Optimal Bidding with Announcement of the Reservation Price

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This paper is aimed at introducing a model of the symmetric Bayes-Nash equilibrium in a FPSB auction, when the auctioneer announces the reservation price known at all bidders. Following the specification of the ways in which the existing literature treats this matter and, in particular Carey (1993), it is established an alternative model based on the "nature's move" tipical of bayesian games. Finally, in the conclusion, there is a treatment of the difference between the optimal bidding strategy which concludes this work and the one of Carey, after that the data have been made homogeneous for ease of comparison. [JEL CODE: D44, C70].

1 - Introduction

For almost twenty years, beginning from the pioneristic work of Vickrey (1961), the contribution of economic theory to auctions has been rather scanty. Generally speaking, the main areas of research moved towards two parallel paths which were, at the same time, tightly linked: on the one hand, they addressed the classification of the main types of auctions and their equivalence (under certain circumstances) and, on the other, they concentrated on the optimal bidding mechanism which maximized the bidder expected profits.

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* <*gianniwalter@yahoo.it*>. A thankyou deeply felt to Prof. Massimo Marrelli for the fundamental role he has played in my life, in my education and for the help as thesis supervisor. I also thank my collegue Francesco Flaviano Russo for helpful comments, and to *Rivista di Politica Economica* e Confindustria for the opportunity provided via the Angelo Costa prize. The final version of this article has greatly benefited from helpful comments of anonymous international and Italian referees.

Only in the early ‘80s, different models started to appear in the economic literature; these were able to analize, in a quite general way, the most important elements which characterize an auction, i.e. the award rule, the payment rule and the economic valuation of the object and were able to take account of the different environments linked with common value, indipendent value and correlated value auctions. The focal point turned from “optimal bidding” to “optimal auction” and considerable importance was dedicated to the normative analisys and to the game’s rules, emphasizing the auctioneer’s crucial role[1].

Within this approach, many papers attributed importance to the means, often very simple, used by the auctioneer to enhance the result which he expect to gain from selling. The simplest of these means, and also the most analysed in theory, is the reservation price, i.e. that value below which the auctioneer is not willing to accept bids[2]. In other words, by anticipating the bidder’s optimal bidding function, the auctioneer chooses the value b_+ which maximizes his own expected revenue, bearing, at the same time, the risk of no-auction[3].

What characterized these models, however, was the circumstance that the optimal bidding mechanism was not analyzed in detail, since the emphasis was mainly put on the normative aspects of the auction and, in particular, on the choice of the optimal reservation price. Furthermore, it was assumed that the only relevant difference with more general models which did not include the presence of a reservation price, was the more restricted space of possible bids, whereas all the other hypotheses held on[4].

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[1] See RILEY J.G. - SAMUELSON W. (1981) and MYERSON R.B. (1981). Further, MILGROM P.R. - WEBER R.J. (1982) first introduced the so called “general model”, which included the principal types of auctions.

[2] Clearly, in the simmetric case of a buyer-auctioneer, it costitutes the maximum value of payment (a ceiling to possible bids).

[3] See in particular MCAFEE R.P. - MCMILLAN J. (1987), and also for optimal bidding.

[4] In particular, the most important hypothesis, i.e. the number of bidders, the common knowledge and the probability function, were unchanged, even if it was necessary an adjustement of the density function, conditioned to the more restricted space of possible types.

More specifically, however, it must be stressed that by introducing a minimum bid known to all competitors, it may happen that not all the potential bidders are still able to compete for the allocation of the good or the service, since some could obtain a negative pay-off if obliged to submit allowable bids, and they would therefore prefer not to take part in the auction. The effective number of bidders, in this case, not only is not clear any more, since it is a variable depending on the level of the reservation price, but it is also possible, as an extreme hypothesis, that no awarding happens.

In the literature, different papers dealing with the problems linked to the effective numerousness of bidders when the auctioneer introduces a reservation price exist, but they usually concentrate upon the effects on the bidders reaction function. A notable exception is Carey (1993)[5] who considers this problem in a model aimed at comparing expected revenue for a buyer auctioneer when adopting two alternative strategies; *a)* to announce a reservation price known at all, or *b)* to announce that a reservation price exists without disclosing its true value. Beyond the comparison between the two alternatives available to the monopsonistic buyer, the paper emphasises that optimal equilibrium strategies take into account the fact that the number of bidders, is not exogenous anymore, but, on the contrary, directly linked to the value of b_+.

