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ESTIMATION AND FORECASTING IN LARGE DATASETS WITH CONDITIONALLY HETEROSKEDASTIC DYNAMIC COMMON FACTORS

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ESTIMATION AND FORECASTING IN LARGE DATASETS WITH CONDITIONALLY HETEROSKEDASTIC DYNAMIC COMMON FACTORS [1]

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In 2009 all ECB publications feature a motif taken from the €200 banknote.

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Abstract

We propose a new method for multivariate forecasting which combines Dynamic Factor and multivariate GARCH models. The information contained in large datasets is captured by few dynamic common factors, which we assume being conditionally heteroskedastic. After presenting the model, we propose a multi-step estimation technique which combines asymptotic principal components and multivariate GARCH. We also prove consistency of the estimated conditional covariances. We present simulation results in order to assess the finite sample properties of the estimation technique. Finally, we carry out two empirical applications respectively on macroeconomic series, with a particular focus on different measures of inflation, and on financial asset returns. Our model outperforms the benchmarks in forecasting the inflation level, its conditional variance and the volatility of returns. Moreover, we are able to predict all the conditional covariances among the observable series.

Keywords: Dynamic Factor Models, Multivariate GARCH, Conditional Covariance, Inflation Forecasting, Volatility Forecasting.

JEL Classification: C52, C53.

Non-technical summary

In this paper we combine Dynamic Factor models and multivariate GARCH models with the aim of explaining and forecasting a large number of variables together with their conditional covariances. The model we propose, named Dynamic Factor GARCH (DF-GARCH), can handle large panels of time series by estimating only a relatively small number of parameters, as it assumes a dynamic factor structure in the data. The dynamic common factors driving the data are precisely those of interest for structural analysis, i.e. they can be identified as structural shocks. Finally, thanks to the factor decomposition, we are able to disentangle the common, or systemic, component and the idiosyncratic component of the first and second, conditional and unconditional moments of each variable in the panel. These features of the DF-GARCH make it suitable for applications in several fields. For instance, large conditional covariance matrices are typically used in finance, e.g. for the construction of optimal portfolios and the pricing of options based on many underlying returns. Moreover, one could use the DF-GARCH to construct systemic credit risk indicators, by using data on leverage or asset volatility and exploiting financial risk measures such as Value-at-Risk and probability of default. Finally, the DF-GARCH could be used in macroeconomics for inflation forecasting, for assessing inflation uncertainty, for measuring upside and downside risks to price stability, and for a structural analysis of the conditional covariance structure between the real and financial sides of the economy. We firstly describe the model and the estimation procedure, together with its consistency properties. Then, we evaluate the potentialities of our model and the goodness of our estimation method by carrying out Monte-Carlo simulations and two empirical applications. In the first application, we employ the DF-GARCH for the forecast of inflation and its time dependent confidence intervals. We model also the conditional covariance matrix of the whole macroeconomic dataset. In this way we are able to detect and predict conditional correlations between inflation and other variables, which contain useful information for monetary policy. In a second application, we consider predictions of volatilities and covolatilities of daily asset returns. The DF-GARCH specification in both cases performs well compared with alternative benchmark specifications.

1 Introduction

It is well known that modelling the conditional variance of inflation may improve the prediction of its level (Engle, 1982). Knowing also the covariance structure between inflation and other macroeconomic quantities may improve the forecast even further (see e.g. Hamilton, 2008). Moreover, assessing inflation uncertainty becomes of paramount importance now that the era of the 'Great Moderation' might have come to an end. Indeed, central banks have to evaluate carefully the risks associated with price stability, which in turn is often considered as the avoidance of excess inflation but also of deflation. Moreover, predicting the covolatility of a large number of variables is crucial not only in macroeconomics. In finance, many issues involve the prediction of large covariance matrices, e.g. the construction of optimal portfolios and the pricing of options based on many underlying returns. For these and many other purposes we need to estimate the conditional covariances of large datasets.

We combine Dynamic Factor models and multivariate GARCH models, proposing a model able to explain and forecast a large number of series together with their conditional covariances. The model is parsimonious as the number of parameters, specifying the dynamics of the few driving factors, is relatively small. We name the model Dynamic Factor GARCH (DF-GARCH). As in Giannone et al. (2004) and Forni et al. (2009), we make use of the static representation of the generalized (or approximate) dynamic factor model by Forni et al. (2005), together with a VAR specification of the static factors. Estimation is based on asymptotic principal components (see Bai, 2003) and on a subsequent VAR estimated on these latter. Finally, we add a multivariate GARCH structure to the vector of dynamic common factors. As a consequence, we can model the conditional covariances of the observable series by simply modelling the conditional covariances of the few underlying dynamic factors. We consider two multivariate GARCH models: the full BEKK and the DCC specifications for which Maximum Likelihood estimators are available (see Engle and Kroner, 1995; Engle, 2002, respectively).

In traditional multivariate volatility models an increase in the cross-sectional dimension corresponds to a much larger increase in the number of the required parameters. Those specifications which bypass this problem, on the other hand, pay a price in terms of a severe loss of generality or high complexity in estimation.[1] Factor models offer a key for dimensionality reduction, which consists in assuming a few latent variables, the common factors, as driving forces for the whole dataset. By summarizing the bulk of the information into a few series, the estimation of multivariate GARCH in large datasets becomes feasible.

The idea of a conditional heteroskedastic factor model is firstly suggested by Engle (1987), where the conditional covariance of the observed series follows a one-factor process. Similarly, Diebold and Nerlove (1989) develop a static heteroskedastic one-factor model for exchange rate series. Estimation is pursued by using a Kalman filter which is preferred to the two-step procedure by Engle et al. (1990) where static factors are extracted from the unconditional covariance matrix before being modelled as univariate GARCH processes. In fact,

[1] See Bauwens et al. (2006) for a survey on multivariate GARCH models, and Harvey et al. (1994) for a survey on Stochastic Volatility models.

Sentana (1998) proves that the latter model is nested in the former. Both models have homoskedastic idiosyncratic component. Harvey et al. (1992) build a modified version of the Kalman filter for models with unobservable heteroskedastic factors and apply it to the case of a dynamic one-factor model (a Structural ARCH). This modified version of the Kalman filter is used also by King et al. (1994) to estimate a static factor model with a diagonal time-varying conditional covariance matrix of the idiosyncratic component. All these models are either static or allowing for just one factor.

The contribution of our model is twofold. First, it is a dynamic model and therefore the underlying factors are really few (in particular, they are less than the factors one would find when estimating a purely static model on a dataset where dynamics are important). These few dynamic factors are those of interest for structural analysis. Second, thanks to the very small dimension of the factor space, we are able to estimate the conditional covariance matrix of very large datasets without estimating any highly parametrized model.

After studying the consistency properties of our estimation procedure for the cross-sectional (n) and sample (T) dimensions going to infinity, we evaluate the goodness of our estimation method and the potentialities of our model by carrying out some Monte-Carlo simulations and two empirical applications.

