NBER WORKING PAPER SERIES

TECHNOLOGICAL GROWTH AND ASSET PRICING

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Working Paper 15340
http://www.nber.org/papers/w15340

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
September 2009

We would like to thank Andy Abel, Ricardo Caballero, Adlai Fischer, Tano Santos, Mungo Wilson, Motohiro Yogo, Lu Zhang and participants of seminars and conferences at Wharton, the Swedish School of Economics (Hagen), the Helsinki School of Economics GSF, the University of Cyprus, the Athens University of Economics and Business, the University of Piraeus, the Frontiers of Finance 2006, CICF 2008, the NBER 2005 EFG Summer Institute, the NBER 2006 Chicago Asset Pricing meeting, Western Finance Association 2006, SED 2006 and the Studienzentrum Gerzensee 2006 for very helpful discussions and comments. The views expressed herein are those of the author(s) and do not necessarily reflect the views of the National Bureau of Economic Research.

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NBER Working Paper No. 15340
September 2009
JEL No. E22,G12

ABSTRACT

In this paper we study the implications of general-purpose technological growth for asset prices. The model features two types of shocks: "small", frequent, and disembodied shocks to productivity and "large" technological innovations, which are embodied into new vintages of the capital stock. While the former affect the economy on impact, the latter affect the economy with lags, since firms need to first adopt the new technologies through investment. The process of adoption leads to cycles in asset valuations and risk premia as firms convert the growth options associated with the new technologies into assets in place. This process can help provide a unified, investment-based view of some well documented phenomena such as the asset-valuation patterns around major technological innovations, the countercyclical behavior of returns, the lead-lag relationship between the stock market and output, and the increasing patterns of consumption-return correlations over longer horizons.

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1 Introduction

Economic historians frequently associate waves of economic activity with the arrival of major technological innovations. The profound changes to manufacturing during the industrial revolution, the expanding network of railroads in the late nineteenth century, electrification, telephony, television, and the internet during the course of the last century are only a small number of well-known examples of a general pattern whereby a new technology arrives, gets slowly adopted and eventually permeates and alters all aspects of production and distribution. Naturally, these technological waves impact numerous markets, and especially asset markets, which reflect anticipations of future growth and facilitate the flow of capital towards innovative activity. The impact of technological waves on asset prices is the focus of this paper.

We build a tractable general-equilibrium model within which we characterize the behavior of asset prices throughout the technology-adoption cycle. We argue that the anticipatory nature of asset prices together with the slow deployment of technological innovations generate the joint properties of returns, output, and consumption documented in the data. We believe that the new mechanisms highlighted by our model can complement and improve on the explanatory power of existing endowment-based theories of return time variation.

Our main point of departure from previous work on asset pricing is that we explicitly allow for the joint presence of two types of technological shocks. The first type of shocks are the usual productivity shocks that are routinely assumed in the production-based asset-pricing literature. These shocks are technology "neutral" or "disembodied", in the sense that they affect the productivity of the entire capital stock irrespective of its vintage. However, this type of shocks do not fundamentally alter the technology used to produce consumption goods. The second type of shocks correspond to (infrequent) arrivals of major technological or organizational innovations, like automobiles, the internet, etc. These shocks do not affect the economy on impact, but only after firms have invested in new vintages of the capital stock that "embody" the technological improvements.

The investment in new capital vintages is assumed to involve a fixed (labor) cost that is irreversible. Firms choose the optimal time to invest in the new capital vintages, which leads to an endogenous lag between the arrival of embodied technological shocks and their eventual effects on output and consumption. This process of technological adoption generates endogenous persistence and *investment-driven cycles*, even though all shocks in the model arrive in an unpredictable i.i.d. fashion.

The link between the macroeconomy and asset pricing in our model revolves around the idea that growth options of firms exhibit a "life cycle" as technologies diffuse. On impact of a major technological shock, growth options emerge in the prices of all securities. These growth options are riskier than assets in place, and hence tend to increase the volatility of equity prices and the risk premia in the economy. In the initial phases of the technological cycle (i.e., when consumption is below its stochastic trend line) expected returns in the stock market are therefore high, simply because most growth options have not yet been exercised. As time passes, firms start converting growth options into assets in place, hence reducing the risk premium on their stock.

We argue that this investment-driven time variation in expected returns provides a natural mechanism to explain slow and countercyclical movements in expected returns (high expected returns when consumption is below its stochastic trend and vice versa). More importantly, the current model provides a unified theory for some additional patterns of the joint time-series evolution of returns and consumption in the data that can be challenging for some leading endowment-based models.[1] Specifically, the model can account for a) the fact that returns lead rather than lag output and consumption and b) the robust pattern that correlations between consumption and returns are weak at short horizons (over a quarter) and become stronger over longer horizons (over 1-3 years). In the model, these additional patterns emerge naturally, since major technological innovations produce consumption gains with a lag, but affect asset valuations immediately. This delayed reaction of consumption

[1]See Yu (2007) for a discussion of the difficulties of existing endowment-based models to account for some of these facts. See also Backus et al. (2008) for further evidence on the lead-lag relationship between excess returns and macroeconomic aggregates.

helps explain empirical observations a) and b) above. Finally, the tractability of the model allows a joint discussion of the time-series and cross-sectional patterns of returns. We show that the model is consistent with both aggregate time-series and cross-sectional patterns (such as the size and the value premium) of returns in general equilibrium.[2]

The paper is related to several strands of the literature. The paper by Carlson et al. (2004) is the most closely related to ours. Carlson et al. (2004) develop the intuition that the exercise of growth options can lead to variation in expected returns in a partial-equilibrium setting. In our paper, a similar mechanism operates in general equilibrium. By making consumption and returns jointly endogenous we are able to discuss a richer set of implications for asset pricing, such as short- and long-run correlations between consumption and returns, lead-lag relationships, aggregate time-series implications for consumption, investment, and returns, etc. Gomes et al. (2003) also analyze a general-equilibrium production-based model and examine the time series and cross sectional properties of returns, as we do. The two most significant differences between their set-up and ours are a) the distinction between embodied and disembodied aggregate technological shocks,[3] and b) the presence of an optimal timing decision concerning the exercise of growth options.[4] Since all shocks in Gomes et al. (2003) are disembodied productivity shocks, they affect the economy on impact and afterwards their effects dissipate. Our model differentiates between technological shocks that affect the economy on impact (disembodied shocks) and shocks that affect the econ-

[2]See Santos and Veronesi (2008) for a discussion of the tensions faced by leading endowment-based general equilibrium models in matching simultaneously time-series and cross-sectional aspects of return predictability.

[3]More generally, the literature on production-based asset-pricing routinely abstracts from this distinction. For contributions to this literature, see Cochrane (1996), Jermann (1998), Berk et al. (1999), Berk et al. (2004), Kogan (2004), Kaltenbrunner and Lochstoer (2006), Gomes et al. (2003), Carlson et al. (2004), Zhang (2005), Cooper (2006), Gourio (2004), Zhang (2005), and Gala (2006) among others. Papanikolaou (2007) draws a distinction between productivity shocks and investment specific shocks, but does not discuss embodied shocks or different capital vintages.

[4]Gomes et al. (2003) follow the seminal paper by Berk et al. (1999) and assume that options arrive in an i.i.d. fashion across firms and have a "take it or leave it" nature. By contrast, in our model all firms have discretion over the timing of their investment.

omy with a lag (embodied shocks). The result is a distinctive set of implications for the joint time-series properties of returns and macroeconomic aggregates (lead-lag relationships, correlation patterns, etc.) and the mechanism that produces return countercyclicality. A further implication is that in Gomes et al. (2003) cycles are driven by a trend-stationary productivity process, which implies a trend-stationary consumption process. In our model, all exogenous shocks follow random walks, while cycles emerge *endogenously* as the result of the economy's adoption of new technological vintages. As a result, consumption preserves a strong random-walk component, which is a salient feature of consumption in the data.

The theoretical literature on expected-return time variation is also related to this paper. We do not attempt to summarize this literature here; instead we refer to Cochrane (2005) for an overview. A leading approach to explaining the time variation and predictability of (expected) returns is to assume countercyclical risk aversion at the level of the "representative" consumer. As Yu (2007) shows, the strengthening of consumption-return correlations with the horizon,[5] as well as the fact that stock-market returns tend to lead consumption and output growth, present challenges for *single-shock, pure-endowment* economies with countercyclical risk aversion. Our approach shows how the interplay of multiple shocks in an investment-based framework can address these issues. An alternative approach in the literature, pioneered by Bansal and Yaron (2004), considers endowment economies with predictable consumption growth and Epstein-Zin-Weil preferences. As Yu (2007) and Backus et al. (2008) show, this type of models can capture the lead-lag relationships between consumption and returns, assuming that consumption volatility is stochastic and appropriately anticipates consumption growth. The present investment-driven approach provides an alternative to explaining these facts with i.i.d. shocks, thus circumventing the need to perform the potentially difficult task of estimating stochastic volatility in consumption and the extent to which it anticipates consumption growth.

Motivated by the events of the late nineties, Pastor and Veronesi (2009) connect the arrival of technological growth with the "bubble"-type behavior of asset prices around these

[5]See also Daniel and Marshall (1999) on this issue.

events.[6] Our model produces similar patterns. However, the focus of the two papers and the mechanisms are different. Our mechanism uses the endogenous exercise of growth options to produce variations in expected returns. Moreover, by considering recurrent arrivals of technological innovations we can discuss implications of the model for the joint stationary distributions of excess returns and macroeconomic aggregates and link technological growth with well documented time-series and cross-sectional patterns of returns.

There is a large literature in macroeconomics and economic growth that analyzes innovation, dissemination of new technologies, and the impact of the arrival of new capital vintages.[7] In contrast to our paper, this literature concentrates on innovation decisions in a deterministic environment, rather than the pricing of risk in a stochastic environment.

A technical contribution of our work is that it provides a tractable solution to a general-equilibrium model in which the micro-decisions are "lumpy" and exhibit optimal-stopping features.[8] In recent work, aspects of this model have been used by Obreja and Telmer (2008) to study long run variations's in Tobin's q and by Iraola and Santos (2009) to study links between news about innovations, the macroeconomy, and asset prices. Hsu (2009) studies empirically the link between the arrival of technological innovations and aggregate risk premia.

The structure of the paper is as follows: Section 2 presents the model and Section 3 the resulting equilibrium allocations. Section 4 presents the qualitative and quantitative implications of the model. Section 5 concludes. All proofs are given in the Appendix.

[6] Other papers that have analyzed the recent upswing in prices include Pastor and Veronesi (2004) and Jermann and Quadrini (2007).

[7] A small sample of papers includes Jovanovic and MacDonald (1994), Jovanovic and Rousseau (2003), Greenwood and Jovanovic (1999), Atkeson and Kehoe (1999), and Helpman (1998).

[8] The micro decision of the firm has a similar structure to the partial-equilibrium model of Abel and Eberly (2005). Just as firms in that paper adapt to the technological frontier at an optimally chosen time, firms in our framework decide on the optimal time to plant new trees. Moreover, by assuming cross sectional heterogeneity only at the beginning of an epoch, we can aggregate over firms in a much simpler way than the existing literature. For other analytically tractable approaches to aggregation see also Gomes et al. (2003), Caballero and Pindyck (1996), and Novy-Marx (2003).

2 Model

2.1 Trees, firms, and technological epochs

There exists a continuum of firms indexed by $j \in [0,1]$, which produce *consumption* goods. Each firm owns a collection of trees that have been planted in different technological epochs, and its total earnings are simply the sum of the earnings produced by the trees it owns. In turn, each tree produces earnings that are the product of three components: a) a time-invariant tree-specific component, b) a time-varying aggregate-productivity component, and c) a vintage-specific component, which is common across all trees of the same technological epoch. To introduce notation, let $Y_{N,i,t}$ denote the earnings stream of tree i at time t that was planted in the technological epoch $N \in (-\infty, +\infty)$. In particular, assume the functional form

$$Y_{N,i,t} = \zeta(i)\theta_t A_N. \tag{1}$$

The first term, $\zeta(i)$, is a positive, strictly decreasing function, mapping the interval $[0,1]$ to $\mathbb{R}^+$. $\zeta(i)$ is time invariant and captures a tree-specific effect. The second term, θ_t, is the common productivity shock and evolves as a geometric Brownian Motion:

$$\frac{d\theta_t}{\theta_t} = \mu dt + \sigma dB_t, \tag{2}$$

where $\mu > 0$ and $\sigma > 0$ are constants and B_t is a standard Brownian Motion. The term A_N captures a vintage-specific effect, which is common to all trees that are planted in epoch N. We make two assumptions about the evolution of A_N. First, $A_{N+1} \geq A_N$, so that vintages of trees planted in epoch $N+1$ are more productive than their predecessors (all else equal). Second, the ratio A_{N+1}/A_N is increasing in the extent of technological adoption that took place in epoch N. Specifically, letting $K_{N,t} \in [0,1]$ denote the mass of trees that were planted in epoch N by time t, and τ_{N+1} the time of arrival of epoch $N+1$, we postulate the following dynamics for A_{N+1}:

$$A_{N+1} = A_N \left(1 + \int_0^{K_{N,\tau_{N+1}}} \zeta(i)\, di\right). \tag{3}$$

Equation (3) reflects a standard assumption in the endogenous growth theory that is sometimes referred to as "standing on the shoulders of giants". The act of planting new trees produces knowledge and stimulates further innovation in future epochs.[9] Accordingly, the rate of increase between A_{N+1} and A_N depends on the investment activity in period N.[10] Technological epochs arrive exogenously at the Poisson rate $\lambda > 0$. Throughout, we denote the arrival time of epoch N as τ_N. Once a new epoch arrives, the index N becomes $N+1$, and every firm gains the option to plant a single tree of the new vintage at a time of its choosing.

Firm heterogeneity is introduced as follows: Once epoch N arrives, each firm j draws a random number $i_{j,N}$ from a uniform distribution on $[0,1]$. This number informs the firm of the type of tree that it can plant in the new epoch N. In particular a firm that drew the number $i_{j,N}$ can plant a tree with tree-specific productivity $\zeta(i_{j,N})$. These numbers are drawn in an i.i.d. fashion across epochs. To simplify the setup, we assume that once an epoch changes, the firm loses the option to plant a tree that corresponds to any previous[11]

[9]For some background on the "standing on the shoulders of giants" assumption, see e.g. the seminal paper of Romer (1990) who assumes that the arrival rate of new blueprints depends on the level of past blueprints. For a textbook treatment, see Jones (1997). Scotchmer (1991) contains a number of concrete examples of the positive effects of past innovation on new innovation ranging from the cotton gin to techniques of inserting genes into bacteria.

[10]From an asset pricing perspective, an advantage of the specification (3) is that it mitigates the predictability of consumption growth. For example, in a previous version of the model where the ratio of A_{N+1}/A_N is equal to a constant, the consumption cycles that are implied by the model are more persistent than under specification (3). Since consumption growth is not very predictable in the data, the specification (3) is advantageous.

[11]The assumption that a firm can plant a tree corresponding only to the current epoch can be relaxed (assuming that a firm can plant one tree each epoch), if we modify equation (3) to $A_{N+1} = A_N \bar{A}\left(1 + \int_0^{K_{N,\tau_{N+1}}} \zeta(i)\, di\right)$, where $\bar{A} \geq \frac{\zeta(0)}{\zeta(1)}$. Under this alternative assumption, for any firm j and any epoch N, we obtain $A_{N+1}\zeta(i_{j,N+1}) \geq A_{N+1}\zeta(1) \geq A_N\bar{A}\zeta(1) \geq A_N\zeta(0) \geq A_N\zeta(i_{j,N})$. Assuming that it costs the same to plant a tree of vintage $N+1$ or of vintage $n \in (-\infty, N]$, firm j would never find it optimal to plant a tree of a previous vintage. However, this model modification adds complexity without any extra insights, and we avoid it for parsimony.

epoch. It can only plant a tree corresponding to the technology of the current epoch. Let

$$X_{j,t} = \sum_{n=1...N} A_n \zeta(i_{j,n}) 1_{\{\widetilde{\chi}_{n,j}=1\}}, \tag{4}$$

where N denotes the technological epoch at time t and $1_{\{\widetilde{\chi}_{n,j}=1\}}$ is an indicator function equal to 1 if firm j decided to plant a tree in technological epoch n and 0 otherwise. A firm's total earnings are then given by $Y_{j,t} = X_{j,t}\theta_t$.

Any given firm determines the time at which it plants a tree in an optimal manner. Planting a tree at time t requires a fixed cost of q_t. This cost represents payments that need to be given to workers who will plant these trees and will be determined in general equilibrium. To keep with the usual assumptions of a Lucas tree economy, we assume that the company finances these fixed payments by issuing new equity.[12]

Assuming complete markets, the firm's objective is to maximize shareholder value. Since the productivity index $i_{j,N}$ is i.i.d. across epochs, there is no linkage between the decision to plant a tree in this epoch and any future epochs. Thus, the option to plant a tree can be studied in isolation in each epoch.

