Maximal outerplanar graphs as chordal graphs, path-neighborhood graphs, and triangle graphs

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Abstract

Maximal outerplanar graphs are characterized using three different classes of graphs. A path-neighborhood graph is a connected graph in which every neighborhood induces a path. The triangle graph $T(G)$ has the triangles of the graph G as its vertices, two of these being adjacent whenever as triangles in G they share an edge. A graph is edge-triangular if every edge is in at least one triangle. The main results can be summarized as follows: the class of maximal outerplanar graphs is precisely the intersection of any of the two following classes: the chordal graphs, the path-neighborhood graphs, the edge-triangular graphs having a tree as triangle graph.

Keywords: maximal outerplanar graph, chordal graph, triangle graph, path-neighborhood graph, elimination ordering

1 Introduction

An outerplanar graph is a planar graph that has a plane embedding such that all vertices lie on the outer cycle. A maximal outerplanar graph is an outerplanar graph such that the number of edges is maximum. Another way to view a maximal outerplanar graph is that it is the triangulation of a plane cycle. Because of their simple and nice structure maximal outerplanar graphs, also known as MOP's, have attracted much attention in the literature. Many structural and computational results are available, for a selection of the literature see [15, 2, 10]. This paper involves characterizations of maximal outerplanar graphs in terms of three different classes of graphs. The first class is that of the *chordal graphs*. They are well-known and well studied: a graph is chordal if it does not contain an induced cycle of length at least 4. They were introduced as rigid circuit graphs by Dirac [5], who gave the fundamental characterization that chordal graphs are precisely the graphs admitting a simplicial elimination ordering. The second class is that of the 'path-neighborhood graphs', introduced in [12]: a *path-neighborhood graph* is a connected graph in which every neighborhood induces a path. The third class involves triangle graphs, introduced by [17], see also [6, 1, 13]. Let G be a graph. The *triangle graph* $T(G)$ of G has the triangles of G as its vertices, and two vertices of $T(G)$ are adjacent whenever as triangles in G they share an edge. The third class we have in mind is the class of graphs G such that every edge of G is in a triangle and $T(G)$ is a tree. The main results of this paper can be summarized as follows. The maximal outerplanar graphs form a proper subclass of each of these three classes, but the intersection of any two of these three classes consists precisely of the class of maximal outerplanar graphs.

2 Maximal outerplanar graphs and three classes

An *outerplanar graph* is a planar graph that allows an embedding in the plane such that all vertices are on the outer face. In the sequel we will always assume that such a plane embedding is given. A *maximal outerplanar graph* is an outerplanar graph with a maximum number of edges. In the plane embedding the boundary of the outer face is then a hamiltonian cycle. All other edges form a triangulation of this outer cycle. Outerplanar graphs occur for the first time in the literature in Harary's classical book [9]. The following theorem appeared in [14]. Its proof is an easy exercise: maximality implies that each induced cycle is a triangle.

Theorem 1 *Let G be a maximal outerplanar graph with its plane embedding, and let v be any vertex. Then the neighborhood of v consists of a path* $v_1 \rightarrow v_2 \rightarrow$

$\ldots \rightarrow v_k$ and the edges vv_1 and vv_k are on the outer face, whereas the edges $vv_2, vv_3, \ldots vv_k$ are interior edges.

A *path-neighborhood graph* is a connected graph in which the neighborhood of each vertex induces a path. Path-neighborhood graphs were introduced in [12] The *k-fan*, or simply *fan*, F_k is the graph consisting of a $(k+2)$-cycle with $k-1$ extra edges sharing a common end, that is, it consists of a P_{k+1} with an extra vertex adjacent to all vertices of the P_{k+1}. In a path-neighborhood graph each vertex together with its neighbors induces a fan. Theorem 1 says that all maximal outerplanar graphs are path-neighborhood graphs.

