MEAN-DISPERSION PREFERENCES AND CONSTANT ABSOLUTE UNCERTAINTY AVERSION

By

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June 2011

COWLES FOUNDATION DISCUSSION PAPER NO. 1805

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
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7 June 2011

Mean-Dispersion Preferences and Constant Absolute Uncertainty Aversion.*

Abstract

We axiomatize, in an Anscombe-Aumann framework, the class of preferences that admit a representation of the form $V(f) = \mu - \rho(d)$, where μ is the mean utility of the act f with respect to a given probability, d is the vector of state-by-state utility deviations from the mean, and $\rho(d)$ is a measure of (aversion to) dispersion that corresponds to an uncertainty premium. The key feature of these *mean-dispersion* preferences is that they exhibit constant absolute uncertainty aversion. This class includes many well-known models of preferences from the literature on ambiguity. We show what properties of the dispersion function $\rho(\cdot)$ correspond to known models, to probabilistic sophistication, and to some new notions of uncertainty aversion.

Keywords: ambiguity aversion, translation invariance, dispersion, uncertainty, probabilistic sophistication

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*This paper subsumes parts of two earlier papers, "Generalized Variational Preferences" (2007) and "Mean-Dispersion Preferences" (2008) which also dealt with the case of non-constant uncertainty aversion. We thank Don Brown, Simone Cerreia-Vioglio, Bob Chambers, Rose-Anne Dana, Eddie Dekel, Jurgen Eichberger, Tzachi Gilboa, David Kelsey, Ali Khan, Peter Klibanoff, Fabio Maccheroni, Massimo Marinacci, Boris Mordukhovich, Sujoy Mukerji, John Quiggin, Aldo Rustichini, Zvi Safra, Chris Shannon, Marciano Siniscalchi, Tomas Strzalecki, Peter Wakker, and Joel Watson.

1 Introduction

In this paper, we consider an agent who dislikes uncertainty in the sense that she dislikes variation in utility across states. Specifically, we investigate preferences over Anscombe-Aumann acts that fit the following model

$$V(f) = \mu(f, \pi) - \rho\left(d\left(f, \pi\right)\right),$$

$\mu\left(f, \pi\right) := E_\pi\left(U \circ f\right)$ is the mean utility of the act f with respect to a probability π, U is a von-Neumann Morgenstern expected utility function, and $d\left(f, \pi\right)$ is the vector of deviations from the mean given by $d_s := U\left(f\left(s\right)\right) - \mu\left(f, \pi\right)$. Where no confusion arises, we will omit the dependence of μ and d on f and π.

The interpretation is that the agent with these preferences dislikes dispersion. Similar to the subjective expected utility agent, she forms a von Neumann-Morgenstern expected-utility function U to evaluate lotteries, forms a prior π and calculates the mean (expected) utility μ of the act according to this prior. But she then deducts from this mean an amount $\rho\left(d\right)$ that depends only on the state-by-state deviations d_s from that mean. We can think of the function ρ as an index of dispersion aversion if $\rho \geq 0$ or dispersion loving if $\rho < 0$. More specifically, for each act f, let x_f be a certainty equivalent, a constant act such that $x_f \sim f$. Then the dispersion measure $\rho\left(d\right)$ for the act f is given by $\mu - U\left(x_f\right)$, the utility difference between the mean and the certainty equivalent. It is the reduction in expected utility the agent would be willing to accept in return for removing all the state-contingent utility uncertainty associated with the act. Drawing an analogy from risk premiums, we can think of $\rho\left(d\right)$ as an absolute uncertainty premium. We call this model *mean-dispersion preferences.*

The standard mean-variance model (expressed in terms of utilities) is an example of such preferences. But we could think of other candidate dispersion measures (standard deviation, mean absolute deviation, generalized Gini indices, etc.) each expressing a different attitude toward dispersion.[1] What all these models have in common is that the uncertainty premium $\rho\left(d\right)$ does not depend on the mean level of utility. In this sense, these models all display *constant absolute*

[1] All these examples are (second-order) probabilistically sophisticated but, as we will see below, mean-dispersion preferences need not be.

uncertainty aversion.

Constant uncertainty aversion might be considered as strong a restriction as is constant risk aversion. As agents become better off overall, we might perhaps expect them to be willing to pay less to remove all uncertainty. Several recent models move away from assuming constant uncertainty aversion, for example Klibanoff et al. (2005), Cerreia-Vioglio et al (forthcoming) and the companion to this paper, Grant et al (2011). But much of the literature at least implicitly assumes constant uncertainty aversion.

Here we axiomatize the entire class of such preferences. This is theorem 1 below. To allow for uncertainty aversion, in common with much of the literature, we drop the independence axiom and replace it with something weaker. In our case, as the discussion above suggests, the key axiom is the analog of assuming constant absolute risk aversion: if a random variable is preferred to a sure outcome then adding the same constant to both maintains the preference. In the context of Anscombe-Aumann acts, the analogous axiom needs to be expressed in terms of mixtures, but the idea is the same. The resulting axiom is very closely related to the key axiom in Maccheroni et al (2006a)'s variational preference model, and indeed we will show that variational preferences are a special case of mean-dispersion preferences.[2]

Many models of ambiguity aversion suggest the appealing interpretation that the agent is unsure as to the true prior and is either cautious or thinks she is playing a game against a malevolent nature. For example, in the multiple-prior model of Gilboa and Schmeidler (1989), the agent evaluates an act according to the criterion

$$V(f) = \min_{p \in C} \sum_s U(f(s))\, p_s$$

where C is a convex subset of possible priors, inviting the interpretation that these are the probabilities that the agent thinks possible and that nature picks the least favorable of these for the given act. In the more general variational preference model, the agent evaluates an act according to the criterion

$$V(f) = \min_{p \in \Delta} \left(\sum_s U(f(s))\, p_s + c(p) \right)$$

[2] For full disclosure, we should say that we were inspired by and started from the case of variational preferences.

where c is a convex, grounded function, inviting the interpretation that if nature picks a less plausible prior p then she must pay a larger compensation $c(p)$ back to the agent.

Our model invites the more prosaic interpretation that the agent simply dislikes variation in utility across states. The shape of the function ρ captures the degrees to which particular forms of dispersion cause disutility to the agent. Thus variance reflects one way in which dispersion harms the agent, and generalized Gini indices reflect others. The shape of the dispersion function may have nothing to do with indeterminacies in the agent's underlying beliefs. In fact, in both these examples the agent is (second-order) probabilistically sophisticated and, as such, might be thought to have no indeterminacy in belief.

But the shape of the dispersion function might reflect belief indeterminacy and hence capture behavior commonly associated with ambiguity. States need not be treated symmetrically. For example, in the Ellsberg single-urn experiment where the agent knows that a third of the balls are red but does not know the proportions of blue and green balls, the mean-dispersion agent might choose a uniform π to calculate the mean utility, and then not care at all about dispersion across the events 'red' or 'not-red' holding the mean fixed, but greatly dislike dispersion across 'blue' and 'not blue'.[3]

Since the dispersion function ρ can be quite general, the mean-dispersion model can accommodate a wide variety of preferences as long they exhibit constant absolute uncertainty aversion. In particular, many known classes of preferences that admit representations with appealing ambiguity interpretations also admit mean-dispersion representations. For example, the multiple-prior, Choquet expected utility, invariant biseparable, variational preference and vector expected-utility models each exhibit constant absolute uncertainty aversion and hence each admits a mean-dispersion representation.[4] In section 3, we show what properties of ρ correspond to each of the preferences listed above and to what axioms on the underlying preferences.

[3] Slightly more formally, $\pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and

$$\rho(d_R, d_B, d_G) := \frac{1}{3} |d_B - d_G| .$$

These preferences are also an example of Siniscalchi's (2009) VEU model. The general relationship is given in corrolary 3 below.

[4] See respectively: Gilboa and Schmeidler (1989); Schmeidler (1989); Ghirardato et al (2004); Maccheroni et al (2006); and Siniscalchi (2009). The latter model is already expressed as a mean and an 'adjustment'.

The fact that these classes of preferences concerning ambiguity can also be represented as mean-dispersion preferences raises issues of interpretation. To recall, one natural cognitive interpretation of multiple-prior preferences is that the agent is unsure as to the true prior and so, cautiously, always evaluates acts according to the least favorable prior that she considers possible. The corresponding mean-dispersion representation suggests another cognitive interpretation of the same preferences: the agent has just one prior but is simply averse to dispersion across states. Her dispersion function ρ just happens to take a particular form that corresponds to multiple priors. That the same data (i.e., preferences) can be explained by two different models that invite different interpretations is perhaps less of a concern if the mean-dispersion model that corresponds to a particular ambiguity model involves a rather baroque or implausible dispersion function. For example, in the dispersion function we sketched above to explain the Ellsberg experiment, it seems implausible that dislike of dispersion would have this particular shape unless it reflected something about underlying ambiguity about the number of green balls. But the restrictions on ρ that correspond to the general models above are relatively natural. For example, the mean-dispersion representations that correspond to general multiple-prior preferences are those with ρ restricted to be non-negative, convex and linear homogeneous; properties shared, for example, by dispersion indices like mean-absolute deviation, standard deviation, the absolute Gini deviation, and all of the 'generalized deviations' introduced into finance by Rockafellar et al. (2006). We present these and other examples in section 4.

A problem with the general mean-dispersion model axiomatized in theorem 1 is that, without putting more structure on the shape of the dispersion functions ρ, it is not possible to uniquely identify the (single) prior π that the agent is using to calculate the mean. We show in section 5, however, that as we add more structure on ρ, some identification of π is possible. In some cases, we can identify π uniquely; for example, when preferences are dispersion averse (i.e., $\rho \geq 0$) and smooth around certainty, and we provide conditions on the underlying preferences to ensure such smoothness.[5]

[5] Another case in which π is uniquely identified is if we impose that ρ is symmetric and hence corresponds to Siniscalchi's (2009) vector expected utility model.

In other cases, we get set identification; for example, when mean-dispersion preferences correspond to the multiple-prior model. Recall that in the multiple-prior model, we might interpret the C as the set of priors which the agent thinks possible. From observing the agent's preferences, if we commit to the multi-prior model, we can uniquely identify the set C. If instead we commit to the corresponding mean-dispersion model (i.e., one with ρ non-negative, convex and linear homogeneous), then we can no longer uniquely identify the single π but we know it lies in the *same* set C. In one model, we point identify the set. In the other, we set identify the point. Put another way, both representations involve indeterminacy about beliefs. In one, the indeterminacy is in the head of the agent. In the other, the indeterminacy is in the head of the econometrician.

One way to construct a mean-dispersion model is to take ρ to be a known measure of dispersion from statistics. Models constructed in this way are (second-order) probabilistically sophisticated. In section 6, we state conditions on the underlying preferences that correspond to probabilistic sophistication in a general mean-dispersion model. These conditions are analogous to those Maccheroni et al (2006a) find for variational preferences. Theirs is an invariance condition on their compensation function c. Ours is an invariance condition on our dispersion function ρ.

Finally, in section 7, we consider various notions of uncertainty or dispersion aversion. We already examined some properties of ρ in section 3 that corresponded to known models in the literature on ambiguity. But if we just think of ρ as a measure of dislike of dispersion then other properties seem natural. If the agent at least has a preference for hedges that hold mean utility constant then ρ needs to be quasi-convex. If inflating all dispersions by a common scalar (holding the mean constant) makes the agent worse off then ρ needs to be ray-monotone. We compare these two notions with Schmeidler's notion of uncertainty aversion, with Chateauneuf and Tallon's (2002) and Ghirardato and Marinacci's (2002) weaker notion and with two notions of Ergin and Gul (2009), second-order risk aversion and issue preference.

2 Set Up

We work in a standard Anscombe-Aumann setting. Let $S = \{s_1, \ldots, s_n\}$ be a finite set of states. Let $\Delta(S)$ be the set of probability vectors on S. Let X be the set of simple probability measures (or *lotteries*) on a set of prizes. An *act* is a function $f : S \to X$. Let $\mathcal{F}$ denote the set of acts. With slight abuse of notation, let X also denote the set of *constant acts*; that is, $x \in X$ is also the act that yields the lottery x in every state s in S. Hence, for any act f in $\mathcal{F}$ and any fixed state s' in S, $f(s')$ denotes both the lottery yielded by f in state s' and the constant act that yields the fixed lottery $f(s')$ in every state s in S. Both the sets X and $\mathcal{F}$ are mixture spaces. In particular, for any pair of acts f and g in $\mathcal{F}$, and any α in $(0,1)$, take $\alpha f + (1-\alpha) g$ to be the act $h \in \mathcal{F}$ in which $h(s) = \alpha f(s) + (1-\alpha) g(s)$, for each s in S.

The decision maker's preferences on $\mathcal{F}$ are given by a binary relation $\succsim$. Let $\succ$ denote the strict preference and $\sim$ denote indifference derived from $\succsim$ in the usual way. For any act $f \in \mathcal{F}$, let $x_f \in X$ be a *certainty equivalent* of f if $x_f \sim f$. For any act $f \in \mathcal{F}$ and any probability vector $\pi \in \Delta(S)$, let $E_\pi(f) \in X$ be *the mean act* of f (with respect to π); that is, the constant act that, in each state, yields the lottery $\sum_s \pi_s f(s)$, the reduction of f using the weights π.

Given a (von Neumann-Morgenstern) expected utility function $U : X \to \mathbb{R}$ on the constant acts, let $U \circ f \in \mathbb{R}^n$ be the state-utility vector given by $(U \circ f)_s = U(f(s))$. Let $e := (1, \ldots, 1) \in \mathbb{R}^n$. We refer to the set $\{ke : k \in \mathbb{R}\}$ as the *certainty line.* In particular, for each constant act x, $U \circ x = U(x)e$. For each probability vector $\pi \in \Delta(S)$ and each state-utility vector $u \in \mathbb{R}^n$, let $E_\pi(u) := \pi \cdot u$ be the expectation of u with respect to π. Notice that $E_\pi(U \circ f) = U(E_\pi(f))$ and hence it can be thought of either as the *mean utility* of the act f (with respect to π) or as the utility of the mean act of f (with respect to π). For each $\lambda \in \mathbb{R}$ let $H_\pi^\lambda := \{u \in \mathbb{R}^n : \pi \cdot u = \lambda\}$ be the hyperplane in state-utility space associated with π and λ: the set of utility vectors that have mean (with respect to π) equal to λ.

Definition A *mean-dispersion representation* is given by a tuple $\langle U, \pi, \rho \rangle$ where $U : X \to \mathbb{R}$ is an expected utility function; $\pi \in \Delta(S)$ is a probability vector; and $\rho : H_\pi^0 \to \mathbb{R}$ is a continuous function with $\rho(0) = 0$, and preferences over acts are given by:

$$V(f) = \mu(f, \pi) - \rho(d(f, \pi)),$$

where $\mu(f, \pi) := E_\pi(U \circ f)$ is the mean utility of the act f with respect to π, and $d(f, \pi)$ is the vector of deviations from the mean given by $d_s := U(f(s)) - \mu(f, \pi)$.

Where no confusion arises, we will omit the dependence of μ and d on f and π. We refer to ρ as a *dispersion function*, and we call preferences *mean-dispersion preferences* if they have a mean-dispersion representation.

The normalization $\rho(0) = 0$ means that the mean-dispersion evaluation of any constant act x is $U(x)$ and hence $V(f) = U(x_f)$. Thus, for all acts f, the associated dispersion $\rho(d)$ is given by $E_\pi(U \circ f) - U(x_f)$, the difference between the mean utility of the act and its certainty-equivalent utility. This motivates thinking of ρ as an uncertainty premium.

For the mean-dispersion preferences $\succsim$ with representation $\langle U, \pi, \rho \rangle$, let $\succsim_u$ be the induced preferences over state-utility vectors; that is, for all u, u' in $\mathbb{R}^n$, $u \succsim_u u'$ if there exists acts f,g $\in \mathcal{F}$ such that $U \circ f = u$, $U \circ g = u'$ and $f \succsim g$. Thus, each mean-dispersion representation $\langle U, \pi, \rho \rangle$ is associated with a function $W : U \circ \mathcal{F} \to \mathbb{R}$ representing the corresponding preferences over state-utility vectors given by $W(u) = E_\pi(u) - \rho(u - (E_\pi(u))e)$. We call a mean-dispersion representation *monotone* if the associated function W is weakly increasing, and *dispersion averse* if $\rho(d) \geq 0$ for all d in H_π^0. We call a mean-dispersion representation *unbounded* if $U \circ \mathcal{F} = \mathbb{R}^n$.

3 Axioms and Main Theorems

A.1 *Order.* $\succsim$ is transitive and complete.

A.2 *Continuity.* For any three acts f, g and h in $\mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$ are closed.

A.3 *Monotonicity.* For any pair of acts f and g in $\mathcal{F}$, if $f(s) \succsim g(s)$ for all $s \in S$ then $f \succsim g$.

A.3* *Substitution.* For any pair of acts f and g in $\mathcal{F}$, if $f(s) \sim g(s)$ for all $s \in S$, then $f \sim g$.

A.4 *Unboundedness.* For any pair of acts f and g in $\mathcal{F}$ and any $\alpha \in (0,1)$, there exist w and z in X satisfying $g \succ \alpha w + (1-\alpha) f$ and $\alpha z + (1-\alpha) g \succ f$.[6]

The first three axioms above are standard. It is usual to use the monotonicity axiom to deliver state independence. However, since there are interesting examples of mean-dispersion preferences

[6] This axiom comes from Kopylov (2007) and is similar to that used by Maccheroni et al (2006a).

that are not monotone (e.g., mean-variance preferences), we also include a weaker substitution axiom A.3* that is sufficient for state independence. Axiom A.4 is not essential to what follows but it simplifies the analysis. It is standard to have a non-degeneracy axiom, but in our case, non-degeneracy is implied by A.4.