All the literature cited above deals with theoretical specifications and different possible estimations of conditionally heteroskedastic factor models with a special focus on financial applications. We believe that these kinds of models have a broader scope in time series analysis. For this reason, in the first application, we employ the DF-GARCH for the forecast of inflation and its time dependent confidence intervals. We model also the conditional covariance matrix of the whole macroeconomic dataset. In this way we are able to detect and predict conditional correlations between inflation and other variables, which contain useful information for monetary policy.

There are three streams of literature related to this first empirical exercise. First, the use of conditionally heteroskedastic models for inflation has originally been suggested by Engle (1982, 1983) when forecasting UK and US inflation series. The importance of modelling the conditional variance when forecasting the levels has been also remarked upon by Stock and Watson (2007), who propose a stochastic volatility model for predicting inflation. Moreover, modelling also the conditional covariances among macroeconomic variables can considerably improve the forecasts of their conditional mean, as recently shown by Hamilton (2008). In this light, the applications of dynamic factor models for forecasting macroeconomic quantities (see Artis et al., 2002; Banerjee and Marcellino, 2006; Heij et al., 2008, among others) can be revisited by taking explicitly into account the possible conditional heteroskedasticity of the underlying dynamic factors. Second, there exists a whole stream of literature, starting from the works by Friedman (1977) and Ball (1992), aiming at testing the hypothesis that higher variability of inflation should lead to lower output, *ceteris paribus*, and higher rates of inflation are generally associated with higher variability of inflation (see Engle, 1983; Grier and Perry, 1998; Kontonikas, 2004, among others). If this hypothesis is true, then higher rates of inflation would also be associated with lower levels of output, which implies a positively sloped Phillips curve. Finally, by considering policy makers and central bankers as risk managers, Kilian and Manganelli (2008) show that for maintaining price stability it is necessary to have a model that provides a measure of the risk associated with inflation falling below or breaching a given threshold.

In a second application, we consider predictions of volatilities and covolatilities of daily asset returns. In this case we are just concerned about volatility forecasts, and not level forecasts, as derived from a factor decomposition of the returns. Together with all the literature cited above, two other models are related to this empirical analysis. First, the Orthogonal GARCH by Alexander (2001), which is typically used for Value-at-Risk modelling and is based on a univariate GARCH specification of the principal components of the financial returns. Second, the Generalized Orthogonal GARCH by van der Weide (2002), in which the linear map that links components and observed data is allowed to be non orthogonal. There are two differences between the DF-GARCH and these two models, which are in fact principal component and not factor models. First, both models are in practice applied only to very small datasets ($n \simeq 10$), while we are able to apply the DF-GARCH to very large datasets ($n \simeq 100$). Second, there is a technical and interpretative difference between the two approaches. Indeed, factor models require more structure since the number of factors is not arbitrary but determined by the data. Moreover, factor models are more flexible than principal components by allowing the idiosyncratic covariance matrix to be non-orthogonal.

The paper is structured as follows. In Section 2 we present the model and its assumptions. In Section 3 we describe the estimation procedure and we prove its consistency. In Section 4 we report the results from Monte-Carlo simulations. In Section 5 we apply the DF-GARCH to a large panel of macroeconomic series in order to forecast inflation's level and its conditional variance and covariance with other relevant series. In Section 6 we compare the predictions of financial volatilities and covolatilities for different specifications of the DF-GARCH. In Section 7 we conclude and discuss possible developments.

2 The Dynamic Factor GARCH

We consider a modification of the model by Forni et al. (2009), which in turn is a special case of the model in Forni and Lippi (2001) and Forni et al. (2005). It is an approximate factor model as it allows for mildly cross-correlated idiosyncratic component. For this reason we need in principle to have an infinite number of cross-sectional units. Closely related models are in Stock and Watson (2002) and Bai (2003).

Denote by $\mathbf{x}$ an $n \times T$ rectangular array of observations:[2]

Assumption 1: $\mathbf{x}$ *is a finite realization of an infinite dimensional real-valued stochastic process defined in* $L_2(\Omega, \mathcal{F}, P)$*, where all the* n*-dimensional vector processes* $\{\mathbf{x}_t = (x_{1t} \dots x_{nt})',$

[2]Throughout the paper, we omit the dependence on n and T to avoid heavy notation. In particular, we do not make explicit the dependence on n when considering quantities at population level. In the same way, we indicate the estimate of a generic quantity $\boldsymbol{B}$, which depends on n and T, simply as $\widehat{\boldsymbol{B}}$.

$t \in \mathbb{Z}\}$, $n \in \mathbb{N}$, *are stationary, with zero mean and finite second-order moments* $\mathbf{\Gamma}_k^x = \mathrm{E}[\mathbf{x}_t \mathbf{x}'_{t-k}]$, $k \in \mathbb{Z}$.

We assume that each variable x_{it} is the sum of two unobservable components: the common component χ_{it} and the idiosyncratic component ξ_{it}. The common component is driven by q common shocks $\mathbf{u}_t = (u_{1t} \dots u_{qt})'$, where q is independent of n and, typically, $q << n$. By defining $\boldsymbol{\chi}_t = (\chi_{1t} \dots \chi_{nt})'$ and $\boldsymbol{\xi}_t = (\xi_{1t} \dots \xi_{nt})'$, we have

$$\mathbf{x}_t = \boldsymbol{\chi}_t + \boldsymbol{\xi}_t = \mathbf{B}(L)\mathbf{u}_t + \boldsymbol{\xi}_t, \tag{1}$$

where:

Assumption 2: $\mathbf{u}_t$ *is a* q*-dimensional orthonormal white noise,* $\mathbf{B}(L)$ *is a one-sided* $n \times q$ *absolutely summable matrix polynomial, i.e. a filter (infinite in general). Moreover, there exists an integer* $r > q$*, an* $n \times r$ *matrix* $\mathbf{\Lambda}$*, a one-sided absolutely summable* $r \times r$ *matrix polynomial (infinite in general)* $\mathbf{N}(L)$*, and a maximum rank* $r \times q$ *matrix* $\mathbf{H}$*, such that*

$$\mathbf{B}(L) = \mathbf{\Lambda}\mathbf{N}(L)\mathbf{H}$$

Defining the $r \times 1$ *vector* $\mathbf{F}_t = \mathbf{N}(L)\mathbf{H}\mathbf{u}_t$*,* (1) *can be written in the static form*

$$\mathbf{x}_t = \mathbf{\Lambda}\mathbf{F}_t + \boldsymbol{\xi}_t. \tag{2}$$

We call static factors the r entries of $\mathbf{F}_t$, while the common shocks $\mathbf{u}_t$ are also referred to as dynamic factors.

Assumption 3: *for all* n*, the vector* $\boldsymbol{\xi}_t$ *is stationary. Moreover,* u_{jt} *is orthogonal to* ξ_{is}*, for all* $i, j \in \mathbb{N}$*, and* $t, s \in \mathbb{Z}$*, i.e. E*$[u_{jt}\xi_{is}] = 0$.

