977 [CRW]), and the notation and definitions of that paper continues here. The covariance between estimation errors for parameter vectors β_s and β_t at times s and t,

$$P_{s,t|T} \equiv E((b_{s|T} - \beta_s)(b_{t|T} - \beta_t)') \quad s = 1,,T, \quad t = 1,,T$$

has previously been reported only for the case where $s = t$ (Rauch, 1963; Rauch, Tung, and Striebel, 1965; Meditch, 1967; and references in the preceding paper). Formulas for these intraperiod variances (denoted by $P_{s|T}$ rather than $P_{s,s|T}$ to simplify notation) were given in CRW. In an earlier work (Rosenberg, 1968), the interperiod estimation error covariances were derived, but in a relatively obscure form. The purpose of this note is to provide easily accessible and computationally convenient formulas.

Inference concerning the historical behavior of parameters generally requires both intraperiod error variances and interperiod covariances. For example, consider the parameter shift between periods s and t, $\beta_t - \beta_s$. The minimum mean square error linear unbiased estimator or MMSLUE (Rosenberg, 1973) is $b_{t|T} - b_{s|T}$, that is, the difference between the smoothed estimators at these two time points. The mean square error of the estimator is:

$$\text{MSE}(\widehat{\beta_t - \beta_s}) = E((b_{t|T} - b_{s|T} - (\beta_t - \beta_s))$$
$$(b_{t|T} - b_{s|T} - (\beta_t - \beta_s))')$$

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$$= E((b_{t|T} - \beta_t) - (b_{s|T} - \beta_s))$$
$$((b_{t|T} - \beta_t) - (b_{s|T} - \beta_s))')$$
$$= P_{t|T} - P_{s,t|T} - P_{t,s|T} + P_{s|T}.$$

Thus, the mean square error is a function of intraperiod error variances and interperiod covariances.

Interperiod estimation error covariances are also needed for inference concerning the values of parameters at two or more times. Let $t_1, \ldots, t_p$ be p given times. The variance matrix of estimation errors for $\beta_{t_1}, \ldots, \beta_{t_p}$ is made up of intraperiod variance and interperiod covariance submatrices: the submatrix in location (i, j) is $P_{t_i t_j|T}$.

2. A RECURSION FOR THE ERROR VARIANCE MATRICES

The estimates for the parameter vectors, $\beta_1, \ldots, \beta_T$, were given by CRW through the "gain" matrices of the "forward" and "backward filters":

(1) (forward) $$K_t = (\Phi' + H_{t|t}\Phi^{-1}Q)^{-1}$$

(2) (backward) $$J_t = \Phi'(I + G_{t|t}Q)^{-1}.$$

The computational procedure involves recursive application of these filters to accumulate the forward information matrices $H_{t|t}$, and filtered variables $f_{t|t}$, $t = 1, \ldots, T$ and the corresponding backward terms $G_{t|t+1}$ and $r_{t|t+1}$, $t = T, \ldots, 1$. Then the information from forward and backward filters is summed to obtain the smoothed estimates $b_{t|T}$ and their estimation variances $P_{t|T}$, $t = 1, \ldots, T$.

The recursive formulas hide the simple relationship between successive estimation error covariances. In fact, the variance matrices are themselves lined by the recursion:

(3) $$P_{t|T} = K_t' P_{t+1|T} J_{t+1}^{-1}.$$

To verify this equality, postmultiply by J_{t+1} and invert both sides of the equality to obtain:

$$P_{t|T} = K_t' P_{t+1|T} J_{t+1}^{-1} \iff J_{t+1}^{-1} P_{t|T}^{-1} = P_{t+1|T}^{-1} K_t'^{-1}.$$

Then substitute the expression (CRW.16) for the information matrices, taking note of the fact that $H_{t+1|t+1} + G_{t+1|t+2} = H_{t+1|t} + G_{t+1|t+1}$.

$$\iff J_{t+1}^{-1}(H_{t|t} + G_{t|t+1}) = (H_{t+1|t} + G_{t+1|t+1})K_t'^{-1}$$

Next, express $H_{t+1|t}$ in terms of $H_{t|t}$ by (CRW.7), and $G_{t|t+1}$ in terms of $G_{t+1|t+1}$ by (CRW.12).

$$\iff J_{t+1}^{-1}(H_{t|t} + J_{t+1}G_{t+1|t+1}\Phi) = (K_t H_{t|t}\Phi^{-1} + G_{t+1|t+1})K_t'^{-1}.$$

Then substitute (1) and (2) for the gain matrices.