In this paper we propose an alternative approach to the issue of symmetrical equilibria and of optimal bidding in the case where the bid-taker announces a ceiling to offers. The model tries to approach this problem within the natural environment of game theory, and therefore, it is quite different from previous models. The most important issue here is the information on the number of bidders and, as a consequence, on how this is perceived by the competitors, and, therefore, embodied in the optimal bidding function.

In this game, indeed, the number of competitors depends

[5] BULOW J. - KLEMPERER P. (1995) show results linked with those presented in the CAREY K. (1993) work.

strictly on the value of the reservation price which is known only *ex-post*. On the other hand, this number, *ex-ante*, is not a deterministic value, but a stochastic one distributed according to a binomial random variable depending on the reservation price and on the initial numerousness of potential bidders; within this informative context, we analyze how the auction game is affected by auctioneer behaviour.

Rather than considering the competitive factor depending exclusively on the average number of bidders, we introduce the idea of "nature's move", a tipical concept of bayesian games. Nature chooses the tree's branch, characterized by the real bidders number, or, in other words, it chooses the path in which competition takes place effectively, but this choice, like any other bayesian scenario, is not visible to bidders; as a consequence, in this context, this information is not common knowledge and it is replaced by the probability function about nature's move.

In modelling the bidders optimal behaviour, we show the first order condition, or, in other words, the necessary condition for optimality which allows to characterize the Bayes-Nash equilibrium of this game, and which, at the end, specifies the optimal bidding strategy.

The final Section contains a comparison between the main result obtained here and those already known in the literature. Some simulations show that Carey overestimates optimal bidding as calculated in our paper.

2. - The Model

Consider a general model of first-price sealed bid auction with IPV. The auctioneer is a monopsonistic buyer and he awards the contract for service or good X to the lowest bid (bidders). Assume further that N is the initial number of bidders, known at all. Each bidder i knows its own cost C_i, but he doesn't know the others' ones. Each competitor, however, perceives that the costs of other bidders are distribuited according a probability function $F(x)$, common knowledge and the same for all bidders so that $F(x) : [\underline{c},$

$\bar{c}] \rightarrow [0, 1]\ \forall i \in \{1, 2, .., n\}$. Finally, payoff to bidder is b_i-C_i, where b_i is the bid, while C_i is the same bidder's cost.

Each bidder is risk-neutral and maximizes his own expected profits, and expects competitors to behave rationally. As widely seen in literature, simmetrical (Bayes-Nash) equilibrium[6] is:

$$(1) \qquad b_i^* = C_i + \frac{\int_{C_i}^{\bar{c}} [1 - F(s)]^{N-1} ds}{[1 - F(C_i)]^{N-1}}$$

Most models, even in the '90s, as seen in the previous Section, did not study in depth all the possible impications and the important consequences upon the optimal bidding of the auctioneer announcing a reservation price $\bar{a}$ (that is a ceiling price), known to all bidders; simply, they continued to maintain the total number of bidders N as an exogenous variable, and so unchanged.

Usually, indeed, optimal bidding was quite similar to the previous formula just, except for a different upper bound, which from $\bar{c}$ became $\bar{a}$, and for the updating of the cumulative probability function[7]:

$$(2) \qquad b_i^* = C_i + \frac{\int_{C_i}^{\bar{a}} \left[1 - \frac{F(s)}{F(\bar{a})}\right]^{N-1} ds}{\left[1 - \frac{F(C_i)}{F(\bar{a})}\right]^{N-1}}$$

In other words, it was implicitly assumed that, after the announcement of the price ceiling, all N bidders were able to overcame the obstacle, and, therefore, to submit bids. In other words,

[6] See *Appendix* for the proof.

[7] We can easily check the strict similarity of the formula *(2)* with the previous one. The foundamental difference, however, is the cumulative probability function which is now conditioned.

all the competitors presented a cost lower that the ceiling to all possible bids. Furthermore, eq. *(2)* implies that this fact is common knowledge among bidders who, therefore, slightly modify the cumulative probability function, and the types' interval, which now is $[\underline{c}, \bar{a}]$.

The following discussion, instead, describes the optimal bidding (and therefore the simmetrical Bayes-Nash equilibrium) in the case where the bid taker announces a ceiling $\bar{a}$ to all allowable bids, known to all bidders, and the final number of competitors is not anymore a deterministic variable, but becomes a random one; to do this, we will introduce both "nature's moves" and a bayesian context characterized by the effective number of competitors on each branch of the auction game tree.