Let $\mathbf{\Gamma}_k^{\chi} = \mathrm{E}[\boldsymbol{\chi}_t \boldsymbol{\chi}'_{t-k}]$ and $\mathbf{\Gamma}_k^{\xi} = \mathrm{E}[\boldsymbol{\xi}_t \boldsymbol{\xi}'_{t-k}]$, and denote by μ_j^{χ} and μ_j^{ξ} the j-th eigenvalue, in decreasing order of $\mathbf{\Gamma}_0^{\chi}$ and $\mathbf{\Gamma}_0^{\xi}$. Moreover, let $\mathbf{\Sigma}^{\chi}(\theta)$ and $\mathbf{\Sigma}^{\xi}(\theta)$ be the spectral density matrix of $\boldsymbol{\chi}_t$ and $\boldsymbol{\xi}_t$, respectively, and denote by $\lambda_j^{\chi}(\theta)$ and $\lambda_j^{\xi}(\theta)$ their eigenvalues as functions of $\theta \in [-\pi, \pi]$, in decreasing order.

Assumption 4: *(a) as* $n \to \infty$*,* $\lambda_q^{\chi}(\theta) \to \infty$ *for* θ *a.e. in* $[-\pi, \pi]$*;*

(b) there exist constants $\underline{c}_j$ *,* $\bar{c}_j$ *,* $j = 1, \dots, r$*, such that* $\underline{c}_j > \bar{c}_{j+1}$*,* $j = 1, \dots, r-1$*, and*

$$0 < c_j < \liminf_{n \to \infty} n^{-1}\mu_j^{\chi} \leq \limsup_{n \to \infty} n^{-1}\mu_j^{\chi} \leq \bar{c}_j.$$

Assumption 5: *there exists a real* K *such that* $\lambda_1^{\xi}(\theta) \leq K$ *for any* $n \in \mathbb{N}$ *and a.e. in* $[-\pi, \pi]$*. This implies that* $\mu_1^{\xi} \leq K$ *for any* $n \in \mathbb{N}$*.*

Under these Assumptions, the number q of dynamic factors and the common component χ_{it} are uniquely identified. In particular, a representation of the form (1) with a different number of dynamic factors is not possible (see Forni and Lippi, 2001). Assumption 4(b) is necessary to identify r. In particular, a static representation of the common component with a different number of static factors is not possible. The existence of the static representation (2) is crucial for the estimation of our model and it is proved in Forni et al. (2009). Finally, we add to our model two other Assumptions.

Assumption 6: *the entries of $\mathbf{N}(L)$ are rational functions, and in particular $\mathbf{N}(L)$ results from inversion of the VAR(m) $\mathbf{F}_t = (\mathbf{I}_r - \mathbf{A}L - \ldots - \mathbf{A}_m L^m)^{-1}\boldsymbol{\epsilon}_t$. For simplicity, we assume $m = 1$, so that $\mathbf{N}(L) = (\mathbf{I}_r - \mathbf{A}L)^{-1}$, where $\mathbf{I}_r$ is the r-dimensional identity matrix, and $\mathbf{A}$ is an $r \times r$ matrix.*

Notice that, according to Assumption 2, $\boldsymbol{\epsilon}_t = \mathbf{H}\mathbf{u}_t$, i.e. the residuals of the VAR have reduced rank q. More precisely, $\boldsymbol{\epsilon}_t \in \overline{\text{span}}\{\mathbf{u}_t\}$, i.e. the residuals belong to a q-dimensional linear space generated by the dynamic factors.

Assumption 7: *(a) the dynamic factors are conditionally heteroskedastic $\mathbf{u}_t|\mathcal{I}_{t-1} \sim \mathrm{N}(\mathbf{0}, \mathbf{Q}_t)$, $\mathbf{Q}_t$ being a nondiagonal time-dependent $q \times q$ matrix. $\mathcal{I}_{t-1}$ contains all the information available at time $t-1$, including previous estimates of the dynamic factors, thus $\mathbf{Q}_t = \mathrm{E}[\mathbf{u}_t\mathbf{u}_t'|\mathcal{I}_{t-1}]$;*

(b) the idiosyncratic component evolves according to a univariate ARMA-GARCH model $\boldsymbol{\xi}_t|\mathcal{I}_{t-1} \sim \mathrm{N}(\boldsymbol{\mu}_t, \mathbf{P}_t)$, where $\boldsymbol{\mu}_t$ is an n-dimensional vector of ARMA specifications and $\mathbf{P}_t$ is an $n \times n$ diagonal matrix containing the conditional variances of each idiosyncratic series.

In practice, we allow for two possible specifications of $\mathbf{Q}_t$. The first one is the full BEKK specification by Engle and Kroner (1995):

$$\mathbf{Q}_t = \mathbf{C}_0'\mathbf{C}_0 + \mathbf{C}_1'\mathbf{u}_{t-1}\mathbf{u}_{t-1}'\mathbf{C}_1 + \mathbf{C}_2'\mathbf{Q}_{t-1}\mathbf{C}_2. \tag{3}$$

As we require $\mathrm{E}[\mathbf{Q}_t] = \mathrm{E}[\mathbf{u}_t\mathbf{u}_t'] = \mathbf{I}_q$, an additional condition applies on the coefficients of the BEKK representation: $\mathbf{C}_0'\mathbf{C}_0 = \mathbf{I}_q - \mathbf{C}_1'\mathbf{C}_1 - \mathbf{C}_2'\mathbf{C}_2$. Alternatively, $\mathbf{Q}_t$ can be modelled according to the DCC specification by Engle (2002):

$$\mathbf{Q}_t = \mathbf{D}_t\mathbf{W}_t\mathbf{D}_t, \qquad \mathbf{W}_t = (\mathbf{V}_t^*)^{-1}\mathbf{V}_t(\mathbf{V}_t^*)^{-1}. \tag{4}$$

$\mathbf{D}_t$ is a diagonal matrix containing on its diagonal the conditional standard deviations for each $\mathbf{u}_t$ obtained from a univariate GARCH(1,1) specification:

$$D_{jjt} = \omega_i + \alpha_i u_{jt-1}^2 + \beta_i D_{jjt-1}, \qquad j = 1, \ldots, q. \tag{5}$$

$\mathbf{W}_t$ is the dynamic correlation matrix and has ones on its diagonal, while its out-of-diagonal elements are given by the corresponding elements of $\mathbf{V}_t$. More precisely, after defining the standardized residuals $v_{jt} = u_{jt}(D_{jjt})^{-1}$ for $j = 1, \ldots, q$, we have

$$\begin{aligned} \mathbf{V}_t &= (1-a-b)\,\mathrm{E}[\mathbf{v}_t\mathbf{v}_t'] + a\,(\mathbf{v}_{t-1}\mathbf{v}_{t-1}') + b\mathbf{V}_{t-1}, \\ V_{jjt}^* &= (V_{jjt})^{1/2} \quad \text{and} \quad V_{jht}^* = 0 \quad \text{for} \quad j \neq h \quad j = 1, \ldots, q. \end{aligned} \tag{6}$$

We indicate $\mathbf{Q}_t(\mathbf{C}_1, \mathbf{C}_2)$ when considering the full BEKK and $\mathbf{Q}_t(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}, a, b)$ when considering the DCC, with $\boldsymbol{\omega} = (\omega_1 \ldots \omega_q)'$, $\boldsymbol{\alpha} = (\alpha_1 \ldots \alpha_q)'$, and $\boldsymbol{\beta} = (\beta_1 \ldots \beta_q)'$.

