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Robust Optimization of Currency Portfolios

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Abstract

We study a currency investment strategy, where we maximize the return on a portfolio of foreign currencies relative to any appreciation of the corresponding foreign exchange rates. Given the uncertainty in the estimation of the future currency values, we employ robust optimization techniques to maximize the return on the portfolio for the worst-case foreign exchange rate scenario. Currency portfolios differ from stock only portfolios in that a triangular relationship exists among foreign exchange rates to avoid arbitrage. Although the inclusion of such a constraint in the model would lead to a nonconvex problem, we show that by choosing appropriate uncertainty sets for the exchange and the cross exchange rates, we obtain a convex model that can be solved efficiently. Alongside robust optimization, an additional guarantee is explored by investing in currency options to cover the eventuality that foreign exchange rates materialize outside the specified uncertainty sets. We present numerical results that show the relationship between the size of the uncertainty sets and the distribution of the investment among currencies and options, and the overall performance of the model in a series of backtesting experiments.

Key words: robust optimization, portfolio optimization, currency hedging, second-order cone programming

1 Introduction

Since Markowitz's seminal work on portfolio optimization and the benefits of diversification [23], academic research in portfolio optimization has received great attention and developed to a mature area of operations research. In recent years, researchers have begun to investigate international investment and portfolios that comprise both national and international assets as a further way to

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increase diversification and reduce risk. It is expected that international assets have a lower correlation with national assets than the latter amongst themselves.

Grubel [13] was the first to describe and quantify the gains from international diversification. He concludes that international diversification of portfolios could bring a new source of gains and at the same time have an important impact on policy making, as international capital movements are a function not only of interest rate differentials, but also depend on the growth rates of asset holdings in both countries. A later study by Levy and Sarnat [17] concludes on the risk reduction gains from international diversification, measured by the variance of a portfolio. The authors suggest to invest in developing countries together with developed ones: although the risk associated with developing countries may be higher, their returns are also less correlated with the returns from developed countries, and therefore allow to minimize the overall portfolio variance.

The first results on gains exclusively from foreign currency holdings were reported by Levy [18]. His work aims at finding an alternative way to reduce the foreign exchange risk by using a portfolio balancing approach, as following the collapse of the Bretton Woods system in the early 70's, exchange rates were now able to float freely. He shows that in the period from January 1971 to July 1973, US investors could have made significant gains by holding foreign currencies only. In a mixed portfolio of currencies and stocks, though generally stocks yielded a higher return than currencies, optimal portfolios would still have a significant proportion of currencies as these had lower standard deviations and lower correlation with stocks, therefore contributing to risk diversification. Recently, in the period following the introduction of the EUR, the USD suffered a constant depreciation against some of the major currencies (see Table 1), which created similar opportunities for US investors to profit from investing in these currencies.

Currencies	Appr. Rate (%)	Std. (%)
EUR	0.46	2.53
GBP	0.01	2.23
JPY	0.43	2.46
CHF	0.45	2.47
CAD	0.30	2.02
AUD	0.28	3.20

Table 1: Monthly average appreciation rate and standard deviation of major foreign currencies against the USD from Jan-02 to Dec-08

However, international portfolios carry an additional risk related to unfavorable movements of the foreign exchange rates. The issue of hedging the currency risk, and consequently of determining the optimal hedge ratio and of deciding on which financial instrument to use became more and more relevant. Black [3] introduced in 1989 the concept of "universal hedging", arguing that investors should always hedge their foreign assets, equally for all countries, but never 100%. His "universal hedging" formula has only three inputs based on averages across countries of the following parameters: i) excess expected return on the world market portfolio; ii) volatility of the world market portfolio; and

iii) exchange rate volatility.

Eun and Resnick [8] argue that the studies from Grubel [13] and Levy and Sarnat [17] overstated the actual gains from international diversification as they do not account for parameter uncertainty that affects the estimation of returns. They argue that the risk inherent to foreign exchange rates can eliminate or reduce substantially the gains of an international portfolio due to their own volatility and their positive correlation with the stock returns. Two methods are proposed to reduce this risk: (i) diversification through the investment in several currencies, and (ii) a hedging strategy that sells the expected foreign currency returns at the forward rate. The effectiveness of this strategy depends on how accurate the investor's estimates are relative to the future returns. The authors conclude that hedged portfolios dominated non-hedged ones. Similar results have also been reported by other authors, see Glen and Jorion [12], Larsen and Resnick [16], and Topaloglou *et al.* [31]. The latter implemented a multi-stage stochastic programming model and jointly determined the asset weights and the corresponding hedge ratios for the international currencies. A survey of the topic may be found in Shawky *et al.* [25].

In all of the approaches mentioned, the hedging instrument was always the forward rate, with little attention given to currency options. In 1983, Giddy [11] studied the application of foreign exchange options, the relationship between forward rates and currency options, and their pricing methodology. He concludes that options were a more adequate hedging instrument than forwards when the future revenues were uncertain. However, he does not test his hypothesis with real market data. In the same year, Garman and Kohlhagen [10] developed a pricing model for currency options based on the Black & Scholes [4] model for stock options. Steil [27] argues that the "Giddy rule" is based on a false premise, as the underlying contingency — receiving or not receiving a future revenue — is not the same as the one underlying the option, which is the foreign exchange rate. He tests both hedging strategies using forwards and options for three different expected utility maximization functions and concluded on the poor performance of options compared to forward rates. Similar conclusions were reached by Topaloglou *et al.* [29, 30], where the Conditional Value-at-Risk is used as a risk measure.

In order to incorporate the uncertainty associated with the estimation of the relevant parameters, we propose to combine robust optimization with currency options to protect against a depreciation of the foreign currencies. We expand on the work of Rustem and Howe [24], who present both a strategical and a tactical model for robust currency hedging. Robust optimization differs from other uncertainty reduction techniques by incorporating uncertainty directly in the model, as returns are not assumed deterministic, but as random variables which may be realized within a prespecified uncertainty set. Already in 1973, Soyster [26] discussed the optimization over a collection of sets, but only in 1998 did the technique gain widespread attention with the simultaneous works of Ben-Tal and Nemirovski [2] and El-Ghaoui [6]. We refer the reader to Ben-Tal *et al.* [1] for a recent survey of robust optimization and its applications.