$$\Leftrightarrow \quad (I + G_{t+1|t+1}Q)\Phi'^{-1}H_{t|t} + G_{t+1|t+1}\Phi$$
$$= ((\Phi' + H_{t|t}\Phi^{-1}Q)^{-1}H_{t|t}\Phi^{-1} + G_{t+1|t+1})(\Phi + Q\Phi'^{-1}H_{t|t}).$$

The terms in $G_{t+1|t+1}$ now cancel out.

$$\Leftrightarrow \quad \Phi'^{-1}H_{t|t} = (\Phi' + H_{t|t}\Phi^{-1}Q)^{-1}H_{t|t}\Phi^{-1}(\Phi + Q\Phi'^{-1}H_{t|t}).$$

Premultiplication by $\Phi' + H_{t|t}\Phi^{-1}Q$ leads to the desired equality

$$\Leftrightarrow \quad (\Phi' + H_{t|t}\Phi^{-1}Q)\Phi'^{-1}H_{t|t} = H_{t|t}\Phi^{-1}(\Phi + Q\Phi'^{-1}H_{t|t})$$
$$\Leftrightarrow \quad H_{t|t} + H_{t|t}\Phi^{-1}Q\Phi'^{-1}H_{t|t} = H_{t|t} + H_{t|t}\Phi^{-1}Q\Phi'^{-1}H_{t|t} \quad \text{Q.E.D.}$$

3. COVARIANCE BETWEEN ESTIMATION ERRORS IN DIFFERENT PERIODS

For any two time periods, s and t, $s < t$, it is shown below that

$$(4) \qquad P_{s,t|T} = K'_sK'_{s+1}\ldots K'_{t-1}P_{t|T} = P_{s|T}J_{s+1}\ldots J_t.$$

Equations (3) and (4) provide a complete recursion among all covariance matrices. For example, by recursive application of (3) to (4), every matrix may be expressed in terms of the variance matrix of the latest estimate:

$$(5) \qquad P_{s,t|T} = K'_sK'_{s+1}\ldots K'_{T-1}P_{T|T}J_T^{-1}J_{T-1}^{-1}\ldots J_{t+1}^{-1} \qquad s \le t \le T$$

The derivation of (4) is rather tedious and will be presented here for the simplified case where $\Phi = I$. Let $H_s = x_sx'_s/\sigma^2$ be the information matrix in period s, and let $h_s = x_sy_s/\sigma^2$. Also, let $K_{ts} = K_{t-1}K_{t-2}\ldots K_s$ and $J_{st} = J_{s+1}J_{s+2}\ldots J_t$ be the sequential products of gain matrices over the interval $s \leftrightarrow t$. (Notice the complicated subscript convention: for $s = t$, $J_{ss} = K_{ss} = I$). The estimation error $(b_{s|T} - \beta_s)$ can be expressed as a function of the stochastic terms in the model by collecting the recursive equations in CRW, as:

$$(6) \quad (b_{s|T} - \beta_s) = P_{s|T}\left[\sum_{j=1}^{s} K_{sj}x_je_j/\sigma^2 - \sum_{j=1}^{s-1} K_{sj}H_{j|j}u_j\right.$$
$$\left. + \sum_{j=s+1}^{T} J_{sj}x_je_j/\sigma^2 + \sum_{j=s}^{T-1} J_{sj}G_{j+1|j+1}u_j\right].$$

Taking the expression (6) for two periods, s and t, and evaluating the product moment, one finds:

$$P_{s,t|T} = P_{s|T}\Bigg[\Bigg(\sum_{j=1}^{s-1} K_{sj}H_jK'_{sj} + \sum_{j=1}^{s-1} K_{sj}H_{j|j}QH_{j|j}K'_{sj}\Bigg)K'_{ts}$$
$$+\Bigg(\sum_{j=s}^{t} K_{sj}H_jK'_{tj} - \sum_{j=s}^{t-1} J_{sj}G_{j+1|j+1}QH_{j|j}K'_{tj}\Bigg)$$
$$+ J_{st}\Bigg(\sum_{j=t+1}^{T} J_{tj}H_jJ'_{tj} + \sum_{j=t}^{T-1} J_{tj}G_{j+1|j+1}QG_{j+1|j+1}J'_{tj}\Bigg)\Bigg]P'_{t|T}. \tag{7}$$