The next axiom captures our notion of constant absolute uncertainty aversion. A standard ordinal way to describe risk attitudes is in terms of the random variables that the agent would prefer to a sure outcome. Constant absolute risk aversion says that if we add or subtract the same constant both to a random variable and to a sure outcome to which it is preferred, then the preference is maintained. For heuristic purposes let us abuse notation and write this as: for all random variables $\tilde{X}$ and degenerate random variables (i.e., sure outcomes) x, y and z, if "$\tilde{X} + x \succsim z + x$" then "$\tilde{X} + y \succsim z + y$". The addition operation used here is not well defined in the context of Anscombe-Aumann acts, but we can define an analogous notion using mixtures.

A.5 *Constant absolute uncertainty aversion.* For any act f in $\mathcal{F}$, any three constant acts x,y and z, and any α in $(0,1)$,

$$\alpha f + (1-\alpha)\,x \succsim \alpha z + (1-\alpha)\,x \Rightarrow \alpha f + (1-\alpha)\,y \succsim \alpha z + (1-\alpha)\,y.$$

Axiom A.5 is implied by Anscombe-Aumann's independence axiom, but it is also implied by many weakenings of independence in the literature such as Schmeidler's (1989) comonotonic independence, Gilboa and Schmeidler's (1989) certainty independence, and even Maccheroni et al's (2006a) weak certainty independence. At first glance constant absolute uncertainty aversion appears weaker than Maccheroni et al's axiom: if we replace the constant act z in the statement of axiom A.5 with a general act g then we obtain Maccheroni et al's axiom. But we show in lemma 19 of the appendix that, in the presence of axioms A1, A2, A3* and A.4, constant absolute uncertainty aversion and weak certainty independence are equivalent.

Together with the first four axioms, constant absolute uncertainty aversion characterizes mean-dispersion preferences.

Theorem 1 (Mean-Dispersion) *The preferences $\succsim$ satisfy axioms A.1, A2, A3*, A4 and A.5* *if and only if they admit an unbounded mean-dispersion representation* $\langle U, \pi, \rho \rangle$*. If axiom A.3* is replaced by A.3, then (in addition) the associated function* W *is weakly increasing.*

Moreover, the utility function U *is unique up to affine transformations with appropriate adjustment to* ρ*: i.e., for all* $a > 0$ *and* $b \in R$*, if* $\hat{U}(x) = aU(x) + b$ *and* $\hat{\rho}(ad) = a\rho(d)$*, then* $\langle U, \pi, \rho \rangle$ *and* $\left\langle \hat{U}, \pi, \hat{\rho} \right\rangle$ *represent the same preferences.*

We state theorem 1 first without the monotonicity axiom, A.3, to emphasize that we can accommodate non-monotonic examples such as mean-variance preferences. Monotonicity plays no role in the construction of the mean-dispersion representations per se: its only role is the obvious one, restricting the representations to be monotone. Notice also that no convexity axiom is required.

Although the representation we obtain differs from that in Maccheroni et al (2006a), parts of the proof build upon their arguments. In particular, axiom A.5 (like their weak certainty independence axiom) is used twice. First, it is used to ensure that preferences over constant acts have an affine utility representation U. Given such a U, the underlying preferences $\succsim$ over acts induces preferences $\succsim_u$ over state-utility vectors. (Axiom A.4 allows us to treat the domain of the induced preferences as the whole of $\mathbb{R}^n$.) The second use of axiom A.5 is to ensure that these induced preferences satisfy a translation invariance property. The last step in the proof adapts an argument of Roberts (1980) from social-choice to show that preferences with this translation invariance admit representations with the appropriate form.

An immediate consequence of theorem 1 is that Choquet expected utility preferences, multiple-prior preferences, variational preferences and many others can be thought of as mean-dispersion preferences. The reason is that all of these models implicitly assume constant absolute uncertainty aversion. Below we will provide specific mean-dispersion versions of these models.

Although theorem 1 captures the whole class of mean-dispersion preferences, it is too general to be very useful. In particular, given just these axioms, *any* probability vector $\pi \in \Delta(S)$ is associated with a mean-dispersion representation of the preferences. Moreover, theorem 1 places no restrictions on the dispersion functions ρ except continuity and the fact that $\rho(0) = 0$. Typically, we will be interested in mean-dispersion preferences that at least partially tie down the admissible probabilities and that put more structure on the dispersion functions. These issues are related.

The next result considers four properties we might want to impose on a dispersion function: dispersion aversion (i.e., $\rho \geq 0$), convexity (i.e., $\rho(\alpha d + (1-\alpha) d') \leq \alpha\rho(d) + (1-\alpha)\rho(d')$ for all $\alpha \in [0,1]$), linear homogeneity (i.e., $\rho(\lambda d) = \lambda\rho(d)$ for all $\lambda > 0$), and symmetry (i.e., $\rho(d) = \rho(-d)$). These four properties are related to the following four axioms. The first is due to Chateauneuf and Tallon (2002).

A.6* *Preference for complete hedges.* For any finite set of acts $f_1, \ldots, f_m$ in $\mathcal{F}$ such that $f_1 \sim f_j$ for all $j = 2, \ldots, m$, and any constant act x in X: if the convex combination $a_1 f_1 + \ldots + a_m f_m = x$ then $x \succsim f_1$.

Axiom A.6* is a weakening of Schmeidler's (1989) convexity axiom, which is often taken as the definition of uncertainty aversion:

A.6 *Convexity.* For any pair of acts f and g in $\mathcal{F}$ and α in $(0,1)$, $f \sim g \Rightarrow \alpha f + (1-\alpha) g \succsim f$.

The intuition behind Schmeidler's convexity axiom is that mixing indifferent acts provides a hedge against subjective uncertainty. This property is sometimes called Schmeidler uncertainty aversity. Axiom A.6* only requires this mixing to be preferred if it provides a 'perfect hedge'; that is, if all subjective uncertainty is removed. It does not require all partial hedges to be preferred.[7]

Both axioms A.6 and A6* arose in the context of the Choquet expected utility model. The next axiom is related to Gilboa & Schmeidler's certainty independence axiom, the key axiom in their multiple-prior model.

A.7 *Certainty betweenness.* For any act f in $\mathcal{F}$, any constant act x and any α in $(0,1)$: $f \sim x \Rightarrow \alpha f + (1-\alpha) x \sim x$.

Axiom A.7 says that if the constant act x is a 'certainty equivalent' of the act f then it is also a certainty equivalent of any combination of the act with itself. Gilboa & Schmeidler's certainty independence axiom immediately implies the conjunction of constant absolute uncertainty aversion (axiom A.5) and certainty equivalent betweenness (axiom A.7). We will show in lemma 19 of the

[7] This is analogous to Yaari's notion of weak risk aversion: the expectation of a lottery is weakly preferred to the lottery itself, but the agent is not necessarily averse to all mean-preserving spreads.

appendix that, in the presence of axioms A1, A2, A3*, and A.4, axioms A.5 and A.7 are equivalent to Gilboa & Schmeidler's axiom.

The last axiom is due to Siniscalchi (2009) and is the key axiom in his vector-expected utility (VEU) model. We first need Siniscalchi's definition of complementary acts.

Definition Two acts f and $\bar{f}$ are *complementary* if and only if, for any two states s and s': $\frac{1}{2}f(s) + \frac{1}{2}\bar{f}(s) \sim \frac{1}{2}f(s') + \frac{1}{2}\bar{f}(s')$. If two acts f and $\bar{f}$ are complementary then $(f, \bar{f})$ is referred to as a *complementary pair.*

Given this definition, Siniscalchi introduces the following axiom.

A.8 *Complementary independence.* For any two complementary pairs of acts $(f, \bar{f})$, $(g, \bar{g})$: if $f \succsim \bar{f}$ and $g \succsim \bar{g}$ then $\alpha f + (1-\alpha) g \succsim \alpha \bar{f} + (1-\alpha) \bar{g}$ for all α in $(0, 1)$.

The following theorem identifies the property of the dispersion function ρ that corresponds to each of these known axioms.

Theorem 2 (Taxonomy) *Given axioms A1-A.5, the preferences $\succsim$ admit a monotone unbounded mean-dispersion representation $\langle U, \pi, \rho \rangle$ in which dispersion function ρ is:*

(a) non-negative if and only if the preferences satisfy axiom A.6;*

(b) non-negative and convex if and only if the preferences satisfy axiom A.6;

(c) linear homogenous if and only if the preferences satisfy axiom A.7; and

(d) symmetric if and only if the preferences satisfy axiom A.8.[8]

Notice in part (b) that axiom A.6, by itself, only implies quasi-concavity of the representation, but when combined with constant absolute uncertainty aversion, it is enough to force the dispersion function to be convex.

Theorem 1 already established that many well-known classes of preferences are mean-dispersion preferences. The following corollary connects several such classes to the corresponding restriction on the dispersion function ρ.

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[8] Given theorem 1, part (4) is essentially a corollary of results in Siniscalchi (2009), but we provide a "low-tech" proof exploiting the finite state space.

Corollary 3 *Suppose that axiom A.4 holds. Then:*

(a) If the preferences $\succsim$ *have a (monotone, unbounded) mean-dispersion representation* $\langle U, \pi, \rho \rangle$*, then* ρ *is non-negative if and only if* $\succsim$ *are* 'weakly ambiguity averse' *in the sense of of Ghirardato and Marinacci [2002]).*

(b) Variational preferences *(Maccheroni et al [2006a]) are the subset of (monotone, unbounded) mean-dispersion preferences that admit a representation in which* ρ *non-negative and convex.*

(c) Invariant biseparable preferences *(Ghirardato et al. [2004]) are the subset of (monotone, unbounded) mean-dispersion preferences that admit a representation in which* ρ *is linear homogenous.*

(d) Vector expected utility (VEU) preferences *(Siniscalchi 2009) are the subset of (monotone, unbounded) mean-dispersion preferences that admit a representation in which* ρ *is symmetric.*

(e) Multiple-prior (MEU) preferences *(Gilboa & Schmeidler [1989]) are the subset of (monotone, unbounded) mean-dispersion preferences that admit a representation in which* ρ *is non-negative, convex and linear homogenous.*[9]

(f) Choquet expected utility (CEU) preferences *(Schmeidler [1989]) are the subset of (monotone, unbounded) mean-dispersion preferences that admit a representation in which* ρ *is comonotonically linear; i.e., for all* $\alpha \in [0,1]$ *and all comonotonic* $d, d' \in H_\pi^0$ $\rho(\alpha d + (1-\alpha) d') = \alpha \rho(d) + (1-\alpha) \rho(d')$.

In the proof of theorem 2 we derive the properties of the function ρ directly from the axioms. A consequence of corollary 3 is that we could, instead, have started from a known representation such as variational preferences and then 'solved' for what ρ must be in the corresponding mean-dispersion representation. We use the direct approach in part because it is simple, and in part to emphasize that we do not need to go via the known forms of representation to derive the mean-dispersion forms.

[9] *Given theorem 1, part (e) can also be shown using results in Safra & Segal (1998), and Chambers & Quiggin (1998).*

4 Examples

Some examples may help illustrate the results from section 3. In each of the examples, let U be an unbounded affine utility function.

Example 1 (mean-standard deviation preferences) *Fix* $\pi \in \Delta(S)$*, let* $\bar{S} := \{s \in S : \pi(s) > 0\}$*, and let the mean-dispersion preferences be defined by* $\rho(d) := \sqrt{\tau}\sigma$ *where* $\sigma = \sqrt{\sum_s \pi_s (d_s)^2}$ *is the standard deviation and* $\tau > 0$ *is such that* $\tau < \min_{s \in \bar{S}} (\pi_s / (1 - \pi_s))$.

We show in appendix B that these preferences are monotone. The dispersion function in example 1 is non-negative, convex, linear homogenous and symmetric. By corollary 3 parts (c), (d) and (e), these preferences are an example of VEU preferences that are Schmeidler uncertainty averse, and hence they must also have a multiple-prior (and hence invariant biseparable) representation: see Grant & Kajii (2007) for the corresponding set of priors.

Example 2 (value at risk preferences) *Fix* $\pi \in \Delta(S)$ *and an* $\alpha \in (0, 1)$*, and let the mean-dispersion preferences be defined by* $\rho(d) := VaR_\alpha(d) = -\inf\{t \in \mathbb{R} : \pi(\{s \in S : d(s) \leq t\}) > \alpha\}$.

Value at risk is a standard measure of risk in finance: see, for example, Föllmer and Schied (2004). Notice that $\rho(d)$ is the negative of the (upper) α-quantile of the random variable d under π, hence the preferences given by the negative of value at risk are (weakly) monotone. This dispersion function is linear homogeneous but not in general non-negative, convex or symmetric. Thus, by corollary 3, these preferences are invariant biseperable but not in general variational, VEU, Schmeidler uncertainty averse or even Ghirardato-Marinacci ambiguity averse. In fact, value at risk preferences are comonotonically linear: we show in appendix B that they have a CEU representation given by the capacity $\mu_{VaR_\alpha}(E) = 0$ if $\pi(E) \leq \alpha$ and $\mu_{VaR_\alpha}(E) = 1$ if $\pi(E) > \alpha$.

Example 3 (conditional value at risk$_\alpha$ preferences) *Fix* $\pi \in \Delta(S)$ *and an* $\alpha \in (0, 1)$*, and let the mean-dispersion preferences be defined by* $\rho(d) := CVaR_\alpha(d) = \frac{1}{\alpha}\int_0^\alpha VaR_\beta(d)\, d\beta$

Conditional value at risk was introduced more recently as a measure of risk in finance: see again, Föllmer and Schied (2004). This 'deviation' form of CVaR is due to Rockafellar et al

(2006). For intuition on the name of this dispersion function, notice that if S were infinite and the distribution function induced by π and d was continuous at VaR_α then we could write $\rho(d) := -E[d \mid d \leq -VaR_\alpha(d)]$. Pflug (2000) shows that conditional value at risk is a "coherent risk measure" in the sense of Artzner et al (1999). It follows that the preferences above are monotone, and that the dispersion function is non-negative, linear homogeneous and convex. Thus, by corollary 3, these preferences have a multiple-prior representation. A slight adaptation of Follmer and Schied's (2004) theorem 4.47 shows that the corresponding multiple-prior set is given by $C_{CVaR_\alpha} = \{p \in \Delta(S) : p(s)/\pi(s) \leq 1/\alpha\}$. Conditional value at risk preferences also inherit the comonotonic linearity of value at risk preferences and hence also admit a CEU representation given by the capacity $\mu_{CVaR_\alpha}(E) = \frac{1}{\alpha}\int_0^\alpha \mu_{VaR_\beta}(E)\,d\beta$.

Still within the finance literature, Rockafellar et al. (2006) extend the properties of standard deviation and conditional value at risk to define what they call "generalized deviations". These correspond to ρ that are non-negative, convex and linear homogenous, but the associated mean-dispersion preferences are not in general monotonic. Other examples come from beyond finance.

Example 4 (generalized Gini deviation preferences) *Fix* $\pi \in \Delta(S)$. *For each* $d \in H^0_\pi$, *let* $(d_{[1]}, \ldots, d_{[n]})$ *denote a reordering of the elements of the vector* d, *in which* $d_{[j]} \geq d_{[j+1]}$, *for all* $j = 1, \ldots, n-1$. *Let the mean-dispersion preferences be defined by*
$\rho(d) := -\left(\sum_{j=1}^n \left[\left(\sum_{k=1}^j \pi_{[k]}\right)^\delta - \left(\sum_{k=1}^{j-1} \pi_{[k]}\right)^\delta\right] d_{[j]}\right)$, *where* $\pi_{[0]} := 0$ *and* $\delta \geq 0$.

The name derives from Donaldson and Weymark's (1980) analogous measures of income inequality: if we view the state-contingent utility vector u as a vector of incomes across society, interpreting the probability π_s as the fraction of the population receiving income u_s, we can view $W(u) = \mu - \rho(d)$ as the "equally distributed equivalent income (EDEI)". For $\delta = 1$, the ρ defined above is 0. For $\delta = 2$, it is the standard absolute Gini inequality index: $\frac{1}{2}\sum_{s=1}^n \sum_{s=1}^n \pi_s \pi_{s'} |d_s - d_{s'}|$. For $\delta = \infty$, $W(u) = \min_s \{u_s\}$, and for $\delta = 0$ $W(u) = \max_s \{u_s\}$. This dispersion function is comonotonically linear (hence linear homogenous). Thus, by corollary 3, part (f), the corresponding mean-dispersion preferences are CEU. The dispersion function is positive and convex for $\delta > 1$, and negative and concave for $\delta < 1$. In appendix B, we show it is symmetric if and only if $\delta = 1$ or $\delta = 2$.

Example 5 (mean-log(mean absolute error) preferences) *Fix* $\pi \in \Delta(S)$*, and let the mean-dispersion preferences be defined by* $\rho(d) := k\left(\log\left(1 + \sum_s \pi_s |d_s|\right)\right)$ *where* $k < 1/4$*.*

These preferences come from Ergin & Gul (2009) who show that they are monotone. This dispersion function is non-negative and symmetric but not convex or linear homogenous. Thus, the preferences are neither invariant biseparable nor variational, but they are an example of VEU preferences which are Ghirardato-Marinacci ambiguity averse but not Schmeidler ambiguity averse.

Example 6 (multiplier preferences) *Fix* $\pi \in \Delta(S)$*, and let the mean-dispersion preferences be defined by* $\rho(d) := \theta \ln\left[\sum_s \pi_s \exp(-d_s/\theta)\right]$ *for* $\theta > 0$*.*

These preferences were introduced by Hansen & Sargent (2001) and were axiomatized by Strzalecki (2011). Their representation takes the variational form $V(f) = \min_{p \in \Delta(S)} \sum_s p_s U(f(s)) + \theta R(p|\pi)$ where $\theta > 0$ and $R(p|\pi)$ is the relative entropy of the probability p with respect to π; that is, $R(p|\pi) = \sum_s p_s \log(p_s/\pi_s)$ if $p \ll \pi$, and ∞ otherwise. Maccheroni et al (2006b) provide a primal representation for these preferences.[10] In appendix B, we show that the Maccheroni et al's primal representation can be re-written in the form above. The dispersion function in example 6 is non-negative and convex but not linear homogenous, and it is not symmetric unless S has just two elements and π is uniform, Hence these preferences are neither invariant biseparable nor VEU.