Although currencies are not commonly seen as investment asset, the added risk of an international portfolio has been thoroughly studied in the literature. The focus of these studies, however, has been on currencies from the perspective of an investor on *assets*, that is, an investor who manages a portfolio of foreign assets and wishes to account for the currency risk and return. In contrast, this paper focuses on portfolios of *currencies* and, in particular, on the problem of hedging against a depreciation of the foreign exchange rates. The main contributions of our work may be summarized as follows:

1. Application of robust optimization to the problem of allocating investments between several currencies with different patterns of risk and return.

2. Analysis of the impact of the triangular relationship between foreign exchange rates in the model, particularly of the convexity issues that arise. Suggestion of solutions in order to overcome those same issues.

3. Description of a hedging strategy that minimizes the currency risk by including currency options, and implementation of a model that combines currency options with robust optimization. We take on a portfolio perspective and simultaneously consider all currencies. That way, we aim to avoid over-conservativeness in the hedging strategy.

4. Presentation of numerical results that describe the relationship between the size of the uncertainty set and the total investment in options. Presentation of a series of backtesting experiments that assess the performance of both strategies — robust optimization with and without currency options — relative to the Markowitz risk minimization approach.

The rest of the paper is organized as follows. Section 2 presents the robust portfolio optimization model, the distinguishing features of a currency portfolio, and the approach followed to guarantee convexity of the model. In Section 3 we extend the model to include currency options and explain how the investor is further insured against depreciations of the foreign currencies. We also show how robust optimization can be used together with currency options as a global hedging strategy. Section 4 presents numerical results that compare both models with and without currency options. We conclude in Section 5.

2 Robust Portfolio Optimization

We consider a portfolio that comprises n different foreign currencies, taking the USD as our base currency. The return on a currency is measured by the ratio between the expected future spot exchange rate and the spot exchange rate today. We denote by E_i and E_i^0 the expected future and the current spot exchange rates, respectively. Both quantities are expressed in terms of the base currency per unit of the foreign currency i. The expected return on a specific currency i is then described by $e_i = E_i/E_i^0$. In the Markowitz framework [23] we would want to maximize our expected portfolio return given some risk measure, in this case the variance of the portfolio. The formulation of our problem would be:

$$\max_{\mathbf{w}\in\mathbb{R}^n} \quad \mathbf{e}'\mathbf{w} \tag{1}$$

$$\begin{aligned} \text{s.t.} \quad \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} &\leq \sigma_{\text{target}} \\ \mathbf{1}'\mathbf{w} &= 1 \\ \mathbf{w} &\geq 0 \end{aligned}$$

The variable $\mathbf{w}$ denotes the vector of currency weights in the portfolio, while the parameter Σ represents the covariance matrix of the currency returns. Parameter σ_{target} denotes the maximum level of risk the investor is willing to take. Throughout this article, variables or parameters in bold face denote vectors. We denote by $\mathbf{1}$ a vector of all ones, whose dimension is clear from the context.

Although the Markowitz model stimulated a significant amount of research, the mean-variance framework has also been subject to criticism due its lack of robustness. Model (1) is deterministic: it assumes that the expected returns are given, and it does not account for their random nature. Small changes in the value of the parameters, however, may pull the solution far from the optimum or even render it infeasible. Robust optimization assumes that there is a degree of uncertainty in these estimates: future returns are not certain, but random, and they may take any value within a predetermined uncertainty set. This uncertainty set represents the investor's expectations about the future currency returns and can be constructed according to some probabilistic measures.

Because we would like our solution to be robust to changes in the parameter values, we will maximize our portfolio return in view of the worst-case currency returns within the specified uncertainty set. We formulate the robust counterpart of problem (1) as:

$$\max_{\mathbf{w}\in\mathbb{R}^n} \min_{\mathbf{e}\in\Theta_e} \quad \mathbf{e}'\mathbf{w} \tag{2}$$

$$\begin{aligned} \text{s.t.} \quad \mathbf{1}'\mathbf{w} &= 1 \\ \mathbf{w} &\geq 0 \end{aligned}$$

Parameter $\mathbf{e}$ designates a random variable that represents the real currency returns, and which we assume are within the uncertainty set Θ. This uncertainty set can be described in several ways, of which the most widely used ones are range intervals and ellipsoids. In our models, we define Θ_e as:

$$\Theta_{\mathbf{e}} = \{\mathbf{e} \geq 0 : (\mathbf{e} - \bar{\mathbf{e}})'\Sigma^{-1}(\mathbf{e} - \bar{\mathbf{e}}) \leq \delta^2\}, \tag{3}$$

which describes an ellipsoid that is centered at the expected returns $\bar{\mathbf{e}}$ and rotated and scaled by the covariance matrix of the returns. Ellipsoidal uncertainty sets were first described by Ben-Tal and Nemirovski [2]. They reflect the idea of a joint confidence region (the differences between the returns and their estimates are weighted by the covariance matrix), as opposed to an individual one like in hyper-rectangular sets. Note that, we optimize our portfolio in view of the worst possible outcome of the currency returns. As a result, we are bound to obtain at least the return exhibited by the objective value as long as the returns are realized within the uncertainty set. This is called the *non-inferiority* property of robust optimization and provides a guarantee to the investor regarding future returns. Although in principle, the covariance matrix is also subject to uncertainty, its statistical estimation is much easier and hence more accurate than the estimation of the returns. Furthermore, mean-variance problems are much less sensitive to deviations from the estimate of the covariance matrix than to estimates of the returns [9]. Therefore, we have not taken into account the uncertainty caused by the estimation of the covariance matrix.