The optimization problem of firm j in epoch N amounts to choosing the stopping time τ that maximizes shareholder value. This amounts to solving the optimal stopping problem

$$P^o_{N,j,t} \equiv \sup_\tau E_t \left\{ 1_{\{\tau < \tau_{N+1}\}} \left[\left(A_N \zeta(i_{j,N}) \int_\tau^\infty \frac{H_s}{H_t} \theta_s ds \right) - \frac{H_\tau}{H_t} q_\tau \right] \right\}, \tag{5}$$

where H_s is the (endogenously determined) stochastic discount factor, τ_{N+1} is the random time at which the next epoch arrives, and $P^o_{N,j,t}$ denotes the value of the (real) option of planting a new tree in epoch N.

Given the setup, a firm's price consists of three components: a) the value of assets in place, b) the value of the growth option in the current technological epoch, and c) the value of the growth options in all subsequent epochs. Letting

$$P^A_{j,t} \equiv X_{j,t} \left(E_t \int_t^\infty \frac{H_s}{H_t} \theta_s ds \right) \tag{6}$$

[12] This assumption is inessential, since the completeness of markets (which we assume shortly) ensures that the Modigliani-Miller theorem holds.

denote the value of assets in place and

$$P^f_{N,t} \equiv E_t\left(\sum_{n=N+1}^{\infty}\frac{H_{\tau_n}}{H_t}P^o_{n,j,\tau_n}\right) \tag{7}$$

the value of all future growth options, the price of firm j, assuming it has not planted a tree (yet) in technological epoch N, is

$$P_{N,j,t} = P^A_{j,t} + P^o_{N,j,t} + P^f_{N,t}. \tag{8}$$

Naturally, for a firm that has planted a tree in the current technological epoch there is no longer a current epoch option $\left(P^o_{N,j,t} = 0\right)$ and hence its value is given by $P_{N,j,t} = P^A_{j,t} + P^f_{N,t}$.

2.2 Aggregation

Since the firms described in Section 2.1 produce consumption goods, the total consumption in the economy at time t is given by the production of all firms:

$$C_t = \int_0^1 Y_t(j)dj = \left(\int_0^1 X_{j,t}dj\right)\theta_t, \tag{9}$$

with $X_{j,t}$ defined in (4). Before proceeding, it is useful to define $F(x) \equiv \int_0^x \zeta(i)di$. Since $\zeta(\cdot)$ is positive and declining, we obtain $F_x \geq 0$, $F_{xx} < 0$, so that $F(x)$ has two key properties of a production function, namely it is increasing and concave. Assuming that firms with more productive trees always plant their trees before firms with less productive trees (we show later that this is indeed the case), and using the dynamics of A_N (equation (3)) and the definition of $K_{N,t}$ gives[13]

$$C_t = A_N\theta_t\left[1 + F\left(K_{N,t}\right)\right]. \tag{10}$$

[13]To see this note that

$$\begin{aligned}\int_0^1 X_{j,t}dj &= \sum_{n\leq N-1} A_n\int_0^{K_{n,\tau_{n+1}}}\zeta(i)di + A_N\int_0^{K_{N,t}}\zeta(i)di \\ &= \sum_{n\leq N-1}\left(A_{n+1} - A_n\right) + A_N\int_0^{K_{N,t}}\zeta(i)di\end{aligned}$$

The term $\sum_{n\leq N-1}\left(A_{n+1} - A_n\right) = A_N\left(1 - \lim_{n\to-\infty}\left(A_n/A_N\right)\right)$ converges to A_N (assuming a strictly positive probability that $\int_0^{K_{n,t}}\zeta(i)di > 0$ for all n). Hence $\int_0^1 X_{j,t}dj = A_N\left(1 + \int_0^{K_{N,t}}\zeta(i)di\right)$.

Aggregate output of consumption goods is thus the product of two terms: (a) the non-stationary stochastic trend $A_N\theta_t$, which captures the joint effects of technological progress due to the arrival of epochs (A_N) and neutral aggregate productivity growth (θ_t), and (b) the component $[1 + F(K_{N,t})]$, which captures the contribution of technological adoption in the current epoch and is a stationary, cyclical component, as we show in the next section.

2.3 Markets

The value of all (positive-supply) assets is given by the total value of the stock market:

$$P_{N,t} = \int_0^1 P_{N,j,t}dj. \tag{11}$$

In additional to shares in all firms, zero-net-supply zero-coupon bonds of arbitrary maturities are available for trade. We assume that markets are complete.[14] Accordingly, the search for equilibrium prices can be reduced to the search for a stochastic discount factor H_t, which will coincide with the marginal utility of consumption for the representative agent. (See Karatzas and Shreve (1998), Chapter 4.)

2.4 Consumer-workers and preferences

The economy is populated by a continuum of identical consumers-workers that can be aggregated into a single re presentative agent. The representative agent owns all the firms in the economy, and is also the (competitive) provider of labor services.

We shall use a preference specification introduced by Abel (1990), which generalizes standard constant relative risk aversion to allow for some degree of external habit formation. Specifically, letting c_t denote the representative agent's consumption, C_t the aggregate

[14]In particular, market completeness requires the existence of markets where agents can trade securities (in zero net supply) that promise to pay 1 unit of the numeraire when technological round N arrives. These markets are redundant in general equilibrium, since agents are able to create dynamic portfolios of stocks and bonds that produce the same payoff as these claims. However, it is easiest to assume their existence in order to guarantee ex-ante that markets are complete.

consumption, and $M_t^C = \max_{s \le t} \{C_s\}$ the running maximum of aggregate consumption, the agent's utility is given by

$$U\left(c_t, M_t^C\right) = \frac{1}{1-\gamma}\left[\left(\frac{c_t}{M_t^C}\right)^{1-\alpha} c_t^{\alpha}\right]^{1-\gamma}, \text{where } \alpha \in [0,1]. \tag{12}$$

This utility specification nests standard constant relative risk aversion preferences (when $\alpha = 1$) and the preferences considered by Abel (1999) and Chan and Kogan (2002) (when $\alpha = 0$) as special cases. Allowing for external habit formation is not crucial for the qualitative implications of the model. However, it is useful for calibration purposes, since it (a) allows matching the low level of interest rates and the high equity premium in the data and b) mitigates the reaction of interest rates to an acceleration of anticipated consumption growth caused by the arrival of a new technological epoch.[15] Importantly, unlike the specification in Campbell and Cochrane (1999), the specification (12) implies a constant relative risk aversion. Even though a certain degree of time varying risk aversion could be introduced into our framework[16], the property of constant relative risk aversion in specification (12) helps us illustrate more clearly the new economic mechanisms that drive our results. Finally, we specify the external habit level as the running maximum of past consumption in order to obtain closed form solutions.[17]

The representative agent is also the sole provider of labor services. Purely for simplicity,

[15]To see why, let c_{t_1} and c_{t_2} denote the consumption of the representative agent at two dates t_1 and t_2, where $t_1 < t_2$ and let $\varpi = \frac{d\ln(c_{t_2}/c_{t_1})}{d\ln\left(U_{c_1}/U_{c_2}\right)}\Big|_{d\ln(c_{t_2}/c_{t_1}) = d\ln(M_{t_2}^C/M_{t_1}^C)}$ denote the intertemporal elasticity of substitution with respect to a change in c_{t_2}/c_{t_1} that is accompanied by an equal change in $M_{t_2}^C/M_{t_1}^C$. Using (12) to compute ϖ yields $\varpi = \frac{1}{\gamma + (\gamma-1)(\alpha-1)}$. When $\alpha = 1$, the representative agent has standard CRRA preferences. In this case $\varpi = \frac{1}{\gamma}$ and we obtain the familiar result that the intertemporal elasticity of substitution is simply equal to the inverse of the risk-aversion parameter. At the opposite extreme $\alpha = 0$ and $\varpi = 1$. Hence, when $\alpha \neq 1$, these preferences exhibit a higher intertemporal elasticity of substitution with respect to variations in the growth rate of an agent's consumption that are accompanied by equal changes to the running maximum of aggregate consumption.

[16]This could be done by either building time varying risk aversion in the preferences of the representative agent as in Campbell and Cochrane (1999) or by assuming investor heterogeneity as in Chan and Kogan (2002) or Gârleanu and Panageas (2007).

[17]As most habit level specifications already proposed in the literature, it has the attractive property that it

we assume that work is not directly useful in the production of *consumption* goods, but it is useful in the production of *investment* goods, i.e., trees.[18] To keep with the Lucas "tree" analogy, we shall therefore refer to workers as "gardeners" who plant the new trees, and we also assume (for parsimony) that planting new trees requires exclusively labor.

Gardeners have a disutility of effort for planting new trees and need to be compensated accordingly. Planting a tree creates a fixed disutility of $U_c(s)\eta(s)$ per tree planted. Hence, the representative agent's utility function is given by

$$\max_{C_s, dl_s} E_t \left[\int_t^{\infty} e^{-\rho(s-t)} U(c_s, M_s^C) ds - \int_t^{\infty} e^{-\rho(s-t)} U_c(c_s, M_s^C) \eta(s) dl_s \right], \tag{13}$$

where $\rho > 0$ is the subjective discount factor and $dl(s) \geq 0$ denotes the increments in the number of trees that the representative consumer / gardener has planted.

This utility specification for the representative agent captures the fact that labor services are sunk in this model, i.e., the effort of planting a tree cannot be reversed. (Furthermore, since $\eta(s)$ can be an arbitrary adapted process, there is no loss in generality from specifying the disutility of labor (per tree planted) as $U_c(c_s, M_s^C)\eta(s)$.) The specification (13) implies that η_t can be interpreted as *a reservation wage*, above which the supply of labor services is perfectly elastic.[19] To see this, let V_W denote the derivative of the gardener's value function with respect to wealth. A gardener has an incentive to plant a tree if and only if

$$q_t V_W \geq \eta_t U_c. \tag{14}$$

Imposing the envelope condition,[20] we obtain $V_W = U_c$. Using this fact inside (14) implies

is "cointegrated" with aggregate consumption in the sense that the difference between $\log(C_t)$ and $\log(M_t^C)$ is stationary.

[18]The idea of modeling the consumption- and the investment-goods sectors of the economy separately is standard in endogenous-growth models. For a nice application see, e.g., Rebelo (1991), and for a finance application see Papanikolaou (2007).

[19]We note in passing that our specification of the disutility of labor is similar to the form advocated by Greenwood et al. (1988), since it isolates any intertemporal considerations and makes the leisure-consumption choice operate exclusively on the intratermporal margin.

[20]The envelope condition follows directly from the first order equations associated with the Bellman equation (see, e.g., Øksendal (2003), Chapter 11).

that another tree is planted only as long as $q_t \geq \eta_t$. Since there is a continuum of gardeners, perfect competition among them drives the price of planting a tree to[21]

$$q_t = \eta_t. \tag{15}$$

The consumer maximizes (13) over consumption plans in a complete market:

$$\max_{c_s, dl_s} E_t \left[\int_t^\infty e^{-\rho(s-t)} U(c_s, M_s^C) ds - \int_t^\infty e^{-\rho(s-t)} U_c(c_s, M_s^C) \eta_s dl_s \right] \tag{16}$$

subject to

$$E_t \left(\int_t^\infty \frac{H_s}{H_t} c_s ds \right) \leq \int_0^1 P_{N,j,t} dj + E_t \left(\int_t^\infty \frac{H_s}{H_t} q_s dl_s \right). \tag{17}$$

To close the model we need to make some functional form assumptions about η_t and $\zeta(i)$. We choose η_t with three goals in mind. The first goal is to reproduce (within the model) the facts documented in King et al. (1988). Specifically, in the data wages are non-stationary and labor income is cointegrated with total output. Furthermore, the hours worked are stationary. As King et al. (1988) have shown, cointegration of labor income and aggregate output can only obtain in equilibrium if the marginal disutility of an additional unit of work is proportional to (or, more generally, co-integrated with) $U_c \times c_t$. Under this assumption income and substitution effects on labor supply cancel, the hours supplied are stationary, and labor income is co-integrated with aggregate consumption. In our framework the marginal disutility of an additional unit of work is given by $U_c(t)\eta_t$, so that the King et al. (1988)

[21] Throughout we speak of a "representative" consumer-worker to expedite the presentation. Since the production of a new tree requires an indivisible amount of labor, we are implicitly following Rogerson (1988), who allows for labor-supply lotteries as one of the tradable contingent claims. Accordingly, even if firms choose a worker randomly to plant a tree, trading between workers allows them to share that risk. We refer to Rogerson (1988) for details. We note that in our setup, one can justify the concept of a representative agent even without labor-supply lotteries, by assuming that the planting of each tree is a divisible task amongst workers. Specifically, if a) planting a single tree takes a continuum of tasks $v \in [0, 1]$, b) each worker incurs a disutility of effort $\eta_t U_c$ per-task performed, and c) any worker can perform any set of tasks in perfect competition, then allocating tasks equally across workers would allocate the proceeds from planting a tree equally across the continuum of workers even in the absence of labor-supply lotteries.

restrictions on preferences require that η_t and c_t share the same common trend, namely $A_N\theta_t$. The second goal is to ensure that in equilibrium η_t grows between epochs, which seems plausible, assuming that trees of later vintages embody more complex ideas. The third goal is to keep the disutility of planting a tree, $U_c(t)\eta_t$, independent of the number of trees ($K_{N,t}$) planted in epoch N. This goal is motivated partly by parsimony and partly by the significant technical simplifications it allows. The process for η_t that achieves the three goals above is

$$\eta_t = \eta M^c_{\tau_N} \left(1 + F\left(K_{N,t}\right)\right)^{\nu}, \tag{18}$$

where $\eta > 0$, $\nu \equiv \gamma - (\gamma - 1)(1 - \alpha)$, and $N = \max\{n \,|\, \tau_n \leq t\}$. A final functional-form specification that facilitates closed-form solutions is

$$\zeta(i) = bp\,(1 + bi)^{p-1}\,, \quad i \in [0, 1], \tag{19}$$

where $b > 0$ and $p \in [0, 1]$ are constants that control the level and the curvature of $\zeta(i)$.

2.5 Equilibrium

The equilibrium definition is standard. It requires that all markets clear and all actions be optimal given prices.

Definition 1 *A competitive equilibrium is a set of stochastic processes* $\langle c_t, C_t, K_{n,t}, H_t, dl_t, q_t \rangle$ *such that*

a) c_t *and* dl_t *solve the optimization problem (16) subject to (17).*

b) Firms determine the optimal time to plant a tree by solving the optimization problem (5).

c) The consumption-good market clears:

$$c_t = C_t = \int_0^1 Y_t(j)dj \qquad \text{for all } t \geq 0, \tag{20}$$

where C_t *denotes aggregate consumption,* $\int_0^1 Y_t(j)dj$ *is given by the right-hand side of (10), and* $K_{n,t}$ *is given by*

$$K_{n,t} = \int_0^1 \widetilde{\chi}_{n,j,t}dj, \tag{21}$$

where $\widetilde{\chi}_{n,j,t}$ is an indicator that takes the value 1 if firm j has planted a tree in epoch n by time t and 0 otherwise.

d) The investment-goods market clears for all n, t :

$$dl_t = dK_{n,t}. \tag{22}$$

e) The markets for all assets clear.

If one could calculate the optimal processes $K_{n,t}$, then the optimal consumption process would be given immediately by (20) and (10), which would in turn determine the equilibrium stochastic discount factor,

$$H_t = e^{-\rho t} U_c. \tag{23}$$

The key challenge in determining an equilibrium is that the stochastic discount factor (23) and the optimal investment process must be determined jointly.

3 Equilibrium Allocations and Technological Cycles

3.1 Investment decisions by firms

To start, it will be convenient to define M_t as the running maximum of θ_t :

$$M_t \equiv \max_{s \leq t} \theta_s. \tag{24}$$

Subject to some technical assumptions, proposition 2 in the Appendix shows that there exists a constant $\Xi^* > 0$ such that firm j in round N finds it optimal to plant a tree at time $\tau^*_{j,N}$ given by

$$\tau^*_{j,N} = \inf_{\tau_N \leq t < \tau_{N+1}} \left\{ t : \frac{\theta_t}{M_{\tau_N}} \geq \frac{\Xi^* \left(1 + F\left(i_{N,j}\right)\right)^{\nu}}{\zeta(i_{N,j})} \right\}. \tag{25}$$

Since $F(i_{N,j})$ is increasing in $i_{N,j}$, $\zeta(i_{N,j})$ is decreasing in $i_{N,j}$, and $\nu > 0$, an implication of equation (25) is that firms with more productive trees always plant their trees before firms with less productive trees: the opportunity cost of waiting is larger for the former.