Example 7 (Hurwicz preferences) *Let* π *be uniform. Fix* $a \in [0,1]$ *and let* ν *be the capacity defined by:* $\nu(\emptyset) = 0$*,* $\nu(S) = 1$*, and* $\nu(E) = \alpha$ *for all* $E \subseteq S$*,* $E \neq \{\emptyset, S\}$*. Let the mean dispersion preferences be defined by* $\rho(d) = -\int (d)\, d\nu$*, the Choquet integral with respect to the capacity* ν*.*

These preferences were introduced by Hurwicz (1951). Their standard representation takes the form: $V(f) = \alpha \max_{s \in S} U(f(s)) + (1-\alpha) \min_{s \in S} U(f(s))$. In appendix B, we show that the preferences also have the above mean-dispersion representation, and hence are CEU. The dispersion function in example 7 is linear homogenous hence the preferences are invariant biseparable. It is not symmetric unless $\alpha = 1/2$ (hence the preferences are not VEU), and it is not convex

[10] See also Dupuis & Ellis (1997, Prop 1.4.2, pp.33-4). We thank Massimo Marinacci for sending helpful notes on this paper.

unless $\alpha = 0$. The dispersion function is non-negative (and hence the preferences are Gharardato-Marinacci ambiguity averse) if and only if $\alpha \leq 1/n$.

5 Identifying beliefs.

In this section, we consider what restrictions are placed on the probability π in a mean-dispersion representation. Equivalently, if an econometrician knew (or made the structural assumption) that an agent had mean-dispersion preferences, to what extent could she identify π from observing those preferences.

To fix ideas, consider multiple-prior preferences. A literal interpretation of the standard MEU representation treats the multiple-prior set as those probabilities that the agent considers possible, and assumes the agent evaluates each act using the minimizer from this set.[11] If an econometrician commits to this model, he can identify exactly the agent's *set* of beliefs from observing her preferences. From corollary 3, we know that any multiple-prior preferences also have a mean-dispersion representation. A literal interpretation of this representation treats the agent as having only one prior, π, and assumes that the agent simply dislikes dispersion of utilities across states with this dislike represented by ρ. If an econometrician commits to this model, he cannot identify the agent's *single* belief from observing her preferences, but (see corollary 5(b) below) the econometrician knows this belief lies in exactly the same set as appeared in the multiple-prior representation. Both representations involve indeterminacy about beliefs. In one, the indeterminacy is in the head of the agent. In the other, the same indeterminacy is in the head of the econometrician.

In general, our ability to identify π will depend on restrictions we place on the mean-dispersion model, in particular on the dispersion function ρ. In the most general version of the model in theorem 1, we only restricted ρ to be normalized and continuous. This yielded no restriction on π: that is, *any* π is admissible in the sense that there exists a mean-dispersion representation of the underlying preferences involving this π. But if we also restrict ρ to be non-negative (i.e., the preferences are dispersion averse), then the set of admissible π corresponds to Ghirardato

[11] See, for example, Gilboa et al, (2010). Of course, the model does not require us to have this literal interpretation.

& Marinacci (2002)'s and Ghirardato et al (2004)'s notion of the core.[12] Recall that $E_\pi(f)$ is the constant act that, in each state, yields the lottery $\sum_s \pi_s f(s)$, and that x_f is the certainty equivalent of f.

Definition A probability vector π is an element of $core(\succsim) \subseteq \Delta(S)$ if: for all acts f and g, $x_f \succsim E_\pi(g)$ implies $f \succsim g$.

Proposition 4 (identifying π) *If preferences $\succsim$ satisfy axioms A.1-A.5 then: a probability vector $\pi \in \Delta(S)$ is in core $(\succsim)$ if and only if there exists an (unbounded affine) utility function U and a* non-negative *dispersion function ρ such that $\langle U, \pi, \rho \rangle$ is a (monotone, unbounded) mean-dispersion representation of the preferences. The set core $(\succsim)$ is closed and convex.*

Given theorem 2, the following are immediate corollaries.[13]

Corollary 5 *Consider the (monotone, unbounded) mean-dispersion preferences $\succsim$.*

(a) core $(\succsim)$ is non-empty if and only if $\succsim$ satisfy A.6 (i.e., there exists a mean-dispersion representation $\langle U, \pi, \rho \rangle$ with $\rho \geq 0$).*

(b) If the preferences admit a MEU representation $min_{p \in \mathcal{P}} E_p(U \circ f)$ for some convex set $\mathcal{P} \subseteq \Delta(S)$, then a probability vector π admits a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ with ρ non-negative, convex and linear homogenous if and only if $\pi \in \mathcal{P}$.

(c) If the preferences admit a CEU representation with capacity ν, then a probability vector π admits a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ with ρ non-negative and comonotonically linear if an only if $\pi \in core(\nu)$.

Geometrically, we can think of the core($\succsim$) as the normal vectors of the supporting hyperplanes of the upper contour sets at the certainty line. Thus π will be unique if there is a unique supporting hyperplane for the upper contour set $\{u : u \succsim_u \lambda e\}$. A sufficient condition for this is for preferences

[12] Ghirardato & Marinacci's (2002, p.268) set $\mathcal{D}(\succsim)$ and GMM's (2004, p.151) notion of the core were defined in the context of the bi-separable model, but the definition here is, we hope, a natural extension to general mean-dispersion preferences.

[13] Parts (b) and (c) also follow from Ghirardato & Marinacci (2002: corollaries 13 and 14).

at certainty to be locally approximated by SEU preferences. To express this in terms of underlying preferences, notice that for SEU preferences there is a probability vector π such that implication of the core condition goes both ways: that is, not only $x_f \succsim E_\pi(g)$ implies $g \succsim f$ but also $E_\pi(g) \succ x_f$ implies $g \succ f$. This suggests the following partial or local converse to the core condition.

A.9 (local smoothness) For all $\pi \in core(\succsim)$ and all acts f and g: if $E_\pi(g) \succ x_f$ then there exists an $\bar{\alpha} \in (0,1]$ such that for all $\alpha \in (0, \bar{\alpha})$: $\alpha g + (1-\alpha) x_f \succ x_f$.[14]

If preferences satisfy this local smoothness condition then π is unique; moreover, this condition is necessary in the case where preferences are convex.

Proposition 6 (uniqueness I) *Suppose that preferences $\succsim$ admit a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ with $\rho \geq 0$ (i.e., they satisfy axioms A.1-A.5, and A.6*). Then the following are equivalent.*

(a) the preferences satisfy A.9

(b) ρ is Gateaux differentiable with $\rho'(0; d) = 0$ for all d; that is, the corresponding W has $\nabla W(0) = \pi$.

In this case, $core(\succsim) = \{\pi\}$; i.e., π is unique. If A.6 is replaced by A.6 then uniqueness is equivalent to conditions (a) and (b) above.*

Two remarks: first, absent the convexity axiom A.6, axiom A.9 is sufficient but not necessary for π to be unique. Consider the case with two states and induced preferences $\succsim_u$, where the hyperplane H^0_π supports the upper contour set $\{u : u \succsim_u 0e\}$ in three points; once at $0e$ and once on either side, say at d and $-d \in H^0_\pi$. At $0e$, the preferences could be 'kinked', but the two other points, d and $-d$ prevent there existing any other supporting hyperplane at $0e$. Second, the differentiability at the certainty line in proposition 6 is 'very local'. In particular, on its own, axiom A.9 does not imply that ρ is continuously or even strictly differentiable at 0.

We next consider two variants of the local smoothness condition, A.9. The first variant provides us with a uniqueness condition for π even when ρ is not restricted to be non-negative (i.e., the core might be empty), but at the cost of imposing an existential condition on π.

[14] A weaker sufficient condition for uniqueness is: for all $\pi \in \Delta(s)$ such that the core condition holds, and for any pair of acts f and g: $E_\pi(g) \succ x_f$ implies there exists an $\alpha \in (0,1]$ such that $\alpha g + (1-\alpha) x_f \succ x_f$.

A.9* (local smoothness*) There exists $\pi \in \Delta(S)$ such that for any pair of acts f and g: $E_\pi(g) \succ x_f$ (resp. $x_f \succ E_\pi(g)$) implies there exists an $\bar{\alpha} \in (0,1]$ such that for all $\alpha \in (0,\bar{\alpha})$, $\alpha g + (1-\alpha)x_f \succ x_f$ (resp. $x_f \succ \alpha g + (1-\alpha)x_f$).

Axiom A.9 yields a unique π for ρ non-negative; that is, when preferences are dispersion averse. Axiom A.9* yields a uniqueness result for (locally) differentiable ρ even when dispersion aversion may fail.[15]

Proposition 7 (uniqueness II) *Preferences* $\succsim$ *satisfy axioms A.1-A.5, and A.9* if and only if there exits a (monotone, unbounded) mean-dispersion representation* $\langle U, \pi, \rho \rangle$ *with* ρ *Gateaux differentiable at* 0 *with* $\rho'(0; d) = 0$ *for all* d*; that is, the corresponding* W *has* $\nabla W(0) = \pi$*. In this case,* π *is unique.*

An alternate route to get unique identification of π without assuming either dispersion aversion or differentiability is to impose symmetry on ρ. In this case, by part (d) of corollary 3, the mean-dispersion preferences correspond to vector-expected utility, and Siniscalchi (2009) shows that the baseline prior π is unique in this case.

To motivate the second variant of axiom A.9, recall that in the case of SEU preferences, we identify beliefs with the marginal rates of substitution (or shadow prices) across states at certainty. Ghirardato et al (2004) generalize this idea in the context of invariant biseparable preferences, and argue for using the *set* of marginal rates of substitution across states at certainty to identify the *set* of beliefs of the agent. If we extend this idea to general mean-dispersion preferences, the set of marginal rates of substitution is larger than the core.[16] But the following weakening of axiom A.9 gives a case where the two sets coincide.[17] Intuitively, whereas axiom A.9 ensures that preferences around certainty are locally approximated by SEU preferences, axiom A.9** ensures that preferences around certainty are locally approximated by MEU preferences.

[15] In the case of dispersion-averse preferences – that is, givenA.1-A.5 and A.6* – A.9 implies A.9*.

[16] Indeed, this is already true for the invariant biseparable case.

[17] Ghirardato et al (2004) use the Clark superdifferential for their set of marginal rates of substitution whereas we will use Dini superdifferential. In general, the Clark superdifferential is larger than the Dini superdifferential (which may be empty). Given axioms A.6* and A.9*, however, the Dini superdifferential is sufficient for our purposes. In particular, the non-empty *core* $(\succsim)$ corresponds to the Dini superdifferential, and the (Dini) directional derivatives are the support functions of the Dini superdifferential (see, Borwein and Lewis p.36 and pp. 125-6).

A.9 (local MEU)** For all acts f and g: if $E_\pi(g) \succ x_f$ for all $\pi \in core(\succsim)$ then there exists an $\bar{\alpha} \in (0,1]$ such that for all $\alpha \in (0,\bar{\alpha})$: $\alpha g + (1-\alpha)x_f \succ x_f$.

This is weaker than axiom A.9 since the condition $E_\pi(g) \succ x_f$ must now hold for all π in the core.

Proposition 8 (local MEU) *Suppose that preferences $\succsim$ admit a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ with $\rho \geq 0$ (i.e., they satisfy axioms A.1-A.5, and A.6*), and let W be the associated representation over the induced preferences $\succsim_u$. Then the following are equivalent.*

*(a) the preferences satisfy A.9***

(b) the Dini directional derivative of W at 0 with respect to $\hat{u}$ exists and is given by $W'(0,\hat{u}) = \hat{\pi} \cdot \hat{u}$ where $\hat{\pi}$ minimizes $\hat{\pi} \cdot \hat{u}$ on $core(\succsim)$.

In this case, $core(\succsim)$ is equal to the Dini superdifferential of W at 0.

6 Probabilistic Sophistication

In this section, we consider conditions such that mean-dispersion preferences are *second-order probabilistically sophisticated.* It is natural when discussing probabilistic sophistication to consider infinite states. Thus, for this section, we will allow the state space S to be infinite. Let Σ be a sigma algebra of S, and let $\mathcal{F}$ now be the set of simple (i.e., finite valued) Σ-measurable acts; that is, for any f in $\mathcal{F}$, $f^{-1}(x) \in \Sigma$, for all $x \in X$ and the set $\{x \in X : \exists s \in S, \text{s.t } f(s) = x\}$ is finite. We will abuse notation throughout the section. Formally, however, let $B_0(\Sigma)$ denote the set of real-valued, Σ-measurable, simple functions. Thus, fixing an act f, for any utility function u, $U(f)$ is a element of $B_0(\Sigma)$. For any probability measure π, the mean utility μ of the act f is given by $\sum_{y\in X} U(y)\pi\left(f^{-1}(y)\right)$. And the associated utility difference function d is an element of $B_0(\Sigma)$ given by $d(s) = U(f(s)) - \mu$. Abusing notation, the domain of the dispersion function ρ will be

$$H^0_\pi := \left\{d \in B_0(\Sigma) : \textstyle\sum_{t\in\mathbb{R}} \pi\left(d^{-1}(t)\right)t = 0\right\}.$$

Given a probability measure π on the events in Σ, each Anscombe-Aumann act f is mapped to the two-stage lottery that assigns first-stage probability $\pi\left(f^{-1}(x)\right)$ to each second-stage lottery

x in X. Preferences are second-order probabilistically sophisticated if they can be described by the ordering they induce on these two-stage lotteries. Formally:

Definition 1 (Second-Order Probabilistic Sophistication) *We say that the preferences $\succsim$ are second-order probabilistically sophisticated with respect to π, if $\pi\left(f^{-1}(x)\right) = \pi\left(g^{-1}(x)\right)$ for all $x \in X$ implies $f \sim g$.*

The following theorem gives conditions for mean-dispersion preferences $\langle U, \pi, \rho \rangle$ to be second-order probabilistically sophisticated with respect to π.

Proposition 9 (Probabilistic Sophistication I) *Let $\succsim$ be mean-dispersion preferences. Then the following are equivalent:*

(a) *The preferences are second-order probabilistically sophisticated with respect to π.*

(b) *The preferences admit a mean-dispersion representation $\langle U, \pi, \rho \rangle$ where ρ has the property that for any pair of utility differences from the mean d and d'*

$$\pi\left(\{s \in S : d(s) \leq t\}\right) = \pi\left(\{s \in S : d'(s) \leq t\}\right) \text{ for all } t \in \mathbb{R} \Rightarrow \rho(d) = \rho(d') \qquad (1)$$

The restriction on ρ in expression (1) is analogous to the restriction in Maccheroni et al's theorem 14 that their 'cost function' c is re-arrangement invariant. In words, condition (1) says that if two differences-from-the-mean-functions d and d' induce the same probability distribution over differences with respect to π then their dispersions $\rho(d)$ and $\rho(d')$ are the same.

A more subtle issue is whether mean-dispersion preferences being second-order probabilistically sophisticated with respect to some π implies that they have a mean-dispersion representation that involves that same π and has the properties on the associated ρ implied by theorem 2. It turns out that this is indeed the case, at least if π is convex-valued (or uniform in the case that S is finite).[18]

[18] A probability measure π is convex-valued if for every event $E \in \Sigma$, and every $r \in (0,1)$, there exists an event $D \subset E$ such that $\pi(D) = r\pi(E)$.

Proposition 10 (Probabilistic Sophistication II) *Suppose $\succsim$ admits a mean-dispersion representation $\langle U, \pi, \rho \rangle$ and is second-order probabilistically sophisticated with respect to π. If π is convex valued (or uniform if S is finite) then the claims of theorem 2 hold for this representation.*

7 Aversion to dispersion.

This section considers different notions of uncertainty aversion or aversion to variation of utility across states: the standard notion introduced by Schmeidler (1989), the two notions introduced by Ergin and Gul (2009), two new notions suggested by the mean-dispersion model, and the notion of Chateauneuf and Talon (2002). In the mean-dispersion model, an agent's attitude toward uncertainty or the dispersion of utilities across states is captured by properties of the function $\rho(.)$, which can be thought of as the utility premium for removing such dispersion. We already know from section 3 that aversion to dispersion (i.e., $\rho \geq 0$) is equivalent to Chateauneuf and Talon's preference for complete hedges, axiom A.6* (which in turn is equivalent to Ghirardato and Marinacci's (2002) weak ambiguity aversion) which is weaker than Schmeidler's uncertainty aversion, axiom A.6.

Schmeidler's notion of uncertainty aversion says that the agent always (weakly) prefers to hedge. From the perspective of mean-dispersion preferences, however, hedging affects both dispersions and means. The trade off between mean and dispersion is captured by a cardinal property of ρ, namely convexity. We might want to assume that hedging reduces dispersion premiums without committing to a particular trade off between means and dispersion. That is, we might want to assume that ρ is quasi-convex but not necessarily convex. Most classical measures of dispersion are indeed quasi-convex.

An agent with mean-dispersion preferences for whom ρ is quasi-convex exhibits uncertainty aversion subject to holding the mean utility fixed. That is, for mean dispersion preferences $\langle U, \pi, \rho \rangle$, ρ is quasi-convex if and only if the following property holds for that π. Recall that $E_\pi(f)$ is the mean act (or reduction) of f with respect to π.

Definition 2 (Common-Mean Uncertainty Aversion) *For any pair of acts f, g in $\mathcal{F}$ such that $E_\pi(f) = E_\pi(g)$ and any α in $(0,1)$, if $f \sim g$ then $\alpha f + (1-\alpha) g \succsim f$.*

An example of mean-dispersion preferences in which ρ is quasi-convex but not convex was given by example 5 in section 3 where, recall, the dispersion is defined to be $\rho(d) := k(\log(1 + \sum_s \pi_s |d_s|))$ where $k < 1/4$.[19] The reason an agent with these preferences is common mean uncertainty averse but not uncertainty averse in Schmeidler's sense is that, loosely speaking, she has 'decreasing marginal disutility of dispersion' reflected in the log. Convexity, by ruling out this kind of example, is saying more than that the agent is averse to dispersion across states: it is saying that the agent has '(weakly) increasing marginal disutility of dispersion' as we increase the dispersion.