For the univariate idiosyncratic conditional variances we use a GARCH(1,1) model:

$$P_{iit}(\pi_{0i}, \pi_{1i}, \pi_{2i}) = \pi_{0i} + \pi_{1i}\xi_{it-1}^2 + \pi_{2i}P_{iit-1}(\pi_{0i}, \pi_{1i}, \pi_{2i}), \qquad i \in \mathbb{N}. \tag{7}$$

We do not define in this paper any particular structure of the ARMA process governing the dynamics of $\boldsymbol{\xi}_t$, although this is in principle possible without any further complication. In the empirical application on financial returns, which are unlikely to have a strong dynamic structure in their levels, we do not consider any ARMA structure of the idiosyncratic component (i.e. $\boldsymbol{\mu}_t = 0$). When instead we forecast inflation, we do not model the idiosyncratic component at all as we are interested in building an indicator for inflation based only on the common component.

Under Assumption 4(b), the space spanned by the static factors and the common component are always identified. However, the r common static factors can be identified only up to a unitary transformation $\mathbf{G}$. Under Assumption 4(a), also the space spanned by the dynamic factors is identified. Assumption 6 implies that also the $\mathbf{u}_t$'s are identified up to a unitary transformation $\mathbf{R}$ (see Proposition 2 in Forni et al., 2009). The roles of $\mathbf{G}$ and $\mathbf{R}$ are however different and are discussed in the next Section.

Finally, for identification purposes we need some techincal assumptions: $\boldsymbol{\Lambda}'\boldsymbol{\Lambda}/n \to \mathbf{I}_r$ as $n \to \infty$, where $\mathbf{I}_r$ is the r-dimensional identity matrix, and $\sum_t \mathbf{F}_t\mathbf{F}_t'/T \to \boldsymbol{\Gamma}_0^F$ as $T \to \infty$ for some positive definite $\boldsymbol{\Gamma}_0^F$ (see Bai, 2003, for details).

3 Estimation and consistency

The estimation of the DF-GARCH is based on Giannone et al. (2004) and Forni et al. (2009). We make use of the static representation (2) together with the VAR(1) specification of the static factors as given in Assumption 6:

$$\mathbf{x}_t = \boldsymbol{\Lambda}\mathbf{F}_t + \boldsymbol{\xi}_t, \tag{8}$$

$$\mathbf{F}_t = \mathbf{A}\mathbf{F}_{t-1} + \boldsymbol{\epsilon}_t, \quad \text{with} \quad \boldsymbol{\epsilon}_t = \mathbf{H}\mathbf{u}_t. \tag{9}$$

This state-space representation is equivalent to the dynamic representation (1), with filters defined as

$$\mathbf{B}(L) = \boldsymbol{\Lambda}(\mathbf{I}_r - \mathbf{A}L)^{-1}\mathbf{H}. \tag{10}$$

Before estimating (8)-(9), we have to specify the number of dynamic factors q and the number of static factors r. Hallin and Liška (2007) and Bai and Ng (2002) provide consistent estimators, as $n, T \to \infty$, of q and r, respectively.

The estimation of the DF-GARCH is in four steps.

STEP 1 Given a consistent estimator of the covariance matrix $\widehat{\mathbf{\Gamma}}_0^x$, the static factors $\mathbf{F}_t$ are consistently estimated as the r largest principal components as in Stock and Watson (2002) and Bai (2003). We have also a consistent estimate of the loadings $\mathbf{\Lambda}$, as proved in the next Proposition.[3]

Proposition 1: *(a) the estimated loadings $\widehat{\mathbf{\Lambda}}$ are $\sqrt{n}$ times the normalized eigenvectors corresponding to the r largest eigenvalues of the sample covariance matrix $\widehat{\mathbf{\Gamma}}_0^x$; the estimated factors are $\widehat{\mathbf{F}}_t^T = \frac{1}{n}\widehat{\mathbf{\Lambda}}'\mathbf{x}_t$, moreover there exists a $r \times r$ unitary matrix $\mathbf{G}$ such that as $n, T \to \infty$*

$$||\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda}\mathbf{G}|| \xrightarrow{P} 0, \qquad ||\widehat{\mathbf{F}}_t - \mathbf{G}'\mathbf{F}_t|| \xrightarrow{P} 0;$$

(b) we have also a consistent estimation of the common and idiosyncratic components: $\widehat{\boldsymbol{\chi}}_t = \widehat{\mathbf{\Lambda}}\widehat{\mathbf{F}}_t$, and $\widehat{\boldsymbol{\xi}}_t = \mathbf{x}_t - \widehat{\boldsymbol{\chi}}_t$. As $n, T \to \infty$

$$\widehat{\chi}_{it} - \chi_{it} \xrightarrow{P} 0, \qquad \widehat{\xi}_{it} - \xi_{it} \xrightarrow{P} 0, \qquad \forall\, i.$$

Proof: see Bai (2003); part (a) is in Theorems 1 and 2, while part (b) is in Theorem 3.[4]

Alternatively, we can estimate the static factors as the r largest generalized principal components as in Forni et al. (2005). In this case we need consistent estimators of $\widehat{\mathbf{\Gamma}}_0^\chi$ and $\widehat{\mathbf{\Gamma}}_0^\xi$, which can be obtained from the spectral decomposition of a periodogram smoothing consistent estimator of the spectral density matrix $\mathbf{\Sigma}^x(\theta)$. A result analogous to Proposition 1 can be proved (see Proposition 1 in Forni et al., 2005). In the first empirical application we use standard principal components, while in the second one we use generalized principal components. The method by Bai (2003) does not require the spectral decomposition used instead in Forni et al. (2005), and it is, in this sense, a static method. Although in theory we may miss some relevant information by computing only static principal components, in practice the evidence is mixed and it has been shown that the two estimation methods deliver similar results in terms of forecasting performance (see e.g. Boivin and Ng, 2005; D'Agostino and Giannone, 2006).

STEP 2 Given an estimate of the static factors $\mathbf{F}_t$ and of the loadings $\mathbf{\Lambda}$, we need to estimate equation (9) in order to have an estimate of the dynamic factors. We just have to run a VAR on the estimated static factors.