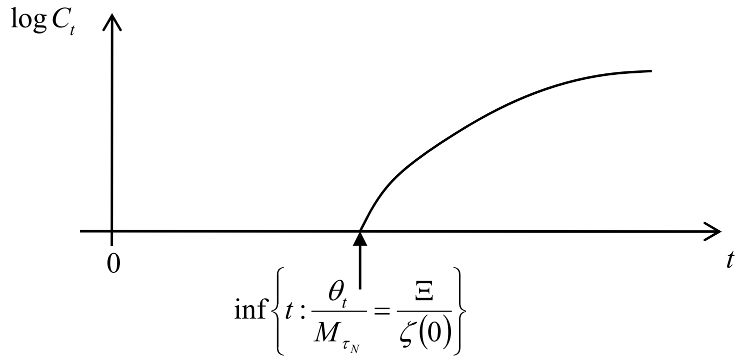

Figure 1: Response of $\log C_t$ to an increase in N

A second implication of policy (25) is that no firm finds it optimal to plant a tree when $t = \tau_N$ (i.e., right at the beginning of the epoch), as long as[22]

$$\frac{\Xi^*}{\zeta(0)} > 1, \tag{26}$$

which we shall assume throughout.

A third and economically important implication of (25) is that in equilibrium there is comovement between the optimal investment decisions of firms. Conditional on $\frac{\theta_t}{M_{\tau_N}}$ reaching the relevant investment threshold $\frac{\Xi^*}{\zeta(0)}$ for the first firm, a number of other firms with $\zeta(i_{j,N}) \approx \zeta(0)$ also find it optimal to invest in close temporal proximity.[23] Figure 1 gives a visual impression of this fact by plotting the impulse response function of an increase in N (i.e., the arrival of a new epoch) on consumption.

As can be seen, in the short run consumption is unaffected, as all firms are waiting to invest. Eventually, however, the firms with the most profitable firms start investing, and hence the most productive investment opportunities are depleted early on. This leaves less attractive investment opportunities unexploited and hence a moderation in the anticipated growth rate of the economy going forward. This delayed reaction of the economy to a major technological shock is consistent with recent findings in the macroeconomic literature (see,

[22]This condition is sufficient to induce waiting because of (25) and $\frac{\theta_t}{M_{\tau_N}} = \frac{\theta_{\tau_N}}{M_{\tau_N}} \leq 1$ at the beginning of epoch N. Hence all firms (even the most productive one) are "below" their investment thresholds.

[23]This is simply because $\zeta(i)$ is a continuous function of i and θ_t is a continuous function of time.

e.g., Vigfusson (2004) and references therein).

3.2 Aggregate consumption and endogenous cycles

Taking logs on both sides of equation (10) gives

$$\log(C_t) = \log(\theta_t) + \log(A_N) + x_t, \tag{27}$$

where x_t is equal to

$$x_t = \log\left(1 + F\left(K_{N,t}\right)\right). \tag{28}$$

Letting

$$m_t \equiv M_t / M_{\tau_N},$$

aggregating across the optimal investment policies implied by (25), and using the expression for $\zeta(i)$ in equation (19) leads to the following closed-form expression[24] for $K_{N,t}$:

$$K_{N,t} = K\left(m_t\right) = \min\left\{\max\left[\left(\frac{1}{b}\left(\frac{bp}{\Xi^*} m_t\right)^{\frac{1}{1-p+\nu p}} - \frac{1}{b}\right), 0\right], 1\right\}. \tag{29}$$

Since the duration between epochs is an exponentially distributed i.i.d. variable, m_t and, consequently, $K_{N,t}$ and x_t are stationary processes. Hence, even though the increments to the exogenous productivity shocks θ_t and the epoch index N are i.i.d., the model produces *endogenous investment-driven cycles,* given by x_t in equation (28).

Figure 2 illustrates the decomposition of log consumption into its components. Letting $E(x_t)$ denote the unconditional expectation of x_t, Figure 2 shows that $x_t - E(x_t)$ can be thought of as a measure of the distance between actual output and its stochastic trend. The figure illustrates how the arrival of a new epoch makes A_N jump upwards, consistent with equation (3). In the short run, this jump in the stochastic trend line is not reflected in the level of consumption, since consumption itself does not jump. However, as time passes and

[24] To see why, observe that the first time that θ_t / M_{τ_N} crosses $\Xi^* / \zeta(i_{N,j})$ is also the first time that m_t crosses $\Xi^* / \zeta(i_{N,j})$.

firms start to invest, the consumption growth rate increases as the most profitable firms exploit their investment opportunities, and slowly decays thereafter. At some point, a new epoch arrives, and this cycle repeats itself.

Figure 2 helps explain the strong negative correlation (-0.9) between shocks to trend and shocks to the cyclical component of consumption, which has been documented in the data by Morley et al. (2002). They interpret this negative correlation as an indication that the economy absorbs permanent innovations with a lag. Our model supports this conclusion: As Figure 2 shows, the arrival of a new epoch implies that the stochastic trend line in the economy jumps up instantaneously. However, the level of consumption remains unchanged. Since (by definition) the cycle is the difference between level and trend, this means that the cyclical component exhibits an offsetting negative jump.[25]

3.3 Equilibrium stochastic discount factor

Differentiating $U(c_t, M_t^C)$ with respect to c_t, and recognizing that $c_t = C_t$, equation (23) implies the following expression for the stochastic discount factor:

$$H_t = e^{-\rho t} C_t^{-\gamma+(\gamma-1)(1-\alpha)} \left(\frac{C_t}{M_t^C}\right)^{(1-\gamma)(1-\alpha)}. \tag{30}$$

Because firms follow threshold policies and A_N and $K_{N,t}$ are non-decreasing, in equilibrium we obtain $C_t/M_t^C = \theta_t/M_t$.[26] Furthermore, the explicit expression for $K_{N,t}$ implies that

$$H_t = e^{-\rho t} \left[A_N \theta_t \left(1 + F(K(m_t))\right)\right]^{-\gamma+(\gamma-1)(1-\alpha)} \left(\frac{\theta_t}{M_t}\right)^{(1-\gamma)(1-\alpha)}. \tag{31}$$

Equation (31) together with equation (29) provide an explicit expression for the stochastic discount factor, in terms of the exogenous shocks to the model. Proposition 3 in the appendix

[25] Of course, as time passes, positive shocks to the trend θ_t make firms invest, and hence translate into positive cyclical shocks, mitigating the negative correlation.

[26] Since both $K_{N,t}$ and A_N are non-decreasing processes we obtain the inequality $M_t^C = \max_{s\leq t} C_s \leq A_N M_t(1 + F(K_{N,t}))$. The threshold form of the optimal investment policies in equation (25) implies that $K_{N,s}$ increases only when $\theta_s = M_s$. Therefore, there always exist some time $s^* \leq t$ such that $C_{s^*} = A_N \theta_{s^*}(1 + F(K_{N,s^*})) = A_N M_{s^*}(1 + F(K_{N,s^*}))$. Combining this last observation with the upper bound $M_t^C \leq A_N M_t(1 + F(K_{N,t}))$ implies that $M_t^C = A_N M_t(1 + F(K_{N,t}))$, and hence $C_t/M_t^C = \theta_t/M_t$.

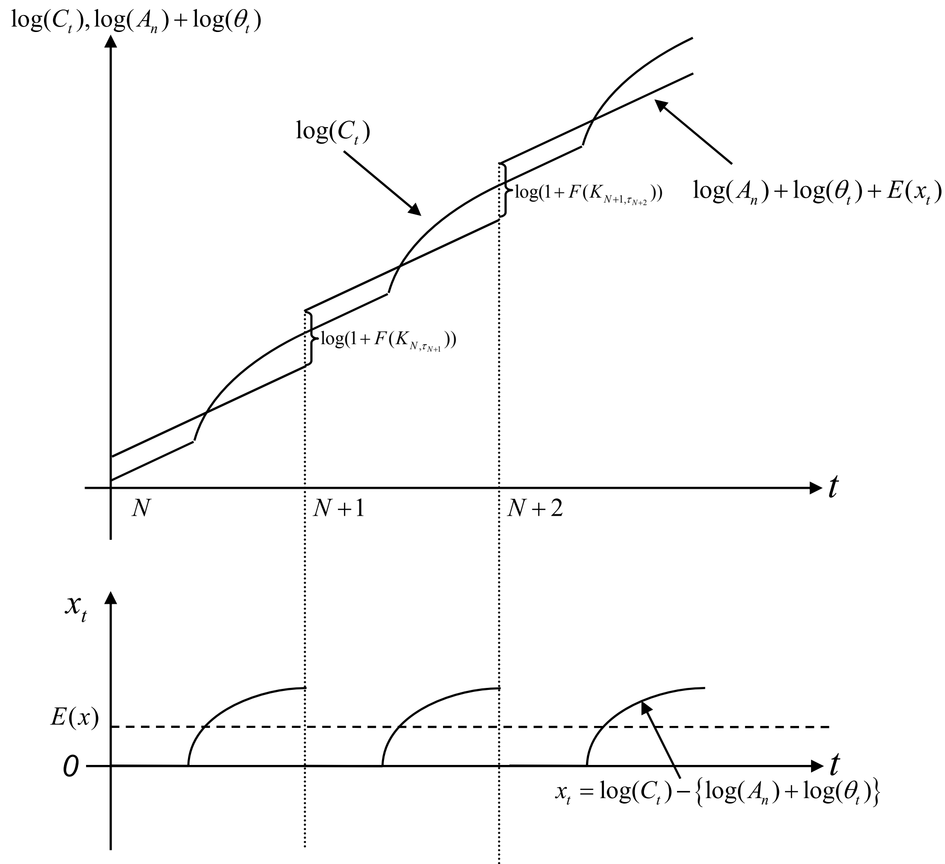

Figure 2: Decomposition into trend $\log(\theta_t) + \log(A_n) + E(x)$ and cycle $x_t - E(x)$. The figure plots consumption for a path with $dB_t = 0$ and $K_{n,\tau_{n+1}} > 0$.

uses this explicit expression to obtain closed form expressions for individual firm prices.

4 Qualitative and Quantitative Implications

In this section we discuss the qualitative implications of the model and illustrate its quantitative ones. We proceed by explaining how we calibrate the model, and in subsequent subsections discuss the cyclical properties of expected returns, the correlation patterns between excess return and consumption growth, excess-return predictability, and cross-sectional

μ	0.012	γ	9	b	0.8
σ	0.030	ρ	0.012	p	0.6
λ	0.1	α	0.1	η	22.6

Table 1: Parameters used for the calibration

implications.

4.1 Calibration

Before presenting some of the quantitative implications of the paper, we discuss here how we chose the model parameters for the simulations. Table 1 presents our choice of the 9 parameters for the baseline calibration exercise. There are three parameters that are related to the distribution of the exogenous shocks (μ, σ, and λ), four parameters that pertain to preferences (ρ, γ, α, and η), and two parameters (p and b) that control the function $\zeta(i)$, i.e., the the degree of heterogeneity across trees that can be planted in a given epoch. We choose μ to match the contribution of (neutral) total factor productivity to aggregate growth. Hulten (1992) computes that number to be 1.17%, which motivates our choice of $\mu = 0.012$. The parameter σ controls the volatility of consumption. We set it to $\sigma = 0.03$, in order to match the volatility of time-integrated consumption data.[27] The parameters $\lambda, p, b,$ and η control the growth contribution of the quality and quantity increase in trees (capital goods), the speed of adoption of new trees, and the time variation in consumption growth rates. We follow Comin and Gertler (2006), who estimate the frequency of technology-driven "medium-run" cycles and set $\lambda = 0.1$. The parameters b and η control (respectively) the contribution of new capital vintages to aggregate growth per epoch and the time it takes until firms start planting trees. As a result they control the total consumption growth rate and the cyclical effects of technology adoption. We choose these parameters to approximately match a) the total consumption growth rate in the data and b) the autocorrelation properties

[27] As is well understood, time integration makes the volatility of time-integrated consumption data lower than the instantaneous volatility of consumption.

of consumption. Finally, the parameter p controls the curvature of the function $\zeta(i)$ and hence the acceleration in consumption growth once firms start adopting new technologies. To measure this acceleration in growth due to adoption of a new technology, we use the difference in consumption growth rates between 1980-1994 and 1995-2000, which is about 1.1%. We choose p to approximately match such a difference in growth rates between the initial stages of the epoch (when no firm invests) and the latter stages of the epoch (when firms start investing). In terms of the preference parameters ρ, γ, and α we choose ρ and α so as to a) match the low level of real interest rates in the data and b) obtain plausible degrees of intertemporal elasticity of substitution (IES) with respect to changes in an agent's consumption that are accompanied by changes in the aggregate habit level. Specifically, as we explain in footnote 15, $\gamma + (\gamma - 1)(\alpha - 1)$ provides a measure of the inverse of the IES with respect to changes in an agent's consumption that are accompanied by changes in the aggregate habit level. With $\alpha = 0.1$, the implied IES with respect to such shocks is about 0.55, well within the reasonable range of values estimated in the literature (see, e.g., Attanasio and Vissing-Jorgensen (2003)). Finally, we choose $\gamma = 9$, which is sufficient to match the average equity premium.

Table 2 compares the model's performance to some unconditional moments in the data. The overall performance of the model in terms of unconditional time-series moments is comparable to the pure-endowment models of external habit formation, such as Abel (1990) and Chan and Kogan (2002). We note that the model manages to reproduce unconditional asset-pricing moments despite the presence of investment, which typically deepens the usual asset-pricing puzzles[28]. The reason is the presence of habit formation (see Jermann (1998) for a discussion), and also the fact that consumption and investment goods are produced with different technologies. This latter property of the model makes it impossible to mitigate the effects of bad productivity shocks on consumption by simply running down the capital

[28] For instance, Kaltenbrunner and Lochstoer (2006) show that it is challenging (with general Epstein-Zin-Weil preferences) to match both the return volatility and the equity premium, if one also insists on the fact that productivity should exhibit a strong random walk component.

	Data	Model
Mean of consumption growth	0.017[a]	0.016
Volatility of consumption growth	0.033[a]	0.027
Mean of 1-year zero coupon yield	0.029[a]	0.010
Volatility of 1-year zero coupon yield	0.03[a]	0.060
Mean of Equity Premium (logarithmic returns)	0.039[a]	0.041
Volatility of Equity Premium	0.18[a]	0.176

Table 2: Unconditional moments of the model and the data (annualized rates). All data are from the long sample (1871-2005) in Campbell and Cochrane (1999), with the exception of the volatility of the 1-year zero coupon yield which is from Chan and Kogan (2002). The unconditional moments for the model are computed from a Monte Carlo Simulation involving 12000 years of data, dropping the initial 1000 to ensure that initial quantities are drawn from their stationary distribution. The time increment dt is chosen to be 1/60. From the simulated paths we time aggregate consumption and dividends and obtain quarterly series for consumption growth, dividend growth, returns and interest rates, which we then convert to annualized rates.

stock (as in the standard production-based model) and hence raises the riskiness of stocks and the equity premium.

Table 3 shows that the quarterly consumption autocorrelations implied by the model are about as large as in the data. Consistent with the data, the autocorrelations implied by the model are small and decay rapidly. The intuition for this finding is that only a small fraction of the variability of consumption comes from the cyclical component x_t.

4.2 Expected returns over the course of a technological epoch

Having studied the properties of aggregate consumption in the model, we now turn to a discussion of the model implications for asset returns, which are the main focus of this paper. In this subsection we start by explaining how the model can reproduce (qualitatively) a pattern of high expected returns at the early stages of a technological adoption cycle,

Quarter	1	2	3	4
Data	0.35	0.20	0.23	0.06
Model	0.32	0.07	0.03	0.02

Table 3: Quarterly consumption autocorrelations - data and model. The consumption data are per capita real consumption expenditures on nondurables and services (1952-2008). Source: St. Louis FED (FRED Database). Simulated data are time-integrated over a quarter.

followed by an investment-driven boom and low subsequent returns. The three subsections that follow assess the implications of this expected return time-variation from a quantitative perspective.

The price of a firm, given by equation (8), consists of three components: 1) the value of assets in place, 2) the value of growth options in the current technological epoch, and 3) the value of growth options in all subsequent technological epochs.

In analogy to an individual firm, the value of the aggregate stock market can be decomposed into the values of assets in place, of current-epoch options, and of future-epoch options. Such a decomposition shows that the relative weight of growth options is countercyclical at the aggregate. When the current level of consumption is below its stochastic trend (i.e., the cycle component x_t is below its unconditional mean), there are a large number of unexploited investment opportunities for firms. Accordingly, the relative weight of growth options is high. In contrast, when consumption is above its trend level, several of the most profitable investment opportunities have been exploited, and the relative importance of growth options is small at the aggregate.

Growth options command a higher expected return than assets in place. Intuitively, a growth option can be viewed as a call option where the underlying is an asset in place. Since a call option is a levered position on the underlying claim, it commands a higher expected return than that claim[29]. Since the expected excess return on the market is a weighted

[29]This basic intuition has been emphasized by Carlson et al. (2004) and Carlson et al. (2006) in a partial equilibrium setting.

average[30] of the excess returns of assets in place and growth options (current and future), the counter-cyclicality of the relative importance of growth options implies the counter-cyclicality of excess returns.