Even though quasi-convexity seems a natural property to put on an index of dispersion aversion, there are well-known preferences introduced in the context of ambiguity that turn out to have a mean-dispersion representation in which ρ is non-negative and yet not quasi-convex. For example, the ρ of the Hurwicz preferences of example 7 (with $\alpha < 1/n$) is not quasi-convex. Similarly, in finance, the ρ of the value-at-risk preferences of example 2 are not quasi-convex. But, to the best of our knowledge, all examples in the literature that admit a mean-dispersion representation with $\rho \geq 0$ satisfy at least ray monotonicity: $\rho(kd) \geq \rho(d)$ for all $d \in H_\pi^0$ and all $k > 1$.[20] This property just says that as we increase statewise differences from the mean by scalar multiplication, holding the mean fixed, then the agent is weakly worse off. All classical measures of dispersion satisfy this property.

An agent with mean-dispersion preferences for whom ρ is ray monotonic prefers to move towards a constant act (holding mean utility fixed). That is, for mean dispersion preferences $\langle U, \pi, \rho \rangle$, ρ is ray monotonic if and only if the following property holds for that π.

Definition 3 (Common-Mean Monotonicity) *For any pair of act f in $\mathcal{F}$, and any α in $(0,1)$, $\alpha f + (1-\alpha) E_\pi(f) \succsim f$.*

Clearly, common-mean uncertainty aversion implies common mean monotonicity. Less clear is how these connect with conditions considered by Ergin and Gul in the context of preferences that are also second-order probabilistically sophisticated.

[19] This ρ is quasi-convex since it is a monotonic transformation of a convex function. Ergin & Gul (2009) show that these preferences are not convex.

[20] This says that the upper contour sets of ρ on H_π^0 are star-shaped with respect to the origin.

Following Ergin and Gul, let π_f be the two-stage lottery generated by π and f; that is, the two-stage lottery that to each lottery x in X assigns the first-stage probability $\pi\left(f^{-1}(x)\right)$. Ergin and Gul show that if π is convex-valued, then an agent who is second-order probabilistically sophisticated with respect to π is averse to mean-preserving spreads in the (first stage of the) induced two-stage lotteries if and only if the following property holds.

Definition 4 (Second-Order Risk Aversion) *For any pair of acts f, g in $\mathcal{F}$ and any α in $(0,1)$, if $\pi_f = \pi_g$ then $\alpha f + (1-\alpha) g \succsim f$.*

Ergin and Gul also introduced a weaker condition for preferences that are second-order probabilistically sophisticated with respect to π. We say that an act is degenerate in the second stage if, for each state s, the lottery $f(s)$ is degenerate. Ergin and Gul's second condition captures the idea that an agent would prefer to bet on the (objective) events that define the lotteries in X than on the (subjective) events in S.

Definition 5 (Issue Preference) *For any acts f, g in $\mathcal{F}$, if g is degenerate in the second stage, and $E_\pi(f) = E_\pi(g)$ then $E_\pi(f) \succsim f \succsim g$.*

The following theorem states that the two notions of aversion to dispersion around a common mean that seem natural in the context of mean-dispersion preferences bound the two notions of aversion to dispersion in first-stage lotteries that seem natural in the context of second-order probabilistic sophistication. Furthermore, all four of these notions are in turn bounded by Schmeidler's uncertainty aversion and Chateauneuf and Tallon's preference for complete hedges.

Theorem 11 (Dispersion Aversions) *Suppose that the (monotone) mean-dispersion preferences $\succsim$ defined by $\langle U, \pi, \rho \rangle$ are second-order probabilistically sophisticated with respect to π, suppose that π is convex valued, and suppose that the final outcome set is rich in that for all lotteries $x \in X$ there exists a degenerate lottery that is indifferent to x. Then uncertainty aversion (A.6) implies common-mean uncertainty aversion which implies second-order risk aversion which implies issue preference which implies common-mean monotonicity which implies preference for complete hedges (A.6*).*

A Appendix: Proofs

Proof of sufficiency of the axioms in theorem 1: mean-dispersion**.** The proof has four steps. First, we show that, for every act f, there exists a constant act x_f such that $x_f \sim f$. Second, we show that there is an expected utility representation U for the preferences restricted to constant acts. Thus we can set $V(f) := U(x_f)$. Third, we define induced preferences over the state-utility vectors generated by U and $\mathcal{F}$, and show that constant absolute uncertainty aversion implies that these induced preferences satisfy a translation-invariance property. Finally, we show that translation invariance allows us, for each π, to express V in a mean-dispersion form.

We first show that certainty equivalents exist for each act: in fact, we show a little more.

Lemma 12 (Certainty Equivalents) *For all acts f in $\mathcal{F}$, all constant acts x and all $\alpha \in [0,1]$, there exists a constant act z such that $\alpha z + (1-\alpha)x \sim \alpha f + (1-\alpha)x$.*

Proof. The case $\alpha = 0$ is trivial. For $\alpha \in (0,1)$, first set $\hat{f} := \alpha f + (1-\alpha)x$ and set $\hat{g} := x$. By unboundedness A.4, there exists a $\hat{z}$ such that $\alpha\hat{z} + (1-\alpha)\hat{g} \succ \hat{f}$. That is, $\alpha\hat{z} + (1-\alpha)x \succ \alpha f + (1-\alpha)x$. Now, reset $\hat{f} := x$ and reset $\hat{g} := \alpha f + (1-\alpha)x$. By unboundedness, there exists a $\hat{w}$ such that $\hat{g} \succ \alpha\hat{w} + (1-\alpha)\hat{f}$. That is, $\alpha\hat{z} + (1-\alpha)x \succ \alpha f + (1-\alpha)x \succ \alpha\hat{w} + (1-\alpha)x$. Taking $\alpha \to 1$ and using continuity we get $\hat{z} \succeq f \succeq \hat{w}$. Applying continuity again, for all $\alpha \in (0,1]$, there exists some $\beta \in [0,1]$, such that $z := \beta\hat{w} + (1-\beta)\hat{z}$ satisfies $\alpha z + (1-\alpha)x \sim \alpha f + (1-\alpha)x$, as required. □

We next find an affine utility representation for the preferences restricted to constant acts.

Lemma 13 (Expected Utility on Constant Acts) *The restriction of preferences to constant acts admits an expected utility representation. That is, there exists an affine utility function $U : X \to \mathbb{R}$ such that, for all x, y in X, $U(x) \geq U(y)$ if and only if $x \succsim y$.*

Proof. The proof is similar to Maccheroni et al (2006a, pp.1477-8) Consider any pair of constant acts x and y be such that $x \sim y$. Suppose there exists a constant act z such that $\frac{1}{2}x + \frac{1}{2}z \nsim \frac{1}{2}y + \frac{1}{2}z$, say, without loss of generality, $\frac{1}{2}x + \frac{1}{2}z \succ \frac{1}{2}y + \frac{1}{2}z$. By constant absolute uncertainty aversion A.5, we can replace z with x to obtain $x \succ \frac{1}{2}y + \frac{1}{2}x$ and we can replace z with y to obtain $\frac{1}{2}x + \frac{1}{2}y \succ y$, a contradiction. Then the hypotheses of Hernstein and Milnor's (1953) mixture space theorem are satisfied. □

Next, we define induced preferences on state-utility vectors. Recall that a given expected utility function U maps the act f to a state-utility vector $U \circ f$. The following lemma due to Kopylov (2007), ensures that the range of this mapping is all of $\mathbb{R}^n$.

Lemma 14 (Unboundedness [Kopylov]) *Assume there exists an expected utility representation U on the constant acts. If the preferences satisfy unboundedness A.4 then $U(X) = (-\infty, +\infty)$.*

With this in place, we can define preferences over all state-utility vectors in $\mathbb{R}^n$ as follows.

Definition (induced preferences) Fix an expected utility function U on the constant acts. Let $\succsim_u$ be the binary relation on $\mathbb{R}^n$ defined by $u' \succsim_u u''$ if there exists a corresponding pair of acts f' and f'' in $\mathcal{F}$ with $U \circ f' = u'$ and $U \circ f'' = u''$, such that $f' \succsim f''$.

The next lemma shows that the induced preference relation is well behaved.

Lemma 15 (State-Utility Preferences) *Let U be an unbounded affine representation of $\succsim$ on X. The induced binary relation $\succsim_u$ inherits order and continuity. In particular, $u' \succsim_u u''$ if and only if for all acts f' and f'' in $\mathcal{F}$ such that $U \circ f' = u'$ and $U \circ f'' = u''$, we have $f' \succsim f''$.*

Proof. Completeness follows from unboundedness via lemma 14. That is, for any u' in $\mathbb{R}^n$, there exists an act f in $\mathcal{F}$ with $U \circ f = u'$. For any pair of acts f and g in $\mathcal{F}$, if $U \circ f = U \circ g$ then $f(s) \sim g(s)$ for all s. Hence, by state independence, $f \sim g$. Hence $U \circ f \succsim_u U \circ g$ if *and only if* $f \succsim g$. Similarly, $U \circ f \succ U \circ g$ if and only if $f \succ g$ and $U \circ f \sim_u U \circ g$ if and only if $f \sim g$. Hence transitivity is inherited by $\succsim_u$. To establish continuity, fix a u in $\mathbb{R}^n$ and consider a sequence of state-utility vectors (u^t) converging to u. We will construct a corresponding sequence of acts, (f^t) with the property $U \circ f^t = u^t$ for all elements of u^t within an ε-neighborhood $\mathcal{N}(u)$ of u such that f^t converge to an act f where $U \circ f = u$. Fix an ε-neighborhood $\mathcal{N}(u)$ of u. Since the state space is finite, we can find a $\bar{u}$ and $\underline{u}$ such that $\bar{u} \geq u'_s \geq \underline{u}$ for all u' in $\mathcal{N}(u)$ and all s in S. By

A.4, there exist constant acts $\bar{x}$ and $\underline{x}$ such that $U(\bar{x}) = \bar{u}$, and $U(\underline{x}) = \underline{u}$. Then, for each state utility vector u^t in $\mathcal{N}(u)$, by expected utility on constant acts, there is a unique act f^t of the form $f^t(s) = \beta_s^t \bar{x} + (1 - \beta_s^t)\underline{x}$ where $\beta_s^t \in [0,1]$ and $u^t = U \circ f^t$. Thus (f^t) converges to an f such that $U \circ f = u$. Thus, since the preferences $\succsim$ over acts are continuous and satisfy state-independence, the induced preferences are continuous. □

The next lemma shows that constant absolute uncertainty aversion applies analogously to indifferent acts.

Lemma 16 (CAUA for indifference) *If $\alpha f + (1-\alpha)x \sim \alpha z + (1-\alpha)x$ then $\alpha f + (1-\alpha)y \sim \alpha z + (1-\alpha)y$.*

Proof. By constant absolute uncertainty aversion, we know that $\alpha f + (1-\alpha)y \succsim \alpha z + (1-\alpha)y$. By expected utility on constant acts (lemma 13) and unboundedness A.4, there exists an act $\hat{z} \succ z$ and for all $\beta \in (0,1)$, $\alpha f + (1-\alpha)x \prec \alpha(\beta\hat{z} + (1-\beta)z) + (1-\alpha)x$. Then by the contrapositive of constant absolute uncertainty aversion, we obtain $\alpha f + (1-\alpha)y \prec \alpha(\beta\hat{z} + (1-\beta)z) + (1-\alpha)y$. Taking $\beta \to 0$ and applying continuity A.2, we obtain $\alpha f + (1-\alpha)y \precsim \alpha z + (1-\alpha)y$, as required. □

We now show that the induced preferences $\succsim_u$ satisfy a translation invariance property.

Definition (translation invariance) For any pair of utility vectors u and u' in $\mathbb{R}^n$, and any $\delta \in \mathbb{R}$, $u \sim_u u' \Rightarrow u + \delta e \sim_u u' + \delta e$.

For the induced preferences $\succsim_u$, translation invariance is implied by constant absolute uncertainty aversion.

Lemma 17 (Translation Invariance) *Let U be an unbounded affine representation of $\succsim$ on X, and let $\succsim_u$ be the induced preferences over state-utility vectors in $\mathbb{R}^n$. Then $\succsim_u$ satisfy translation invariance.*[21]

[21] This result is essentially the same as part of MMR's lemma 28 except that they use their weak certainty independence in place of constant absolute uncertainty aversion. MMR refer to translation invariance as vertical invariance.

Proof . Fix a state-utility vector u. By lemma 12, there exists a $\lambda \in \mathbb{R}$ such that $u \sim_u \lambda e$. Recall that constant acts are mapped to constant state-utility vectors. By unboundedness and lemma 13, there are constant acts that correspond to each point in this line. In particular, there is a constant act x_0 in X, such that $U \circ x_0 = 0e$. Fix $\alpha \in (0,1)$ and let f be given by $\alpha U \circ f + (1-\alpha) U \circ x_0 = u$. The existence of f is guaranteed by unboundedness. By lemma 12, there exists a constant act z such that $\alpha f + (1-\alpha) x_0 \sim \alpha z + (1-\alpha) x_0$. By construction $U(z) = \lambda/\alpha$. Let y be the constant act, $U \circ y = \delta e/(1-\alpha)$. Again, the existence of such a constant act is guaranteed by unboundedness. By constant absolute uncertainty aversion and lemma 16, we have $\alpha f + (1-\alpha) y \sim \alpha z + (1-\alpha) y$. Hence, $\alpha U \circ f + (1-\alpha) U \circ y \sim_u \alpha U \circ z + (1-\alpha) U \circ y$. But $\alpha U \circ f + (1-\alpha) U \circ y = u + \delta e$ and $\alpha U \circ z + (1-\alpha) U \circ y = \lambda e + \delta e$. By transitivity, any $u' \sim_u u$ is also indifferent to λe. Hence, by a similar construction, $u' + \delta e \sim_u \lambda e + \delta e$. The conclusion follows by transitivity. □

Although the induced preferences are not necessarily monotonic, they are increasing in the direction parallel to the certainty line.

Lemma 18 (Certainty Monotonicity) *Let U be an unbounded affine representation of of $\succsim$ on X, and let $\succsim_u$ be the induced preferences over state-utility vectors in $\mathbb{R}^n$. Then, for all state-utility vectors u in $\mathbb{R}^n$ and all $\delta > 0$, $u + \delta e \succ_u u$.*

Proof. If u is a constant act, then the conclusion follows from the construction of $\succsim_u$. Fix a state-utility vector u. By lemma 12, there exists a $\lambda \in \mathbb{R}$ such that $u \sim_u \lambda e$. By translation invariance, $u + \delta e \sim_u \lambda e + \delta e$. But the constant act $\lambda e + \delta e \succ_u \lambda e$. The conclusion follows by transitivity. □

Completing the proof that the axioms are sufficient. We adapt a standard argument (see, for example, Mas-Colell, Whinston and Green [1995, p..834]). Applying lemmas 13 and 14, fix an unbounded affine utility function U that represents the preferences on constant acts. Let $\succsim_u$ be the associated induced preferences on state-utility vectors. Applying lemma 12, set $V(f) := U(x_f)$. For all u in $\mathbb{R}^n$, define $W(u) \in \mathbb{R}$ by $u \sim_u W(u) e$. That is, by construction, $V(f) = W(U \circ f) = U(x_f)$, and W is continuous.

It is enough to show that for any $\pi \in \Delta(S)$ there exists a continuous function $\rho : H_\pi^0 \to \mathbb{R}$ such that $W(u) = \pi \cdot u - \rho(u - (\pi \cdot u)e)$. That is, we need to show that $W(u) - \pi \cdot u$ depends only on the vector of differences $u - (\pi \cdot u)e$. Or equivalently, if $u - (\pi \cdot u)e = u' - (\pi \cdot u')e$ then $W(u) - \pi \cdot u = W(u') - \pi \cdot u'$. But if $u - (\pi \cdot u)e = u' - (\pi \cdot u')e$ then, by translation invariance (lemma 17),

$$u' = u + (\pi \cdot u' - \pi \cdot u)e \sim_u W(u)e + (\pi \cdot u' - \pi \cdot u)e = [W(u) + (\pi \cdot u' - \pi \cdot u)]e$$

and therefore, $W(u') = W(u) + (\pi \cdot u' - \pi \cdot u)$, as required. This completes the proof of sufficiency

□

Proof of necessity of the axioms in theorem 1. For an unbounded mean-dispersion representation $\langle U, \pi, \rho \rangle$, it is immediate that the associated preferences over acts satisfy axioms A.1, A.2 and A.3*.

To show unboundedness, A.4, fix two acts f and g. Without loss of generality, assume $V(f) \geq V(g)$. Fix an $\alpha \in (0,1)$. Using the fact that U is unbounded, let $v \in \mathbb{R}$ such that $V(g) > \alpha v + (1-\alpha)V(f)$. Let $w \in X$ be a constant act such that $U(w) = v$. By the representation, since w is a constant act, $V(\alpha w + (1-\alpha)f) = \alpha v + (1-\alpha)V(f)$. The construction of z is similar. Since α was arbitrary, we are done.

The next lemma shows that linear mean-dispersion preferences satisfy Maccheroni et al's weak certainty independence axiom.