However, foreign exchange rates have a particular feature that distinguishes them from other investment assets such as stocks or bonds. If we define two exchange rates relative to a base currency, for example, the USD versus the EUR (USD/EUR) and the USD versus the GBP (USD/GBP), then we automatically define an exchange rate between the EUR and the GBP as well. This triangular relationship between exchange rates must be observed at all times, since otherwise arbitrage opportunities would arise and the market mechanisms would drive this relationship back to its equilibrium. Robust optimization, on the other hand, takes into account all possible returns within the uncertainty set. Hence, we need to add a new constraint to the model which enforces this triangular relationship to be respected. With n currencies in the model, the number of cross exchange rates is $n(n-1)/2$. If we define as X_{ij} the cross exchange rate between E_i and E_j, that is, X_{ij} is the number of units of currency i that equals one unit of currency j, then:

$$E_i \cdot \frac{1}{E_j} \cdot X_{ij} = 1 \tag{4}$$

In analogy to our previous notation, X_{ij}^0 denotes the current spot cross exchange rate, while x_{ij} is the return on the cross exchange rate. We may modify this equation to express the future exchange rates in terms of the currency returns and the spot exchange rates:

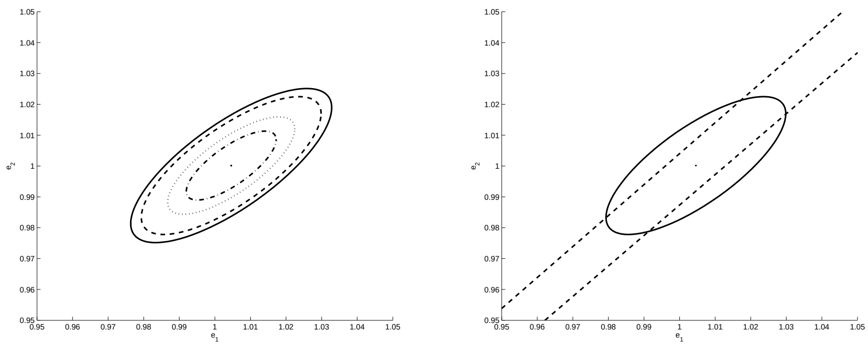
$$\begin{aligned}
& E_i^0 e_i \cdot \frac{1}{E_j^0 e_j} \cdot X_{ij}^0 x_{ij} = 1 \\
\Leftrightarrow \quad & [E_i^0 \cdot \frac{1}{E_j^0} \cdot X_{ij}^0] \cdot [e_i \cdot \frac{1}{e_j} \cdot x_{ij}] = 1 \\
\Leftrightarrow \quad & e_i \cdot \frac{1}{e_j} \cdot x_{ij} = 1
\end{aligned}$$

Including this constraint, however, will make the problem nonconvex. Note that although we need to model and estimate the future returns of the cross exchange rates, they do not have a direct impact on our objective function. In fact, the only effect of the cross exchange rates is to constrain further the uncertainty set originally defined for the exchange rates, that is, to render the model less conservative. We express the uncertainty associated with the returns of the cross exchange rates as intervals centered at the estimate, and subsequently make use of the triangular relationship to simplify the expression and eliminate the cross exchange rate returns from the model. The choice of intervals for the uncertainty sets of the cross exchange rates is merely for the sake of exposition. We could have also chosen ellipsoidal uncertainty sets, as these can be efficiently approximated by polyhedra. We are then able to reformulate the triangulation constraint as a linear expression and maintain the convexity of the model. We define $\Theta_{\mathbf{x}}$ as the uncertainty set associated with the returns of the cross exchange rates, where:

$$\Theta_{\mathbf{x}} = \{\mathbf{x} \geq 0 : \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\} \tag{5}$$

The returns on the cross exchange rates may be replaced by their corresponding ratio and the nonlinear expression may be simplified to a linear one:

$$\begin{array}{rrcl} & l_{ij} \leq & x_{ij} & \leq u_{ij}, \quad \forall i,j = 1,\ldots,n, i \leq j \\ \Leftrightarrow & l_{ij} \leq & \frac{e_j}{e_i} & \leq u_{ij} \\ \Leftrightarrow & l_{ij}e_i \leq & e_j & \leq u_{ij}e_i \end{array}$$



(a) Size of the uncertainty sets depending on the parameter δ

(b) Restriction of the ellipsoidal uncertainty set due to the triangulation requirement

Figure 1: Uncertainty sets. Both figures were constructed with the Ellipsoidal Toolbox developed by Kurzhanskiy [15].

This simplification leads to the transformation of $n(n-1)/2$ nonconvex inequalities into $n(n-1)$ linear ones. For two currencies, Figure 1a represents different uncertainty sets for different values of the parameter δ. Including the triangulation constraint in the model restricts the size of the uncertainty set as shown in Figure 1b. We define $\mathbf{A}$ as the coefficient matrix reflecting all the triangular relationships between the foreign exchange rates. $\mathbf{A}$ consists of $n(n-1)$ rows and n columns. We redefine our uncertainty set $\Theta_{\mathbf{e}}$ to include this new constraint:

$$\Theta_{\mathbf{e}} = \{\mathbf{e} \geq 0 : (\mathbf{e} - \bar{\mathbf{e}})'\Sigma^{-1}(\mathbf{e} - \bar{\mathbf{e}}) \leq \delta^2 \wedge \mathbf{A}\mathbf{e} \geq 0\} \tag{6}$$

Robust optimization uses duality theory to reformulate the inner minimization problem of model (2) as a maximization problem for a fixed vector $\mathbf{w}$ of weights. The inner minimization problem determines the worst possible outcome of the currency returns and may be formulated as:

$$\min_{\mathbf{e}\in\mathbb{R}^n} \quad \mathbf{e}'\mathbf{w} \tag{7}$$

$$\begin{array}{rrcl} \text{s.t.} & \|\Sigma^{-1/2}(\mathbf{e} - \bar{\mathbf{e}})\| & \leq & \delta \\ & \mathbf{A}\mathbf{e} & \geq & 0 \\ & \mathbf{e} & \geq & 0, \end{array}$$

where the operator $\|\cdot\|$ denotes the Euclidean two-norm. Problem (7) is a second-order cone program [5], and its dual may be written as, [5, 19]:

$$\max_{\mathbf{k},\mathbf{y},v} \quad \bar{\mathbf{e}}'(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y}) - \delta v \tag{8}$$

$$\begin{array}{rcl} \text{s. t.} \quad \|\Sigma^{1/2}(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y})\| & \leq & v \\ \mathbf{k}, \mathbf{y}, v & \geq & 0 \end{array}$$