Let $\mathcal{P}$ be a property of a vertex in a graph $G = (V, E)$ of order $|V| = n$. A $\mathcal{P}$-*elimination ordering* of G is an ordering $v_1, v_2, \ldots, v_n$ of V such that v_i has property $\mathcal{P}$ in the subgraph of G induced by $v_i, v_{i+1}, \ldots, v_n$, for $i = 1, 2, \ldots n-1$. For instance, a *simplicial vertex* is a vertex, the neighborhood of which is a clique. Then a *simplicial elimination ordering* of G is a $\mathcal{P}$-elimination ordering in which property $\mathcal{P}$ is "being simplicial". Similarly, a *degree-two elimination ordering* of G is a $\mathcal{P}$-elimination ordering in which property $\mathcal{P}$ is "having degree 2". With such a property we have to amend: in the last step of the elimination order v_{n-1} does not have degree 2 anymore for obvious reasons: there are only two vertices left. So now we require that v_{n-1} has degree as close to 2 as possible, so $v_{n-1}v_n$ is an edge. Note that a graph admitting a degree-two elimination ordering necessarily is connected.

A *chordal graph* is a graph without induced cycles of length at least 4. In 1961 Dirac [5] proved the classical result that a graph is chordal if and only if it admits a simplicial elimination ordering. For more information on chordal graphs see [8].

As observed above, each induced cycle in a maximal outerplanar graph is a triangle. This implies the well-known fact that a maximal outerplanar graph is chordal. Furthermore, any simplicial vertex having degree 3 or more will be part of a K_4 subgraph, a well-known obstruction for outerplanar graphs, see [4]. In light of these considerations, we can formulate another result, which is part of folklore.

Theorem 2 *A maximal outerplanar graph is necessarily chordal, admitting a simplicial elimination ordering which is at the same time a degree-two elimination ordering.*

Let us call an elimination ordering that is both simplicial and degree-two a *triangle elimination ordering*. This seems appropriate because by eliminating such a vertex we destroy a triangle by deleting a vertex from the triangle that has no neighbors outside the triangle. Closely related is the concept of a *2-tree*, see [16], namely, a graph constructed by beginning with K_2 and at each iteration adding a new vertex v, joining it to two existing, adjacent vertices, thereby forming a new triangle. While it is not essential to our main results, we next note, in passing, a strengthening of the above result. In essence, this result is proved earlier, from an algorithmic point of view, and within the proof of a different result, see [16]. We offer our version for clarity and brevity.

Theorem 3 *A graph is maximal outerplanar if and only if it admits a triangle elimination ordering and does not contain a* $K_{1,1,3}$.

Proof. Let G be a maximal outerplanar graph. Then obviously G does not contain a $K_{1,1,3}$: the obstruction $K_{2,3}$ is a subgraph of $K_{1,1,3}$. The existence of the elimination ordering follows from Theorem 2.

Conversely, let G be $K_{1,1,3}$-free having a triangle elimination scheme. We proceed by induction on the number n of vertices. For $n \leq 3$ it is obvious that G is maximal outerplanar. So let $n \geq 4$, and let v be a simplicial vertex of degree 2 in G. Let x and y be the neighbors of v, so that xy is an edge. By induction $G - v$ is maximal outerplanar. It suffices to prove that xy is on the outer-cycle in a plane embedding of $G - v$. Assume to the contrary that xy is a chord of the outer-cycle. Then we can find vertices p and q such that x, y, p and x, y, q induce triangles on different sides of xy in the plane embedding of $G - v$. But now v, x, y, p, q induce a $K_{1,1,3}$ in G. This impossibility completes the proof. □ □ □

As observed a maximal outerplanar graph is a chordal graph as well as a path-neighborhood graph. The following result states that together these properties suffice. A related (but not identical) result, with a somewhat longer proof, is that G is maximal outerplanar if and only if G is a path-neighborhood graph that is '2-degenerate,' namely, every subgraph of G has a vertex of degree 2 or less, see [11].

Theorem 4 *A graph G is a chordal path-neighborhood graph if and only if G is a maximal outerplanar graph.*

Proof. Theorems 1 and 2 tell us that a maximal outerplanar graph is a chordal path-neighborhood graph.