We maintain unchanged the general hypoteses explained at the beginning of this Section[8]. Suppose that $\bar{a}$ is the ceiling price. Clearly $\bar{a} \in [\underline{c}, \bar{c}]$, otherwise it wouldn't be economically significant. It is not true that all initial bidders are still able to compete. Indeed, those, whose $C_i > \bar{a}$, would obtain a negative payoff if submitted bids; only types whose $C_i \leq \bar{a}$ $\forall i$ can then take part in the auction.

The number of effective sellers, in this case, is a random variable R which takes on an integer value between 0 and N and distributes according to a binomial probability with parameters $\bar{a}$ and N. R is unknown to bidders, who could formulate their bidding strategies based on the final value of the final expected number of competitors, $\bar{a}N$[9]. In this paper instead, we propose a model based on non cooperative bayesian games, in which the "nature's move" chooses types, and so the number of those who can submit bids.

From the point of view of bidder i, he knows whether his cost is less than the ceiling $\bar{a}$. Assume it is. The final number of residual competitors follows the same binomial distribution but now parameters are $\bar{a}$ and N-1. R[10] is a r.v. with probability

[8] This also to facilitate the comparison between the two different functional specification: Carey's one and *(6)* concluding this Section.

[9] This idea is exactly the same which Carey's equilibrium is modelled on.

[10] For semplicity, for all the treatment, R will be, the expected number of residual bidders.

$$\binom{N-1}{R} F(\bar{a})^R [1 - F(\bar{a})]^{N-1-R}$$

where $F(\bar{a})$ is the probability that C_j which is in the interval [$\underline{c}$, $\bar{a}$] can partecipate.

For each value of R, there exists not only a certain probability associated to it, but also a standard auction game in which the number of bidders is well defined and related to the "nature's move". Incomplete information on the number becomes now incomplete information on the kind of game (or the branches of tree), and probabilities on the effective number R become now probabilities on the branch type.

Payoffs, for the bidder i, are then:

$(b_i - C_i)$ if there are no competitors (with probability $[1 - F(\bar{a})]^{N-1}$);

$(b_i - C_i)$ [Prob($b_i < b_j$)] — when competitors are i and j (with probability $(N - 1)\ F\ (\bar{a})\ [1 - F(\bar{a})]^{N-2}$).

Clearly, this sequence of expected payoffs ends when R is equal to N-1.

$(b_i - C_i)$ [Prob($b_i < b_j\ \forall i \neq j$)] — when all bidders N partecipate (with probability $F(\bar{a})^{N-1}$).

Each payoff on the left side is the expected one of a traditional auction game equivalent to ours, and in which the number of bidders is known with certainty and correspondes to one of the possible realization of the stochastic variable R; there are as many games as possible situation about noumerousness, from 1 to N. Remembering that nature chooses the game or the branch, as just explained, with a certain probability, we have to maximize the following expected profit.

$$(b_i - C_i)\begin{Bmatrix} [1 - F(\bar{a})]^{N-1} + Prob(b_i < b_j)(N-1)F(\bar{a})[1 - F(\bar{a})]^{N-2} + \\ +Prob(b_i < b_j \forall i \neq j)F(\bar{a})^{N-1} \end{Bmatrix}$$

It's easy to show that bid b_i, obtained by maximizing the above equation, is a strictly increasing function in C_i[11]. This means[12] that, given the relationship $b(C_i) \neq b(C_j) \Leftrightarrow C_j \neq C_i$, we can clearly obtain the inverse of the bidding function, i.e. $C_i = b^{-1}(b_i)$.

The profit Π (b_i, C_i) then becomes

$$(b_i - C_i)\left\{\begin{bmatrix}[1-F(\bar{a})]^{N-1} + \left[1-\dfrac{F(b^{*-1}(b_i))}{F(\bar{a})}\right](N-1)F(\bar{a})[1-F(\bar{a})]^{N-2} + \\ +\ldots + F(\bar{a})^{N-1}\left[1-\dfrac{F(b^{*-1}(b_i))}{F(\bar{a})}\right]^{N-1}\end{bmatrix}\right\}$$

where $b^{*-1}(b_i)$ is the inverse function of $b^*(.)$, whose existence is guaranted by class E of Riley-Samuelson.