[3]Given a matrix $\boldsymbol{B}$, $||\boldsymbol{B}||$ denotes the spectral norm of $\boldsymbol{B}$, thus $||\boldsymbol{B}|| = \sqrt{\mu_1(\boldsymbol{B}\boldsymbol{B}')}$, where $\mu_1(\boldsymbol{B}\boldsymbol{B}')$ is the largest eigenvalue of $\boldsymbol{B}\boldsymbol{B}'$. If $\boldsymbol{B}$ is a row matrix, then $||\boldsymbol{B}||$ is the euclidean norm.

[4]In this and the following Propositions we do not study the rate of convergence, which for the common component is typically equal to $\min(\sqrt{T}, \sqrt{n})$.

Proposition 2: *the coefficients of equation* (9) *and the covariance matrix of the residuals are estimated as:*

$$\widehat{\mathbf{A}} = \widehat{\mathbf{\Gamma}}_1^F \left(\widehat{\mathbf{\Gamma}}_0^F\right)^{-1}, \qquad \widehat{\mathbf{\Gamma}}_0^{\epsilon} = \left(\widehat{\mathbf{\Gamma}}_0^F - \widehat{\mathbf{A}}\widehat{\mathbf{\Gamma}}_0^F\widehat{\mathbf{A}}'\right);$$

these estimates are consistent up to the unitary transformation $\mathbf{G}$ *defined in Proposition 1, i.e., as* $n, T \to \infty$*,*

$$||\widehat{\mathbf{A}} - \mathbf{G}'\mathbf{A}\mathbf{G}|| \xrightarrow{P} 0, \qquad ||\widehat{\mathbf{\Gamma}}_0^{\epsilon} - \mathbf{G}'\mathbf{\Gamma}_0^{\epsilon}\mathbf{G}|| \xrightarrow{P} 0, \qquad ||\widehat{\boldsymbol{\epsilon}}_t - \mathbf{G}'\boldsymbol{\epsilon}_t|| \xrightarrow{P} 0.$$

Proof: see the Appendix.

STEP 3 Since the estimated residuals $\widehat{\boldsymbol{\epsilon}}_t$ have reduced rank, as they belong to the space spanned by the q dynamic factors, we can simply use principal components in order to estimate an orthogonal linear basis for this space. Therefore we have:

Proposition 3: *(a) the columns of estimated matrix* $\widehat{\mathbf{H}}$ *are* $\sqrt{r}$ *times the normalized eigenvectors corresponding to the* q *largest eigenvalues of the covariance matrix* $\widehat{\mathbf{\Gamma}}_0^{\epsilon}$*; the estimated dynamic factors are* $\widehat{\mathbf{u}}_t = \frac{1}{r}\widehat{\mathbf{H}}'\widehat{\boldsymbol{\epsilon}}_t$*; moreover there exists a* $q \times q$ *unitary matrix* $\mathbf{R}$ *such that, as* $n, T \to \infty$*,*

$$||\widehat{\mathbf{H}} - \mathbf{G}'\mathbf{H}\mathbf{R}|| \xrightarrow{P} 0, \qquad ||\widehat{\mathbf{u}}_t - \mathbf{R}'\mathbf{u}_t|| \xrightarrow{P} 0,$$

where $\mathbf{G}$ *is defined in Proposition 1;*

(b) we have also a consistent estimation of the filters (10)*; define* $\mathbf{B}(L) = \sum_{k=0}^{s} \mathbf{B}_k L^k$*, then, as* $n, T \to \infty$*,*

$$||\widehat{\mathbf{B}}_k - \mathbf{B}_k\mathbf{R}|| \xrightarrow{P} 0, \qquad \forall\, k \geq 0.$$

Proof: see the Appendix.

Notice that, since the static factors are unobserved and therefore estimated up to a unitary transformation $\mathbf{G}$, then the reduced rank matrix $\mathbf{H}$ is estimated up to the same transformation with the addition of a $q \times q$ unitary transformation $\mathbf{R}$ that comes from principal component analysis. However, Proposition 3 proves that the estimated dynamic factors and the associated filters do not depend on $\mathbf{G}$, but only on $\mathbf{R}$. If we interpret the dynamic factors as structural shocks, then $\mathbf{R}$ could be identified by imposing economic meaningful restrictions. This is the procedure adopted in Forni et al. (2009): if we were able to identify the dynamic factors according to some economic criterion, then we could first estimate $\mathbf{R}$ and then estimate the conditional covariance matrix of $\mathbf{R}'\widehat{\mathbf{u}}_t$ interpreting these as the "true" dynamic factors. If we want to give a structural interpretation to the dynamic factors, we can estimate $\mathbf{R}$ with standard techniques used in the Structural VAR literature (e.g. long run restrictions). The use of dynamic factor models for structural analysis is however beyond the scope of this paper, mainly because in a forecasting context there is no objective criterion that can lead us to identify $\mathbf{R}$. In this respect $\mathbf{R}$ does not represent an issue for us, therefore we assume

$\mathbf{R} = \mathbf{I}_q$.[5] It is important to remark that the estimate of the dynamic factors does not depend on $\mathbf{G}$. Indeed, this matrix would be of difficult interpretation as no economic meaning could be attached to the unobserved static factors.

STEP 4 We impose heteroskedasticity on the estimated principal components of $\boldsymbol{\epsilon}_t$ and in Assumption 6 we define the "true" dynamic factors as these principal components. Therefore, $\widehat{\mathbf{u}}_t$ is a consistent estimate of $\mathbf{u}_t$ and $\mathbf{R} = \mathbf{I}_q$. Under this condition, in the next Proposition we use Maximum Likelihood (ML) to consistently estimate conditional covariances.

Proposition 4: *(a) in the DCC specification, the estimated conditional covariance matrix of the dynamic factors, given in* (4), (5)*, and* (6)*, is estimated as* $\widehat{\mathbf{Q}}_t(\widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{a}, \widehat{b})$*, where*

$$(\widehat{\omega}_j, \widehat{\alpha}_j, \widehat{\beta}_j) = \arg\max_{\omega_j, \alpha_j, \beta_j} -\frac{1}{2}\sum_{t=1}^{T}\left(\log D_{jjt}(\omega_j, \alpha_j, \beta_j) + \frac{\widehat{u}^2_{jt}}{D_{jjt}(\omega_j, \alpha_j, \beta_j)}\right), \quad j = 1, \ldots q,$$

and

$$(\widehat{a}, \widehat{b}) = \arg\max_{a,b} -\frac{1}{2}\sum_{t=1}^{T}\left(\log\left(\det \mathbf{W}_t(a,b)\right) + \widehat{\mathbf{v}}_t'\mathbf{W}_t^{-1}(a,b)\widehat{\mathbf{v}}_t\right),$$

where $\widehat{v}_{jt} = \widehat{u}_{jt}(\widehat{D}_{jjt})^{-1}$ *for any* $j = 1, \ldots, q$*;*