We prove the converse by induction on the number of vertices n. For $n \leq 3$, the theorem is trivial. So assume that $n \geq 4$. Let v be the first vertex in a simplicial elimination ordering, making v simplicial and $G - v$ still chordal. Since the neighborhood of v is both an induced path and an induced clique, it must be an edge xy. Then $N(x)$ induces the path $P_x = v \to y \to \ \dots$, and $N(y)$ induces the path $P_y = v \to x \to \ \dots$. So in $G - v$ the neighborhood of x induces the path

$P_v - v = y \rightarrow \ldots$, and the neighborhood of y induces the path $P_y - v = x \rightarrow \ldots$. Hence $G - v$ is again a chordal, path-neighborhood graph, so that, by induction, $G - v$ is a maximal outerplanar graph. By Theorem 1, the edge xy is on its outer cycle. This implies that G is also maximal outerplanar. □ □ □

The *triangle graph* $T(G)$ of a graph G is the graph with the triangles of G as vertices, and two such vertices are joined in $T(G)$ if, as triangles in G, they share an edge. Triangle graphs were first introduced in a different context by Pullman [17]. They were introduced later independently a couple of times, see e.g. [6, 1, 13]. The 3-*sun* consists of a 6-cycle with three chords that form a triangle. It is sometimes also called a *trampoline* or *Hajós graph*. The following facts follow easily from the definitions.

Fact 5 *An induced $K_{1,3}$ in $T(G)$ comes from a 3-sun in G that is not necessarily induced.*

Fact 6 *$K_{1,4}$ does not occur in $T(G)$ as an induced subgraph.*

From the viewpoint of constructing the triangle-graph $T(G)$ of a graph G, any vertex or edge in G not contained in a triangle is irrelevant, and may be deleted. Therefore we restrict ourselves to graphs in which every edge is contained in a triangle. We call such a graph *edge-triangular*, for want of a better term.

Let G be an edge-triangular graph, and let u be a vertex of G. Its neighborhood $N(u)$ consists of all neighbors of u. Assume that $N(u)$ induces a disconnected graph, consisting say of two disjoint subgraphs N_1 and N_2 with no edge joining N_1 and N_2. Let H be the graph obtained from G by replacing u in G by two new vertices u_1, u_2 and joining u_i to all vertices in N_i, for $i = 1, 2$. Note that in H the distance between u_1 and u_2 is at least 4. We say that H is obtained from G by *splitting* u. Clearly, we have $T(G) \cong T(H)$. By successive splittings we can get an edge-triangular graph $\widehat{G}$ from G in which all neighborhoods are connected. In fact, as we show in the next result, $\widehat{G}$ is independent of the order of the splittings and is hence unique. Clearly we have $T(G) \cong T(\widehat{G})$.

Proposition 7 *The splitting operation on an edge-triangular graph is order independent.*

Proof. It suffices to show that any splitting operation preserves the connected components of neighborhoods for all vertices.

Assume to the contrary that in splitting a vertex u, replacing it with u_1 and u_2, some other vertex w has adjacent vertices x and y which were in the same connected component of $N(w)$ prior to the split, but afterwards are in different ones. But then, prior to the split, x was connected to y via a path in $N(w)$. This path was destroyed in splitting u. So this path must have contained u. No edges are destroyed in splitting u. So u_1w and u_2w both are edges after u is split, which is impossible. □ □ □

By the above proposition, for any edge triangular graph G, it makes sense to define $\widehat{G}$ to be the unique graph obtained from G by successive splittings and having all its neighborhoods connected. On the class of connected edge-triangular graphs we may also define the relation $\sim$ by $G \sim H$ if $\widehat{H} \cong \widehat{G}$. The relation $\sim$ is clearly an equivalence relation: two equivalent graphs have the same triangle-graph; and any equivalence class contains a unique graph with connected neighborhoods, viz. $\widehat{G}$, for any graph G in the class.

Let G be a maximal outerplanar graph. Then it is easy to see that $T(G)$ is a tree of maximum degree 3. It can also be obtained from the dual graph of G by deleting the vertex that represents the outer face of G. This graph is the so-called *weak dual* of G, see [7]. It can be obtained from G as follows: the interior faces of G are the vertices of the weak dual G^*, two vertices in G^* being adjacent whenever as faces in G they share an edge in there boundaries. The weak dual was used in [2] to construct recognition algorithms for outerplanar graphs. And it was used in [10] to study maximal outerplanar graphs and their interior graphs: the graph obtained by deleting the edges on the exterior face of the maximal outerplanar graph.