An additional implication of our analysis concerns the lead-lag relationship between investment in new trees and excess returns at the aggregate. Once the first firm invests, several other firms with productivity close to the most productive firm will follow in close succession. Since this investment-driven boom signifies the exercise of the most profitable growth options, it coincides with a decline in the expected excess returns going forward. This negative relation between investment and future excess returns has been documented in the empirical literature (see, e.g., Lamont (2000)).

To summarize, our model implies a theory for the countercyclical behavior of aggregate returns that is driven by the composition of growth options and assets in place in the firm's value and is linked to the investment decisions of firms. In the next subsection we examine some empirical implications of this investment-based view of predictability.

4.3 The correlation between consumption and returns

Besides providing a theory for the countercyclicality of expected returns, the investment-based view of return predictability helps explain two additional salient patterns in the data: a) the correlation between excess return and consumption growth increases with the horizon and b) excess returns lead aggregate growth.

Table 4 illustrates the first pattern in the data. The correlation between consumption growth and (excess) returns in the data is 1.76 times higher for 3-year intervals than for quarterly intervals. Specifically, it increases from 0.17 to 0.3 as one moves from quarterly to 3-year intervals. As Daniel and Marshall (1999) document, this is a manifestation of a more general phenomenon: the correlation between consumption growth and excess returns increases at lower frequencies. The second row of table 4 illustrates this finding. Applying a Baxter and King (1999) filter to isolate cycles that last less than 1.5 years (high frequency

[30] The weights are given by the fractions of stock market value that are due to each of the three components.

Correlations	Data	Model
$\frac{\text{corr. cons. growth and returns (3-year-intervals)}}{\text{corr. cons. growth and returns (quarterly)}}$	1.76	1.63
$\frac{\text{Bandpass filtered returns and consumption (low frequency)}}{\text{Bandpass filtered returns and consumption (high frequency)}}$	2.93	2.60
Lead-Lag Relationships		
p-value (Consumption does not Granger-cause returns)	0.45	0.48
p-value (Excess returns do not Granger-cause consumption)	4×10^{-4}	0

Table 4: Correlations between consumption growth and excess returns and lead-lag relationships. Consumption data include the full post WWII sample on non-durables and services as provided by the St. Louis FED, and returns are value weighted CRSP returns. The first row reports the ratio of excess-return and consumption correlation over 3-year intervals divided by the respective correlation over a quarter. The second row reports the ratio of correlation between excess return and consumption at low frequencies divided by the respective correlation at high frequencies. We computed this ratio by using the Baxter and King (1999) filter to isolate "high frequencies" (swings smaller than 1.5 years) and "low frequencies" (swings between 1.5 and 8 years) in both consumption and excess returns, and computed the respective correlations. (See Daniel and Marshall (1999) for details.) The third and fourth rows contain the results of standard Granger-causality tests (using 2 autoregressive lags and quarterly data.)

movements) and cycles that last between 1.5 and 8 years (low-frequency movements), we find that the low-frequency correlation (0.44) is almost 3 times as high as the high-frequency correlation (0.15).

As Yu (2007) shows, this increasing correlation between consumption and excess returns at longer horizons/lower frequencies presents a challenge for leading single-shock, endowment-based asset pricing models that derive return predictability exclusively from time variation in risk aversion. (See, e.g., Campbell and Cochrane (1999).) Such models typically produce the opposite pattern (higher correlations at shorter horizons / higher frequencies).[31]

The present model helps address this limitation by introducing two types of technological shocks. Shocks to θ_t affect both consumption and returns on impact. However, the arrival of technological epochs produces different reactions in consumption and returns in the short run and in the long run. In the short run, the arrival of a new epoch raises expected returns, as the new growth options raise the riskiness of the stock market. However, average consumption growth declines in the short run, since the old growth options become obsolete and it is not profitable to plant the new vintages yet. It is only after the passage of some time that the new technology boosts consumption growth. The interplay of these two shocks helps explain why consumption is weakly correlated with returns in the short run, whereas the correlation becomes stronger in the long run.

Table 4 illustrates these effects, by comparing correlations in the data with the equivalent correlations in simulated data. Similar to the data, the model is able to reproduce the increase in correlation as one moves to longer horizons (lower frequencies).[32]

The model can also provide an explanation for the fact that asset returns tend to lead aggregate growth rates, consistent with the data. This fact is illustrated in the last two rows of table 4, which show that the data reject the hypothesis that excess returns do not Granger-cause consumption, but do not reject the reverse (namely that consumption does

[31] Intuitively, the reason is that such models impose a negative correlation between excess returns and a smooth average of past consumption. This negative correlation attenuates the correlation between consumption and excess returns at low frequencies, since the moving average of past consumption acts as a low-pass filter that isolates low-frequency movements in consumption.

[32] Even though not important for our analysis, we note that the model produces higher correlations between consumption and excess returns compared to the data. For instance the quarterly correlation between consumption and excess returns in the model is 0.55, while the high-frequency correlation is 0.33. (The respective numbers in the data are 0.17 and 0.15.) This is driven by the fact that the only mechanism that separates consumption and dividends inside the model is the presence of investment. Adding labor as an additional factor of production for consumption goods and a countercyclical labor share (as in Gârleanu and Panageas (2007)) or allowing entry of new firms (as in Gârleanu et al. (2009)) are simple ways to reduce these correlations, but we do not pursue them here for parsimony.

not Granger-cause returns). Yu (2007) and Backus et al. (2008) provide further evidence on these lead-lag patterns, and Yu (2007) shows that this pattern presents a challenge for some leading, single-shock, pure-endowment based models. Our model can reproduce these patterns in the data for a simple reason: Slightly before the onset of new-technology adoption, expected returns are high, since the aggregate amount of outstanding growth options is large. As time passes, and θ_t grows, firms start planting trees and consumption growth accelerates. Therefore, high (expected) asset returns anticipate an acceleration of consumption growth.

We conclude this subsection by noting that our analysis does not deny the importance of other mechanisms for return time variation (such as time varying risk aversion), nor does our model preclude their inclusion into our framework. Our analysis simply illustrates the benefits from augmenting the commonly used, single-shock, pure-endowment asset pricing models to allow for both embodied and disembodied shocks in an investment framework.

4.4 P/D Predictability

We conclude the discussion of the time-series properties of returns by performing the usual predictability regressions of aggregate excess returns on the aggregate log P/D ratio. Table 5 tabulates the results of these regressions, and compares them to the data. Because of well documented small-sample issues in return-predictability regressions, we simulate one thousand independent samples of 100-year-long paths of artificial data. We run predictability regressions for each of these samples and report the average coefficient along with a 95% distribution band. We then compare these simulations to the equivalent point estimates in the data.

The coefficients in the simulations have the right sign, but are about one third of their empirical counterparts. Most of the empirical point estimates, however, are within the 95% distribution band according to the model.

It is useful to relate our results to Chan and Kogan (2002), who study an endowment economy with preferences similar to equation (12) and show that in the absence of risk-aversion heterogeneity, the (log) P/D ratio predicts excess returns with a positive rather

P/D Predictive Ability

	Data		Model	
Horizon(years)	Coefficient	R-square	Coefficent	R-square
1	-0.120	0.040	-0.051	0.005
			(-0.363, 0.049)	(0.000, 0.112)
2	-0.300	0.100	-0.105	0.013
			(-0.431, 0.108)	(0.000, 0.100)
3	-0.350	0.110	-0.156	0.017
			(-0.500, 0.152)	(0.000, 0.114)
5	-0.640	0.230	-0.208	0.022
			(-0.720, 0.225)	(0.000, 0.172)
7	-0.730	0.250	-0.247	0.023
			(-0.894, 0.279)	(0.000, 0.240)

Table 5: Results of predictive Regressions. Excess returns in the aggregate stock market between t and $t+T$ for $T = 1, 2, 3, 5, 7$ are regressed on the P/D ratio at time t. A constant is included but not reported. The data column is from Chan and Kogan (2002). The simulations were performed by drawing 1000 time series of a length equal to 100 years. We report the means of these simulations next to the respective point estimates in the data. The numbers in parentheses are the 95% confidence interval of the estimates obtained in the simulations.

than a negative sign[33]. Indeed, in the absence of investment, our model would share the same features as the model of Chan and Kogan (2002) with homogenous risk aversion; specifically, increases in the surplus ratio $\frac{C_t}{M_t^C} = \frac{\theta_t}{M_t}$ would raise both the P/D ratio and excess returns. With investment, however, increases in consumption also impact the relative importance of aggregate growth options. In particular, during the onset of a cycle this weight increases and hence expected excess returns increase. Simultaneously, anticipations of increased consumption growth raise interest rates and lower P/D ratios.[34] Once real-

[33] This leads Chan and Kogan (2002) to consider a model with agents that have heterogenous risk aversion.

[34] As we show in footnote 15, the magnitude of the interest rate reaction depends on the value of α. That

options start being exercised, on the other hand, the economy experiences a combination of increased investment, higher P/D ratios (due to lower interest rates) and lower expected excess returns. Thus, the presence of investment counteracts the effect identified by Chan and Kogan (2002), and allows the model to match the negative relationship between (log) P/D ratios and expected excess returns.

In results that we do not report here to save space, we also ran regressions similar to Duffee (2005). He finds a negative relation between conditional excess returns and the conditional covariance of consumption and excess returns (as predicted by the stock price to consumption ratio and other instruments) over quarterly horizons. Using the stock price to consumption ratio as an instrument, we found similar results over quarterly horizons, even though in our model return predictability is driven by time variation in the instantaneous covariance between consumption growth and excess returns. The reason is that the true conditional covariances and the true conditional returns are unobserved; hence one needs to use imperfect and persistent instruments over a short sample to predict time variation in returns, and conditional covariances, which can attenuate or reverse the true link between the two quantities.

4.5 Cross-sectional implications

The expected-return patterns that we have described so far are more pronounced for the firms that can plant the most productive trees, i.e., the firms that are likely to profit most from the new technology. Specifically, for firms whose value is comprised mostly of growth options, the model predicts a pattern of high expected returns up to the time of investment followed by a discontinuous drop in expected returns as growth options are converted into assets in place. This return pattern has been derived in a partial equilibrium setting by Carlson et al. (2004) and continues to manifest itself in our general-equilibrium framework.

footnote also shows that the intertemporal elasticity of substitution is always bounded below one in our model, so that the decline in the P/D ratio due to increased discount rates is stronger than the increase in the P/D ratio due to anticipations of increased dividend growth.

This observation has implications for two well-documented patterns in the cross section of expected returns: the size premium and the value premium.

To see why the model is able to produce a size premium, it is easiest to consider a firm j that has a higher market value of equity (size) than firm j', so that $P_{N,j,t} > P_{N,j',t}$. To simplify the analysis, assume further that both of these firms have exercised their growth option in the current epoch, so that $P^o_{N,j,t} = P^o_{N,j',t} = 0$. Since the future growth options are the same for both firms, the relative importance of growth options for firm j must be smaller, and hence firm j must therefore have a lower expected return. Hence, assuming that one could safely ignore current epoch growth options,[35] a sorting of companies based on size would produce a size premium.

The model is also consistent with the value premium. This may seem counterintuitive at first, since one would expect that firms with a high market-to-book ratio should have a substantial fraction of their value tied up in growth options, and hence should be riskier. The resolution of the puzzle is similar to Gomes et al. (2003). Specifically, trees are heterogenous in the model, and accordingly the market-to-book ratio of a given firm reflects primarily the average productivity of its existing trees.

The easiest way to see how tree heterogeneity helps account for the value premium is to consider two firms j and j' that have planted a tree in every single epoch, including the current one. As a result, the two firms have identical book values and identical growth options. However, suppose that firm j has always been "luckier" than firm j' in terms of the productivity of the trees it has had the opportunity to plant. Then the market value of firm j will be higher than the market value of firm j', because the value of its assets in place is higher. The growth options of the two firms are identical, and hence the total value of firm j is larger than the total value of firm j', and firm j has a *smaller* fraction of its value tied up

[35] The presence of current epoch growth options distorts the perfect ranking of expected returns implied by size. Intuitively, high market values may be associated with a valuable current-period growth option (in which case expected returns should be high) instead of numerous assets in place (in which case expected returns should be low). For the calibrations that we consider, however, we find that current-epoch growth options are not quantitatively important enough to affect the size effect.

in growth options. Accordingly, firm j has a lower expected return than firm j'. Since the book values of the two firms are identical, firm j has a lower book-to-market ratio than firm j'. This is consistent with the well known fact that firms with a low book-to-market ratio have a low expected return (the value premium).[36]

Even though not at the core of our analysis, we note that the model is also consistent with additional cross-sectional properties of the data. Thus, since high-size (and high-growth) firms typically have trees with higher productivity on average, the model is consistent with the empirical evidence reported in Fama and French (1995) that sorting on size and value produces predictability for a firm's profitability (earnings-to-book ratio). The model is also consistent with the evidence that small firms tend to grow faster than large firms.[37] Finally, the model also predicts that firms with a low book-to-market ratio (high Tobin's q) tend to exhibit stronger investment activity (as measured by the growth in the book value of assets). The intuition for this is simple: A high Tobin's q (low book to market) reflects a) the productivity of existing trees, but also b) the magnitude of growth options compared to the current capital stock of the firm. The first component drives expected returns down as we showed above, but is irrelevant for predicting the growth rate in the capital stock. However, the second component predicts the growth in the capital stock. The interplay of these two forces can help explain the joint presence of a value premium along with a weak positive correlation between Tobin's q and the investment-to-capital ratio.

Table 6 reports results on the cross sectional predictability of returns. In order to match more accurately the cross-sectional distribution of size and book-to-market dispersion, we introduced idiosyncratic (disembodied) tree-specific shocks. To motivate such shocks, we note that so far we have made the assumption that technology is fully embodied in the new

[36]Of course, the presence of current-period growth options distorts the ranking of expected returns implied by the above argument. As we show below, in a calibrated version of the model this distortion is not powerful enough to substantially affect the value effect.

[37]The reason is mean reversion: In expectation all firms have the same book value of trees (after detrending by $A_{\tau_N}\theta_t$) in the long run. Hence firms who are below that stationary value at a given time can be expected to grow faster and vice versa.

Portfolios formed on Size (Stationary Distribution)										
Deciles	1	2	3	4	5	6	7	8	9	10
log(Size) – Data	-2.45	-1.36	-0.82	-0.40	0.00	0.40	0.82	1.34	1.99	3.51
log(Size) – Sim.	-2.04	-1.28	-0.79	-0.39	0.00	0.38	0.77	1.16	1.72	3.61
Returns(Size) – Data	13.91	11.72	11.63	11.07	10.53	10.44	9.88	9.13	8.53	7.00
Returns(Size) – Sim.	7.96	6.55	5.79	5.72	5.79	5.85	6.00	6.17	5.77	5.63
Portfolios formed on book to market (Stationary Distribution)										
Deciles	1	2	3	4	5	6	7	8	9	10
log(BM) – Data	-1.47	-0.88	-0.59	-0.38	-0.20	-0.03	0.14	0.34	0.60	1.22
log(BM) – Sim.	-2.94	-1.68	-0.98	-0.47	-0.04	0.32	0.66	1.03	1.52	2.50
Returns(BM) – Data	6.65	7.86	8.05	7.73	8.53	9.01	9.21	11.05	12.00	12.67
Returns(BM) – Sim.	5.71	5.77	5.77	5.87	6.04	6.28	6.19	6.15	6.30	7.17

Table 6: Portfolios sorted by size and book to market – model and data. The data are from the website of Kenneth French. Time period: 1927-2009. Average returns per decile are based on monthly data, which are converted to annualized rates. We subtract 3.09% from all returns to account for the average CPI inflation between 1927 and 2009. The median (log) firm size is normalized to zero.

trees. However, in reality new technological paradigms also affect the internal organization of firms, their marketing practices, and, potentially the way existing technologies are used in the production process. Hence, the arrival of a new epoch may affect the profitability of *existing* trees. To account for this possibility, we allowed for the presence of tree-specific shocks $Z(i,t)$, so that the time-t output of tree $i \in [0,1]$ that is planted at time s in epoch N is given by $A_N \zeta(i) Z(i,t) \theta_t$. The shock $Z(i,t)$ is equal to one at the time s that the tree is planted — i.e., $Z(i,s) = 1$ — thereafter stays constant within each epoch — i.e., $Z(i,t) = Z(i,\tau_N),\ t \in [\tau_N, \tau_{n+1})$ — and jumps between epochs so that $Z(i,\tau_{N+1}) = Z(i,\tau_N) u(i,\tau_{N+1})$, where $u(i,\tau_{N+1})$ is i.i.d. across trees and epochs, distributed lognormally with mean 1 and variance $\sigma^2_{u(\tau_{N+1})}$, and independent of all other shocks in the model.