In the case of second-order cone programs, strong duality holds, that is, as long as both problems are feasible, the value of the objective function of the dual problem is equal to the value of the objective function in the primal problem. Our problem now becomes:

$$\max_{\mathbf{w}} \quad \max_{\mathbf{k},\mathbf{y},v} \quad \bar{\mathbf{e}}'(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y}) - \delta v \tag{9}$$

$$\begin{array}{rcl} \text{s. t.} \quad \|\Sigma^{1/2}(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y})\| & \leq & v \\ \mathbf{1}'\mathbf{w} & = & 1 \\ \mathbf{w}, \mathbf{k}, \mathbf{y}, v & \geq & 0, \end{array}$$

which simplifies to:

$$\max_{\mathbf{w},\mathbf{k},\mathbf{y}} \quad \bar{\mathbf{e}}'(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y}) - \delta\|\Sigma^{1/2}(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y})\| \tag{10}$$

$$\begin{array}{rcl} \text{s. t.} \quad \mathbf{1}'\mathbf{w} & = & 1 \\ \mathbf{w}, \mathbf{k}, \mathbf{y} & \geq & 0 \end{array}$$

Problems (2) and (10) are equivalent, but (10) constitutes a tractable formulation that can be easily computed with a modern conic optimization software.

Note how this formulation is similar to the original Markowitz formulation [23]. In problem (10), however, we penalize the expected returns objective via the standard deviation of the portfolio returns instead of the variance. The advantage of the robust approach is that the parameters determining the size of the uncertainty sets may be chosen to reflect some probabilistic measures. El-Ghaoui *et al.* [7] shows that:

$$\max_{w}\{-\mathbf{w}'\mathbf{e} \mid (\mathbf{e} - \bar{\mathbf{e}})'\mathbf{\Sigma^{-1}}(\mathbf{e} - \bar{\mathbf{e}}) \leq \delta_{\omega}^{\mathbf{2}}\}$$

is equivalent to finding the worst-case ω-Value-at-Risk over all exchange rate return distributions whose first two moments coincide with $\bar{\mathbf{e}}$ and Σ, if δ is set to $\sqrt{(1-\omega)/\omega}$. However, an investor may wish to minimize the risk while at the same time demanding a minimum expected return. In that case, we may include a further constraint in the problem:

$$\mathbb{E}[e] = \mathbf{w}'\bar{\mathbf{e}} \geq e_{\text{target}} \tag{11}$$

Maximizing in view of the worst possible outcome of the future returns ensures the investor with a guarantee that the portfolio value at maturity date will always be at least as high as the objective value of (10). The investor is protected against any depreciation of the foreign exchange rates that materializes within the uncertainty set, and hence robust optimization provides guarantees against the currency risk without the need to enter into any hedging agreement. The main disadvantage of this approach is that it only protects the portfolio value for fluctuations *inside* the uncertainty set. If the future spot exchange rates fall *outside* this set, robust optimization does not provide any guarantees. In the next section we present an additional strategy which includes investing in currency options to hedge against the possibility of the foreign exchange rates falling outside the uncertainty set.

3 Hedging and Robust Optimization

Although robust optimization insures the investor against exchange rate fluctuations within the uncertainty set, the investor is left without any guarantees if the exchange rates materialize outside the uncertainty set. Insurance against the latter case can be obtained by using currency options, which allow to lock in *a priori* chosen exchange rates.

Although the focus of our work is on the use of currency options, forwards and currency futures are also popular hedging instruments. The latter, however, are binding agreements and do not offer the investor the flexibility to move away from it. They are therefore more appropriate for situations when the amount to be paid or received in the future is known with certainty, see Giddy [11].

Options entitle the investor to a right, and not to an obligation, to buy (call) or sell (put) a particular asset at a specified strike price at a certain point in the future, see Hull [14]. Currency options are similar to other options, but the strike price considered here is a foreign exchange rate. Buying a put option on EUR versus USD with a strike price of \$1.25 gives the right to transform EUR into USD at the rate of \$1.25 at the maturity date. Whether the investor chooses to exercise the option will depend on the spot exchange rate at maturity. We consider only European options, therefore options may only be exercised at maturity.

We assume that for each currency the investor has a set of m available put and call options with different premiums and strike prices. We denote by E_i the future spot exchange rate and by K_{il} the strike price of the lth option on the ith currency. We can compute the payoff V_{il} of the lth option on currency i versus the USD as:

$$\begin{aligned} V_{il}^{\text{call}} &= \max\{0, E_i - K_{il}\} && (12) \\ V_{il}^{\text{put}} &= \max\{0, K_{il} - E_i\} && (13) \end{aligned}$$

Assume now that a portfolio comprises of one unit of currency i and one put option on currency i with a strike price K_{il}. At maturity date, the payoff of the portfolio would be:

$$\begin{aligned} V_{\text{port}} &= E_i + \max\{0, K_{il} - E_i\} \\ &= \max\{E_i, K_{il}\} && (14) \end{aligned}$$

Hence, by including a put option corresponding to currency i in the portfolio, we are able to lock the foreign exchange rate at K_{il}. The aim of including currency options in the portfolio is therefore to guarantee a minimum return for the extreme cases where the exchange rates materialize outside the uncertainty set. If the realized exchange rate is higher than the strike price, the option will not be exercised and the investor may still benefit from the corresponding appreciation. This flexibility is a differentiating characteristic of options relative to other instruments such as forward contracts and futures: the latter two are binding agreements that lock the investor into a predefined exchange rate.