Theorem 8 *Let G be a edge-triangular graph. Then $T(G)$ is a tree if and only if $\widehat{G}$ is a maximal outerplanar graph.*

Proof. As observed above, if $\widehat{G}$ is a maximal outerplanar graph, then $T(G) \cong T(\widehat{G})$ is a tree.

Conversely, let G be an edge-triangular graph such that $T(G)$ is a tree. Recall that $T(G) \cong T(\widehat{G})$. In $\widehat{G}$ all neighborhoods are connected. We prove by induction on the number of vertices n of $T(\widehat{G})$ that $\widehat{G}$ is a maximal outerplanar graph. First note that, by Fact 6, the maximum degree in $T(\widehat{G})$ is 3.

If $n = 1$, then $\widehat{G}$ is a triangle, and we are done. So assume that $n \geq 2$. Then $T(\widehat{G})$ contains a pendant vertex x adjacent to a vertex y. Then y is a vertex of degree 1, 2, or 3. So in the tree $T(\widehat{G}) - x$ it is a vertex of degree 0, 1, or 2. Let x represent the triangle in $\widehat{G}$ on a, b, c, and let y represent the triangle in $\widehat{G}$ on b, c, d. Then the edges ab and ac in $\widehat{G}$ are not contained in any other triangle, so a is a vertex of degree 2 in $\widehat{G}$. Moreover, edge bc is contained only in the triangles representing x and y. So in $\widehat{G} - a$ edge bc is contained in a unique triangle. Now

$T(\widehat{G}) - x$ is the triangle-graph of $\widehat{G} - a$. Moreover, $\widehat{G}$ being an edge-triangular graph with connected neighborhoods, it follows that $\widehat{G} - a$ is again such. So, by induction, $\widehat{G} - a$ is a maximal outerplanar graph. Since edge bc is in a unique triangle in $\widehat{G} - a$, it must be on the outer face of $\widehat{G} - a$. Hence, if we add a back on, then $\widehat{G}$ remains a maximal outerplanar graph. □ □ □

A *snake* is a maximal outerplanar graph in which every triangle shares an edge with the outer face. Otherwise formulated, it is a 3-sun-free maximal outerplanar graph. Clearly, its triangle-graph is a path.

Corollary 9 *Let G be a edge-triangular graph. Then $T(G)$ is a path if and only if $\widehat{G}$ is a snake.*

The following new characterization of maximal outerplanar graphs is an easy consequence of Theorem 8.

Theorem 10 *A graph G is a path-neighborhood graph with a tree as its triangle-graph if and only if G is a maximal outerplanar graph.*

Proof. If G is a path-neighborhood graph, then $\widehat{G} = G$, and Theorem 8 tells us that, $T(G)$ being a tree, G is maximal outerplanar.

The converse follows from Theorems 8 and 1. □ □ □

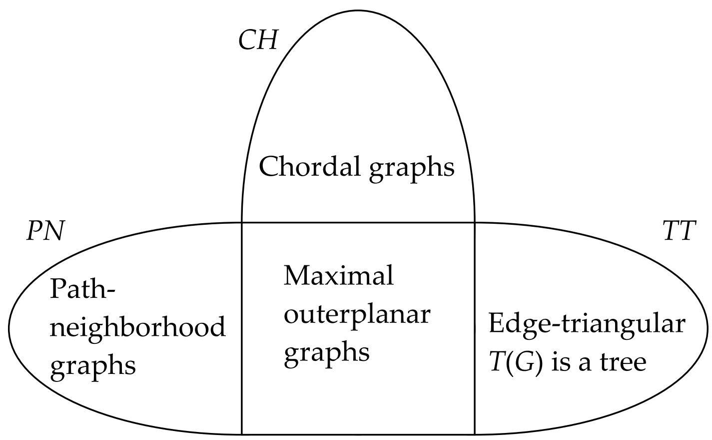
Finally, we consider the intersection of the class of chordal graphs and that of the graphs with a tree as triangle graph.

Theorem 11 *A graph G is a chordal graph with a tree as its triangle-graph if and only if G is a maximal outerplanar graph.*

Proof. Let G be a maximal outerplanar graph. Then it follows from Theorems 4 and 8 that G is chordal and has a tree as its triangle-graph.