By their construction, the idiosyncratic shocks $Z(i,t)$ do not affect a firm's optimal stopping problem, the stochastic discount factor, or any other aggregate quantity. Hence, they do not affect any of the conclusions of the paper so far. They simply add more variability to the stationary cross-sectional distribution of size and book-to-market ratios, so as to allow matching these distributions more accurately. With this goal in mind, we choose the variance $\sigma_{u(\tau_{N+1})} = 2$, thus approximately matching the deciles of each of the two distributions.[38] Table 6 shows that returns sorted by book-to-market and size replicate the qualitative patterns in the data. The magnitudes, however, are smaller.

Having illustrated that our model is consistent with some well documented cross-sectional asset-pricing puzzles, we would also like to make it clear that its parsimonious structure and focus on one mechanism has its limitations. One such limitation is that, in calibrations, sorting on one of the two effects (size or value) drives out the other. This is linked to the fact that within the model only one source of risk is reflected in the stochastic discount factor. Therefore, as long as one of the two sorting procedures leads to a satisfactory ranking of the conditional betas, the other sorting procedure adds little.[39] Gârleanu et al. (2009) propose a model in which the stochastic discount factor rewards multiple sources of risk because of a lack of intergenerational risk sharing and rivalry between technological innovations. Within such a model value and size premia could potentially be obtained as independent effects, but such an extension is beyond the scope of the current paper.

[38] A technical condition to ensure stationarity of the cross sectional size distribution is $lim_{N\to\infty}\sigma^2_{u(\tau_{N+1})} = 0$. In the simulations we enforce this condition by simply assuming that the idiosyncratic shocks have constant variance σ_u^2 for M epochs after the tree is planted and zero variance thereafter. We chose σ_u^2 and M as free parameters to match as closely as possible the 20 cross sectional moments of the size and the book to market distribution. Specifically, we choose $\sigma_u = 2$ and $M = 2$.

[39] In this connection we also note that (unconditional) market beta cannot explain the dispersion in excess returns, since they do not generate sufficient conditional-beta variation.

5 Conclusion

We proposed a model of technological change that posits, in addition to the usual small, embodied shocks, major disembodied ones that affect output only following new investment. Whereas it takes a while for the investment in the new technologies to become viable and thus translate into higher output, asset prices react immediately to their emergence, giving rise to the type of lead-lag relationship between returns and consumption that has been documented in the data. Related, the correlations between returns and consumption growth increase with the horizon, also as in the data. During the early stages of the adoption cycle consumption growth is low, while excess returns, driven by the relatively numerous real options, are high; the pattern reverses once investment increases the growth rate of consumption and the ratio between the values of assets in place and growth options. This investment-driven countercyclicality of discount rates can also generate the positive predictability of excess returns by the aggregate P/D ratio and is also consistent with such cross-sectional phenomena as the value and size premia.

This investment-based approach to expected-return time variation is distinct from existing endowment-based approaches that build on either countercyclical risk aversion or stochastic consumption volatility that anticipates long-run consumption growth. Besides providing an intuitive mechanism for expected-return variability, the investment-based approach is consistent with a host of joint time-series properties of returns and consumption; it also is parsimonious, since it assumes that all underlying shocks are i.i.d.. Finally, it helps explain the ability of investment to predict returns both in the time series[40] and the cross section.[41]

Our goal in this paper was to isolate the mechanism that links adoption of new technologies and return time variation. In order to facilitate exposition, we intentionally suppressed other channels that could also lead to return time variation within our framework. We recognize, however, that combining our setup with, for instance, some degree of time-varying risk

[40]See, e.g., Lamont (2000) and Hsu (2009).

[41]See, e.g., Titman et al. (2004).

aversion or Epstein-Zin-Weil preferences could help further strengthen the model's ability to explain asset-pricing data. The latter extension seems particularly promising in our framework because disembodied shocks that affect the economy with a lag are a natural source of "long-run risk", i.e., low-frequency consumption-growth predictability. Because such an extension does not allow closed-form solutions and introduces a series of new insights and issues, we leave it for future work.

A Appendix

A.1 Propositions and Proofs

In this appendix we prove that there exists an appropriate constant $\Xi = \Xi^*$ such that if a firm perceives the equilibrium process for $K_{N,t}$ to be given by (29) and the stochastic discount factor to be given by (31) then that firm will optimally plant a tree the first time that θ_t reaches the threshold value given by equation (25). We also provide closed-form expressions for the equilibrium value of any firm j in round N at time t.

We start by defining some constants and functions that appear repeatedly in the proof. Specifically, let constants γ_1, and γ_1^* be defined as

$$\gamma_1 \equiv \frac{\sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2(\rho+\lambda)} - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2} > 0,$$

$$\gamma_1^* \equiv \frac{\sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2\rho} - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2} > 0,$$

and the constants β_1 and β_2 as

$$\beta_1 \equiv \frac{1}{\rho + \lambda - \mu(1-\gamma) + \gamma(1-\gamma)\frac{\sigma^2}{2}},$$

$$\beta_2 \equiv \frac{-\nu p}{1 - p + \nu p} - \nu - \gamma_1 < 0,$$

We assume that $\beta_1 > 0$. Furthermore, let the functions $g_1(x)$, and $\tilde{g}_1(x)$, be given by

$$g_1(x) \equiv \beta_2 + \gamma_1 + \gamma + x,$$

$$\tilde{g}_1(x) \equiv \frac{\alpha(1-\gamma)p}{1-p+\nu p} + (\gamma - 1)(1-\alpha) + x,$$

the functions $g_2(x)$ and $\tilde{g}_2(x)$ be defined as

$$g_2(x) \equiv \frac{(\gamma-1)(1-\alpha) + x}{\alpha\gamma - \alpha + \gamma_1} + \frac{g_1(x)}{1+\beta_2},$$

$$\tilde{g}_2(x) \equiv \frac{(\gamma-1)(1-\alpha) + x}{\alpha\gamma - \alpha + \gamma_1} + \frac{\tilde{g}_1(x)}{1+\beta_2 + \frac{p}{1-p+\nu p}},$$

and $g_3(x)$ be given by

$$g_3(x) \equiv \frac{\lambda}{(\rho+\lambda) + \frac{\sigma^2}{2}(\gamma+x)(1-\gamma-x) - \mu(1-\gamma-x)}.$$

A useful first result is contained in the following Lemma.

Lemma 1 *Fix a constant $\Xi \geq bp$ and suppose that $K_{N,t}$ is given by*

$$K_{N,t} = K\left(\frac{M_t}{M_{\tau_N}}\right) = \min\left\{\max\left[\frac{\left[\left(\frac{bp}{\Xi}\right)\frac{M_t}{M_{\tau_N}}\right]^{\frac{1}{1-p+\nu p}} - 1}{b}, 0\right], 1\right\}, \tag{32}$$

C_t is given by $C_t = \theta_t X_{\tau_n}(1 + bK_{N,t})^p$, and H_t is given by $H_t = e^{-\rho t} C_t^{-\gamma+(\gamma-1)(1-\alpha)} \left(\frac{\theta_t}{M_t}\right)^{(1-\gamma)(1-\alpha)}$. Define $m_t \equiv \frac{M_t}{M_{\tau_{N_t}}}$, and also let

$$\begin{aligned} g_4(x) &\equiv g_3(x)\left[\frac{(\gamma-1)(1-\alpha)+x}{\alpha(\gamma-1)+\gamma_1} + \left(\frac{bp}{\Xi}\right)^{\alpha(\gamma-1)+\gamma_1} g_2(x)\left[(1+b)^{(1-p+\nu p)(1+\beta_2)} - 1\right]\right], \\ \tilde{g}_4(x) &\equiv g_3(x)\left[\frac{(\gamma-1)(1-\alpha)+x}{\alpha(\gamma-1)+\gamma_1} + \left(\frac{bp}{\Xi}\right)^{\alpha(\gamma-1)+\gamma_1} \tilde{g}_2(x)\left[(1+b)^{(1-p+\nu p)(1+\beta_2)+p} - 1\right]\right]. \end{aligned}$$

Furthermore, let the constants α_1 and m^* de defined as

$$\begin{aligned} \alpha_1 &= \left[\frac{(\gamma-1)(1-\alpha)}{\alpha(\gamma-1)+\gamma_1} + \left(\frac{bp}{\Xi}\right)^{\alpha(\gamma-1)+\gamma_1} g_2(0)\left[(1+b)^{(1-p+\nu p)(1+\beta_2)} - 1\right]\right]\beta_1, \\ m^* &= \frac{\Xi}{bp}(1+b)^{1-p+\nu p}, \end{aligned}$$

and the constants Δ_1 and Δ_2 be given by

$$\Delta_1 = -\frac{\alpha_1 + \frac{\beta_1}{1-\lambda\beta_1} g_4(0)}{g_4(1-\gamma-\gamma_1^*)}, \tag{33}$$

$$\Delta_2 = \frac{\beta_1}{1-\lambda\beta_1}. \tag{34}$$

We assume throughout that $\Delta_1 > 0$. Finally, let χ_t denote the following conditional expectation:

$$\chi_t \equiv E_t \int_t^\infty e^{-\rho(s-t)} \left(\frac{C_s}{C_t}\right)^{-\nu} \left(\frac{\frac{M_s}{\theta_s}}{\frac{M_t}{\theta_t}}\right)^{(\gamma-1)(1-\alpha)} \frac{\theta_s}{\theta_t} ds. \tag{35}$$

Then $\chi_t = \chi\left(\frac{\theta_t}{M_t}, m_t\right)$, where

$$\begin{aligned} \chi\left(\frac{\theta_t}{M_t}, m_t\right) &= \Delta_2\left\{1 + \left(\frac{\theta_t}{M_t}\right)^{\gamma-1+\gamma_1}\left[\frac{(\gamma-1)(1-\alpha)}{\alpha(\gamma-1)+\gamma_1} + \left(\frac{bpm_t}{\Xi}\right)^{\alpha(\gamma-1)+\gamma_1} g_2(0)\left((1+b)^{(1-p+\nu p)(1+\beta_2)} - 1\right)\right]\right\} \\ &\quad + \Delta_1\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\left\{1 + \left(\frac{\theta_t}{M_t}\right)^{\gamma_1-\gamma_1^*}\left[\begin{array}{c} -\frac{\alpha(\gamma-1)+\gamma_1^*}{\alpha(\gamma-1)+\gamma_1} + \left(\frac{bpm_t}{\Xi}\right)^{\alpha(\gamma-1)+\gamma_1} \times \\ g_2(1-\gamma-\gamma_1^*)\left[(1+b)^{(1-p+\nu p)(1+\beta_2)} - 1\right] \end{array}\right]\right\} \end{aligned}$$

when $m_t \leq \frac{\Xi}{bp}$,

$$\chi\left(\frac{\theta_t}{M_t}, m_t\right) = \Delta_2 \left\{ 1 + \left(\frac{\theta_t}{M_t}\right)^{\gamma-1+\gamma_1} \left[-\frac{g_1(0)}{1+\beta_2} + \left(\frac{bpm_t}{\Xi}\right)^{-(1+\beta_2)} (1+b)^{(1-p+\nu p)(1+\beta_2)} g_2(0) \right] \right\}$$
$$+\Delta_1 \left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1} \left\{ 1 + \left(\frac{\theta_t}{M_t}\right)^{\gamma_1-\gamma_1^*} \left[\begin{array}{c} -\frac{g_1\left(1-\gamma-\gamma_1^*\right)}{1+\beta_2} + \left(\frac{bpm_t}{\Xi}\right)^{-(1+\beta_2)} \times \\ (1+b)^{(1-p+\nu p)(1+\beta_2)} g_2\left(1-\gamma-\gamma_1^*\right) \end{array} \right] \right\}$$

when $\frac{\Xi}{bp} \leq m_t \leq m^*$, *and finally*

$$\chi\left(\frac{\theta_t}{M_t}, m_t\right) = \Delta_2 \left\{ 1 + \left(\frac{\theta_t}{M_t}\right)^{\gamma-1+\gamma_1} \frac{(\gamma-1)(1-\alpha)}{\alpha(\gamma-1)+\gamma_1} \right\} + \Delta_1 \left(\frac{\theta_t}{M_t}\right)^{-\left(1-\gamma-\gamma_1^*\right)} \left\{ 1 - \frac{\alpha(\gamma-1)+\gamma_1^*}{\alpha(\gamma-1)+\gamma_1} \left(\frac{M_t}{\theta_t}\right)^{\gamma_1^*-\gamma_1} \right\}$$

when $m_t \geq m^*$.

Proof of Lemma 1. To save space we only give a sketch of the argument. As a first step, let $Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$ be given as

$$Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right) \equiv E_{\tau_n} \int_{\tau_n}^{\infty} e^{-\rho(s-\tau_n)} \left(\frac{C_s}{C_{\tau_n}}\right)^{-\nu} \frac{\left(\frac{M_s}{\theta_s}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}} \frac{\theta_s}{\theta_{\tau_n}} ds.$$

$Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$ satisfies the recursive relationship

$$Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right) = E_{\tau_n} \int_{\tau_n}^{\tau_{n+1}} e^{-\rho(s-\tau_n)} \left(\frac{C_s}{C_{\tau_n}}\right)^{-\nu} \frac{\left(\frac{M_s}{\theta_s}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}} \frac{\theta_s}{\theta_{\tau_n}} ds \tag{36}$$
$$+E_{\tau_n}\left[e^{-\rho(\tau_{n+1}-\tau_n)} \left(\frac{C_{\tau_{n+1}}}{C_{\tau_n}}\right)^{-\nu} \frac{\left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}} \left(\frac{\theta_{\tau_{n+1}}}{\theta_{\tau_n}}\right) Z\left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right) \right].$$

Let $\omega_t \equiv \frac{\theta_t}{\theta_{\tau_n}}$ and let $\xi(\omega_t, m_t)$ be defined as

$$\xi(\omega_t, m_t) \equiv E_t \int_t^{\tau_{n+1}} e^{-\rho(s-t)} \left(\frac{C_s}{C_{\tau_n}}\right)^{-\nu} \frac{\left(\frac{M_s}{\theta_s}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}} \frac{\theta_s}{\theta_{\tau_n}} ds$$
$$= E_t \int_t^{\tau_{n+1}} e^{-\rho(s-t)} \left(1+F(K(m_t))\right)^{-\nu} \omega_t^{1-\gamma} m_t^{(\gamma-1)(1-\alpha)} ds, \tag{37}$$

where the second line follows the definitions of ω_t and m_t and from $C_t = \theta_t X_{\tau_n}\left(1+F(K(m_t))\right)$. To provide a closed form solution for $\xi(\omega_t, m_t)$ we solve the ordinary differential equation (ODE)

$$\frac{\sigma^2}{2}\omega^2 \xi_{\omega\omega} + \mu\omega\xi_\omega - (\rho+\lambda)\xi + \left(1+F(K(m_t))\right)^{-\nu} \omega_t^{1-\gamma} m_t^{(\gamma-1)(1-\alpha)} = 0 \tag{38}$$

subject to the boundary conditions

$$\xi_m\left(\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)m_t, m_t\right) = 0, \quad \lim_{\omega_t \to 0} \frac{\xi\left(\omega_t, m_t\right)}{(1+F(K\left(m_t\right)))^{-\nu} m_t^{-\alpha(\gamma-1)} \left(\frac{\omega_t}{m_t}\right)^{1-\gamma}} < \infty. \tag{39}$$

By the results in Heinricher and Stockbridge (1991), a continuously differentiable function (in ω_t) that solves (38) and (39) is the solution to (37).[42] The function that solves (38) subject to (39) is given by

$$\xi\left(\omega_t, m_t\right) = \begin{cases} \beta_1 m_t^{-\alpha(\gamma-1)} \left(\frac{\omega_t}{m_t}\right)^{1-\gamma} \\ \times \left\{1 + \left(\frac{\theta_{\tau_n}\omega_t}{M_{\tau_n} m_t}\right)^{\gamma_1+\gamma-1} \left[\begin{array}{c} \frac{(\gamma-1)(1-\alpha)}{\alpha(\gamma-1)+\gamma_1} + \left(\frac{bpm_t}{\Xi}\right)^{\alpha(\gamma-1)+\gamma_1} \\ \times g_2\left(0\right)\left[(1+b)^{(1-p+\nu p)(1+\beta_2]} - 1\right) \end{array}\right]\right\}; & m_t \leq \frac{\Xi}{bp} \\ \beta_1 m_t^{-\alpha(\gamma-1)} \left(\frac{bpm_t}{\Xi}\right)^{\frac{-\nu p}{1-p+\nu p}} \left(\frac{\omega_t}{m_t}\right)^{1-\gamma} \\ \times \left\{1 + \left(\frac{\theta_{\tau_n}\omega_t}{M_{\tau_n} m_t}\right)^{\gamma_1+\gamma-1} \left[-\frac{g_1(0)}{1+\beta_2} + \left(\frac{bpm_t}{\Xi}\right)^{-(1+\beta_2)} (1+b)^{(1-p+\nu p)(1+\beta_2)} g_2\left(0\right)\right]\right\}; & m_t \in \left[\frac{\Xi}{bp}, m^*\right] \\ \beta_1 m_t^{-\alpha(\gamma-1)} (1+b)^{-\nu p} \left(\frac{\omega_t}{m_t}\right)^{1-\gamma} \left\{1 + \left(\frac{\theta_{\tau_n}\omega_t}{M_{\tau_n} m_t}\right)^{\gamma_1+\gamma-1} \frac{(\gamma-1)(1-\alpha)}{\alpha(\gamma-1)+\gamma_1}\right\}; & m_t \geq m^*, \end{cases} \tag{40}$$

which can be verified by direct substitution into (38) and (39).