The price of this increased flexibility is the premium of the option, which must be paid upfront and which is incurred independently from the exercise of the option. Currency options are priced by the Garman-Kohlhagen model [10], which can be derived from the Black-Scholes model [4] by assuming that currencies are equivalent to stocks with a known dividend yield, namely the risk free rate prevailing at the foreign country.

In the subsequent analysis, we follow the notation in Lutgens [21] and the approach in Zymler *et al.* [33]. We define as $\mathbf{e^d}$ the vector of returns and as $\mathbf{w^d}$ the vector of weights of the options. If p_{il} is the price of the lth put option on currency i, then its return can be calculated as:

$$e^d_{il} = \max\left\{0, \frac{K_{il} - E_i}{p_{il}}\right\} \tag{15}$$

The value of the future spot exchange rate may be rewritten as a function of the return on the ith currency e_i by taking into account the relationship $E_i = E^0_i e_i$,

$$e^d_{il} = f(e_i) = \max\left\{0, \frac{K_{il} - E^0_i e_i}{p_{il}}\right\}, \tag{16}$$

which leads to a simplified expression, that we will be using in the following formulations of our model:

$$e^d_{il} = f(e_i) = \max\{0, a^{il} + b^{il} e_i\} \quad \text{with} \quad a^{il} = \frac{K_{il}}{p_{il}} \quad \text{and} \quad b^{il} = -\frac{E^0_i}{p_{il}} \tag{17}$$

Similarly, if c_{il} is the price of the lth call option on currency i, its return may be expressed as:

$$e^d_{il} = f(e_i) = \max\{0, a^{il} + b^{il} e_i\} \quad \text{with} \quad a^{il} = -\frac{K_{il}}{c_{il}} \quad \text{and} \quad b^{il} = \frac{E^0_i}{c_{il}} \tag{18}$$

As in the previous section, our investor wishes to maximize the portfolio return in view of the worst-case currency returns, while assuming that these will materialize within the uncertainty set $\Theta_{\mathbf{e}}$ as defined in (6).

$$\max_{\mathbf{w}, \mathbf{w^d} \in \mathbb{R}^n} \min_{\substack{\mathbf{e} \in \Theta_{\mathbf{e}} \\ \mathbf{e^d} = f(\mathbf{e})}} \quad \mathbf{e}'\mathbf{w} + \mathbf{e^d}'\mathbf{w^d} \tag{19}$$

$$\begin{aligned} \text{s.t.} \quad \mathbf{1}'(\mathbf{w} + \mathbf{w^d}) &= 1 \\ \mathbf{w}, \mathbf{w^d} &\geq 0 \end{aligned}$$

Note that the option returns are written as a function of the currency returns. Following the same procedure as in the previous section, we will reformulate the inner minimization problem as a maximization problem by using duality theory.

The minimization problem is concerned with finding the worst-case currency returns.

$$\min_{\mathbf{e},\mathbf{e^d}} \quad \mathbf{e}'\mathbf{w} + \mathbf{e^d}'\mathbf{w^d} \tag{20}$$

$$\begin{aligned} \text{s. t.} \quad \|\Sigma^{-1/2}(\mathbf{e} - \bar{\mathbf{e}})\| &\leq \delta \\ \mathbf{Ae} &\geq 0 \\ \mathbf{e^d} &\geq \mathbf{a} + \mathbf{be} \\ \mathbf{e}, \mathbf{e^d} &\geq 0 \end{aligned}$$

The dual of problem (20) may be formulated as:

$$\max_{\mathbf{k},\mathbf{y},\mathbf{u},v} \quad \bar{\mathbf{e}}'(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y} + \mathbf{b}'\mathbf{u}) - \delta v + \mathbf{a}'\mathbf{u} \tag{21}$$

$$\begin{aligned} \text{s. t.} \quad \|\Sigma^{1/2}(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y} + \mathbf{b}'\mathbf{u})\| &\leq v \\ \mathbf{u} &\leq \mathbf{w^d} \\ \mathbf{k}, \mathbf{y}, \mathbf{u}, v &\geq 0 \end{aligned}$$

Strong duality holds as problem (7) is a second-order cone program, which means that as long as they are feasible, the primal and dual problem have the same objective function values. Hence, we can replace the inner minimization problem in problem (19):

$$\max_{\mathbf{w},\mathbf{w^d},\mathbf{k},\mathbf{y},\mathbf{u}} \quad \bar{\mathbf{e}}'(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y} + \mathbf{b}'\mathbf{u}) - \delta\|\Sigma^{1/2}(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y} + \mathbf{b}'\mathbf{u})\| + \mathbf{a}'\mathbf{u} \tag{22}$$

$$\begin{aligned} \text{s. t.} \quad \mathbf{1}'(\mathbf{w} + \mathbf{w^d}) &= 1 \\ \mathbf{u} &\leq \mathbf{w^d} \\ \mathbf{w}, \mathbf{w^d}, \mathbf{k}, \mathbf{y}, \mathbf{u} &\geq 0 \end{aligned}$$

By using robust optimization, the investor is protected against any depreciation of the foreign exchange rates within the uncertainty set. Adding currency options to the model provides a "cap" on the value of the future foreign exchange rates. In the event of the foreign exchange rates materializing outside the uncertainty, robust optimization provides no guarantees. However, if the put options held in the portfolio correspond to the same number of units of foreign currency as the portfolio holdings (hedge ratio of 1), the investor is guaranteed with a minimum return given by the strike price of the put options. If this is not the case, however, the investor is left without any guarantees.