Conversely, let G be a chordal graph with $T(G)$ a tree. Then, $T(G)$ being a tree, K_4 and $K_{1,1,3}$ do not occur in G. Hence a simplicial vertex in G is of degree 2, and the triangle containing the simplicial vertex is a pendant vertex in $T(G)$. We use induction on the number n of vertices. For $n \leq 3$ the assertion is trivial. So assume that $n \geq 4$, and let v be a simplicial vertex of G with neighbors x and y. Then v, x, y form a triangle representing by a pendant vertex p in $T(G)$. Let its neighbor q in $T(G)$ represent the triangle in G on x, y, z. Now $G - v$ is a chordal graph with $T(G) - p$ as its triangle graph. So, by induction, $G - v$ is maximal outerplanar. If edge xy is on the outer face of $G - v$, then G is outerplanar as well. Consider the neighborhood $N(x)$ of x in $G - v$. Since $G - v$ is maximal outerplanar, $N(x)$ induces a path P_x. If y would be internal vertex of this path, say, with neighbors z and w, then v, x, y, z, w would produce a $K_{1,1,3}$ in G. So y is not an internal vertex of P_x. So, by Theorem 1, the edge xy is on the outer face, and we are done. □ □ □

3 Concluding Remarks



$$MOP = PN \cap CH = PN \cap TT = CH \cap TT$$

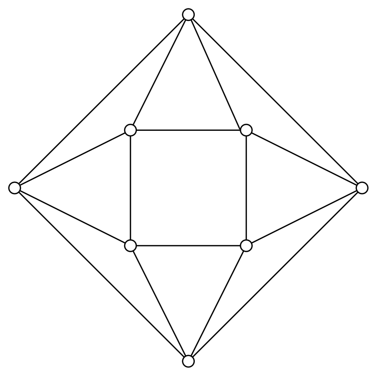
Figure 1: Maximal outerplanar graphs and the three classes

In this paper we have considered three classes of graphs, namely, chordal graphs, path-neighborhood graphs, and edge-triangular graphs having a tree as triangle-graph. Note that, by definition, a path-neighborhood is connected. Moreover, if a graph G is edge-triangular, then its triangle graph being connected implies that G itself is also connected. In the previous sections we have proved that the intersection of any two of these classes constitutes precisely the class of maximal outerplanar graphs. This is depicted in Figure 1. As part of our concluding remarks, we present examples that show that the class of maximal outerplanar graphs is properly contained in each of the three classes.

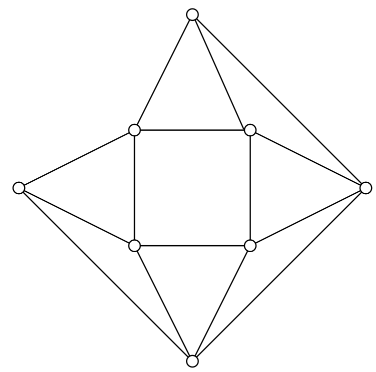
Any complete graph with more than three vertices is chordal but not a path-neighborhood graph and its triangle-graph contains a K_4.

The triangulated band Z_n in [12] is a path-neighborhood graph, but it is not chordal, and its triangle-graph is the n-cycle. For a picture of Z_4 see Figure 2.

Take a snake, in which the vertices of degree 2 are at distance at least 4. Glue the two vertices of degree 2 together. This is not a chordal graph and not a path-neighborhood graph, but its triangle-graph is a path. See Figure 2 for an example.



The triangulated band Z_4



$T(G) = P_7$

Figure 2:

There is a rich and still growing literature on chordal graphs. The path-neighborhood graphs and the triangle graphs are not as well studied. It seems that here there are still many interesting open problems.

Jointly with chordal graphs, there is a rich literature on simplicial elimination orderings. But there are many more interesting elimination orderings, which will yield many interesting problems. For instance, 'find a nice characterization of the graphs admitting degree-two elimination orderings,' to mention but one.

In [3] the existence was shown of so-called 3-simplicial vertices in planar graphs: a vertex is 3-*simplicial* if its neighborhood can be edge-covered by at most three cliques. Here the analogue for outerplanar graphs is rather trivial. An outerplanar graph always contains a vertex of degree at most two, which is 2-simplicial, if it has two non-adjacent neighbors, and otherwise it is simplicial.

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