We next take a number $\delta \geq 1-\gamma-\gamma_1$, and compute the function $\Phi\left(\omega_t, m_t; \delta\right)$, defined as

$$\begin{aligned} &\Phi\left(\omega_t, m_t; \delta\right) && (41) \\ &\equiv E_t\left[e^{-\rho(\tau_{n+1}-t)} \left(\frac{C_{\tau_{n+1}}}{C_{\tau_n}}\right)^{-\nu} \frac{\left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}} \left(\frac{\theta_{\tau_{n+1}}}{\theta_{\tau_n}}\right) \left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right)^{\delta}\right] \\ &= \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta} \cdot B\left(\omega_t, m_t, \delta\right), && (42) \end{aligned}$$

where

$$B\left(\omega_t, m_t; \delta\right) \equiv E_t\left[e^{-\rho(\tau_{n+1}-t)} \left((1+F(K(m_{\tau_{n+1}^-})))\right)^{-\nu} m_{\tau_{n+1}^-}^{(\gamma-1)(1-\alpha)+\delta} \omega_{\tau_{n+1}^-}^{1-\gamma-\delta}\right].$$

[42] A sketch of the argument follows: Apply Ito's Lemma to $e^{-(\rho+\lambda)t}\xi\left(\omega_t m_t\right)$ to obtain

$$\begin{aligned} E\left(e^{-(\rho+\lambda)(T-t)}\xi\left(\omega_T, m_T\right)\right) - \xi\left(\omega_t.m_t\right) &= E\int_t^T e^{-(\rho+\lambda)(s-t)}\left(\frac{\sigma^2}{2}\omega^2\xi_{\omega\omega} + \mu\omega\xi_\omega - (\rho+\lambda)\xi\right)ds \\ &\quad + E\int_t^T e^{-(\rho+\lambda)(s-t)}\xi_m\left(\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)m_s, m_s\right)dm_s, \end{aligned}$$

where the second line of the above display uses the fact that m_t increases whenever $\theta_t = M_t$, i.e., whenever $\omega_t = \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)m_t$. Now, let $T \to \infty$, and use (38) together with (39) to arrive at (37).

The last line of equation (41) follows from the definitions of ω_t and m_t and from $C_t = \theta_t X_{\tau_n}(1 + F(K(m_t)))$. The expressions $m_{\tau_{n+1}^-}$ and $\omega_{\tau_{n+1}^-}$ denote the values of m_t and ω_t at the end of epoch n (i.e., an "instant" before the epoch changes).

To determine the expression for $B(\omega_t, m_t; \delta)$, we repeat the same argument as for $V(\omega_t, m_t)$. Specifically, $B(\omega_t, m_t)$ satisfies the ODE

$$\frac{\sigma^2}{2}\omega^2 B_{\omega\omega} + \mu\omega B_\omega - (\rho+\lambda) B + (1 + F(K(m_t)))^{-\nu} m_t^{(\gamma-1)(1-\alpha)+\delta} \omega_t^{1-\gamma-\delta} = 0 \tag{43}$$

subject to the boundary conditions,

$$B_m\left(\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right) m_t, m_t\right) = 0, \quad \lim_{\omega_t \to 0} \frac{B(\omega_t, m_t; \delta)}{\omega_t^{1-\gamma-\delta}} < \infty. \tag{44}$$

It can be verified by direct substitution that the solution to (43) and (44) is given by

$$B(\omega_t, m_t; \delta) = \begin{cases} g_3(\delta)\left(\frac{\omega_t}{m_t}\right)^{1-\gamma-\delta} m_t^{-\alpha(\gamma-1)} \\ \times\left\{1 + \left(\frac{M_{\tau_n} m_t}{\theta_{\tau_n}\omega_t}\right)^{1-\gamma-\gamma_1-\delta}\left[\begin{array}{c} \frac{(\gamma-1)(1-\alpha)+\delta}{\alpha(\gamma-1)+\gamma_1} + \left(\frac{bpm_t}{\Xi}\right)^{\alpha(\gamma-1)+\gamma_1} \\ \times g_2(\delta)\left[(1+b)^{(1-p+\nu p)(1+\beta_2)} - 1\right]\end{array}\right]\right\}; m_t \le \frac{\Xi}{bp} \\ g_3(\delta)\left(\frac{bpm_t}{\Xi}\right)^{\frac{-\nu p}{1-p+\nu p}}\left(\frac{\omega_t}{m_t}\right)^{1-\gamma-\delta} m_t^{-\alpha(\gamma-1)} \\ \times\left\{1 + \left(\frac{M_{\tau_n} m_t}{\theta_{\tau_n}\omega_t}\right)^{1-\gamma-\gamma_1-\delta}\left[\begin{array}{c} -\frac{g_1(\delta)}{1+\beta_2} + \left(\frac{bpm_t}{\Xi}\right)^{-(1+\beta_2)} \\ \times(1+b)^{(1-p+\nu p)(1+\beta_2)} g_2(\delta)\end{array}\right]\right\}; \quad m_t \in [\frac{\Xi}{bp}, m^*] \\ g_3(\delta)(1+b)^{-\nu p}\left(\frac{\omega_t}{m_t}\right)^{1-\gamma-\delta} m_t^{-\alpha(\gamma-1)} \\ \times\left[1 + \frac{(\gamma-1)(1-\alpha)+\delta}{\alpha(\gamma-1)+\gamma_1}\left(\frac{M_{\tau_n} m_t}{\theta_{\tau_n}\omega_t}\right)^{1-\gamma-\gamma_1-\delta}\right]; \quad m_t \ge m^*. \end{cases} \tag{45}$$

Hence, at the beginning of epoch, $\omega_t = 1$ and $m_t = 1$, and therefore

$$B(1,1;\delta) = g_3(\delta) + g_4(\delta)\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1-\delta}. \tag{46}$$

where the function $g_3(\delta)$ and $g_4(\delta)$ are given in the statement of the Lemma. Combining (46) with (42), it follows that

$$\Phi(1,1;\delta) = g_3(\delta)\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta} + g_4(\delta)\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1}. \tag{47}$$

To complete the computation of $Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$, we employ a "guess and verify" approach. We first guess that $Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$ can be written as

$$Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right) = \Delta_1\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta_1} + \Delta_2 \tag{48}$$

for some appropriate constants δ_1, Δ_1, and Δ_2. Using (48) inside the recursive equation (37), recalling that at the beginning of the epoch $\omega_{\tau_n} = 1$, $m_{\tau_n} = 1$, and using the definition of $\Phi\left(\omega_{\tau_n}, m_{\tau_n}; \delta\right)$ (equation [41]), we obtain

$$\begin{aligned}
&\Delta_1 \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta_1} + \Delta_2 \qquad (49)\\
&= \xi\left(\omega_{\tau_n}, m_{\tau_n}\right) + \Delta_1 \Phi\left(\omega_{\tau_n}, m_{\tau_n}; \delta_1\right) + \Delta_2 \Phi\left(\omega_{\tau_n}, m_{\tau_n}; 0\right)\\
&= \beta_1 + \alpha_1 \left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1} + \Delta_1 \left[g_3\left(\delta_1\right)\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta_1} + g_4\left(\delta_1\right)\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1}\right]\\
&\quad + \Delta_2 \left[g_3(0) + g_4(0)\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1}\right]\\
&= \left[\beta_1 + \Delta_2 g_3(0)\right] + \left[\alpha_1 + \Delta_1 g_4\left(\delta_1\right) + \Delta_2 g_4(0)\right]\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1} + \Delta_1 g_3\left(\delta_1\right)\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta_1}.
\end{aligned}$$

Conjecture (48) is true if the coefficients on the left- and the right-hand sides of (49) match. Matching free-term coefficients, i.e., setting

$$\Delta_2 = \beta_1 + \Delta_2 g_3(0) = \beta_1 + \Delta_2 \lambda \beta_1,$$

gives Δ_2 as in (34). The value $\delta_1 = 1 - \gamma - \gamma_1^*$ follows from equating the coefficients of $\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{\delta_1}$:

$$\Delta_1 = \Delta_1 g_3(\delta_1).$$

Finally, the term that pre-multiplies $\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{1-\gamma-\gamma_1}$ needs to equal zero,

$$0 = \alpha_1 + \Delta_1 g_4(\delta_1) + \Delta_2 g_4(0),$$

which leads to Δ_1 as in (33). This completes the computation of $Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$.

Having determined $Z\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)$, we observe next that χ_t in equation (35) can be written as

$$\begin{aligned}
\chi\left(\frac{\theta_t}{M_t}, m_t\right) &= E_t\left(\int_t^{\tau_{n+1}} e^{-\rho(s-t)}\left(\frac{C_s}{C_t}\right)^{-\nu}\left(\frac{\frac{M_s}{\theta_s}}{\frac{M_t}{\theta_t}}\right)^{(\gamma-1)(1-\alpha)} \frac{\theta_s}{\theta_t} ds\right)\\
&\quad + E_t\left[e^{-\rho(\tau_{n+1}-t)}\left(\frac{C_{\tau_{n+1}}}{C_t}\right)^{-\nu}\left(\frac{\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}}{\frac{M_t}{\theta_t}}\right)^{(\gamma-1)(1-\alpha)} \frac{\theta_{\tau_{n+1}}}{\theta_t}\right] \times Z\left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right),
\end{aligned}$$

which implies that

$$\chi\left(\frac{\theta_t}{M_t}, m_t\right)$$

$$= \left(\frac{C_{\tau_n}}{C_t}\right)^{-\nu} \frac{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)}} \frac{\theta_{\tau_n}}{\theta_t} \times E_t\left(\int_t^{\tau_{n+1}} e^{-\rho(s-t)}\left(\frac{C_s}{C_{\tau_n}}\right)^{-\nu}\left(\frac{\frac{M_s}{\theta_s}}{\frac{M_{\tau_n}}{\theta_{\tau_n}}}\right)^{(\gamma-1)(1-\alpha)} \frac{\theta_s}{\theta_{\tau_n}} ds\right)$$

$$+\left(\frac{C_{\tau_n}}{C_t}\right)^{-\nu} \frac{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)}} \frac{\theta_{\tau_n}}{\theta_t}$$

$$\times E_t\left\{\left[e^{-\rho(\tau_{n+1}-t)}\left(\frac{C_{\tau_{n+1}}}{C_{\tau_n}}\right)^{-\nu}\left(\frac{\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}}{\frac{M_{\tau_n}}{\theta_{\tau_n}}}\right)^{(\gamma-1)(1-\alpha)} \frac{\theta_{\tau_{n+1}}}{\theta_{\tau_n}}\right] \times\left(\Delta_1\left(\frac{M_{\tau_{n+1}}}{\theta_{\tau_{n+1}}}\right)^{\delta_1}+\Delta_2\right)\right\}.$$

Therefore,

$$\chi\left(\frac{\theta_t}{M_t}, m_t\right) = \left(\frac{C_{\tau_n}}{C_t}\right)^{-\nu} \frac{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)}} \frac{\theta_{\tau_n}}{\theta_t} \cdot \xi\left(\omega_t, m_t\right) \tag{50}$$

$$+\left(\frac{C_{\tau_n}}{C_t}\right)^{-\nu} \frac{\left(\frac{M_{\tau_n}}{\theta_{\tau_n}}\right)^{(\gamma-1)(1-\alpha)}}{\left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)}} \frac{\theta_{\tau_n}}{\theta_t}\left[\Delta_1 \Phi\left(\omega_t, m_t ; \delta_1\right)+\Delta_2 \Phi\left(\omega_t, m_t ; 0\right)\right].$$

Plugging the expressions for $\xi\left(\omega_t, m_t\right)$ and $\Phi\left(\omega_t, m_t ; \delta_1\right)$ into equation (50) and simplifying the resulting expression we arrive at the expression for $\chi\left(\frac{\theta_t}{M_t}, m_t\right)$ given in the statement of the Lemma. ■

Corollary 1 *The value of assets in place for firm j is given by*

$$P_{j,t}^A = X_{j,t}\theta_t \cdot \chi\left(\frac{\theta_t}{M_t}, m_t\right).$$

Proof of Corollary 1. Combine the definition of χ and (6). ■

With this Lemma we are now in a position to discuss the solution to the firm's optimization problem. The option to plant a tree in epoch N does not affect the option to plant a tree in any subsequent epoch.

The individual firm takes the processes for new trees ($K_{N,t}$) and consumption (C_t), and hence the stochastic discount factor H_t and the costs of planting a tree (equation (18)), as given. For the remainder of the proof we consider a firm that expects $K_{N,t}$ to behave as in (32). Such a firm solves the problem

$$J(\theta_t, M_t) = \sup_{\tau} E_t \left[1_{\{\tau < \tau_{N+1}\}} e^{-\rho(\tau - t)} \left(\zeta(i_{N,j}) G(\theta_\tau, M_\tau) - \eta M_{\tau_N} X_{\tau_N}^{-\nu} \theta_\tau^{-\nu} \left(\frac{M_\tau}{\theta_\tau} \right)^{(\gamma-1)(1-\alpha)} \right) \right] \tag{51}$$

with $G(\theta_t, M_t)$ defined as

$$\begin{aligned} G(\theta_t, M_t) &\equiv E_t \int_t^{\infty} e^{-\rho(s-\tau)} C_s^{-\gamma} \left(M_s^C\right)^{(\gamma-1)(1-\alpha)} \theta_s ds \\ &= \left[X_{\tau_N}\left(1 + F\left(K(m_t)\right)\right)\right]^{-\nu} \theta_t^{\alpha(1-\gamma)} \left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)} \cdot \chi\left(\frac{\theta_t}{M_t}, m_t\right). \end{aligned} \tag{52}$$

Hence, the firm's optimization problem is

$$\begin{aligned} &J(\theta_t, M_t) \\ &= \sup_{\tau} E_t \left[1_{\{\tau < \tau_{N+1}\}} e^{-\rho(\tau - t)} \left(\begin{array}{c} \zeta(i_{N,j}) \left[X_{\tau_N}\left(1 + F\left(K(m_t)\right)\right)\right]^{-\nu} \theta_\tau^{\alpha(1-\gamma)} \left(\frac{M_\tau}{\theta_\tau}\right)^{(\gamma-1)(1-\alpha)} \cdot \chi\left(\frac{\theta_\tau}{M_\tau}, m_\tau\right) \\ -\eta M_{\tau_N} X_{\tau_N}^{-\nu} \theta_\tau^{-\nu} \left(\frac{M_\tau}{\theta_\tau}\right)^{(\gamma-1)(1-\alpha)} \end{array} \right) \right]. \end{aligned} \tag{53}$$

To solve the optimization problem inside the square brackets we proceed in two steps. First, we derive the optimal policy in a heuristic way by constraining attention to the class of "trigger strategies." Such strategies assume that the firm invests the first time that θ_t (and hence M_t) crosses an (optimally determined) threshold $\bar{\theta}$. Formally, the stopping times associated with these strategies are given by

$$\tau_{\bar{\theta}} = \inf\{s \geq t : \theta_s \geq \bar{\theta}\}. \tag{54}$$

Additionally, we assume that the optimal $\bar{\theta}$ lies in the interval $\left[\frac{\Xi}{bp} M_{\tau_N}, m^* M_{\tau_N}\right]$.[43] We let Θ denote the class of such trigger strategies. We do not attempt to justify ex-ante why the optimal strategy should lie in this class. Instead, in a second step, we verify the optimality of these strategies via a standard verification theorem for optimal stopping (Proposition 2).