We would like to insure our portfolio further, even in a situation of a sharp depreciation of the foreign exchange rates, that is, if they were to materialize outside the uncertainty set. We reformulate our model in order to include an additional constraint guaranteeing a minimum return, expressed as a percentage of the worst-case portfolio return, for all the possible values of the currency returns such that $\mathbf{e} \geq 0$. We change the formulation of our problem in order to include this new constraint:

$$\max_{\mathbf{w},\mathbf{w^d},\phi} \quad \phi \tag{23a}$$

$$\begin{aligned}
\text{s.t.} \quad \mathbf{e}'\mathbf{w} + \mathbf{e^d}'\mathbf{w^d} &\geq \phi, \quad \forall \mathbf{e} \in \Theta_{\mathbf{e}}, \mathbf{e^d} = f(\mathbf{e}) && \text{(23b)}\\
\mathbf{e}'\mathbf{w} + \mathbf{e^d}'\mathbf{w^d} &\geq \rho\phi, \quad \forall \mathbf{e} \geq 0, \mathbf{e^d} = f(\mathbf{e}) && \text{(23c)}\\
\mathbf{1}'(\mathbf{w} + \mathbf{w^d}) &= 1 && \text{(23d)}\\
\mathbf{w}, \mathbf{w^d} &\geq 0 && \text{(23e)}
\end{aligned}$$

We have already seen how to reformulate the inner minimization problem corresponding to constraint (23b) as a maximization problem. We will follow the same approach for constraint (23c).

$$\min_{e,e^d} \quad \mathbf{e}'\mathbf{w} + \mathbf{e^d}'\mathbf{w^d} \tag{24}$$

$$\begin{aligned}
\text{s.t.} \quad \mathbf{e^d} &\geq \mathbf{a} + \mathbf{b}\mathbf{e}\\
\mathbf{e}, \mathbf{e^d} &\geq 0
\end{aligned}$$

The dual of this linear problem can be formulated as:

$$\max_{t} \quad \mathbf{a}'\mathbf{t} \tag{25}$$

$$\begin{aligned}
\text{s.t.} \quad \mathbf{w} + \mathbf{b}'\mathbf{t} &\geq 0\\
\mathbf{t} &\leq \mathbf{w^d}\\
\mathbf{t} &\geq 0
\end{aligned}$$

As strong duality also holds for linear problems, we may replace the objective function of problem (25) in our original problem (23a), already including the reformulation of constraint (23b) as well:

$$\max_{\mathbf{w},\mathbf{w^d},\mathbf{k},\mathbf{y},\mathbf{u},\mathbf{t}} \quad \phi \tag{26}$$

$$\begin{aligned}
\text{s.t.} \quad \bar{\mathbf{e}}'(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y} + \mathbf{b}'\mathbf{u}) - \delta\|\Sigma^{1/2}(\mathbf{w} - \mathbf{A}'\mathbf{k} - \mathbf{y} + \mathbf{b}'\mathbf{u})\| + \mathbf{a}'\mathbf{u} &\geq \phi\\
\mathbf{a}'\mathbf{t} &\geq \rho\phi\\
\mathbf{w} + \mathbf{b}'\mathbf{t} &\geq 0\\
\mathbf{1}'(\mathbf{w} + \mathbf{w^d}) &= 1\\
\mathbf{u} &\leq \mathbf{w^d}\\
\mathbf{t} &\leq \mathbf{w^d}\\
\mathbf{w}, \mathbf{w^d}, \mathbf{k}, \mathbf{y}, \mathbf{u}, \mathbf{t} &\geq 0
\end{aligned}$$

Note that neither the currency returns nor the currency option returns enter in the final formulation (26). This is a tractable problem which can be solved efficiently by any second-order cone optimization software.

As before, if the investor wishes to move away from the minimum risk solution, a constraint on the expected return may be added to the model:

$$\mathbb{E}[e] = \mathbf{w}'\bar{\mathbf{e}} \geq e_{\text{target}}.$$

We have chosen not to include the options return in this constraint as this would cause a distortion on our solution. On the one hand, our goal when including currency options is from a hedging strategy point of view, that is, we want to protect the portfolio return from depreciations in the foreign exchange rates and not to speculate on options. On the other hand, because options are leveraged assets and we are optimizing in view of the worst possible outcome of the currency returns, the optimal solution would be to invest the full budget on "in-the-money" options and not on currencies. Note how the hedging strategy presented has a portfolio point of view and it does not focus on any individual currency. The investor does not limit the weights of the currency options to the weights of the respective currency holdings. The guaranteed portfolio when the foreign exchange rates materialize outside the uncertainty set is defined by the investor and does not depend on the individual depreciation of any currency.

In the next section, we present numerical results for the models with and without considering currency options and assess their performance.

4 Numerical Results

The theoretical framework presented in Sections 2 and 3 will now be used to compute optimal currency portfolios based on real market data. We assume an US investor who wishes to invest in six foreign currencies: EUR, GBP, JPY, CHF, CAD and AUD. The models were implemented using the modelling language YALMIP (Lofberg [20]) together with the second-order cone solver SDPT3 (Toh [28], Tutuncu [32]). Both the expected returns on the foreign exchange rates and the covariance matrix are constructed from 7 years of monthly data between January 2002 and December 2008, see Table 2.

	Annual Ret. (%)	Std. (%)	Correl.					
EUR	5.64	8.75	1.00					
GBP	0.18	7.74	0.77	1.00				
JPY	5.32	8.51	0.42	0.16	1.00			
CHF	5.52	8.55	0.91	0.69	0.62	1.00		
CAD	3.61	7.00	0.56	0.51	0.01	0.41	1.00	
AUD	3.43	11.09	0.69	0.65	0.04	0.53	0.78	1.00

Table 2: Distributional parameters of monthly currency returns against the USD (Jan-02 to Dec-08)

We start by studying the composition of the portfolio and the distribution of weights between currencies and options for different levels of risk, defined by ω, and different levels of hedging, defined by ρ.

4.1 Portfolio Composition

In the following analysis, we will designate problem (10) as the robust problem and problem (26) as the hedging problem. We start by comparing the robust model with the Markowitz minimum risk model. In our robust model, the size of the uncertainty set defined by δ_ω, with $\delta = \sqrt{(1-\omega)/\omega}$, can be interpreted as a risk measure, namely, the worst-case VaR [7]. It is expected that as ω increases, the risk associated with the portfolio increases as well. If we measure the risk of the portfolio as its variance, we are able to conclude that for higher values of ω there is an increased value of the variance of the portfolio. The portfolio composition of problem (10) reflects this increase, as for higher levels of ω the optimizer concentrates its investment on a single currency. This a similar behavior to the Markowitz model, with the difference that in this case the focus is not on the currency with the highest estimated return rate, but on the one with the highest worst case return rate.