Lemma 19 (weak certainty independence) *Fix a mean-dispersion representation $\langle U, \pi, \rho \rangle$. The associated preferences over acts $\succsim$ satisfy: for any two acts f and g in $\mathcal{F}$, any two constant acts x and y, and any α in $(0,1)$, $\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \Rightarrow \alpha f\,(1-\alpha)y \succsim \alpha g + (1-\alpha)y$.*

Proof of lemma 19. Given $\langle U, \pi, \rho \rangle$, it is immediate that the induced preferences $\succsim_u$ over state-utility vectors satisfy translation invariance: in fact, for any pair of utility vectors u and u' in $\mathbb{R}^n$, and any $\delta \in \mathbb{R}$, $u \succsim_u u' \Rightarrow u + \delta e \succsim_u u' + \delta e$. Fix acts f and g in $\mathcal{F}$, constant acts x and y in X

and α in $(0,1)$. Set $\delta := (1-\alpha)(U \circ y(s) - U \circ x(s))$ and notice that

$$\begin{aligned} & [\alpha U \circ f + (1-\alpha) U \circ y] - [\alpha U \circ f + (1-\alpha) U \circ x] \\ = \; & [\alpha U \circ g + (1-\alpha) U \circ y] - [\alpha U \circ g + (1-\alpha) U \circ x] \\ = \; & \delta e. \end{aligned}$$

Hence $\alpha U \circ f + (1-\alpha) U \circ x \succsim_u \alpha U \circ g + (1-\alpha) U \circ x$ implies $\alpha U \circ f + (1-\alpha) U \circ y \succsim_u \alpha U \circ g + (1-\alpha) U \circ y$. Therefore $\alpha f + (1-\alpha) x \succsim \alpha g + (1-\alpha) x$ implies $\alpha f + (1-\alpha) y \succsim \alpha g + (1-\alpha) y$, as required. □

Since weak certainty independence implies constant absolute uncertainty aversion, this completes the proof of necessity. □

Strengthening state independence A.3* to monotonicity. Fix an unbounded mean dispersion representation of the preferences $\langle U, \pi, \rho \rangle$, and let W be the associated functions representing the corresponding preferences over state-utility vectors. It is immediate that $u \geq u'$ implies $W(u) \geq W(u')$. We need to show that $u \gg u'$ implies $W(u) > W(u')$. Suppose by way of contradiction that there exists a $u \gg u'$ such that $W(u) = W(u')$. Then there exists $\delta > 0$ such that $u \geq u' + \delta e$. By lemma 18, $W(u' + \delta e) > W(u')$ and hence $W(u' + \delta e) > W(u)$, contradicting monotonicity. Conversely, it is immediate that W increasing implies monotonicity of the underlying preferences.

Uniqueness of U and ρ. From the steps above, we have established $V(f) = \pi \cdot (U \circ f) - \rho((U \circ f) - (\pi \cdot ((U \circ f)))e)$, represents $\succsim$, where, by lemma 13, U is an affine function that represents $\succsim$ on constant acts. As is well-known, if we fix $a > 0$, and b in R, then the function $\hat{U}(x) = aU(x) + b$, is also an affine representation of $\succsim$ restricted to the set of constant acts. Set $\hat{W}(\hat{u}) := \pi \cdot \hat{u} - \hat{\rho}(\hat{u} - (\pi \cdot \hat{u})e)$, where $\hat{\rho}(ad) = \rho(d)$, for all d in H^0_π. And set $\hat{V}(f) := \hat{W}\left(\hat{U} \circ f\right)$. By construction we have $\hat{V}(f) \geq \hat{V}(g)$ if and only if $\pi \cdot \left(\hat{U} \circ f\right) - \hat{\rho}(\hat{U} \circ f - \left(\pi \cdot \left(\hat{U} \circ f\right)\right)e) \geq \pi \cdot \left(\hat{U} \circ g\right) - \hat{\rho}(\hat{U} \circ g - \left(\pi \cdot \left(\hat{U} \circ g\right)\right)e)$ if and only if $\pi \cdot (a(U \circ f) + be) - \hat{\rho}(a(U \circ f) - a(\pi \cdot ((U \circ f)))e) \geq \pi \cdot (a(U \circ g) + be) - \hat{\rho}(a(U \circ g) - a(\pi \cdot ((U \circ g)))e)$ if and only if $a[\pi \cdot (U \circ f) - \rho((U \circ f) - (\pi \cdot ((U \circ f)))e)] \geq a[\pi \cdot (U \circ g) - \rho((U \circ g) - (\pi \cdot ((U \circ g)))e)]$ if and only if $V(f) \geq V(g)$. ■

Proof of the sufficiency of the axioms in theorem 2 part (1): ρ non-negative. Unlike the proof of theorem 1, we will not be able to use any $\pi \in \Delta(S)$. The key to the proof will be to construct a particular probability vector π such that there exists a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ of the preferences in which the dispersion function ρ is non-negative. We will use axiom A.6* to construct a π that has this property.

Fix an unbounded affine U that represents the preferences over the constant acts, and let $\succsim_u$ be the associated induced preferences over state-utility vectors. From the proof of theorem 1, we know that these induced preferences satisfy ordering, continuity, monotonicity and translation invariance.

The proof proceeds by two lemmas.

Lemma 20 *For any finite set of acts $f_1, \ldots, f_m$ in $\mathcal{F}$ that are indifferent (that is, $f_1 \sim f_j$ for all $j = 2, \ldots, m$): if for the convex combination $a_1 f_1 + \ldots + a_m f_m$, the state-utility vector $U \circ (a_1 f_1 + \ldots + a_m f_m) \in \mathbb{R}^S$ is a constant vector, then there exists a corresponding set of acts $\hat{f}_1, \ldots, \hat{f}_m$ in $\mathcal{F}$ with $U \circ \hat{f}_j = U \circ f_j$ for all $j = 1, \ldots, m$, (and hence, by the affinity of U, $U \circ (a_1 f_1 + \ldots + a_m f_m) = U \circ \left(a_1 \hat{f}_1 + \ldots + a_m \hat{f}_m\right)$), such that $a_1 \hat{f}_1 + \ldots + a_m \hat{f}_m$ is a constant act.*

Proof. Let $\bar{u} = U \circ (a_1 f_1 + \ldots + a_m f_m)$. By unboundedness, we can find two lotteries $\bar{z}$ and $\underline{z}$ such that $\bar{z} \succ f_j(s) \succ \underline{z}$ for all s in S and all $j = 1, \ldots, m$. Define $\hat{f}_j$ by $\hat{f}_j(s) = \beta_s^{f_j} \bar{z} + \left(1 - \beta_s^{f_j}\right) \underline{z} \sim f_j(s)$. By state independence A.3* (which is implied by monotonicity), $\hat{f}_j \sim f_j$ for each j. Furthermore, since the preferences over constant acts respect independence, $[a_1 f_1 + \ldots + a_m f_m](s) \sim \left[a_1 \hat{f}_1 + \ldots + a_m \hat{f}_m\right](s)$ for all s in S. Hence, $U \circ (a_1 f_1 + \ldots + a_m f_m) = U \circ \left(a_1 \hat{f}_1 + \ldots + a_m \hat{f}_m\right) = \bar{u}$. But by construction, for each s in S, $a_1 \hat{f}_1(s) + \ldots + a_m \hat{f}_m(s) = \left[\alpha_1 \beta_s^{f_1} + \ldots + \alpha_m \beta_s^{f_m}\right] \bar{z} + \left[\alpha_1 \left(1 - \beta_s^{f_1}\right) + \ldots + \alpha_m \left(1 - \beta_s^{f_m}\right)\right] \underline{z}$. Hence

$$
\begin{aligned}
& U \circ \left(\left[\alpha_1 \beta_s^{f_1} + \ldots + \alpha_m \beta_s^{f_m}\right] \bar{z} + \left[\alpha_1 \left(1 - \beta_s^{f_1}\right) + \ldots + \alpha_m \left(1 - \beta_s^{f_m}\right)\right] \underline{z}\right) \\
= \quad & \left[\alpha_1 \beta_s^{f_1} + \ldots + \alpha_m \beta_s^{f_m}\right] U(\bar{z}) + \left[\alpha_1 \left(1 - \beta_s^{f_1}\right) + \ldots + \alpha_m \left(1 - \beta_s^{f_m}\right)\right] U(\underline{z}) = \bar{u},
\end{aligned}
$$

for every s in S. But since $U(\bar{z}) > U(\underline{z})$, there is a unique λ in $[0,1]$, satisfying, $\bar{u} = \lambda U(\bar{z}) + (1 - \lambda) U(\underline{z})$. Hence, $\alpha_1 \beta_s^{f_1} + \ldots + \alpha_m \beta_s^{f_m} = \lambda$ for all s in S. Thus $a_1 \hat{f}_1 + \ldots + a_m \hat{f}_m =$

$\lambda \bar{z} + (1-\lambda)\underline{z}$, is a constant act as required. $\square$

For any state-utility vector u', let $\mathcal{U}(u') = \{u'' \in \mathbb{R}^n : u'' \succsim_u u'\}$ be the (weak) upper contour set of u' with respect to the induced preference relation $\succsim_u$. Our axioms do not imply that these upper contour sets are convex, hence some utility vectors might not lie in supporting hyperplanes of their upper contour sets. The next lemma shows, however, that all *constant* utility vectors lie on supporting hyperplanes of their upper contour sets.

Lemma 21 (supporting hyperplane for constant acts) *Given preference for complete hedges A.6*, for all constant acts x in X, the constant vector $U \circ x$ lies in a supporting hyperplane of its weak upper contour set $\mathcal{U}(U \circ x)$.*

Proof. Suppose not. Consider the convex hull of the upper contour set of $U \circ x$. Since $U \circ x$ does not lie in a supporting hyperplane of the upper contour set, $U \circ x$ must lie in the interior of this convex hull. Therefore, there exists a constant act $y \prec x$ such that the utility vector $U \circ y$ also lies in the convex hull. By definition, $U \circ y$ is not in the upper contour set. Using a mild extension of Caratheodory's Theorem (see Rockafellar (1970, p.155)), we claim that there exist $n+1$ points on the boundary of the upper contour set (i.e., points indifferent to $U \circ x$) such that $U \circ y$ is a convex combination of these points.

By Caratheodory's Theorem, we know that any u' in the convex hull of a set $S \subset \mathbb{R}^n$ can be expressed as a convex combination of at most $n+1$ points in the set. Suppose (as in our case) the set S is closed and the point u' is not the set itself. Let $k+1 \leq n+1$ be the number of distinct points in S are used to construct u' as a convex combination each with positive weight. That is, $u' = \alpha_1 u_1 + \ldots + \alpha_k u_k + \alpha_{k+1} u_{k+1}$. Without loss of generality, we will show that we can choose another point $\hat{u}_{k+1}$ on the boundary of S such that the convex combination $\hat{\alpha}_1 u_1 + \ldots + \hat{\alpha}_k u_k + \hat{\alpha}_{k+1} \hat{u}_{k+1} = u'$. Let $u'' := \beta_1 u_1 + \ldots + \beta_k u_k$ where $\beta_j = \alpha_j / (1-\alpha_{k+1})$. By construction, $(1-\alpha_{k+1}) u'' + \alpha_{k+1} u_{k+1} = u'$. Suppose u_{k+1} was not on the boundary of S. Since S is closed and $u' \notin S$, there must exist a $\gamma \in (\alpha_{k+1}, 1)$ such that $\hat{u}_{k+1} := (1-\gamma) u'' + \gamma u_{k+1}$ lies on the boundary of S. Set $\hat{\alpha}_{k+1} := \alpha_{k+1}/\gamma$ and, for each $j = 1, \ldots, k$, set $\hat{\alpha}_j := \alpha_j (1-\hat{\alpha}_{k+1}) / (1-\alpha_{k+1})$. By construction $(1-\hat{\alpha}_{k+1}) u'' + \hat{\alpha}_{k+1} \hat{u}_{k+1} = u'$. And $\hat{\alpha}_1 u_1 + \ldots + \hat{\alpha}_k u_k + \hat{\alpha}_{k+1} \hat{u}_{k+1} = u'$. Repeating this step $k+1$ times completes the argument.

Since $u \circ y$ can be expressed as a convex combination of a finite number of points indifferent to $u \circ x$, by lemma 20, there exist a finite number of acts indifferent to x such that a convex combination of those indifferent acts yields a constant act indifferent to $y \prec x$. But this contradicts preference for complete hedges, A.6*. □

Completing the proof that the axioms are sufficient in theorem 2 part 1. By lemma 21, there exists a supporting hyperplane H_π^0 of the upper contour set through 0. By monotonicity, $\pi \geq 0$. By the mean-dispersion theorem 1, we can find a representation $\langle U, \pi, \rho \rangle$. Let $W(u) = \pi \cdot u - \rho(u - \pi \cdot u)$ be the associated mean-dispersion representation of the induced preferences $\succsim_u$. Suppose that ρ is not non-negative. Then there exists a difference vector $d \in H_\pi^0$, such that $\rho(d) < 0$. Then $W(d) = -\rho(d) > 0$, contradicting the definition of H_π^0. □

Proof of the necessity of the axioms in theorem 2 part (1). Fix a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ with ρ non-negative, and let W be the associated function representing the corresponding preferences $\succsim_u$ over state-utility vectors. Notice that since $\rho(0) = 0$ and $\rho \geq 0$, for all $\mu \in R$ and state-utility vectors $u' \in R^n$: if $u' \sim_u \mu e$ then $\pi \cdot u' \geq \mu$. Therefore, if $\mu' e$ is a convex combination of utility vectors that are indifferent to μe, then $\mu' \geq \mu$, and hence $W(\mu' e) \geq W(\mu e)$. Given lemma 15, this implies axiom A.6*. □

Proof of sufficiency of the axioms in theorem 2 part (2): ρ non-negative and convex. Since axiom A.6 implies axiom A.6*, we know from part (1) that that there exists a (monotone unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ with ρ non-negative.

To show that the convexity axiom A.6 implies that ρ is convex, suppose that ρ is not convex. Then there exists two difference vectors d and d' in H_π^0 and an α in $(0,1)$ such that

$$\rho(\alpha d + (1-\alpha) d') > \alpha \rho(d) + (1-\alpha) \rho(d').$$

Let μ and μ' be such that $\mu - \rho(d) = \mu' - \rho(d')$. Let u be the utility vector such that $\mu = \pi \cdot u$ and $d = u - \mu e$. Similarly, let u' be the utility vector such that $\mu' = \pi \cdot u'$ and $d' = u' - \mu' e$. By construction, $W(u) = W(u')$, but

$$\begin{aligned} W(\alpha u + (1-\alpha) u') &= [\alpha\mu + (1-\alpha)\mu'] - \rho(\alpha u + (1-\alpha) u') \\ &< [\alpha\mu + (1-\alpha)\mu'] - [\alpha\rho(d) + (1-\alpha)\rho(d'')] \\ &= \alpha W(u) + (1-\alpha) W(u'), \end{aligned}$$

which contradicts convexity of the preferences $\succsim_u$. But the convexity axiom A.6 implies that $\succsim_u$are convex. □

Proof of necessity of the axioms in theorem 2 part (2). Fix a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ with ρ convex. Convexity of the underlying preferences follows immediately. □

Proof of sufficiency of the axioms in theorem 2 part (3): ρ linear homogenous. By theorem 1, there exists a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$. Let W be the associated representation of preferences over state utility vectors. For any difference vector $d \in H_\pi^0$. We have $W(d) = -\rho(d)$; that is, $d \sim -\rho(d)e$. By certainty equivalent betweenness, axiom A.7, for any $\alpha \in (0,1)$, $W(\alpha d + (1-\alpha)(-\rho(d)e)) = -\rho(d)$. But $W(\alpha d + (1-\alpha)(-\rho(d)e)) = -(1-\alpha)(\rho(d)) - \rho(\alpha d)$. Hence $\rho(\alpha d) = \alpha\rho(d)$ for all $\alpha \in (0,1)$. To show linear homogeneity for $\lambda > 1$, fix d', let $d = \lambda d'$ and $\alpha = 1/\lambda$. Hence $\rho(\lambda d') = \rho(d) = (1/\alpha)\rho(\alpha d) = \lambda\rho(d/\lambda) = \lambda\rho(d')$. Thus ρ is linearly homogenous. □

Proof of necessity of the axioms in theorem 2 part (3). Fix a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ with ρ linear homogenous. It is enough to show that, for the induced preferences over state-utility vectors, for all state utility vectors $u \in \mathbb{R}^n$ and $\alpha \in (0,1)$, $u \sim_u \lambda e$ implies $\alpha u + (1-\alpha)\lambda e \sim_u \lambda e$. Let $d := u - (\pi \cdot u)e$ By the representation, $W(u) = \pi \cdot u - \rho(d) = \lambda$. And $W(\alpha u + (1-\alpha)\lambda e) = \alpha\pi \cdot u + (1-\alpha)\lambda - \rho(\alpha d) = \alpha\pi \cdot u - \alpha\rho(d) + (1-\alpha)\lambda = \lambda$, where the second last equality uses linear homogeneity of ρ. □

Proof of sufficiency of the axioms in theorem 2 part (4): ρ symmetric. As in part (1), the key to the proof will be to construct a probability vector $\pi \in \Delta(S)$ such that there exists a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ of the preferences in which the dispersion function ρ is symmetric. We will use axiom A.8 to construct the π that has this property.

Fix an unbounded affine U that represents the preferences over the constant acts, and let $\succsim_u$ be the associated induced preferences over state-utility vectors. From the proof of theorem 1, we know that these induced preferences satisfy ordering, continuity, monotonicity and translation invariance.

We first find the property of state utility vectors that corresponds to complementarity of the underlying acts, and the property of preferences over state-utility vectors that corresponds to axiom A.8.

Lemma 22 (complementary vectors) *The acts f and $\bar{f}$ are complementary if and only if $U \circ f + U \circ \bar{f} = ke$ for some $k \in \mathbb{R}$.*

Proof. Since f is complementary to $\bar{f}$, $\frac{1}{2}U(f(s)) + \frac{1}{2}U(\bar{f}(s)) = \frac{1}{2}U(f(s')) + \frac{1}{2}U(\bar{f}(s'))$ for all s. Set $k/2$ as the constant to which all these terms are equal, hence $U(f(s)) + U(\bar{f}(s)) = k$, for all s. That is, $U \circ f + U \circ \bar{f} = ke$. Conversely, if two utility vectors are such that $u + \bar{u} = ke$ then any two acts f and $\bar{f}$ such that $U \circ f = u$ and $U \circ \bar{f} = \bar{u}$, have the property that $U(f(s)) + U(\bar{f}(s)) = k$, for all s. Hence, $\frac{1}{2}f(s) + \frac{1}{2}\bar{f}(s) \sim \frac{1}{2}f(s') + \frac{1}{2}\bar{f}(s')$ for all pairs of states s and s'. Hence the underlying acts are complementary. $\square$

>From here on, we will define two utility vectors u and $\bar{u}$ as complementary if $u + \bar{u} = ke$ for some $k \in \mathbb{R}$.