Rewriting the previous formula we obtain:

$$\text{(3)} \quad \Pi(.,.) = (b_i - C_i)\left[\sum_{R=0}^{N-1}\binom{N-1}{R}F(\bar{a})^R[1-F(\bar{a})]^{N-1-R}\left[1-\frac{F(b^{*-1}(b_i))}{F(\bar{a})}\right]^R\right]$$

Given IPV model's assumptions, to get the optimal bidding strategy characterising the Bayes-Nash equilibrium of this game, we write I[st] order conditions:

$$\left[\sum_{R=0}^{N-1}\binom{N-1}{R}F(\bar{a})^R\left[1-F(\bar{a})\right]^{N-1-R}\left[1-\frac{F(b^{*-1}(b_i))}{F(\bar{a})}\right]^R\right]+$$

$$+(b_i - C_i)\sum_{R=0}^{N-1}\binom{N-1}{R}F(\bar{a})^R\left[1-F(\bar{a})\right]^{N-1-R}R\left[1-\frac{F(b^{*-1}(b_i))}{F(\bar{a})}\right]^{R-1} *$$

$$*\left(\frac{-f(b^{*-1}(b_i))}{F(\bar{a})}\right)\left(\frac{1}{db^*/dC_i}\right) = 0$$

[11] See a good handbook of Game Theory for the complete proof. For example, FUDENBERG D. - TIROLE J. (1991).
[12] See *Appendix*.

Remembering that in equilibrium $b^{*-1}(b_i) = C_i$

$$\frac{db^*}{dC_i}\left[\sum_{R=0}^{N-1}\binom{N-1}{R}F(\bar{a})^R\left[1-F(\bar{a})\right]^{N-1-R}\left[1-\frac{F(C_i)}{F(\bar{a})}\right]^R\right]+$$

$$+(b_i-C_i)\left[\sum_{R=0}^{N-1}\binom{N-1}{R}F(\bar{a})^R\left[1-F(\bar{a})\right]^{N-1-R}R\left[1-\frac{F(C_i)}{F(\bar{a})}\right]^{R-1}\left(\frac{-f(C_i)}{F(\bar{a})}\right)\right]=0$$

The differential equation is the first necessary condition for optimality.

Further, initial condition is that marginal bidder must be indifferent whether winning or non-winning the auction[13]. It derives that $b^*(\bar{a}) = \bar{a}$. Moreover, this must be assured for each possible value of individual cost $C_j \in [\underline{c}, \bar{c}]$. Integrating and taking account of the initial condition, we have:

$$(4)\quad b_i^* = C_i + \frac{\int_{C_i}^{\bar{a}}\left[\sum_{R=0}^{N-1}\binom{N-1}{R}F(\bar{a})^R\left[1-F(\bar{a})\right]^{N-1-R}\left[1-\frac{F(s)}{F(\bar{a})}\right]^R\right]ds}{\sum_{R=0}^{N-1}\binom{N-1}{R}F(\bar{a})^R\left[1-F(\bar{a})\right]^{N-1-R}R\left[1-\frac{F(C_i)}{F(\bar{a})}\right]^R}$$

To simplify and help the interpretation of the differences between the two different models, we consider a probability function uniformly distributed in the interval [0.1]. The first of the following equations shows optimal bidding as seen in Carey's work, while the second (our optimal bidding function) summarizes our model, for the same distribution.

$$(5)\qquad b_i^* = C_i + \frac{1}{N} - \frac{C_i}{N\bar{a}}$$

[13] Marginal bidder is the one who has the worst cost signal (in this case his cost is equal to the maximum value of allowable bids).

Rewriting formula *(4)*[14], we obtain:

$$b_i^* = C_i + \frac{\sum_{R=0}^{N-1}\left[\dfrac{\binom{N-1}{R}\bar{a}^R\left[1-\bar{a}\right]^{N-1-R}\left[\bar{a}\right]\left[1-\dfrac{C_i}{\bar{a}}\right]^{R+1}}{R+1}\right]}{\sum_{R=0}^{N-1}\left[\binom{N-1}{R}\bar{a}^R\left[1-\bar{a}\right]^{N-1-R}R\left[1-\dfrac{C_i}{\bar{a}}\right]^{R}\right]} \qquad (6)$$

In the final and concluding section, we start from this result and show how optimal biddings differ in the two approaches, by a simple simulation.

3 - Conclusion

As already said, this paper intends to compare the two above mentioned results. By simulating the different trends of the two bidding functions with $N \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50\}$ and $\bar{a} \in \{0,5; 0,6; 0,7; 0,8; 0,9; 1,0\}$, a certain regularity in their differences can be shown.