The three expressions are the respective contributions to error from stochastic terms up to period t, between s and t, and beyond t. The second summation in each expression can be rewritten as the negative of the first summation plus one or two additional terms. For example, since

$$K_j^{-1} = I + H_{j|j}Q$$

by (1), it follows that

$$\sum_{j=1}^{s-1} K_{sj}H_{j|j}QH_{j|j}K'_{sj} = \sum_{j=1}^{s-1} K_{sj}(K_j^{-1} - I)H_{j|j}K'_{sj}$$
$$= \sum_{j=1}^{s-1} K_{s,j+1}(H_{j|j} - K_jH_{j|j})K'_{sj}.$$

Regrouping terms and adding and subtracting $K_{s1}H_{0|0}K'_{s1}$ yields:

$$= -K_{s1}H_{0|0}K'_{s1} + \sum_{k=1}^{s-1} K_{sk}(-H_{k|k} + H_{k-1|k-1}K'_{k-1})K'_{sk} + K_{ss}H_{s-1|s-1}.$$

Furthermore, since $H_{0|0} = 0$, $K_{ss} = I$, and $H_{k|k} = H_{k-1|k-1}K'_{k-1} + H_k$ from (CRW, 10), the preceding expression reduces to:

$$= -\sum_{k=1}^{s-1} K_{sk}H_kK'_{sk} + H_{s-1|s-1}. \tag{8}$$

Similar simplifications for the second and third expressions yield:

$$\sum_{j=s}^{t-1} J_{sj}G_{j+1|j+1}QH_{j|j}K'_{tj} = H_{s-1|s-1}K'_{ts} + \sum_{j=s}^{t} K_{sj}H_jK'_{tj} - J_{st}H_{t|t}, \tag{9}$$

$$\sum_{j=t}^{T} J_{tj}G_{j+1|j+1}QG_{j+1|j+1}J'_{tj} = G_{t|t+1} - \sum_{j=t+1}^{T} J_{tj}H_jJ'_{tj}. \tag{10}$$

Substitution of these expressions into (7) results in:

$$(11)\quad P_{s,t|T} = P_{s|T}[(H_{s+1|s+1}K'_b) + (-H_{s+1|s+1}K'_b + J_{st}H_{t|t}) + (J_{st}G_{t|t+1})]P_{t|T}$$

$$= P_{s|T}J_{st}(H_{t|t} + G_{t|t+1})P_{t|T} = P_{s|T}J_{st}P_{t|T}^{-1}P_{t|T} = P_{s|T}J_{st}.$$

This is the second equality in (4). The first equality then follows by repeated application of (3) to this equation. Q.E.D.

4. Mean Square Estimation Error for a Parameter Shift

It is often of interest to compare values of the parameter vector at two points in time by means of the estimated parameter shift $(b_{t|T} - b_{s|T})$. From (3) and (4), the mean square estimation error matrix is:

$$(12)\quad \mathrm{MSE}((\hat{b}_{t|T} - \hat{b}_{s|T}) - (\beta_t - \beta_s)) = P_{t|T} - P_{s,t|T} - P'_{s,t|T} + P_{s|T}$$

$$= P_{t|T} - K'_b P_{t|T} - P_{t|T}K_b + P_{s|T}.$$

5. Computation

These results show an elegant relationship among the estimation error covariances. The smoothing procedure proposed in CRW, which does not follow this pattern, involves $3T$ matrix inversions: (T operations to compute the forward gain matrices, by formula (1) above; T to compute the backward gain matrices by formula (2) above; and T to compute the variance matrices by formula (16) in CRW). Once these matrices have been computed and stored, no further matrix inversions are required to compute all intertemporal covariance matrices via formula (4).

Simplifying formulas do exist that reduce computational difficulty by exploiting the structure implicit in equation (3). The procedure derived in Rosenberg (1968, chapters 4 and 5) entailed no matrix inversions in the filtering state and required only a single matrix inversion overall. Alternatively, recursive application of formula (3), as a substitute for (CRW, 17) reduces matrix inversions to $2T + 1$. However, despite the gain in computational efficiency, these algorithms seem to be worthless because of numerical instability. The CRW filtering formulas are inherently stable because the errors in computation are attenuated due to subsequent multiplication by gain matrices with eigenvalues smaller than unity. Formula (3) uses the product of one matrix with eigenvalues smaller than unity, with another having eigenvalues greater than unity, to extrapolate information from one period to the next. The errors in such a procedure inevitably build up over a long sample series and, in our experience, have proved uncontrollable. The only practical usefulness of formula (3) above is as an error check to validate an algorithm based on (CRW, 16). Formula (4) is the efficient method to compute intertemporal estimation error covariance.

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