(b) in the full BEKK specification, the estimated conditional covariance matrix of the dynamic factors, given in (3)*, is* $\widehat{\mathbf{Q}}_t(\widehat{\mathbf{C}_1}, \widehat{\mathbf{C}_2})$ *such that*

$$(\widehat{\mathbf{C}_1}, \widehat{\mathbf{C}_2}) = \arg\max_{\mathbf{C}_1, \mathbf{C}_2} -\frac{1}{2}\sum_{t=1}^{T}\left(\log\left(\det \mathbf{Q}_t(\mathbf{C}_1, \mathbf{C}_2)\right) + \widehat{\mathbf{u}}_t'\mathbf{Q}_t^{-1}(\mathbf{C}_1, \mathbf{C}_2)\widehat{\mathbf{u}}_t\right);$$

(c) the estimated conditional variance of the i*-th idiosyncratic component, given in* (7)*, is* $\widehat{P}_{iit}(\widehat{\pi}_{0i}, \widehat{\pi}_{1i}, \widehat{\pi}_{2i})$*, where* .

$$(\widehat{\pi}_{0i}, \widehat{\pi}_{1i}, \widehat{\pi}_{2i}) = \arg\max_{\pi_{0i}, \pi_{1i}, \pi_{2i}} -\frac{1}{2}\sum_{t=1}^{T}\left(\log P_{iit}(\pi_{0i}, \pi_{1i}, \pi_{2i}) + \frac{\widehat{\xi}^2_{it}}{P_{iit}(\pi_{0i}, \pi_{1i}, \pi_{2i})}\right), \quad i = 1, \ldots n$$

(d) $\widehat{\mathbf{Q}}_t$ *and* $\widehat{\mathbf{P}}_t$ *are consistently estimated, i.e., as* $n, T \to \infty$*,*

$$||\widehat{\mathbf{Q}}_t - \mathbf{Q}_t|| \xrightarrow{P} 0, \qquad ||\widehat{\mathbf{P}}_t - \mathbf{P}_t|| \xrightarrow{P} 0.$$

Proof: see the Appendix.

If we model also the conditional mean of the idiosyncratic component then also an ARMA model should be estimated for each series, in order to obtain an estimate of the conditional mean $\widehat{\boldsymbol{\mu}}_t$. No complication is added as the properties of the ML estimate of a univariate

[5] Notice also that simulated and empirical results show that even without taking into account the indeterminacy due to $\mathbf{R}$, the performance of our model is remarkably good.

ARMA-GARCH are well known and can be generalized to a two-step procedure as in part (d) of Proposition 4.

We then have a complete estimate of the DF-GARCH. The dynamics of $\mathbf{x}_t$ are specified through an unobserved component model as in (8)-(9), while the conditional heteroskedasticity of $\mathbf{x}_t$ enters the model through (3) or (4)-(6), and (7). Therefore, we can consistently estimate the common component as stated in Proposition 1 and its conditional covariance as proved in the following Proposition.

Proposition 5: *the estimated conditional covariance matrix of the common component is consistently estimated as* $\widehat{\mathbf{\Gamma}}_t^{\chi} = \widehat{\mathbf{\Lambda}}\widehat{\mathbf{H}}\widehat{\mathbf{Q}}_t\widehat{\mathbf{H}}'\widehat{\mathbf{\Lambda}}'$*, i.e., as* $n, T \to \infty$*,*

$$||\widehat{\mathbf{\Gamma}}_t^{\chi} - \mathbf{\Gamma}_t^{\chi}|| \xrightarrow{P} 0.$$

Proof: see the Appendix.

Some remarks are necessary.

Remark 1 The conditional covariance matrix of $\mathbf{x}_t$ is consistently estimated as

$$\widehat{\mathbf{\Gamma}}_t^{x} = \widehat{\mathbf{\Lambda}}\widehat{\mathbf{H}}\widehat{\mathbf{Q}}_t\widehat{\mathbf{H}}'\widehat{\mathbf{\Lambda}}' + \widehat{\mathbf{P}}_t.$$

This matrix is positive definite by construction. Indeed $\widehat{\mathbf{P}}_t$ and $\widehat{\mathbf{Q}}_t$ come respectively from n univariate GARCH and a multivariate GARCH, and the first term on the right-hand-side is a quadratic form.

Remark 2 In traditional factor models the h-steps ahead forecast of the common component is obtained by projecting the observed series onto the space spanned by $\widehat{\mathbf{F}}_T$. The coefficients of this projection are computed by means of the estimated lagged covariance matrix of $\mathbf{x}_t$ (in Stock and Watson, 2002) or of $\boldsymbol{\chi}_t$ (in Forni et al., 2005). The main limit of this forecasting method is due to the fact that no dynamic model is estimated for the static factors, therefore we can rely only on their in-sample estimation. Following Giannone et al. (2004), in equation (9) we specify the dynamic evolution of the static factors. The h-step ahead forecast of the common component is obtained as

$$\widehat{\boldsymbol{\chi}}_{T+h|T} = \widehat{\mathbf{\Gamma}}_h^{x}\widehat{\mathbf{\Lambda}}(\widehat{\mathbf{\Lambda}}'\widehat{\mathbf{\Gamma}}_0^{x}\widehat{\mathbf{\Lambda}})^{-1}\widehat{\mathbf{A}}\widehat{\mathbf{F}}_{T+h-1|T}.$$

Remark 3 The proposed estimation method is a multi-step procedure, therefore it suffers from loss of efficiency, i.e. the asymptotic covariance matrix will be larger than the inverse of the Fisher information matrix (see Newey and McFadden, 1994) and given the cumbersome expression it will take we do not attempt here to compute it. Moreover, efficiency is lost because the static factors are unobserved, therefore an additional term expressing the uncertainty in the estimate $\widehat{\mathbf{F}}_t$ should be added to the asymptotic covariance matrix.

Remark 4 Kalman filter estimation could provide us with new, and in principle more efficient, estimates of static factors. Moreover, as prediction strongly depends on the last time-period estimation, the implementation of a Kalman filter may increase forecasting accuracy. We could follow Doz et al. (2006) and use Quasi Maximum Likelihood to estimate both the parameters and the static factors with an EM algorithm. We keep things simpler and propose in Appendix B a modified version of the Kalman filter proposed by Harvey et al. (1992). While the factors, together with their conditional covariances, are reestimated by the filter, we keep the parameters of the linear part of the model fixed. The filter can only be quasi-optimal because, at each step, past disturbances are not observable, and therefore we are not sure that the distribution of current disturbances is conditionally Gaussian. In the following Sections, we use this iterative procedure only when dealing with forecasts, while in-sample estimations are computed using the four-step the method explained above.