To help understand the regularity in these differences, it is necessary a brief analisys of the principal characteristics and of the behaviour of these two functions. It can fairly easily be seen that the first formula is a strictly increasing function in types and, characterized, above all, by the peculiarity that the claim of the marginal bidder is the same of his cost $\bar{a}$, exactly as in the second formula; however, the difference lies in the fact that the first shows a trend which is linear in costs, i.e. a straight line; on the contrary, formula *(6)*, still being a strictly increasing function, is convex in costs.

[14] By considering a probability function uniformly distributed in the interval $[\underline{c}, \bar{c}] = [0, 1]$, the ceiling $\bar{a}$ will be a value between 0 and 1. Moreover, it can be easily seen that each $F(\bar{a})$ corrisponds to $\bar{a}$.

On the base of these considerations it is possible to simulate the trend of the two optimal bidding. The result underlines the circumstance that the bidders should underbid with respect to Carey's strategy; in other words, they must be careful when competing against each other and bid in a less cost-diverging way.

In other words, for equivalent values of bidders' noumerousness and of reservation price, optimal behaviour shown in Carey (1993) overestimates the effective bid. However, an exception occurs in the case of N=2, since for each value $\bar{a}$, there always exists a critical value of C_i, under which our optimal bidding gives rise to a bid function higher than Carey's, and *viceversa* in the upper interval. This point exists also for other values of N, but it can be ignored, since, when it exists, it assumes very small values (with N=3 it is equal to 10^{-3} and even smaller for higher N).

We can therefore conclude that there exists a unique interval of types, the one $[\underline{c}, \bar{a}]$, i.e the original one corrected for the reservation price, and in which the differences between the two results show that Carey's bid is overestimated.

APPENDIX

Proof of formula *(1):*

In this Section, we start directly with the consideration that the optimal bidding is a strictly increasing function in types C_i[15].

Given a certain strategy $b_i(C_i)$, indeed, we obtain the inverse function $\Rightarrow C_i = b^{-1}(b_i)$.

The problem of the indefinite bidder becomes

$$\text{Max}_{b_i} E\ [\pi\ (b_i,\ C_i)] = (b_i - C_i)\ [Prob\ (\forall j \neq i,\ b^{-1}\ (b_i) < C_j)] =$$
$$= (b_i - C_i)\ [1 - F\ (b^{-1}\ (b_i))]^{N-1}$$

Given first order conditions, we obtain:

$$[1 - F(b^{-1}(b_i))]^{N-1} +$$
$$+(b_i - C_i)\left[(N-1)\left[1 - F(b^{-1}(b_i))\right]^{N-2}\left[-f(b^{-1}(b_i))\right]\frac{1}{b_i^{'}(b^{-1}(b_i))}\right] = 0$$

In equilibrium $b^{*-1}(b_i) = C_i$, and so:

$$\frac{\partial b^*}{\partial C_i}\left[1 - F(C_i)\right]^{N-1} - (b_i^* - C_i)(N-1)[1 - F(C_i)]^{N-2} f(C_i) = 0$$

This differential equation must be verified for each value of $C_i \in [\underline{c}, \overline{c}]$ and the initial condition states that the marginal bidder, i.e the one who has the worst cost signal, in equilibrium must be indifferent whether winning or not-winning the auction; $b^*(\overline{c}) = \overline{c}$

[15] v. FUDENBERG D. - TIROLE J. (1991).

$$\int_{C_i}^{\bar{c}} \frac{\partial b^*}{\partial C}[1-F(C)]^{N-1}dC + \int_{C_i}^{\bar{c}} (b^*(C)-C)(N-1)[1-F(C)]^{N-2}(-f(C))dC = 0$$

With some simple algebra, the previous equation becomes:

$$\int_{C_i}^{\bar{c}} \frac{\partial b^*}{\partial C}[1-F(C)]^{N-1}dC + \int_{C_i}^{\bar{c}} -[1-F(C)]^{N-1}\left(\frac{\partial b^*}{\partial C}-1\right)dC +$$

$$+\Big[[1-F(C)]^{N-1}(b^*(C)-C)\Big]_{C_i}^{\bar{c}} = 0$$

finally, solving for $b^*(C_i)$

$$b^*(C_i) = C_i + \frac{\int_{C_i}^{\bar{c}} [1-F(s)]^{N-1}ds}{[1-F(C_i)]^{N-1}}$$

where s is the integration variable.

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