Definition (additivity for complementary pairs) For any two pairs of complementary utility vectors $(u, \bar{u})$ and $(u'\bar{u}')$: if $u \sim_u \bar{u}$ and $u' \sim_u \bar{u}'$ then $\lambda u + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$ for all $\lambda, \gamma \geq 0$.

Lemma 23 (complementarity $\Leftrightarrow$ additivity) *The induced preferences $\succsim_u$ exhibit 'additivity for complementary pairs' if and only if the underlying preferences $\succsim$ satisfy axiom A.8, complementary independence.*

Proof. (Sufficiency) Fix two pairs of complementary utility vectors $(u, \bar{u})$ and $(u', \bar{u}')$ and fix

$\lambda, \gamma \geq 0$. Suppose $u \sim_u \bar{u}$ and $u' \sim_u \bar{u}'$. Since the pairs are complementary, by lemma 22, there exist k, k' in $\mathbb{R}$, such that $u = k - \bar{u}$ and $u' = k' - \bar{u}'$.

So consider the two pairs of acts $(f, \bar{f})$ and $(g, \bar{g})$, satisfying $U \circ f = (\lambda + \gamma)\, u$, $U \circ \bar{f} = (\lambda + \gamma)\, \bar{u}$, $U \circ g = (\lambda + \gamma)\, u'$ and $U \circ g = (\lambda + \gamma)\, \bar{u}'$. These pairs are complementary since $U \circ f + U \circ \bar{f} = (\lambda + \gamma)(u + \bar{u}) = (\lambda + \gamma)\, k$ and similarly, $U \circ g + U \circ \bar{g} = (\lambda + \gamma)\, k'$. Let x_0 be the constant act for which $U(x_0) = 0$.

We first show that $f \sim \bar{f}$ and $g \sim \bar{g}$. There are two cases.

Case *(i)* $\lambda + \gamma \geq 1$. Suppose $f \sim \bar{f}$ fails, and without loss of generality suppose $f \succ \bar{f}$. By continuity there exists $\beta > 0$, and an act $\hat{f}$, satisfying $U \circ \hat{f} = U \circ f - \beta(\lambda + \gamma)\, e$ and $\hat{f} \sim \bar{f}$. By construction, $\hat{f}$ is complementary with $\bar{f}$. Since (x_0, x_0) is trivially a complementary pair with $x_0 \sim x_0$, applying axiom A.8 we have

$$\frac{1}{\lambda + \gamma}\hat{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma}x_0 \sim \frac{1}{\lambda + \gamma}\bar{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma}x_0.$$

But since $U \circ \left[\frac{1}{\lambda+\gamma}\hat{f}\right] = \frac{1}{\lambda+\gamma}\left(U \circ f - \beta(\lambda + \gamma)\, e\right)$, we have

$$\begin{aligned} U \circ \left[\frac{1}{\lambda + \gamma}\hat{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma}x_0\right] &= u - \beta e, \\ \text{and } U \circ \left[\frac{1}{\lambda + \gamma}\hat{f} + \frac{\lambda + \gamma - 1}{\lambda + \gamma}x_0\right] &= \bar{u}. \end{aligned}$$

Thus, the above indifference implies, $u - \beta e \sim_u \bar{u} \sim_u u$ contradicting the monotonicity of $\succsim_u$.[22] The argument is the same for g and $\bar{g}$.

Case *(ii)* $\lambda + \gamma < 1$. Let $(f'', \bar{f}'')$ be the complementary pair of acts for which $U \circ f'' = u$ and $U \circ \bar{f}'' = \bar{u}$. Hence $f'' \sim \bar{f}''$. Recall (x_0, x_0) is trivially a complementary pair with $x_0 \sim x_0$. So, by applying A.8 complementary independence, we have

$$(\lambda + \gamma)\, f'' + (1 - \lambda - \gamma)\, x_0 \sim (\lambda + \gamma)\, \bar{f}'' + (1 - \lambda - \gamma)\, x_0$$

But $U \circ [(\lambda + \gamma)\, f'' + (1 - \lambda - \gamma)\, x_0] = (\lambda + \gamma)\, U(f'') = U \circ f$ and $U \circ \left[(\lambda + \gamma)\, \bar{f}'' + (1 - \lambda - \gamma)\, x_0\right]$ $= U \circ \bar{f}$. Thus, the above indifference implies $f \sim \bar{f}$. Similarly, it follows $g \sim \bar{g}$.

Applying complementary independence to $(f, \bar{f})$ and $(g, \bar{g})$ for $\alpha = \lambda / (\lambda + \gamma)$ yields $\alpha f +$

[22] In fact, even without monotonicity, this contradicts the conclusion of lemma 18.

$(1-\alpha)\, g \sim \alpha \bar{f} + (1-\alpha)\, \bar{g}$. And since

$$\begin{aligned} U \circ (\alpha f + (1-\alpha)\, g) &= \frac{\lambda}{\lambda+\gamma} U \circ f + \frac{\gamma}{\lambda+\gamma} U \circ g = \lambda u + \gamma u' \\ \text{and } U \circ \left(\alpha \bar{f} + (1-\alpha)\, \bar{g}\right) &= \frac{\lambda}{\lambda+\gamma} U \circ \bar{f} + \frac{\gamma}{\lambda+\gamma} U \circ \bar{g} = \lambda \bar{u} + \gamma \bar{u}', \end{aligned}$$

we have $\lambda u + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$, as required.

(**Necessity.**) Fix a pair of complementary acts $\left(f, \bar{f}\right)$, $(g, \bar{g})$ and α in $(0,1)$. Set $\lambda := \alpha$ and $\gamma := (1-\alpha)$. Set $u := U \circ f$, $\bar{u} := U \circ \bar{f}$, $u' := U \circ g$ and $\bar{u}' := U \circ \bar{g}$. Without loss of generality, suppose $u \succsim_u \bar{u}$ and $u' \succsim_u \bar{u}'$.

If $u \sim_u \bar{u}$ and $u' \sim_u \bar{u}'$ then, by additivity for complementary pairs $\lambda u + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$. Hence for the underlying preferences, we have: $\alpha f + (1-\alpha)\, g \sim \alpha \bar{f} + (1-\alpha)\, \bar{g}$, as required. If either of the preferences are strict, for example, say $u \succ_u \bar{u}$, then by monotonicity and continuity of $\succsim$ there exists an act $\hat{f}$ and $\beta > 0$, such that $U \circ \hat{f} = u - \beta e \sim_u \bar{u}$. By construction $\hat{f}$ is complementary to $\bar{f}$, so by additivity for complementary pairs we have $\lambda\, (u - \beta e) + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$. Hence by monotonicity of $\succsim_u$ it follows that $\lambda u + \gamma u' \succ_u \lambda \bar{u} + \gamma \bar{u}'$ and thus for the underlying preferences we have: $\alpha f + (1-\alpha)\, g \succ \alpha \bar{f} + (1-\alpha)\, \bar{g}$, as required. □

We now select the π that we will use in the mean-dispersion representation. Following Siniscalchi, we select the (unique) probability vector π such that for any two complementary utility vectors u and $\bar{u}$, $u \succsim \bar{u}$ if and only if $\pi u \geq \pi \bar{u}$. To show that such a vector exists, we will construct it explicitly.

By lemma 22, for any utility vector u, the set of vectors complementary to u is given by $\{(ke - u) \in \mathbb{R}^n : k \in \mathbb{R}\}$. By continuity and monotonicity of $\succsim_u$, there is a unique complementary vector $(\bar{k}e - u)$ such that $(\bar{k}e - u) \sim u$.

In particular, let e^s denote the unit vector with a 1 in the sth position and zeros elsewhere. For each state s, there is a unique k^s such that $k^s e - e^s \sim_u e^s$. That is, $k^s e - e^s$ is the unique vector that is both complementary to and indifferent to e^s. Set $\pi_s := k^s/2$. By monotonicity of $\succsim_u$, we have $\pi_s \in [0,1]$.

To show that $\sum_{s=1}^{n} \pi_s = 1$, notice that (by construction) for each s, the pair $(2\pi_s e - e^s, e^s)$ is a complementary pair and the elements are indifferent to each other. Therefore by lemma 23 we can apply the 'additivity for complementary pairs' $n-1$ times to obtain: $\sum_{s=1}^{n}(2\pi_s e - e^s) \sim_u \sum_{s=1}^{n} e^s$. That is, $\left[\sum_{s=1}^{n} 2\pi_s - 1\right] e \sim_u e$ which implies $\sum_{s=1}^{n} \pi_s = 1$.

Lemma 24 (complementarity and equal mean) *For the π constructed above and any two complementary vectors u and $\bar{u}$: $u \succsim_u \bar{u}$ if and only if $\pi \cdot u \geq \pi \cdot \bar{u}$.*[23]

Proof. Fix a vector u, and index the set of complementary vectors by k using the definition $\bar{u}^k := ke - u$. By monotonicity, $\bar{u}^k \succsim_u \bar{u}^{k'}$ if and only if $k \geq k'$. Moreover, $\pi \cdot \bar{u}^k = k - \pi \cdot u$ therefore $\pi \cdot \bar{u}^k \geq \pi \cdot u$ if and only if $k \geq 2(\pi \cdot u)$.

Recall that, by construction, $(2\pi_s e - e^s, e^s)$ is a complementary pair and the two state-utility vectors are indifferent to each other. Applying additivity for complementary pairs $n-1$ times (using u_s as λ) we obtain:

$$\begin{aligned} & \sum_{s=1}^{n} u_s (2\pi_s e - e^s) \sim_u \sum_{s=1}^{n} u_s e^s \\ \Rightarrow \quad & 2(\pi \cdot u) e - u \sim_u u. \end{aligned}$$

Therefore, $\bar{u}_k \succsim_u u$ if and only if $k \geq 2(\pi \cdot u)$. □

(To see that this π constructed above is the unique probability vector with this property, recall that for the utility vector e^s, there is a unique complementary vector $k^s e - e^s$ such that $e^s \sim k^s e - e^s$. But, for any $\tilde{\pi} \in \Delta(S)$, if $\tilde{\pi} e^s = \tilde{\pi}(k^s e - e^s)$ then $\tilde{\pi}_s = 2k^s$; that is, $\tilde{\pi} = \pi$.)

To complete the proof of sufficiency: let $\langle U, \pi, \rho \rangle$ be the mean-dispersion representation of the underlying preferences corresponding to the U we fixed and the π we just constructed. It remains to show that corresponding ρ is negatively symmetric: that is $\rho(d) = \rho(-d)$ for all $d \in H_\pi^0$. To see this, let W be the corresponding representation of the induced preferences $\succsim_u$: that is, $W(u) = \pi \cdot u - \rho(u - (\pi \cdot u) e)$. Thus, for all $d \in H_\pi^0$, $\rho(d) = -W(d)$. For all $d \in H_\pi^0$, $\pi \cdot d = 0 = \pi \cdot (-d)$. Therefore, by lemma 24, $W(d) = W(-d)$, and hence $\rho(d) = \rho(-d)$, as required. □

Proof of necessary of the axioms in theorem 2 part (4): Fix a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ for which $\rho(d) = \rho(-d)$, for all $d \in H_\pi^0$. It is enough to show that the associated preferences over state-utility vectors, $\succsim_u$, satisfy the property 'additivity

[23] *This result is essentially Siniscalchi's observation 1.*

for complementary pairs' since lemma 23 shows that this means the underlying preferences $\succsim$ satisfy axiom A.8. Therefore, consider the associated preferences $\succsim_u$ over state-utility vectors, which are given by: $W(u) = \pi \cdot u - \rho(u - (\pi \cdot u) e)$.

Fix a vector u, and let $d := u - (\pi \cdot u) e$. As in the proof of lemma 24, index the set of vectors complementary to u by k using the definition $\bar{u}^k := ke - u$. For all complementary vectors $\bar{u}^k$, notice that $\pi \cdot \bar{u}^k = k - \pi \cdot u$ and that $\bar{d}^k := \bar{u}^k - \left(\pi \cdot \bar{u}^k\right) e = (ke - u) - (k - \pi \cdot u) e = -d$. Thus, since $\rho(d) = \rho(-d)$, $W(u) \geq W\left(\bar{u}^k\right)$ if and only if $\pi \cdot \bar{u}^k = \pi \cdot u$.

Fix $\lambda > 0$ and $\gamma > 0$, and two pairs of complementary utility vectors $(u, \bar{u})$ and $(u' \bar{u}')$ such that $u \sim_u \bar{u}$ and $u' \sim_u \bar{u}'$. We need to show that $\lambda u + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$. From the argument above, we know that $\pi \cdot \bar{u} = \pi \cdot u$ and $\pi \cdot \bar{u}' = \pi \cdot u'$. Thus $\pi \cdot (\lambda u + \gamma u') = \pi \cdot (\lambda \bar{u} + \gamma \bar{u}')$. Let k and k' be such that $ke = u + \bar{u}$, and $k'e = u' + \bar{u}'$. Notice that $(\lambda u + \gamma u') + (\lambda \bar{u} + \gamma \bar{u}') = (\lambda k + \gamma k') e$, therefore $\lambda u + \gamma u'$ and $\lambda \bar{u} + \gamma \bar{u}'$ are complementary vectors. Thus, by the argument above, $\lambda u + \gamma u' \sim_u \lambda \bar{u} + \gamma \bar{u}'$ as required. ■

Proof of corollary 3, part (1) (weak ambiguity aversion). The concept of "weak ambiguity aversion" was introduced by Ghirardato & Marinacci (2002). Informally, an agent is weakly ambiguity averse if they are more ambiguity averse than some subjective expected utility maximizer. More formally,

Definition 6 (Weak Ambiguity Aversion (Ghirardato & Marinacci [2002])) *The preferences $\succsim$ are weakly ambiguity averse if there exists a $\pi \in \Delta(S)$ and an expected utility function U on constant acts such that for all constant acts x and all acts f, $U(x) \geq \pi \cdot (U \circ f) \Longrightarrow x \succsim f$ and $U(x) > \pi \cdot (U \circ f) \Longrightarrow x \succ f$.*

To show that the (monotone, unbounded) mean-dispersion preferences $\langle U, \pi, \rho \rangle$ with ρ non-negative satisfy weak ambiguity aversion, let the subjective expected utility preferences be given by the utility function U and the prior π. Fix a constant act x. By construction, the upper contour set at $U \circ x$ in state utility space induced by the subjective expected utility preferences is $H_\pi^{U(x)}$. But since $\rho \geq 0$, this same hyperplane is a supporting hyperplane of the upper contour set at $U \circ x$ of the preferences $\succsim_u$ induced by the mean-dispersion preferences $\langle U, \pi, \rho \rangle$. Since this holds for any constant act, the conclusion follows.

To show that "weak ambiguity aversion" implies A.6* (and hence, by theorem 2(1), implies that ρ is non-negative), let U and π define a subjective expected utility relation with respect to which the preferences $\succsim$ are weakly ambiguity averse. Notice that, by definition, U represents the preferences over constant acts. Fix a collection of indifferent acts $f_1, \ldots, f_m$ and let $\hat{x}$ be the constant act such that some convex combination of the indifferent acts $a_1 f_1 + \ldots + a_m f_m = \hat{x}$. We need to show that $\hat{x} \succsim f_1$. Let x_{f_1} be a certainty equivalent of f_1 (and hence of $f_2, \ldots, f_m$), and let $u' = U(x_{f_1})$. By ambiguity aversion, $\pi \cdot (U \circ f_j) \geq U(x_{f_1}) = u'$. That is, for each j, the utility vector $U \circ f_j$ lies above the hyperplane $H_\pi^{u'}$ that is the induced indifference set (according to the subjective expected utility preferences defined by U and π) through $u'e = U \circ x_{f_1}$. Therefore, their convex combination $U \circ \hat{x}$ also lies above $H_\pi^{u'}$. Hence $\hat{x} \succsim x_{f_1} \sim f_1$. □

Part (2) (Variational preferences) The equivalence of (monotone, unbounded) mean-dispersion preferences that admit a representation in which ρ non-negative and convex with variational preferences follows from the fact that Maccheroni et al.'s axioms characterizing these preferences are the same as those in part (2) of theorem 2 except that Maccheroni et al. have weak certainty independence in place of the slightly weaker constant absolute uncertainty aversion. However, lemma 19 shows that our axioms imply weak certainty independence. □

Part (3) (Invariant biseparable preferences). The equivalence of (monotone, unbounded) mean-dispersion preferences with a ρ that is linear homogeneous with invariant biseparable preferences follows from the fact that Ghirardato et al's axioms characterizing these preferences are the same as those in part (3) of theorem 2 except they have certainty independence in place constant absolute uncertainty aversion A.5 and certainty equivalent betweenness A.6. It is immediate that certainty independence implies A.5 and A.6. Lemma 19 shows that our axioms imply Maccheroni et al's weak certainty independence. And it is immediate that weak certainty independence combined with A.6 implies certainty independence. □

Part (4) (Vector Expected Utility) The equivalence of (monotone, unbounded) mean-dispersion preferences with a ρ that is symmetric with vector expected utility follows from the fact that Siniscalchi's axioms characterizing these preferences (in an unbounded domain) are the same as those in part (4) of theorem 2 except he has weak certainty independence in place of constant absolute uncertainty aversion A.5. But Lemma 19 shows that our axioms imply weak certainty independence. □

Part (5) (Multiple Priors). The equivalence of (monotone, unbounded) mean-dispersion preferences with a ρ that is non-negative, convex and linear homogeneous with multiple priors follows from the fact that Gilboa-Schmeidler axioms combine those of variational preferences (part 2) and invariant biseparable preferences (part 3). □

Part (6) (Choquet Expected utility). The equivalence of (monotone, unbounded) mean-dispersion preferences with a ρ that is linear homogeneous and additive for comonotonic dispersion vectors, with Choquet expected utility follows from the fact that Choquet expected utility is the subclass of invariant biseparable preferences that are additive for comonotonic acts. Schmeidler (1989) showed that such preferences admit a representation given by the Choquet integral of the induced state-utility vector with respect to a capacity υ, which is a function that assigns to each event a number in $[0,1]$. The capacity is normalized (that is, $\upsilon(\varnothing) = 0$ and $\upsilon(S) = 1$) and monotonic with respect to set-inclusion (that is, $A \subset B$ implies $\upsilon(A) \leq \upsilon(B)$.) Formally, the Choquet integral of a state-utility vector u' with respect to the capacity υ, is defined as:

$$\int u' d\upsilon := \int_{-\infty}^{0} -\left(1 - \upsilon\left(\{s \in S : u'_s \geq u\}\right)\right) du + \int_{0}^{\infty} \upsilon\left(\{s \in S : u'_s \geq u\}\right) du.$$