[43]This implies that equation (32) simplifies to

$$K_{N,t} = \frac{\left[\left(\frac{bp}{\Xi}\right) \frac{M_t}{M_{\tau_N}}\right]^{\frac{1}{1-p+\nu p}} - 1}{b}.$$

To start, let $\tilde{V}(\theta_t, M_t)$ denote the value function for $\tau_{\bar{\theta}} \in \Theta$:

$$\tilde{V}(\theta_t, M_t) \tag{55}$$
$$\equiv \sup_{\tau_{\bar{\theta}}\in\Theta} E_t\left[1_{\{\tau<\tau_{N+1}\}}e^{-\rho(\tau_{\bar{\theta}}-t)}\left(\begin{array}{c}\zeta(i_{N,j})\left[X_{\tau_N}\left(1+F\left(K(m_t)\right)\right)\right]^{-\nu}\theta_{\tau_{\bar{\theta}}}^{\alpha(1-\gamma)}\left(\frac{M_{\tau_{\bar{\theta}}}}{\theta_{\tau_{\bar{\theta}}}}\right)^{(\gamma-1)(1-\alpha)}\cdot\chi\left(\frac{\theta_{\tau_{\bar{\theta}}}}{M_{\tau_{\bar{\theta}}}}, m_{\tau_{\bar{\theta}}}\right)\\ -\eta M_{\tau_N}X_{\tau_N}^{-\nu}\theta_{\tau_{\bar{\theta}}}^{-\nu}\left(\frac{M_{\tau_{\bar{\theta}}}}{\theta_{\tau_{\bar{\theta}}}}\right)^{(\gamma-1)(1-\alpha)}\end{array}\right)\right].$$

We first observe that $\tau_{\bar{\theta}} \in \Theta$ implies that $\theta_{\tau_{\bar{\theta}}} = M_{\tau_{\bar{\theta}}}$ and also $1 + F(K(m_t)) = [1 + bK(m_t)]^p = \left[\left(\frac{bp}{\Xi}\right) m_t\right]^{\frac{p}{1-p+\nu p}}$. Furthermore by Øksendal (2003), p. 210-211 we obtain

$$E_t\left[1_{\{\tau<\tau_{N+1}\}}e^{-\rho(\tau_{\bar{\theta}}-t)}\right] = \left(\frac{\theta_t}{\bar{\theta}}\right)^{\gamma_1}.$$

Accordingly, letting

$$\varphi\left(\frac{\bar{\theta}}{M_{\tau_N}}\right) \equiv \left(\frac{\bar{\theta}}{M_{\tau_N}}\right)^{\beta_2}\left[\frac{\bar{\theta}}{M_{\tau_N}}\zeta(i_{N,j})\chi\left(1, \frac{\bar{\theta}}{M_{\tau_N}}\right) - \eta\left(\frac{bp\bar{\theta}}{\Xi M_{\tau_N}}\right)^{\frac{\nu p}{1-p+\nu p}}\right],$$

equation (55) can be re-written as

$$\tilde{V}(\theta_t, M_t) = \left(\frac{bp}{\Xi}\right)^{\frac{-\nu p}{1-p+\nu p}} M_{\tau_N}^{\alpha(1-\gamma)-\gamma_1}\theta_t^{\gamma_1}X_{\tau_N}^{-\nu} \times \sup_{\bar{\theta}\in\left[\frac{\Xi}{bp}M_{\tau_N}, m^*M_{\tau_N}\right]}\varphi\left(\frac{\bar{\theta}}{M_{\tau_N}}\right). \tag{56}$$

By the assumption $\tau_{\bar{\theta}} \in \Theta$, $\frac{\Xi}{bp} \leq m_{\tau_{\bar{\theta}}} \leq m^*$ and hence Lemma 1 implies that

$$\begin{aligned}\varphi\left(\frac{\bar{\theta}}{M_{\tau_N}}\right) &= \left(\frac{\bar{\theta}}{M_{\tau_N}}\right)^{\beta_2+1}\zeta(i_{N,j})\left\{\begin{array}{c}\Delta_2\left\{1-\frac{g_1(0)}{1+\beta_2}+\left(\frac{bp\bar{\theta}}{\Xi M_{\tau_N}}\right)^{-(1+\beta_2)}(1+b)^{(1-p+\nu p)(1+\beta_2)}g_2(0)\right\}+\Delta_1\times\\ \left\{1-\frac{g_1(1-\gamma-\gamma_1^*)}{1+\beta_2}+\left(\frac{bp\bar{\theta}}{\Xi M_{\tau_N}}\right)^{-(1+\beta_2)}(1+b)^{(1-p+\nu p)(1+\beta_2)}g_2(1-\gamma-\gamma_1^*)\right\}\end{array}\right\}\\ &\quad -\left(\frac{\bar{\theta}}{M_{\tau_N}}\right)^{\beta_2}\eta\left(\frac{bp\bar{\theta}}{\Xi M_{\tau_N}}\right)^{\frac{\nu p}{1-p+\nu p}}.\end{aligned}$$

Assuming an interior solution and setting $\varphi'\left(\frac{\bar{\theta}}{M_{\tau_N}}\right) = 0$, we obtain

$$\frac{\bar{\theta}}{M_{\tau_N}} = \frac{\left(\frac{\nu p}{1-p+\nu p}+\beta_2\right)\eta\left(\frac{bp\bar{\theta}}{\Xi M_{\tau_N}}\right)^{\frac{\nu p}{1-p+\nu p}}}{\zeta(i_{N,j})\left[(1+\beta_2-g_1(0))\Delta_2+(1+\beta_2-g_1(1-\gamma-\gamma_1^*))\Delta_1\right]}. \tag{57}$$

Notice that policy (57) has the same form as policy (25), which is given by

$$\frac{\bar{\theta}}{M_{\tau_N}} = \frac{\Xi\left(1+F\left(i_{N,j}\right)\right)^{\nu}}{\zeta(i_{N,j})} = \frac{\Xi\left(\frac{bp\bar{\theta}}{\Xi M_{\tau_N}}\right)^{\frac{\nu p}{1-p+\nu p}}}{\zeta(i_{N,j})}. \tag{58}$$

Combining (57) with (58) implies that

$$\Xi = \frac{\eta\left(\beta_2+\frac{\nu p}{1-p+\nu p}\right)}{\Delta_2\left(1+\beta_2-g_1(0)\right)+\Delta_1\left(1+\beta_2-g_1\left(1-\gamma-\gamma_1^*\right)\right)}. \tag{59}$$

Notice that Δ_1 is a function of Ξ, although other parameters are independent of Ξ. Hence, equation (59) is a non-linear equation in Ξ. We shall denote the solution to this equation as Ξ^* and assume that parameters are such[44] that $\Xi^* \geq bp$.

Now note that if all other firms follow trigger strategies of the form (57) with $\Xi = \Xi^*$, then the resulting process for K_t is given by (32) with $\Xi = \Xi^*$, confirming the conjecture of firm j about the behaviour of $K_{N,t}$. Assuming that the optimal stopping policy of any firm j lies in the interior of the "trigger" class Θ, firm j behaves optimally by following policy (58) evaluated at $\Xi = \Xi^*$.

The next proposition shows that, if all firms $j' \neq j$ follow policies of the form (58) with $\Xi = \Xi^*$, then the optimal stopping strategy for firm j (across all possible stopping strategies) indeed takes the form (58). We use the notation $x \wedge y$ for $\min(x, y)$ and $x \vee y$ for $\max(x, y)$.

Proposition 2 *Assume $\phi \equiv \rho + \lambda + \mu\gamma - (\gamma + 1)\gamma\frac{\sigma^2}{2} > 0$ and $\gamma + \gamma_1 > 1$. Let Ξ^* denote the solution to (59), $K(m_t)$ be given by (32) with $\Xi = \Xi^*$, and $G(\theta_t, M_t)$ by (52). Define $\bar{\theta}(M_t)$ as the solution to the equation*

$$\bar{\theta}(M_t) = \arg\max_{\bar{\theta}} \left(\frac{1}{\bar{\theta}}\right)^{\gamma_1} \left[\zeta(i_{N,j}) G\left(\bar{\theta}, M_t \vee \bar{\theta}\right) - \eta M_{\tau_N} \bar{\theta}^{-\nu} X_{\tau_N}^{-\nu} \left(\frac{\bar{\theta}}{M_t} \wedge 1\right)^{(1-\gamma)(1-\alpha)}\right]. \quad (60)$$

Then it is optimal for firm j in epoch N to plant a tree the first time that $\theta_t \geq \bar{\theta}(M_t)$.

Proof of Proposition 2. The marginal firm solves the optimal stopping problem specified by (51). For any C^1 function $f : \mathbb{R} \to \mathbb{R}$ that is twice-differentiable a.e. define the infinitesimal operator $\mathcal{A}(f) \equiv \frac{\sigma^2}{2}\theta^2 f_{\theta\theta} + \mu\theta f_\theta - (\rho + \lambda) f$. Next, note that Lemma 1 implies that the function $G(\theta_t, M_t)$ can be written as

$$\begin{aligned} G(\theta_t, M_t) &= \left[X_{\tau_N}\left(1 + F(K_t)\right)\right]^{-\nu} M_t^{(\gamma-1)(1-\alpha)} \\ &\quad \times \left[\Delta_2 \theta_t^{1-\gamma} + \Delta_1 \theta_t^{\gamma_1^*} \left(\frac{1}{M_t}\right)^{\gamma+\gamma_1^*-1} + Const(m_t) \cdot \theta_t^{\gamma_1}\right] \end{aligned} \quad (61)$$

where $Const$ depends on m_t but is independent of θ_t. Since $\mathcal{A}(\theta_t^{\gamma_1}) = 0$, it is straightforward to check that

$$\mathcal{A}G(\theta_t, M_t) = -\left[X_{\tau_N}\left(1 + F(K_t)\right)\right]^{-\nu} \left[\frac{1}{1 - \beta_1\lambda}\theta_t^{1-\gamma}(M_t)^{(\gamma-1)(1-\alpha)} + \lambda\Delta_1\theta_t^{\gamma_1^*} M_t^{-\alpha(\gamma-1)-\gamma_1^*}\right]. \quad (62)$$

[44] This can be achieved by assuming a large enough value for η.

Furthermore, by the construction of $G(\theta_t, M_t)$ (see Lemma 1) we also obtain

$$G_M(M_t, M_t) = 0. \tag{63}$$

With these observations, let $\bar{\theta}(M_t)$ be defined as in equation (60) and define the "candidate" value function $V(\theta_t, M_t)$ as

$$V(\theta_t, M_t) = \left(\frac{\theta_t}{\bar{\theta}}\right)^{\gamma_1}\left[\zeta(i_{N,j})G\left(\bar{\theta}, M_t \vee \bar{\theta}\right) - \eta M_{\tau_N}\bar{\theta}^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\bar{\theta}}{M_t} \wedge 1\right)^{(1-\gamma)(1-\alpha)}\right] \tag{64}$$

whenever $\theta_t \leq \bar{\theta}(M_t)$ and

$$V(\theta_t, M_t) = \zeta(i_{N,j})G(\theta_t, M_t) - \eta M_{\tau_N}\theta_t^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\theta_t}{M_t}\right)^{(1-\gamma)(1-\alpha)} \tag{65}$$

whenever $\theta_t > \bar{\theta}(M_t)$. In what follows we show the following four properties of the function $V(\theta_t, M_t)$:

$$V(\theta_t, M_t) \geq \zeta(i_{N,j})G(\theta_t, M_t) - \eta M_{\tau_N}\theta_t^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\theta_t}{M_t}\right)^{(1-\gamma)(1-\alpha)}, \tag{66}$$

$$V(\theta_t, M_t) \text{ is continuously differentiable in } \theta_t, \tag{67}$$

$$V_M(\theta_t, M_t) \leq 0 \text{ for } \theta_t = M_t, \tag{68}$$

$$\mathcal{A}V(\theta_t, M_t) \leq 0\,. \tag{69}$$

Property (66) is immediate for $\theta_t \geq \bar{\theta}(M_t)$, and for $\theta_t \leq \bar{\theta}(M_t)$ it follows from

$$\zeta(i_{N,j})G(\theta_t, M_t) - \eta M_{\tau_N}\theta_t^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\theta_t}{M_t}\right)^{(1-\gamma)(1-\alpha)} =$$

$$\begin{aligned}
&= \zeta(i_{N,j})G(\theta_t, M_t \vee \theta_t) - \eta M_{\tau_N}\theta_t^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\theta_t}{M_t} \wedge 1\right)^{(1-\gamma)(1-\alpha)} \\
&= \left(\frac{\theta_t}{\theta_t}\right)^{\gamma_1}\left[\zeta(i_{N,j})G(\theta_t, M_t \vee \theta_t) - \eta M_{\tau_N}\theta_t^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\theta_t}{M_t} \wedge 1\right)^{(1-\gamma)(1-\alpha)}\right] \\
&\leq \theta_t^{\gamma_1}\max_{\bar{\theta}}\left(\frac{1}{\bar{\theta}}\right)^{\gamma_1}\left[\zeta(i_{N,j})G\left(\bar{\theta}, M_t \vee \bar{\theta}\right) - \eta M_{\tau_N}\bar{\theta}^{-\nu}X_{\tau_N}^{-\nu}\left(\frac{\bar{\theta}}{M_t} \wedge 1\right)^{(1-\gamma)(1-\alpha)}\right] \\
&= V(\theta_t, M_t)\,.
\end{aligned}$$

To show property (67) consider first the case $\theta_t \leq \overline{\theta}(M_t)$. Differentiating (64) gives

$$\frac{\partial V}{\partial \theta} = \gamma_1 \frac{1}{\theta_t} V(\theta_t, M_t)\,, \tag{70}$$

which is a continuous function. Furthermore, when $\theta_t \to \overline{\theta}(M_t)$, we obtain:

$$\begin{aligned}
&\lim_{\theta_t \to \bar{\theta}(M_t)} \frac{\partial V(\theta_t, M_t)}{\partial \theta_t} \qquad (71)\\
&= \frac{\gamma_1}{\bar{\theta}(M_t)} \left[\zeta(i_{N,j}) G\left(\bar{\theta}(M_t), M_t \vee \bar{\theta}(M_t)\right) - X_{\tau_N}^{-\nu} \left[\bar{\theta}(M_t)\right]^{-\nu} \left(\frac{\bar{\theta}(M_t)}{M_t} \wedge 1\right)^{(1-\gamma)(1-\alpha)} \eta M_{\tau_N} \right].
\end{aligned}$$

Turning next to the case where $\theta_t > \bar{\theta}(M_t)$, direct differentiation of (65) shows that the partial derivative of $V(\theta_t, M_t)$ with respect to θ_t is a continuous function, whose value at $\theta_t = \bar{\theta}(M_t)$ is given by

$$\begin{aligned}
&\lim_{\theta_t \to \bar{\theta}(M_t)} \frac{\partial V(\theta_t, M_t)}{\partial \theta} \qquad (72)\\
&= \zeta(i_{N,j}) G_\theta\left(\bar{\theta}(M_t), M_t\right) + \gamma \left[\bar{\theta}(M_t)\right]^{-\nu} \frac{1}{\bar{\theta}(M_t)} \left(\frac{M_t}{\bar{\theta}(M_t)}\right)^{(\gamma-1)(1-\alpha)} \eta X_{\tau_N}^{-\nu} M_{\tau_N}.
\end{aligned}$$

To establish (67), we need to show that the "left" hand side derivative (equation [71]) and the "right" hand side derivative (equation [72]) coincide. Note that this statement is meaningful only when $\bar{\theta}(M_t) \leq M_t$, for otherwise $\theta_t \leq M_t < \bar{\theta}(M_t)$. Then the necessary condition for optimality (first order condition) of equation (60) implies that:

$$\begin{aligned}
0 &= \left(\frac{1}{\bar{\theta}}\right)^{\gamma_1} \left[\zeta(i_{N,j}) G_\theta\left(\bar{\theta}, M_t\right) - (-\gamma)\, \bar{\theta}^{-\nu} \left(\frac{M_t}{\bar{\theta}}\right)^{(\gamma-1)(1-\alpha)} \frac{1}{\bar{\theta}} \eta X_{\tau_N}^{-\nu} M_{\tau_N} \right] \\
&\quad -\gamma_1 \left(\frac{1}{\bar{\theta}}\right)^{\gamma_1} \frac{1}{\bar{\theta}} \left[\zeta(i_{N,j}) G\left(\bar{\theta}, M_t\right) - \bar{\theta}^{-\nu} \left(\frac{M_t}{\bar{\theta}}\right)^{(\gamma-1)(1-\alpha)} \eta X_{\tau_N}^{-\nu} M_{\tau_N} \right] \qquad (73)
\end{aligned}$$

Dividing both sides of equation (73) by $\left(\frac{1}{\theta}\right)^{\gamma_1}$, we obtain that the right hand side of equation (71) and the right hand side of equation (72) are identical, so that $\frac{\partial V(\theta_t, M_t)}{\partial \theta}$ is continuous at $\theta_t = \overline{\theta}(M_t)$.

To establish (68), consider two cases. When $M_t \geq \overline{\theta}(M_t)$, then whenever $\theta_t = M_t$, equation (65) along with (63) leads to

$$\begin{aligned}
V_M(M_t, M_t) &= \zeta(i_{N,j}) G_M(M_t, M_t) - (\gamma - 1)(1-\alpha) M_t^{(\gamma-1)(1-\alpha)-\gamma} \left(\frac{M_t}{M_t}\right)^{(\gamma-1)(1-\alpha)} \frac{M_{\tau_N}}{M_t} \eta \\
&= -\eta (\gamma - 1)(1-\alpha) M_t^{-\nu-1} M_{\tau_N} \leq 0.
\end{aligned}$$

When $M_t \leq \bar{\theta}(M_t)$, $M_t \vee \bar{\theta} = \overline{\theta}$ and hence whenever $\theta_t = M_t$, $V(\theta_t, M_t)$ is given by

$$\left(\frac{\theta_t}{\bar{\theta}}\right)^{\gamma_1} \cdot \left[\zeta(i_{N,j}) G\left(\bar{\theta}, \bar{\theta}\right) - \bar{\theta}^{(\gamma-1)(1-\alpha)-\gamma} \eta X_{\tau_N}^{(\gamma-1)(1-\alpha)-\gamma} M_{\tau_N} \right],$$

which is independent of M_t. Hence, $V_M(M_t, M_t) = 0$.