4 Simulation results

In order to assess the validity of our estimation method we apply it to simulated panels that differ in the cross and time dimensions, in the number of dynamic and static factors and in the amount of variance explained by the common component with respect to the total.[6] As possible values we choose $n = 75, 150$ and $T = 250, 500, 750$ and for every chosen (n, T) combination we take 2 or 3 dynamic factors, loaded with 2 or 4 lags, and an average variance ratio (VR) between idiosyncratic and common components of 0.3 or 0.5 (the noise-to-signal ratio). We simulate $\mathbf{u}_t$ as a multivariate GARCH following the full BEKK as in (3). Idiosyncratic components are simply simulated as univariate GARCH(1,1) as in (7). Parameters of GARCH and full BEKK are extracted from uniform distributions with range determined according to empirical estimates. Namely, $\mathbf{C}_1$ has diagonal elements in $[0.1, 0.5]$ and off-diagonal elements in $[-0.2, 0.2]$; $\mathbf{C}_2$ has diagonal elements in $[0.8, 0.95]$ and off-diagonal elements in $[-0.15, 0.15]$; and for any i, π_{1i} has values in $[0, 0.1]$ and π_{2i} has values in $[0.8, 0.95]$. At each extraction of the parameters, positive definiteness of the simulated conditional variances has been checked before proceeding.

We simulate the dynamic factor model as in (1) instead of simulating it in its static form (8). Indeed, we consider (1) as the real data-generating-process, while (8)-(9) is just a possible way to represent the data, which is necessary for estimation. Such a choice also avoids the simulation of the static factors. Finally, we simulate the filters (10) by extracting them from a standard normal distribution. All these loadings are then renormalized in such a way that on average $\mathbf{x}_t$ has unit variance and zero mean, and the chosen VR is on average respected. For each parameter set we repeat the simulation of the data and the estimation of the DF-GARCH model 250 times.

Figure 1 shows the confidence interval at 90% level. Figures 2.1, 2.2, 3.1 and 3.2 show four examples of estimated and simulated conditional variances and covariances. We notice the remarkably good performance of our estimation method.

[6] We performed all computations and simulations by using the standard Matlab software packages (v.7.0) plus the freely available toolboxes `MATNEM` by Christian T. Brownlees and `ucsd_garch` by Kevin K. Sheppard.

At every Monte-Carlo replication, we run Mincer-Zarnowitz regressions to evaluate the performance of the estimation. For each series i, we regress the entries of the simulated common component or of its variance-covariance matrix on the estimated ones. First, we consider only the diagonal elements of the conditional covariance matrix, thus obtaining a measure of the goodness-of-fit relative to estimated volatilities. Then, we consider only the out-of-diagonal elements in the upper-triangular part of the conditional covariance matrix, in order to measure the goodness-of-fit relative to estimated covolatilities. We thus run the regressions using all the $n(n-1)/2$ elements of the upper-triangular part of $\widehat{\mathbf{\Gamma}}_t^{\chi}$. We focus upon the R^2 coefficient, which roughly measures the amount of variability of the estimated quantities that can be explained by the model, thus giving a general idea of its potentialities. Table 1 and 2 report the average R^2 over all series and over all 250 replications. Notice the improvement achieved on the estimate of the last in-sample observations, when using the Kalman filter. This supports the use of a modified Kalman filter for making accurate out-of-sample forecasts. Notice also from both Tables that, as n and T increase, the estimation improves.

5 Forecasting Inflation and its Volatility

We consider a panel of 130 US macro time series (from the Global Insight Database) with monthly observations from December 1986 to November 2006.[7] We focus on four price indexes: Personal Consumption Expenditure (PCE) for all items or core (i.e. excluding oil and food), Consumer Price Index (CPI) for all items or core. We transform all series in order to obtain stationarity, and, in particular, for every price index p_t we define the percentage monthly inflation rate (on an annual basis) as

$$\pi_t = 1200 \log \left(\frac{p_t}{p_{t-1}} \right).$$

We consider all values greater than five standard deviations (in a univariate setting) as outliers and we replace them with the mean of the series. The criterion by Hallin and Liška (2007) suggests the presence of four dynamic factors. In order to choose the number of static factors we use a heuristic argument (see D'Agostino and Giannone, 2006). We include as many static factors as it is necessary to explain the same amount of variance as explained by the selected common dynamic factors. With $q = 4$ dynamic factors we explain 64% of the total variance and we need $r = 12$ static factors to explain the same amount of variance.

We estimate the model using a rolling window of 13 years, leaving the remaining 7 years for out-of-sample prediction. As we compute predictions from 1 to 12 months ahead, we repeat estimation and prediction 72 times (i.e. 6 years). Each time we repeat the forecast, we reestimate all the parameters of every considered model and the factor decomposition, except

[7] This dataset is often used in factor models literature (see Stock and Watson, 2002; Giannone et al., 2004; D'Agostino and Giannone, 2006, among others). A complete list of the considered variables is available upon request.

for the number of dynamic factors which is estimated once and for all at the beginning of the forecasting exercise.

As expected, when testing for ARCH effects on the standardized residuals of an AR model for inflation, we find little evidence of heteroskedasticity. This property becomes evident only when considering a longer time-span (i.e only for $T \sim 200$) and longer horizons (more than 12 lags) than those considered here. Indeed, also Engle (1983) finds evidence of ARCH effects only at one and two-year horizons. Given the low degree of heteroskedasticity in recent years inflation, we do not expect that a model which considers heteroskedasticity improves level forecasts with respect to its homoskedastic counterpart. We would be satisfied with a heteroskedastic model that forecasts levels with the same accuracy of a homoskedastic one, but that is also able to provide reliable forecasts of conditional variances that can be used as a proxy of inflation uncertainty. In addition, our model is able to provide forecasts of the conditional covariances, which might be very useful for the implementation of monetary policy.

The lack of heteroskedasticity in inflation is not in contrast with the hypotheses of our model. Indeed, if $\mathbf{u}_t$ is a multivariate GARCH process, then the static factors $\mathbf{F}_t$, which are contemporaneous linear combinations of $\mathbf{u}_t$, are Weak GARCH processes and so are the $\mathbf{x}_t$ (see Nijman and Sentana, 1996; Drost and Nijman, 1993, for details). Therefore, the hypothesis of heteroskedastic dynamic factors is consistent with the observed weak heteroskedasticity of inflation. Once we have an estimate of $\mathbf{u}_t$, we test for GARCH effects by means of the ARCH test. Results for the first and the last 13-year windows of our sample are in Table 3. At least two of the four dynamic factors display ARCH effects, confirming our initial hypothesis. This result suggests to apply the multivariate GARCH on the dynamic factors and not directly on the observable series.

We compute the average correlation between the series of a given group of series and the four estimated dynamic factors. The largest factor is anticorrelated with industrial production growth rates (-0.38) and mildly anticorrelated with inflation (-0.12). Also the second largest factor is correlated with inflation series (0.38), while the fourth largest factor seems to be strongly correlated with asset returns (0.79). Nothing can be said about the third largest factor. According to the ARCH test, the first, second and fourth largest factors display conditional heteroskedasticity and are correlated with typically conditional heteroskedastic series as inflation and asset returns.