As is well-known, if two state-utility vectors u' u'' are comonotonic, (that is, for any pair of states s and $\hat{s}$, $(u'_s - u'_{\hat{s}})(u''_s - u''_{\hat{s}}) \geq 0$), then the Choquet integral of the sum of these two vectors with respect to the capacity υ, satisfies

$$\int (u' + u'')\, d\upsilon = \int u' d\upsilon + \int u'' d\upsilon.$$

Since any constant utility vector λe is comonotonic with respect to any other state-utility vector, it follows that for *any* probability π:

$$\begin{aligned} \int u' d\upsilon &= \int \left(u' - (\pi \cdot u')\, e + (\pi \cdot u')\, e\right) d\upsilon \\ &= (\pi \cdot u') + \int \left(u' - (\pi \cdot u')\, e\right) d\upsilon. \end{aligned}$$

Set $\rho(d) := -\int (d)\, dv$, and by construction, we have for all comonotonic $d, d' \in H_\pi^0$, and all $a > 0$,

$$\rho(d + d') = \rho(d) + \rho(d') \text{ and } \rho(ad) = a\rho(d),$$

as required. □■

Proof of Proposition 4. Sufficiency: Suppose $\pi \in \Delta(S)$ satisfies the core condition. Let U be an (unbounded) affine utility function that represents $\succsim$ restricted to constant acts, and let x^0 be the constant act for which $U \circ x^0 = 0$. We shall show that H_π^0 is a supporting hyperplane for the upper contour set of $\succsim_u$ at 0. To see this, for each $u \in H_\pi^0$, pick an act g for which $U \circ g = u$. By construction, $E_\pi(U \circ g) = 0$, and so for the constant act $E_\pi(g)$, we have $U(E_\pi(g)) = 0$. Thus, $x^0 \succsim E_\pi(g)$ and so by the core condition, $x^0 \succsim g$, which in turn implies $0 \succsim_u u$, as required. Repeating the steps after the proof of Lemma 21 of the proof of the sufficiency of the axioms in theorem 2 part 1, establishes we can use the hyperplane H_π^0 to construct a mean-dispersion representation with a non-negative ρ. □

Necessity: Suppose $\langle U, \pi, \rho \rangle$ is a mean-dispersion representation of the preferences with ρ non-negative. Fix a pair of acts f and g, such that $f \succsim E_\pi(g)$. By definition $V(g) = E_\pi(U \circ g) - \rho(U \circ g - (E_\pi(U \circ g))\, e) \leq E_\pi(U \circ g)$, since $\rho(.)$ is non-negative, which equals $V(E_\pi(g))$. Hence $f \succsim g$. As this holds for any f and g, such that $f \succsim E_\pi(g)$, we have that π satisfies the core condition, as required.

By construction, *core* $(\succsim)$ is the intersection of $\Delta(S)$ with the dual cone of the upper contour set $\{u : u \succsim_u 0e\}$, hence *core* $(\succsim)$ is closed and convex. ■

Proof of proposition 6. (a)⇒(b). Consider the graph of ρ in $H_\pi^0 \times \mathbb{R}$. (Notice that, up to a change of bases, the epigraph of ρ is the upper contour set $\{u : u \succsim_u 0e\}$ in $\mathbb{R}^n$.) It is enough to show that, for any vector d in H_π^0, $\lim_{\lambda \downarrow 0} [\rho(\lambda d)/\lambda]$ exists and is equal to zero. Since $\rho \geq 0$, the quotient in the brackets is bounded below by 0. Therefore, fix a $d \in H_\pi^0$ and, without loss of generality, let $\|d\| = 1$. By way of contradiction, suppose $\limsup_{\lambda \downarrow 0} [\rho(\lambda d)/\lambda] = 2b > 0$. Consider the vector (d, b) in $H_\pi^0 \times \mathbb{R}$. By axiom A.9, there exists an $\bar{\alpha} \in (0, 1]$ such that $\alpha b > \rho(\alpha d)$ for all $\alpha \in (0, \bar{\alpha})$. That is, $\rho(\alpha d)/\alpha < b$ for all $\alpha < \bar{\alpha}$, but this contradicts $\limsup_{\lambda \downarrow 0} [\rho(\lambda d)/\lambda] = 2b > 0$. Since the choice of d was arbitrary, this proves that all directional derivatives are equal to 0; that is, $\rho'(0) = 0$. Since $W(u) := \pi \cdot u + \rho(u - (\pi \cdot u)\, e)$, it follows that that W is Gateaux differentiable at 0, and that $\nabla W(0) = \pi$.

(b)⇒(a). Suppose that axiom A.9 does not hold. That is, there exists a $d \in H^0_\pi$ (without loss of generality, we can let $\|d\| = 1$) and a $b > 0$ such that, for all $\alpha' \in (0, 1]$ there exists an $\alpha \in (0, \alpha')$ such that $0e \succ_u \alpha\,(be + d)$; that is, $\rho\,(\alpha d) \geq \alpha b$. But then there is a sequence of $\alpha_t \to 0$ such that $\rho\,(\alpha_t d)\,/\alpha_t \geq b > 0$. But this contradicts $\lim_{\lambda \downarrow 0}\,[\rho\,(\lambda d)\,/\lambda] = 0$ and hence contradicts $\rho'(0) = 0$.

To prove uniqueness of π, fix a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho \rangle$ with $\rho \geq 0$ and consider the corresponding preferences $\succsim_u$ over state utility vectors. Suppose that there exists a $\hat{\pi} \neq \pi$ and $\hat{\rho}$ such that $\hat{\rho}$ is non-negative and $\langle U, \hat{\pi}, \hat{\rho} \rangle$ represents the same preferences. Then both hyperplanes H^0_π and $H^0_{\hat{\pi}}$ must support the upper contour set $\{u : u \succsim_u 0e\}$. Since $\hat{\pi} \neq \pi$, there exists a state-utility vector u such that $\hat{\pi} \cdot u > 0 > \pi \cdot u$. By axiom A.9 (using $U \circ g = u$, $U\,(x_f) = 0$, and $\hat{\pi}$) there exists an $\alpha \in (0, 1)$ such that $\alpha u \succ_u 0e$. But $0 > \pi \cdot \alpha u$, which violates H^0_π being a supporting hyperplane of $\{u : u \succsim_u 0e\}$.

To prove the converse for the case of A.6 holds in place of A.6*, let π be the unique probability vector admissible with a mean-dispersion representation with $\rho \geq 0$. By theorem 2 we know that ρ is convex, hence W is concave. The graph of W in $\mathbb{R}^{n+1}$ contains the line λe for $\lambda \in \mathbb{R}$ and $e \in \mathbb{R}^{n+1}$ (this is true whether or not ρ is convex). Therefore, if H^0_π is the only supporting hyperplane of $\{u : u \succsim_u 0e\}$, there is a unique supporting hyperplane to the hypograph of W, and hence W is Gateaux differentiable at 0 with gradient $\nabla W(0) = \pi$. This implies ρ is Gateaux differentiable with $\rho'(0) = 0$, so we are done.

In the convex case, we can also replace the (b)⇒(a) proof above by a constructive argument. Fix a g such that $x\,(g, \pi) \succ x_f$, let $u = U \circ g$ and, without loss of generality (because of translation invariance), let $u\,(x_f) = 0$: that is, $\pi \cdot u > 0$. It is enough to show that there exists an $\bar{\alpha} \in (0, 1]$ such that for all $\alpha \in (0, \bar{\alpha})$ $W\,(\alpha u) > 0$. Let $\beta^* := \arg\max_{\beta \in [-1,1]} W\,(\beta u)$. Since $\nabla W(0) = \pi$ and u does not lie in H^0_π, $\beta^* > 0$. But then, by the concavity W, $W\,(\beta u) > 0$ for all $\beta \in (0, \beta^*)$; and, setting $\bar{\alpha} := \beta^*$, we are done.. ■

Proof of proposition 7. Let π satisfy the condition in axiom A.9*. By theorem 1, there exists a (unbounded monotone) mean-dispersion representation $\langle U, \pi, \rho \rangle$ involving this π. Consider the graph of ρ in $H^0_\pi \times \mathbb{R}$. (Notice that, up to a change of bases, the epigraph of ρ is the upper contour set $\{u : u \succsim_u 0e\}$ in $\mathbb{R}^n$.) It is enough to show that, for any vector d in H^0_π, $\lim_{\lambda\downarrow 0}[\rho(\lambda d)/\lambda]$ exists and is equal to zero. Therefore, fix a $d \in H^0_\pi$ and, without loss of generality, let $\|d\| = 1$. By way of contradiction, suppose $\limsup_{\lambda\downarrow 0}[\rho(\lambda d)/\lambda] = 2b > 0$. Consider the vector (d, b) in $H^0_\pi \times \mathbb{R}$. By axiom A.9*, there exists an $\bar{\alpha} \in (0, 1]$ such that $\alpha b > \rho(\alpha d)$ for all $\alpha \in (0, \bar{\alpha})$. That is, $\rho(\alpha d)/\alpha < b$ for all $\alpha < \bar{\alpha}$, but this contradicts $\limsup_{\lambda\downarrow 0}[\rho(\lambda d)/\lambda] = 2b > 0$. Now suppose by way of contradiction, suppose $\liminf_{\lambda\downarrow 0}[\rho(\lambda d)/\lambda] = 2b < 0$. Consider the vector (d, b) in $H^0_\pi \times \mathbb{R}$. By axiom A.9*, there exists an $\hat{\alpha} \in (0, 1]$ such that $\alpha b < \rho(\alpha d)$ for all $\alpha \in (0, \hat{\alpha})$. That is, $\rho(\alpha d)/\alpha > b$ for all $\alpha < \bar{\alpha}$, but this contradicts $\liminf_{\lambda\downarrow 0}[\rho(\lambda d)/\lambda] = 2b < 0$. Since the choice of d was arbitrary, this proves that all directional derivatives are equal to 0; that is, $\rho'(0) = 0$. Since $W(u) := \pi \cdot u + \rho(u - (\pi \cdot u)e)$, it follows that that W is Gateaux differentiable at 0, and that $\nabla W(0) = \pi$.

Conversely, given the representation $\langle U, \pi, \rho\rangle$, it is enough to show that π satisfies the condition in axiom A.9*. Suppose not. Then there exists a $d \in H^0_\pi$ (without loss of generality, we can let $\|d\| = 1$) and a $b \neq 0$ (without loss of generality take $b > 0$) such that, for all $\alpha' \in (0, 1]$ there exists an $\alpha \in (0, \alpha')$ such that $0e \succ_u \alpha(be + d)$; that is, $\rho(\alpha d) \geq \alpha b$. But then there is a sequence of $\alpha_t \to 0$ such that $\rho(\alpha_t d)/\alpha_t \geq b > 0$. But this contradicts $\lim_{\lambda\downarrow 0}[\rho(\lambda d)/\lambda] = 0$ and hence contradicts $\rho'(0) = 0$.

To prove uniqueness of π, fix a (monotone, unbounded) mean-dispersion representation $\langle U, \pi, \rho\rangle$ with $\rho \geq 0$ and consider the corresponding preferences $\succsim_u$ over state-utility vectors. Suppose that there exists a $\hat{\pi} \neq \pi$ and $\hat{\rho}$ such that $\hat{\rho}$ has the same desired properties as ρ, and $\langle U, \hat{\pi}, \hat{\rho}\rangle$ represents the same preferences. By the argument in the previous paragraph, both π and $\hat{\pi}$ satisfy the condition in axiom A.9*. Since $\hat{\pi} \neq \pi$, there exists a state-utility vector u such that $\hat{\pi} \cdot u > 0 > \pi \cdot u$. By axiom A.9*, there exists an $\bar{\alpha} \in (0, 1)$ such that for all $\alpha \in (0, \bar{\alpha})$, $\alpha u \succ_u 0e$, and there exists an $\hat{\alpha} \in (0, 1)$ such that for all $\alpha \in (0, \hat{\alpha})$, $0e \succ_u \alpha u$: a contradiction. ■

Proof of Proposition 8. **Preliminaries**. Notice that, fixing U, for any other mean-dispersion representation $\langle U, \hat{\pi}, \hat{\rho}\rangle$ of the preferences with corresponding $\hat{W}$, we have $W(u) = \hat{W}(u) = E_{\hat{\pi}}(u) - \hat{\rho}(u - E_{\hat{\pi}}(u)e)$ since W and $\hat{W}$ represent the same underlying preferences with the same normalization: $W(\lambda e) = \hat{W}(\lambda e) = \lambda$ for all $k \in \mathbb{R}$. Therefore, if it exists, the (Dini) directional derivative of W at 0 in direction $\hat{u}$, $W'(0,\hat{u})$, is equivalently given by $\hat{\pi} \cdot \hat{u} - \rho'\left(0, \hat{d}\right)$ where $\hat{d} = \hat{u} - E_{\hat{\pi}}(\hat{u})\, e$. That is, $W'(0,\hat{u})$ exists if and only if $\rho'\left(0, \hat{d}\right)$ exists.

(a)⇒(b). Let $\hat{u}$ be a state-utility vector. Let $\hat{\pi}$ minimize $\pi \cdot \hat{u}$ over $core(\succsim)$. Since $core(\succsim)$ is closed and convex, $\hat{\pi} \in bd(core(\succsim))$. If $u' := \lambda\hat{u} + \beta e$ for some $\lambda > 0$ and $\beta \in \mathbb{R}$, then $\hat{\pi}$ also minimizes $\pi \cdot u'$ over $core(\succsim)$.

Since $\hat{\pi}$ is in the core, by proposition 4, the preferences also admit a (monotone, unbounded) mean-dispersion representation $\langle U, \hat{\pi}, \hat{\rho}\rangle$ with $\hat{\rho} \geq 0$. Let $\hat{d} \in H^0_{\hat{\pi}}$ be given by $\hat{u} - E_{\hat{\pi}}(\hat{u})e$. Without loss of generality, let $\left\|\hat{d}\right\| = 1$. It is enough to show that $\lim_{\lambda\downarrow 0}\left[\hat{\rho}\left(\lambda\hat{d}\right)/\lambda\right]$ exists and is equal to zero. Similar to the proof of proposition 6, consider the graph of $\hat{\rho}$ in $H^0_{\pi} \times \mathbb{R}$. Notice that, up to a change of bases, the epigraph of $\hat{\rho}$ is the upper contour set $\{u : u \succsim_u 0e\}$ in $\mathbb{R}^n$. Since $\hat{\rho} \geq 0$, the quotient $\left[\hat{\rho}\left(\lambda\hat{d}\right)/\lambda\right]$ is bounded below by 0. By way of contradiction, suppose $\limsup_{\lambda\downarrow 0}\left[\hat{\rho}\left(\lambda\hat{d}\right)/\lambda\right] = 2\hat{b} > 0$. Consider the vector $\left(\hat{d}, \hat{b}\right)$ in $H^0_{\pi} \times \mathbb{R}$; that is, the state-utility vector $\hat{d} + \hat{b}e$. Notice that $\hat{d} + \hat{b}e = \hat{u} + \hat{\beta}e$ for $\hat{\beta} := (\hat{b} - E_{\pi}(\hat{u}))$ and hence $\pi \cdot \left(\hat{d} + \hat{b}e\right)$ achieves its minimum over $core(\succsim)$ at $\hat{\pi}$. Since $\hat{\pi} \cdot \hat{d} = 0$ and $\hat{\pi} \cdot e = 1$, that minimum is $\hat{b} > 0$. Hence $E_{\pi}(\hat{d} + \hat{b}e) \succ_u 0$ for all $\pi \in core(\succsim)$. Thus, by axiom A.9**, there exists an $\bar{\alpha} \in (0,1]$ such that $\alpha\left(\hat{d} + \hat{b}e\right) \succ 0$ (or equivalently, $\alpha\hat{b} > \hat{\rho}\left(\alpha\hat{d}\right)$) for all $\alpha \in (0,\bar{\alpha})$, That is, $\hat{\rho}\left(\alpha\hat{d}\right)/\alpha < \hat{b}$ for all $\alpha < \bar{\alpha}$, but this contradicts $\limsup_{\lambda\downarrow 0}\left[\hat{\rho}\left(\lambda\hat{d}\right)/\lambda\right] = 2\hat{b} > 0$. Thus, the directional derivative $\rho'\left(0;\hat{d}\right)$ exists and is equal to zero, and hence the directional directional derivative $W'(0;\hat{u}) = \hat{\pi} \cdot \hat{u}$. □

(b)⇒(a). Suppose axiom A.9** fails. Then there exists acts f and g such that $E_{\pi}(g) \succ x_f$ for all $\pi \in core(\succsim)$ but for all $\alpha' \in (0,1]$ there exists an $\alpha \in (0,\alpha']$ such that $x_f \succsim \alpha g + (1-\alpha)\, x_f$. Without loss of generality, let $U \circ f = 0$, and let $\hat{u} := U \circ g$. Let $\hat{\pi} \in bd(core(\succsim))$ minimize $\pi \cdot \hat{u}$ over $core(\succsim)$ and again consider the representation $\langle U, \hat{\pi}, \hat{\rho}\rangle$. Let $\hat{b} := E_{\hat{\pi}}(\hat{u})e > 0$ where the last inequality follows from $E_{\pi}(g) \succ x_f$. Let $\hat{d} \in H^0_{\hat{\pi}}$ be given by $\hat{u} - E_{\hat{\pi}}(\hat{u})e$ and, without loss of generality, let $\left\|\hat{d}\right\| = 1$. Therefore, for all $\alpha' \in (0,1]$ there exists an $\alpha \in (0,\alpha']$ such that $0e \succsim_u \alpha\left(\hat{d} + \hat{b}e\right)$; that is, $\hat{\rho}\left(\alpha\hat{d}\right) \geq \alpha b$. But then there is a sequence $\alpha_t \to 0$ such that $\hat{\rho}\left(\alpha_t\hat{d}\right)/\alpha_t \geq b > 0$. Therefore $\limsup_{\lambda\downarrow 0}\left[\hat{\rho}\left(\lambda\hat{d}\right)/\lambda\right] > 0$. Thus, if it exists, the Dini directional derivative of $W'(0,\hat{u})$ exceeds $\hat{\pi} \cdot \hat{u}$. □

[$core(\succsim) \subset Dini$ superdifferential] By (b), for all $\hat{u}$, $W'(0, \hat{u})$ exists and equals $\hat{\pi} \cdot \hat{u}$ where $\hat{\pi}$ minimizes $\pi \cdot \hat{u}$ over the core. Therefore, if π is in $core(\succsim)$ then $\langle \pi, \cdot \rangle \geq W'(0e, \cdot)$.