We would like to assess the impact of adding currency options to the portfolio, and how the insurance provided by the options relates to the guarantees provided by robust optimization. We consider 50 put options and 50 call options available in the market, with strike prices ranging between 75% and 125% of the current spot prices. In the experiments described below, we include a budget constraint and we do not allow short-selling. Compared to the robust model, there is a change in the weights allocation between the different currencies, in favour of the currencies with the highest worst possible returns. In our first set of experiments we have not considered a minimum expected return, and we have studied how the worst-case return and the total investment in options changes relative to the size of the uncertainty set defined by ω. For higher values of ω (that is, smaller uncertainty sets), the optimal portfolio is comprised mainly of currencies and not of options. As the uncertainty set increases in size, the percentage allocated to put options reaches almost 20%, with the remaining budget distributed among the currencies. Protection against the currency risk in this situation is made through the acquisition of deep "in-the-money" options, while for small uncertainty sets this is done by currency diversification. The worst-case return is constant at 1.0025 (annual rate of 3%) for $\omega \leq 80\%$. Investment in options is actively "capping" the maximum portfolio loss.

We now add an expected return constraint of an annual average return of 5%. Because this constraint does not include options, a larger percentage must be allocated to foreign currency holdings to meet this constraint. In this situation, not only is the weight of the options in the portfolio considerably lower, but also the options chosen to invest on are "at-the-money". In contrast to the previous case, the worst-case return degrades to values below 0, that is, the worst-case implies a loss for the investor of about 3%.

Figure 2 shows the trade-off between the two different sets of guarantees provided by robust optimization and by the currency options. For the same level of desired hedging of the currency risk (expressed by parameter ρ) a higher value of ω (i.e., a smaller uncertainty set) leads to an increase of the worst-case returns. For smaller values of ω, the uncertainty set converges to the full support of the currency returns, which leads to overly conservative portfolios. We may then conclude that the worst-case return monotonically increases with ω. In contrast, for the same size of the uncertainty set ω, a higher level of hedging (given by ρ) leads to a decrease of the worst-case return. This is because options are expensive assets, and a higher hedging demand may only be satisfied if at the same time the worst-case is smaller. Therefore, the worst-case portfolio return has an inverse relationship with ρ.

The results obtained from our experiments lead us to conclude that the

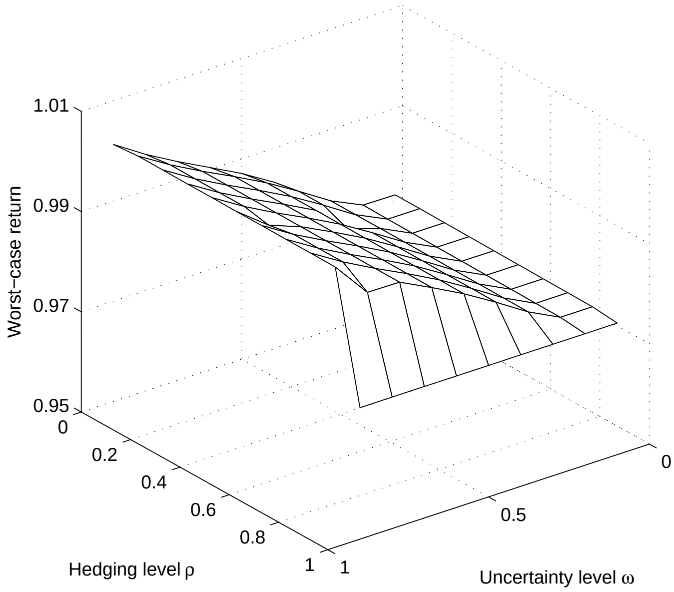


Figure 2: Worst-case return for different values of ω and ρ

constraint on the minimum guaranteed return outside the uncertainty (23c) is not a binding constraint. This conclusion, however, may be flawed due to estimation problems of the option prices. We have used the Garman-Kohlhagen model [10] to obtain the option prices. The model assumes that the implied volatility is constant and neither depends on the strike price of the option nor on its time to maturity. In reality, however, the volatility depends on the strike price of the option and exhibits what is known as a "smile", that is, it is higher for "out-of-the-money" and "in-the-money" options, while it is lower for "at-the-money" options, [14]. By considering the same volatility for all the 50 strike prices tested, we underestimate the option prices, thus the model may choose to either invest in "deep-in-the-money" options or to generally over-invest in options, given their low prices. This would make the minimum guaranteed return constraint (23c) redundant. Given the reasons described above, however, we have chosen to keep this constraint still in the model when performing the historical backtesting, which is described next.

4.2 Model Evaluation With Historical Market Prices

We want to assess the performance of our model under real market conditions by computing the portfolio returns over a long period of time. To this end, we consider the real currency returns in the period from January 2002 to March 2009 and conduct a backtest with a rolling horizon of twelve months. Every month we compute the estimated average returns $\bar{e}$, based on the historical returns from the previous twelve months, and calculate the optimal portfolio weights. The covariance matrix Σ and the triangulation matrix $\mathbf{A}$ are assumed to remain the same throughout the time series. At the end of each month, the portfolio return is computed based on the materialized returns, and the options are exercised or left to expiry depending on the spot rate. This procedure is repeated until March 2009 and the accumulated returns are calculated.