To show property (69), we start by noting that when $\theta_t < \bar{\theta}(M_t)$, $V(\theta_t, M_t)$ is given by (64). Hence $\mathcal{A}(V) = 0$, since $\mathcal{A}(\theta^{\gamma_1}) = 0$. When $\theta_t \geq \bar{\theta}(M_t)$, $V(\theta_t, M_t)$ is given by (65). Using (62) we obtain

$$\begin{aligned}\mathcal{A}V &= X_{\tau_N}^{(\gamma-1)(1-\alpha)-\gamma}\left(\frac{M_t}{\theta_t}\right)^{(\gamma-1)(1-\alpha)}\theta_t^{-\alpha(\gamma-1)-1} \\ &\quad\times\left[\phi\eta M_{\tau_N} - \zeta(i_{N,j})\left[1+F(K_t)\right]^{-\nu}\theta_t\left(\frac{1}{1-\lambda\beta_1} + \lambda\Delta_1\left(\frac{M_t}{\theta_t}\right)^{1-\gamma-\gamma_1^*}\right)\right].\end{aligned} \tag{74}$$

Hence, we only need to show that the term inside square brackets in (74) is non-positive for $\theta_t \geq \bar{\theta}(M_t)$. This amounts to showing that

$$\frac{\phi\eta M_{\tau_N}\left[(1+F(K_t))\right]^{\nu}}{\zeta(i_{N,j})} \leq \theta_t \cdot \left(\frac{1}{1-\lambda\beta_1} + \lambda\Delta_1\left(\frac{M_t}{\theta_t}\right)^{1-\gamma-\gamma_1^*}\right). \tag{75}$$

Since the right hand side of (75) is increasing in θ_t, and $\theta_t \geq \bar{\theta}(M_t)$ it suffices to show that

$$\frac{\phi\eta M_{\tau_N}\left[1+F(K(M_t))\right]^{\nu}}{\zeta(i_{N,j})} \leq \bar{\theta}(M_t)\left(\frac{1}{1-\lambda\beta_1} + \lambda\Delta_1\left(\frac{M_t}{\bar{\theta}(M_t)}\right)^{1-\gamma-\gamma_1^*}\right). \tag{76}$$

Since $M_t \geq \theta_t \geq \bar{\theta}(M_t)$, equation (73) can be re-written as

$$\begin{aligned}0 &= \zeta(i_{N,j})\left[-\gamma_1\frac{1}{\bar{\theta}}G(\bar{\theta}, M_t) + G_\theta(\bar{\theta}, M_t)\right] + \gamma_1\frac{1}{\bar{\theta}}\bar{\theta}^{-\nu}\left(\frac{\bar{\theta}}{M_t}\right)^{(1-\gamma)(1-\alpha)}\eta X_{\tau_N}^{-\nu}M_{\tau_N} \\ &\quad+\gamma\frac{1}{\bar{\theta}}\bar{\theta}^{-\nu}\left(\frac{\bar{\theta}}{M_t}\right)^{(1-\gamma)(1-\alpha)}\eta X_{\tau_N}^{-\nu}M_{\tau_N}.\end{aligned} \tag{77}$$

By (61),

$$\begin{aligned}&-\gamma_1\frac{1}{\bar{\theta}}G(\bar{\theta}, M_t) + G_\theta(\bar{\theta}, M_t) \\ &= \left[X_{\tau_N}(1+F(K_t))\right]^{-\nu}M_t^{(\gamma-1)(1-\alpha)} \times \\ &\quad\left[-\gamma_1\Delta_2\bar{\theta}^{-\gamma} - \gamma_1\Delta_1\bar{\theta}^{\gamma_1^*-1}\left(\frac{1}{M_t}\right)^{\gamma+\gamma_1^*-1} + \Delta_2(1-\gamma)\bar{\theta}^{-\gamma} + \gamma_1^*\Delta_1\bar{\theta}^{\gamma_1^*-1}\left(\frac{1}{M_t}\right)^{\gamma+\gamma_1^*-1}\right].\end{aligned} \tag{78}$$

Combining equations (77) and (78) and simplifying yields

$$\bar{\theta}(M_t) = \frac{\phi\eta M_{\tau_N}(1+F(K(M_t)))^{\nu}}{\zeta(i_{N,j})}\frac{(\gamma_1+\gamma)}{\phi\left(\frac{(\gamma+\gamma_1-1)}{1-\lambda\beta}\beta + \Delta_1(\gamma_1-\gamma_1^*)\left(\frac{M_t}{\bar{\theta}(M_t)}\right)^{1-\gamma-\gamma_1^*}\right)}.$$

Hence, to show equation (76), we only need to verify that

$$\begin{aligned}&\frac{\phi\eta M_{\tau_N}(1+F(K_t))^{\nu}}{\zeta(i_{N,j})}\frac{(\gamma_1+\gamma)\left(\frac{1}{1-\lambda\beta_1} + \lambda\Delta_1\left(\frac{M_t}{\bar{\theta}(M_t)}\right)^{1-\gamma-\gamma_1^*}\right)}{\phi\left(\frac{(\gamma+\gamma_1-1)}{1-\lambda\beta}\beta_1 + \Delta_1(\gamma_1-\gamma_1^*)\left(\frac{M_t}{\bar{\theta}(M_t)}\right)^{1-\gamma-\gamma_1^*}\right)} \\ &\geq \frac{\phi\eta M_{\tau_N}\left[1+F(K(M_t))\right]^{\nu}}{\zeta(i_{N,j})}.\end{aligned}$$

To this end, we only need to show

$$\frac{\gamma_1+\gamma}{1-\lambda\beta_1}+\lambda\Delta_1\left(\gamma_1+\gamma\right)\left(\frac{M_t}{\bar{\theta}\left(M_t\right)}\right)^{1-\gamma-\gamma_1^*}\geq\phi\beta_1\frac{\gamma+\gamma_1-1}{1-\lambda\beta_1}+\phi\Delta_1\left(\gamma_1-\gamma_1^*\right)\left(\frac{M_t}{\bar{\theta}\left(M_t\right)}\right)^{1-\gamma-\gamma_1^*}. \quad (79)$$

Define γ_2 as

$$\gamma_2\equiv\frac{-\sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2+2\sigma^2\left(\rho+\lambda\right)}-\left(\mu-\frac{\sigma^2}{2}\right)}{\sigma^2}<0.$$

Then,

$$\begin{aligned}\gamma_1\gamma_2 &= -\frac{2\left(\rho+\lambda\right)}{\sigma^2}\\ \gamma_1+\gamma_2 &= \frac{-2\left(\mu-\frac{\sigma^2}{2}\right)}{\sigma^2}.\end{aligned}$$

Hence,

$$\phi=-\left[\left(\gamma+1\right)\gamma\frac{\sigma^2}{2}-\mu\gamma-\rho-\lambda\right]=-\frac{\sigma^2}{2}\left(\gamma_1+\gamma\right)\left(\gamma+\gamma_2\right)>0, \quad (80)$$

which implies that $\gamma+\gamma_2<0$ (recall that by assumption $\phi>0$). Direct algebra gives

$$\beta_1\phi=\frac{\gamma+\gamma_2}{\gamma+\gamma_2-1}\times\frac{\gamma+\gamma_1}{\gamma+\gamma_1-1}\leq\frac{\gamma_1+\gamma}{\gamma+\gamma_1-1}.$$

Consequently,

$$\frac{\gamma_1+\gamma}{1-\lambda\beta_1}\geq\phi\beta_1\frac{\gamma+\gamma_1-1}{1-\lambda\beta_1}. \quad (81)$$

Therefore, to show inequality (79), equation (81) implies that we only need to show

$$\lambda\Delta_1\left(\gamma_1+\gamma\right)\left(\frac{M_t}{\bar{\theta}}\right)^{1-\gamma-\gamma_1^*}\geq\phi\Delta_1\left(\gamma_1-\gamma_1^*\right)\left(\frac{M_t}{\bar{\theta}}\right)^{1-\gamma-\gamma_1^*},$$

which is equivalent to showing that $\lambda\left(\gamma_1+\gamma\right)\geq\phi\left(\gamma_1-\gamma_1^*\right)$. Direct algebra shows that

$$\frac{2\lambda}{\sigma^2}=\left(\gamma_1^*-\gamma_2\right)\left(\gamma_1-\gamma_1^*\right). \quad (82)$$

By (80) and (82),

$$\begin{aligned}\lambda\left(\gamma_1+\gamma\right) &= \frac{\sigma^2}{2}\left(\gamma_1+\gamma\right)\left(\gamma_1^*-\gamma_2\right)\left(\gamma_1-\gamma_1^*\right) \quad (83)\\ &= -\frac{\phi}{\gamma+\gamma_2}\left(\gamma_1^*-\gamma_2\right)\left(\gamma_1-\gamma_1^*\right).\end{aligned}$$

Furthermore,

$$\left(\gamma_1^* - \gamma_2\right) \geq -\left(\gamma + \gamma_2\right), \tag{84}$$

since $\gamma_1^* + \gamma \geq 0$. Given that $\gamma_1 > \gamma_1^*$ and $\gamma + \gamma_2 < 0$, (84) and (83) yield the desired conclusion, namely $\lambda\left(\gamma_1 + \gamma\right) \geq \phi\left(\gamma_1 - \gamma_1^*\right)$. This completes the proof of (69).

The rest of the proof follows steps similar to Øksendal (2003), Chapter 9. For completeness we give a brief sketch omitting technical details. Take any stopping time τ and apply Ito's Lemma to $e^{-(\rho+\lambda)t} V\left(\theta_t, M_t\right)$ to obtain

$$\begin{aligned} Ee^{-(\rho+\lambda)(\tau-t)} V\left(\theta_\tau, M_\tau\right) - V\left(\theta_t, M_t\right) &= E_t \int_0^\tau e^{-(\rho+\lambda)(s-t)} \mathcal{A} V\left(\theta_s, M_s\right) ds + \\ &\quad + E_t \int_t^\tau e^{-(\rho+\lambda)(s-t)} V_M\left(M_s, M_s\right) dM_s. \end{aligned} \tag{85}$$

Re-arranging (85) and using (66)-(69) yields

$$\begin{aligned} V\left(\theta_t, M_t\right) &\geq Ee^{-(\rho+\lambda)(\tau-t)} V\left(\theta_\tau, M_\tau\right) \\ &\geq Ee^{-(\rho+\lambda)(\tau-t)} \left[\zeta(i_{N,j}) G\left(\theta_\tau, M_\tau\right) - \eta M_{\tau_N} \left[\theta_\tau\right]^{-\nu} X_{\tau_N}^{-\nu} \left(\frac{\theta_\tau}{M_\tau}\right)^{(1-\gamma)(1-\alpha)}\right]. \end{aligned}$$

Since τ is arbitrary, $V\left(\theta_t, M_t\right)$ provides an upper bound to the value function for all feasible policies. Furthermore, this bound is attainable if the firm plants a tree the first time that $\theta_t = \bar{\theta}\left(M_t\right)$. Hence $V\left(\theta_t, M_t\right)$ is the value function for firm j in round N and planting a tree once $\theta_t = \bar{\theta}\left(M_t\right)$ is optimal.

■

Proposition 2 shows that if firms perceive the equilibrium stochastic discount factor to be given by (31), then it is optimal for them to plant a tree according to equation (25). Furthermore, Corollary 1 gives the equilibrium value of assets in place for firm j in round N at time t. To complete the determination of the value of a firm, the following proposition provides the equilibrium value of "current epoch" growth options and "future epoch" growth options.

Proposition 3 *Let $K(m_t)$ be given by (32) with $\Xi = \Xi^*$. Then, the price of firm j in technological epoch N is given by (8) where the asset in place P_t^A is given by*

$$P_{j,t}^A = X_{j,t} \theta_t \chi\left(\frac{\theta_t}{M_t}, m_t\right), \tag{86}$$

the current "epoch" growth option at time t for firm j is

$$
\begin{aligned}
P^{o}_{N,j,t} &= X_{\tau_N}\theta_t\left(\frac{\theta_t}{M_t}\right)^{\gamma_1+\gamma-1}\left(\frac{M_t}{M_{\tau_N}}\right)^{\gamma_1+\alpha(\gamma-1)}\left(1+F\left(K\left(m_t\right)\right)\right)^{\nu} \\
&\quad\times\left(\frac{bp}{\Xi^*}\right)^{\frac{-\nu p}{1-p+\nu p}-\beta_2}\Xi^* C^{ind}_{op}\left(i_{N,j}\right)\left(1-1_{\{\tilde{\chi}_{N,j}=1\}}\right),
\end{aligned}
\tag{87}
$$

where the constant $C^{ind}_{op}\left(i_{N,j}\right)$ *is given by*

$$
C^{ind}_{op}\left(i_{N,j}\right)=\left(1+bi_{N,j}\right)^{(1-p+\nu p)\beta_2+\nu p}\left(\begin{array}{c} -\frac{\eta}{\Xi^*}\frac{1-\frac{\nu p}{1-p+\nu p}}{1+\beta_2}+\left(1+bi_{j,N}\right)^{-(1-p+\nu p)(1+\beta_2)} \\ \times\left(1+b\right)^{(1-p+\nu p)(1+\beta_2)}\left[\Delta_2 g_2\left(0\right)+\Delta_1 g_2\left(1-\gamma-\gamma_1^*\right)\right]\end{array}\right).
$$

Finally, define the constants C_{op} *and* $\tilde{\Delta}_1$ *as*

$$
\begin{aligned}
C_{op} &= \left[\frac{bp}{\Xi^*}\right]^{\frac{-\nu p}{1-p+\nu p}-\beta_2}\Xi^*\left(\begin{array}{c}\left(1+b\right)^{[1-p+\nu p](1+\beta_2)}\left[\Delta_2 g_2\left(0\right)+\Delta_1 g_2\left(1-\gamma-\gamma_1^*\right)\right]\frac{(1+b)^p-1}{bp} \\ -\frac{\eta}{\Xi^*}\frac{\left(1-\frac{\nu p}{1-p+\nu p}\right)}{1+\beta_2}\frac{(1+b)^{[1-p+\nu p]\beta_2+\nu p+1}-1}{b([1-p+\nu p]\beta_2+\nu p+1)}\end{array}\right), \\
\tilde{\Delta}_1 &= -\frac{C_{op}}{\tilde{g}_4\left(1-\gamma-\gamma_1^*\right)}.
\end{aligned}
$$

Then the value of all "future epoch" growth options is given by

$$
\begin{aligned}
&P^{f}_{N,t} \\
&= \tilde{\Delta}_1 X_t\theta_t\left\{\begin{array}{ll}
\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\left\{1+\left(\frac{\theta_t}{M_t}\right)^{\gamma_1-\gamma_1^*}\left[\begin{array}{c}-\frac{\alpha(\gamma-1)+\gamma_1^*}{\alpha(\gamma-1)+\gamma_1}+\left(\frac{bpm_t}{\Xi^*}\right)^{\alpha(\gamma-1)+\gamma_1}\times \\ \tilde{g}_2\left(1-\gamma-\gamma_1^*\right)\left[\left(1+b\right)^{(1-p+\nu p)(1+\beta_2)+p}-1\right]\end{array}\right]\right\}; & m_t\leq\frac{\Xi^*}{bp} \\
\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\left\{1+\left(\frac{\theta_t}{M_t}\right)^{\gamma_1-\gamma_1^*}\left[\begin{array}{c}\frac{-\tilde{g}_1(1-\gamma-\gamma_1^*)}{1+\beta_2+\frac{p}{1-p+\nu p}}+\left(\frac{bpm_t}{\Xi^*}\right)^{-(1+\beta_2)-\frac{p}{1-p+\nu p}}\times \\ \left(1+b\right)^{(1-p+\nu p)(1+\beta_2)+p}\tilde{g}_2\left(1-\gamma-\gamma_1^*\right)\end{array}\right]\right\}; & m_t\in\left[\frac{\Xi^*}{bp},m^*\right] \\
\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\left[1-\frac{\alpha-\alpha\gamma-\gamma_1^*}{\alpha-\alpha\gamma-\gamma_1}\left(\frac{\theta_t}{M_t}\right)^{\gamma_1-\gamma_1^*}\right]; & m_t\geq m^*.
\end{array}\right.
\end{aligned}
\tag{88}
$$

Proof. The proof of (86) is given in Corollary 1. The proof of (87) follows upon computing expression (64) explicitly. The proof of equation (88) follows similar steps to that of Lemma 1 and is omitted to save space. ■

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