[$Dini$ superdifferential$\subset$ $core(\succsim)$]. Recall that $core(\succsim)$ is convex and closed. Therefore, if π is not in the core, we can find a $\hat{u}$ such that $\pi \cdot \hat{u} < \hat{\pi} \cdot \hat{u}$ where $\hat{\pi}$ minimizes $\hat{\pi} \cdot \hat{u}$ on $core(\succsim)$. We know $W'(0e, \hat{u}) = \hat{\pi} \cdot \hat{u}$ hence $\pi \cdot \hat{u} < W'(0e, \hat{u})$ hence π is not in Dini-Superdifferential ■

Proof of proposition 9 (probabilistic sophistication I)

(b)⇒(a) To see that the condition given in expression (1) is sufficient, suppose that $\pi\left(f^{-1}(x)\right) = \pi\left(g^{-1}(x)\right)$ for all $x \in X$. We need to show $f \sim g$. But since the probability distributions over utilities induced by π and U from f and g are the same, their mean utilities with respect to π are the same, and the probability distribution of differences from the mean are the same. Hence by expression (1), the dispersion measures will be the same. Hence they are indifferent.

(a)⇒(b). We know from theorem 1 there exists a mean dispersion representation $\langle U, \pi, \rho \rangle$ (that is, one involving π). Recall from the construction ρ in the proof of theorem 1, for each d in H^0_π, $\rho(d) := -W(d)$ where $W(d)\, e \sim_u d$. Next consider two state-utility functions d and d' in H^0_π with the property given on the left side of expression (1). That is, these two state utility functions induce the same distribution over utilities with respect to π, hence by probabilistic sophistication, they must indifferent. Since d and d' both have mean 0, it follows that $\rho(d) = \rho(d')$. ■

Proof of proposition 10 (probabilistic sophistication II)

To show that ρ is non-negative given axiom A.6* it is enough to show that H^0_π is a supporting hyperplane of $\succsim_u$ at 0. Suppose not. Then there exists a state-utility function $\tilde{u}$ in H^0_π such that $\tilde{u} \succ_u 0$. Since π is nonatomic (or uniform) we can approximate $\tilde{u}$ by another state-utility function u' measurable with respect to some N-element partition $\{E_1, \ldots E_N\}$ such that $\pi(E_j) = 1/N$ for all $j = 1, \ldots, N$, and such that $u' \succ_u 0$. We can think of any state utility function that is measurable with respect to this partition as an Ndimensional vector of "event-utilities". Abusing notation, let u' also refer to its corresponding vector. By probabilistic sophistication and theorem 9, the vector u' is indifferent to all its permutations (since each induces the same distribution of utilities according to π), and hence each such permutation $\tilde{u}'$ is strictly preferred to 0. Since the distribution on the N element partition is uniform, each of these permutation vectors $\tilde{u}'$ lie in H^0_π. Hence the convex hull of these permutation vectors includes 0. By part 1 of theorem 2, there exists $\hat{\pi}$ such that $0 \succsim_u u''$ for all u'' in $H^0_{\hat{\pi}}$, measurable with respect to the partition. Thus, for each of the permutation vectors $\tilde{u}'$ their expectation with respect to this $\hat{\pi}$ is greater than 0; that is, $\sum_{t \in \mathbb{R}} \pi\left([\tilde{u}']^{-1}(t)\right) t > 0$ for all $\tilde{u}'$ which is a permutation of u'. But this contradicts 0 lying in the convex hull of the permutation vectors.

The properties of ρ in parts 2 and 3 of theorem 2 are immediate.

To show ρ is symmetric, consider some N-element partition $\{E_1, \ldots E_N\}$ such that $\pi(E_j) = 1/N$ for all $j = 1, \ldots, N$. Consider the utility-function $\hat{u}$ measurable with respect to this partition, which is equal to 1 on states in E_j, -1 on states in $E_{j'}$, and zero elsewhere. By construction $\hat{u}$ and $-\hat{u}$ are both lie in H^0_π, and by probabilistic sophistication they are indifferent. Moreover, since $\hat{u} + (-\hat{u}) = 0$, the underlying pair of acts constitute a symmetric pair. Let $\hat{\pi}$ be the probability associated with the Vector Expected Utility representation. The construction of $\hat{\pi}$, and the fact that $\hat{u} \sim_u u$ implies that $\hat{\pi}(E_j) = \hat{\pi}(E_{j'})$. Since the uniform partition and the elements of the partition were arbitrary it follows that $\pi = \hat{\pi}$, and so $\rho(d) = \rho(-d)$. ■

Proof of theorem 11 (Dispersion Aversions) It is immediate that uncertainty aversion implies common-mean uncertainty aversion. Example 5 illustrates that the converse is false. Ergin & Gul (2009, Theorem 2 and by example after the proof of theorem 2) show that second-order risk aversion implies issue preference but not the converse.

To see that common-mean uncertainty aversion implies second-order risk aversion, notice that if an agent is second-order probabilistically sophisticated with respect to π, then $\pi_f = \pi_g$ implies $f \sim g$. Moreover, if two acts f and g induce the same two-stage lottery with respect to π then they have the same mean act, i.e., $E_\pi(f) = E_\pi(g)$, and hence $E_\pi(U \circ f) = E_\pi(U \circ g)$; that is f and g have a common mean with respect to π. Therefore if f and g satisfy the conditions of second-order risk aversion they also satisfy those for common-mean uncertainty aversion.

To see that issue preference implies common-mean monotonicity, fix f and (using the fact that the outcome space is rich) construct an act g that is degenerate in each second stage and that induces the same utility vector as f. By our our substitution axiom A.4*, $g \sim f$. By construction,

$U(E_\pi(f)) = E_\pi(U \circ f) = E_\pi(U \circ g) = U(E_\pi(g))$ and (again by our substitution axiom A.4*), $\alpha g + (1-\alpha) E_\pi(g) \sim \alpha f + (1-\alpha) E_\pi(f)$. Since $E_\pi(\alpha g + (1-\alpha) E_\pi(g)) = E_\pi(g)$, applying issue preference, we get $\alpha g + (1-\alpha) E_\pi(g) \succsim g$. Hence $\alpha f + (1-\alpha) E_\pi(f) \succsim f$ as required.

For mean-dispersion preferences, common mean monotonicity implies that $\rho(d) \geq \rho(\alpha d)$ for all $\alpha \in (0,1)$ and any d in H_π^0. Therefore, by continuity, $\rho(d) \geq \rho(0) = 0$, and, by part (a) of theorem 10 this implies axiom A.6*. ∎

B Appendix: Examples

Example 1 (mean-standard deviation preferences). Recall for some fixed $\pi \in \Delta(S)$, we set $\bar{S} := \{s \in S : \pi(s) > 0\}$. A monotone mean-dispersion preference relation can be constructed by taking $\rho(d) := \sqrt{\tau}\sigma$ where $\sigma = \sqrt{\sum_s \pi_s (d_s)^2}$ is the standard deviation and $\tau > 0$ is such that $\tau < \min_{s \in \bar{S}} (\pi_s / (1-\pi_s))$. It is immediate that the dispersion function is non-negative, linearly homogeneous, symmetric and convex. From this it follows that the upper contour sets of the induced preferences over state-contingent utility vectors are convex cones each with its vertex on the constant-utility line. Hence, to show the preferences are monotonic it is sufficient to show that for any pair of outcomes $x \succ y$, $x_{\{s\}} y \succ y$, for all s in $\bar{S}$. But straightforward calculation yields

$$\begin{aligned} V\left(x_{\{s\}} y\right) &= \pi_s U(x) + (1-\pi_s) U(y) - \sqrt{\tau}\sqrt{\pi_s (1-\pi_s)} (U(x) - U(y)) \\ &= U(y) + \left(\pi_s - \sqrt{\tau}\sqrt{\pi_s (1-\pi_s)}\right) (U(x) - U(y)) \end{aligned}$$

Thus, $x_{\{s\}} y \succ y$ if and only if $\pi_s - \sqrt{\tau}\sqrt{\pi_s (1-\pi_s)} > 0$, that is, $\tau < \pi_s / (1-\pi_s)$. ∎

Example 2 (value at risk preferences). To show these have a CEU representation: fix α in $(0,1)$ and consider the capacity

$$\mu_{VaR_\alpha}(E) = \begin{cases} 0 & \text{if } \pi(E) \leq \alpha \\ 1 & \text{if } \pi(E) > \alpha \end{cases}.$$

Since the state space is finite and hence d is a finite dimensional vector, the Choquet integral of d wrt μ_α may be expressed as:

$$\int (d)\, d\mu_{VaR_\alpha} = \sum_t \left[\mu_{VaR_\alpha}(\{s \in S : d(s) \geq t\}) - \mu_{VaR_\alpha}(\{s \in S : d(s) > t\})\right] \times t$$

Notice that:

(i) for any $t < \inf\left\{\hat{t} \in \mathbb{R} : \pi\left(\left\{s \in S : d(s) \leq \hat{t}\right\}\right) > \alpha\right\}$,

$$\mu_{VaR_\alpha}\left(\{s \in S : d(s) \geq t\}\right) - \mu_{VaR_\alpha}\left(\{s \in S : d(s) > t\}\right) = 1 - 1 = 0;$$

(ii) for $t = \inf\left\{\hat{t} \in \mathbb{R} : \pi\left(\left\{s \in S : d(s) \leq \hat{t}\right\}\right) > \alpha\right\}$,

$$\mu_{VaR_\alpha}\left(\{s \in S : d(s) \geq t\}\right) - \mu_{VaR_\alpha}\left(\{s \in S : d(s) > t\}\right) = 1 - 0 = 1;$$

(iii) for any $t > \inf\left\{\hat{t} \in \mathbb{R} : \pi\left(\left\{s \in S : d(s) \leq \hat{t}\right\}\right) > \alpha\right\}$,

$$\mu_{VaR_\alpha}\left(\{s \in S : d(s) \geq t\}\right) - \mu_{VaR_\alpha}\left(\{s \in S : d(s) > t\}\right) = 0 - 0 = 0.$$

Hence, $\int (d)\, d\mu_{VaR_\alpha} = \inf\{t \in \mathbb{R} : \pi(\{s \in S : d(s) \leq t\}) > \alpha\} = -VaR_\alpha(d)$.

Example 3 (conditional value at risk preferences). To show these have a CEU representation: fix α in $(0,1)$. We established above that

$$VaR_\alpha(d) = \int (d)\, d\mu_{VaR_\alpha}.$$

Hence,

$$\begin{aligned} CVaR_\alpha(d) &= \frac{1}{\alpha}\int_0^\alpha \mu_{VaR_\beta}(E)\, d\beta \\ &= \frac{1}{\alpha}\int_0^\alpha \left[\int (d)\, d\mu_{VaR_\beta}\right] d\beta \\ &= \int (d)\left[d\left(\frac{1}{\alpha}\int_0^\alpha \mu_{VaR_\beta} d\beta\right)\right] \\ &= \int (d)\, d\mu_{CVaR_\alpha}, \end{aligned}$$

as required.

Example 4 (generalized Gini dispersion preferences.) As we observed in our discussion of these preferences in section 3 above, they admit a CEU representation. For any vector u in $\mathbb{R}^n$, if we let $\left(u_{[1]}, \ldots, u_{[n]}\right)$ denote a reordering of the elements of this vector, for which $u_{[j]} \geq u_{[j+1]}$, for $j = 1, \ldots, n-1$, then it readily follows that corresponding $W(u)$ for these mean-dispersion preferences may be expressed as the Choquet integral with respect to the capacity ν, given by $\nu(E) := \left(\sum_{s\in E}\pi_s\right)^{\delta}$. That is,

$$W(u) = \sum_{j=1}^{n}\left[\left(\sum_{k=1}^{j}\pi_{[k]}\right)^{\delta} - \left(\sum_{k=1}^{j-1}\pi_{[k]}\right)^{\delta}\right]u_{[j]}.$$

Siniscalchi (2009, Table II, p826) provides the following necessary and sufficient condition for a CEU representation to admit a symmetric mean-dispersion representation. For all $E \subset S$, $\nu(E) - \nu(E^c) = 2\left(\sum_{s\in E}\pi_s\right) - 1$. For the capacity above, this reduces to:

$$\left(\sum_{s\in E}\pi_s\right)^{\delta} - \left(1 - \sum_{s\in E}\pi_s\right)^{\delta} = 2\left(\sum_{s\in E}\pi_s\right) - 1.$$

This functional equation only holds for $\delta = 1$ and $\delta = 2$. ■

Example 6 (multiplier preferences.) Maccheroni, Marinacci and Rustichini (2006b) show that we can write Hansen-Sargent multiplier preferences as $V(f) = -\theta\ln\left[\sum_s \exp\left(-U(f(s))/\theta\right)\pi_s\right]$. To convert this into a mean-dispersion representation, add and subtract $E_\pi(U\circ f)$ to the right hand side, to obtain,

$$E_\pi(U\circ f) - \theta\ln\left[\sum\nolimits_s \exp\left(\frac{-d_s}{\theta}\right)\pi_s\right], \qquad (2)$$

where, as usual, $d \in H^0_\pi$ is the vector of 'differences from the mean' given by $d_s := U(f(s)) - E_\pi(U\circ f)$. Notice that by Jensen's inequality,

$$\rho(d) := \theta\ln\left[\sum\nolimits_s \exp\left(\frac{-d_s}{\theta}\right)\pi_s\right] \geq \theta\ln\left(\exp\left(\frac{-d\cdot\pi}{\theta}\right)\right) = 0,$$

and $\rho(0) = 0$. Hence (2) constitutes a mean-dispersion representation of the multiplier preferences.

To see that it is not symmetric, recall that the third-order Taylor approximation of $\exp(t)$ around $t = 0$, is given by

$$\exp(t) \approx 1 + t + t^2/2 + t^3/6.$$

Hence

$$\begin{aligned}\rho(d) - \rho(-d) &\approx \theta\left[\ln\left(\sum\nolimits_s \pi_s\left(1 + d_s + d_s^2/2 + d_s^3/6\right)\right) - \ln\left(\sum\nolimits_s \pi_s\left(1 - d_s + d_s^2/2 - d_s^3/6\right)\right)\right] \\ &= \theta\ln\left[\frac{1 + \sum_s \pi_s\left(3d_s^2 + d_s^3\right)}{1 + \sum_s \pi_s\left(3d_s^2 - d_s^3\right)}\right].\end{aligned}$$

As this is in general not equal to zero, it follows that for any d for which $\|d\|$ is sufficiently small and $\sum_s \pi_s d_s^3 \neq 0$, $\rho(d) \neq \rho(-d)$. ■

Example 7 (Hurwicz α-MEU preferences). Consider the CEU preferences characterized by U and the 'Hurwicz capacity' ν given by

$$\nu(E) = \begin{cases} 0 & \text{if } E = \varnothing \\ \alpha & \text{if } E \notin \{\varnothing, S\} \\ 1 & \text{if } E = S \end{cases}, \text{ where } \alpha \leq 1/n.$$

This capacity ν is not convex since for any pair of distinct states s' and s'', we have $\nu(\{s'\}) + \nu(\{s''\}) = 2\alpha > \alpha = \nu(\{s', s''\})$. Straightforward calculation yields

$$\begin{aligned} W_\alpha^H(u') &= \int u' d\nu = \alpha \max_{s\in S} u_s + (1-\alpha)\min_{s\in S} u_s \\ &= \mu + \alpha \max_{s\in S} d_s + (1-\alpha)\min_{s\in S} d_s \\ &= \mu + \int (d)\, d\nu, \end{aligned}$$

where, $\mu = \left(\sum_{s\in S} u'_s\right)/n$ and $d = u' - \mu e$. It remains to show that the Choquet integral of the d with respect to the Hurwicz capacity ν is non-positive. To see this, consider the uniform probability vector, $\hat{\pi} = (1/n, \ldots, 1/n)$. Since $\alpha \leq 1/n$, the vector $\hat{\pi}$ is an element of the core $\mathcal{C}$ associated with ν since for any $A \subseteq S$, $\sum_{s\in A} \hat{\pi}_s = |A|/n \geq \nu(A)$. Hence we have, for every $d \in H_{\hat{\pi}}^0$,

$$\begin{aligned} 0 = \hat{\pi}\cdot d &\geq \frac{1}{n}\max_{s\in S} d_s + \frac{n-1}{n}\min_{s\in S} d_s \\ &\geq \alpha \max_{s\in S} d_s + (1-\alpha)\min_{s\in S} d_s = \int (d)\, d\nu. \end{aligned}$$

To see that for $n > 2$, this dispersion function is not symmetric, consider the dispersion vector d, for which $d_1 = \varepsilon$, $d_2 = d_3 = -\varepsilon/2$ and $d_s = 0$ for all $s > 3$. From above it follows:

$$\begin{aligned} \rho(d) &= -\left(\alpha \max_{s\in S} d_s + (1-\alpha)\min_{s\in S} d_s\right) \\ &= -\alpha\varepsilon + \varepsilon/2 - (\alpha\varepsilon)/2 = \frac{\varepsilon}{2}(1-3\alpha), \\ \text{and } \rho(-d) &= -(\alpha\varepsilon)/2 + \varepsilon - \varepsilon\alpha = \frac{\varepsilon}{2}(2-3\alpha). \end{aligned}$$

■

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