Given that currency options are traded mainly over-the-counter, there are no records of historical prices, but only of three different volatilities that may be used to construct the volatility smile and compute the option price. Contrary to the assumptions of the Black & Scholes and the Garman-Kohlhagen models, the volatility is not constant throughout the spectrum of the strike prices, but is higher for "out-of-the-money" and for "in-the-money" options, while it is lower for "at-the-money" options. Moreover, it has been also verified empirically that options with the same exercise price but with different maturities exhibit different implied volatilities, designated as the term structure [14]. The probability distribution of the currency returns, consequently, is not lognormal, but has heavier tails, making it more likely for extreme variations of the returns. The volatility associated to a given strike price may be calculated from the volatility smile, for which there is an approximate expression, Malz [22]:

$$\sigma(\delta, T) = \sigma_{ATM,T} - 2rr_T\left(\delta - \frac{1}{2}\right) + 16str_T\left(\delta - \frac{1}{2}\right)^2 \tag{27}$$

where

$$\delta = e^{-r_d T}\Phi\left[\frac{\ln(S/K) + (r_d - r_f + \sigma^2/2)T}{\sigma\sqrt{T}}\right] \tag{28}$$

Expression (28) corresponds to the delta of a call option and is used in the Garman-Kohlhagen model. The quadratic approximation to the volatility smile (27) includes three different volatilities: i) σ, corresponding to the implied volatility of an at-the-money option (delta = 50); ii) risk reversal (rr), the difference in volatilities between a long out-of-the-money call option and a short out-of-the-money put option (delta = 25); and iii) strangle (str), the average of the volatility of two long out-of-the-money call and put options (delta = 25) minus the volatility of the at-the-money option. The volatility obtained by this expression can then be used in the Garman-Kohlhagen model to calculate the option price. We considered an annual risk free rate of 3.32% for the US investor (based on LIBOR annual rates for the same period).

We have run all of the three models — minimum risk, robust and hedging — over the period considered, rebalancing the portfolio every month and measuring the cumulative gains for different values of the parameters ω and ρ. While the minimum risk model yields an average annual return of 2.8%, the robust model consistently yields a higher return, from 5.7% ($\omega = 80\%$) to 3.9% ($\omega = 30\%$). As the uncertainty set increases the average returns move closer to the values exhibited by the minimum risk model.

Figure 3 depicts the accumulated wealth when optimizing the portfolio with the three different models, taking $\omega = 80\%$ and $\rho = 50\%$. For this particular parameter choice, the minimum risk model is dominated by both the robust and the hedging model, while the hedging model clearly outperforms the robust model, with average annual returns of 14.5% and 5.7% respectively. The hedging model (26) outperforms the robust model (10) for smaller values of the parameter ρ. Without any restriction on the minimum return guarantees outside the uncertainty set (23c), we may choose expensive, "deep-in-the-money" options,

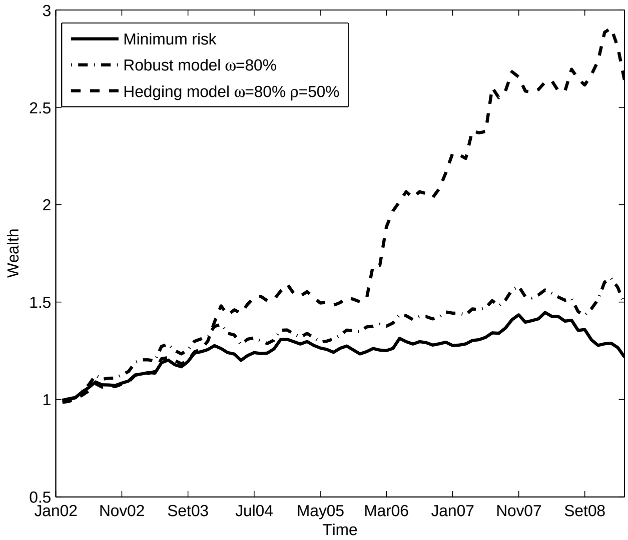


Figure 3: Accumulated wealth over the period from Jan02 to Mar09

although only a small number of units. These options will be exercised with high probabillity and yield a high return per unit as well. In contrast, as we impose a higher restriction on the minimum return, that, is, as ρ increases (23c), we also choose options less expensive (i.e.,more "at-the-money") to be able to buy the necessary number of units to satisfy the constraint. These options will have a 50% chance of being exercised and therefore returns are potentially lower. Table (3) illustrates this relationship. Note that the high returns yielded by some of the models are mainly in the same period where most of the currencies suffered severe losses, that is, from March 2006 onwards. Options may have played an important role in this period in protecting the portfolio from depreciations of the foreign exchange rates.

5 Conclusion

In this paper, we apply robust optimization techniques to a currency only portfolio. We show that, due to the triangular relationship between foreign exchange rates, a new non-arbitrage constraint must be added to the model, which seemingly renders the model non-convex. Given that the cross exchange rates do not have an impact on portfolio return, we may simplify the triangulation constraint by eliminating the variables referring to the cross exchange rates and obtain a set of linear constraints. We further extend the robust model to include currency options as a hedging instrument. We rely on put options to guarantee a minimum value of the foreign exchange rates and therefore providing a "cap" to the worst-case portfolio return. The resulting model provides the investor with two different sets of complementary guarantees: i) robust optimization provides a *non-inferiority* guarantee as long as the realized currency returns are within

ω (%)	ρ (%)	Annual Ret. (%)	ω (%)	ρ (%)	Annual Ret. (%)
50	0	11.4	70	0	15.2
	10	17.1		10	19.3
	20	24.6		20	21.4
	30	17.1		30	29.9
	40	11.9		40	22.9
	50	8.5		50	12.8
	60	4.9		60	12.5
	70	3.2		70	3.9
60	0	15.5	80	0	15.1
	10	11.4		10	14.9
	20	22.8		20	25.3
	30	30.5		30	22.2
	40	13.0		40	13.8
	50	11.7		50	14.5
	60	5.7		60	8.4
	70	5.3		70	2.7

Table 3: Average annual return rates for different values of the parameters ω and ρ

the uncertainty set; ii) put options limit the portfolio losses by "stopping" the depreciation of the foreign exchange up to the value of the strike price.

The suggested approach to the hedging problem has the advantage of being more flexible than the standard hedging strategies, as it relies on options and robust optimization and not on forwards or futures. The backtesting experiment conducted with real market data seems to point towards the overall better performance of the robust and of the hedging model when compared to the Markowitz minimum risk model. Moreover, we observe that when the imposition on the guaranteed portfolio return for the entire support of the currency returns is not too restrictive, the hedging model outperforms the robust model.

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