We compare all results with a univariate AR-GARCH(1,1) and a simple AR for inflation series, where the autoregressive order is always computed by means of the Akaike Information Criterion. Moreover, we test the predictive performance of the DF-GARCH against the static factor model by Stock and Watson (2002) and its homoskedastic counterpart by Giannone et al. (2004). The Root Mean Squared Error is defined as

$$\text{RMSE}_h = \sqrt{\frac{1}{72}\sum_{k=1}^{72}\left(\pi_{T+k+h-1} - \widehat{\pi}_{T+k+h-1|T}\right)^2} \qquad h = 1, \dots, 12,$$

where π_{T+h} is the true value of inflation while $\widehat{\pi}_{T+h|T}$ is the h-steps-ahead forecast of inflation, given all the available information at time T. When using factor models, we consider the estimate of the common component of inflation as an estimate of inflation itself, i.e. $\widehat{\pi}_t \equiv \widehat{\chi}_{\pi t}$. In the DF-GARCH, forecasts are obtained with the Kalman filter procedure described in Appendix B. Table 4 summarizes the results in terms of RMSE ratios: a ratio smaller than one indicates that the forecast based on the model of interest on average yields smaller forecast errors than the forecast based on the benchmark model, i.e. the univariate AR. The DF-GARCH and the model by Giannone et al. (2004) outperform all the other models at least at horizons larger than one, which are those of major interest for policy makers, given the lag between monetary policy interventions and their effects on inflation. Therefore a simple VAR specification of the dynamics of the static factors is enough to achieve better forecast performances with respect to the Stock and Watson (2002) model, where the behavior of the static factors is not modeled. As expected, the heteroskedastic models do not improve significantly over their homoskedastic counterparts.

In Table 5 we show the results of the Diebold-Mariano test of equal predictive accuracy (see Diebold and Mariano, 1995, for details). Although some of the models we are comparing may be considered as nested, this test is already useful to make a first distinction between them. When the null hypothesis of equal predictive accuracy is rejected with high significance levels, then, no matter if the models are nested, we already have an indication of which one is better. According to this test, the model by Stock and Watson (2002) delivers forecasts which are not significantly better than those of the univariate AR-GARCH. Also the DF-GARCH and the model by Giannone et al. (2004) turn out to be equally informative. Both improve over the univariate AR-GARCH especially for long horizons and for the core variables, which are actually the least volatile inflation measures. When we cannot reject the null hypothesis of equal predictive accuracy for two nested models, we consider also the test by Clark and West (2007), who suggest to add a correction term to the RMSE of the DF-GARCH to account for the possible errors made in estimating more parameters. Results comparing the univariate AR-GARCH and the DF-GARCH are reported in Table 6.

The R^2 of the Mincer-Zarnowitz regressions confirm the previous results. Values (not reported here) are however quite small. This fact can be explained by Figure 4, where we show the actual CPI inflation level and the forecasts made with the DF-GARCH at 1 and 12 months horizons. DF-GARCH predictions are slightly lagging and have lower variance than the actual series. The prediction is smoother because we are considering only the common component, as it is always the case when using a factor based index to forecast a variable. Although the variance of the univariate GARCH forecasts is higher, this does not improve the performance, on the contrary sometimes the estimated series misses completely the variations in the actual data.

In Figure 5 we show, for CPI inflation, the 90% confidence intervals forecasted by the DF-GARCH with $h = 1$ and $h = 12$. In Figure 6 we plot the 90% confidence intervals estimated in-sample. The performance of our model seems qualitatively good, and indeed the observations are contained in the 90% confidence intervals 90% of the times (as expected under the assumption of normality). The confidence intervals (not shown) for the core variables, which are less volatile, are quite flat and this is imputable again to the lack of heteroskedasticity.

To assess upside/downside risks to price stability, we compute the 5% and 95% Value-at-Risk (VaR) measures for inflation, which are simply the 5-th and 95-th conditional percentiles of the distribution of inflation. The simplest method to determine the adequacy of a Value-at-Risk measure is to test the hypothesis that the proportion of violations is equal to the expected one 9under the assumption of normality). Kupiec (1995) develops the likelihood ratio statistic

$$LR = 2\log\left[\left(1-\frac{\tau}{T}\right)^{T-\tau}\left(\frac{\tau}{T}\right)^{\tau}\right] - 2\log\left[(1-p)^{T-\tau}p^{\tau}\right] \sim \chi^2_1,$$

under the null hypothesis that the observed exception frequency, τ/T , equals the expected one, p, where τ is the number of violations over a period of length T. Results for the 5-th and 95-th percentiles are in Table 7. Both the GARCH and the DF-GARCH perform well, although the DF-GARCH tends to overpenalize the 95-th percentile for noncore variables, and to underpenalize the 5-th percentile for core variables.[8]

Finally, as we do not have an observable proxy of inflation's conditional variance, we build an indicator as the one proposed by Engle (1983). We fit an AR model, with 3, 6 or 12 lags, on the first five years of data (i.e. 60 observations) and we compute the standard error of the regression which we interpret as the standard deviation of the estimation. Then we drop the first observation and we add a new one at the end of the sample. We reestimate the AR, again with five years of observations. In this way, we obtain a series of standard deviation estimates under the assumption that the model and its variance are constant for the preceding five years.[9] RMSEs between this proxy and the estimates obtained with the GARCH or the DF-GARCH are in Table 8. For noncore variables the DF-GARCH performs better than the AR-GARCH when using a low number of lags in the AR specification of Engle's measure of inflation volatility. The viceversa holds for core variables.

Summing up, the intervals predicted by the DF-GARCH contain the expected proportion of observations and for noncore variables follow quite well the fluctuations of the series. Moreover, if we consider the proxy of inflation's conditional variance by Engle (1983), the DF-GARCH has a comparable performance with respect to a univariate GARCH. We thus have a model that forecasts inflation levels better than univariate AR and traditional factor models (Stock and Watson, 2002), and equally well when compared to factor models estimated in state-space form (Giannone et al., 2004). Moreover, our model forecasts also conditional variances at least as well as the univariate GARCH.

Being multivariate, the DF-GARCH provides also forecasts and estimates of conditional covariances. In Figure 7 we show estimates of conditional covariances for the last window of the rolling scheme. We consider economically interesting couples of series, in particular we show the conditional covariances between CPI inflation and total industrial production growth

[8]A more sophisticated version is the test proposed by Christoffersen (1998), which allows also to examine whether the violations are randomly distributed through time.

[9]As noted by Engle: "[...] the statistical properties of this procedure are not clear as the assumptions are continually changing, but the interpretation is quite straightforward".

rate (Δy_t) or unemployment rate (v_t). While the former is positive, the latter is negative, a result which is in line with the estimated unconditional covariances and with economic theory. Indeed, for our dataset we have $\text{cov}(\Delta y_t, \pi_t) = 0.14$ and $\text{cov}(v_t, \pi_t) = -0.06$. The performance of the DF-GARCH is remarkable in estimating the right sign of conditional covariances and in following the peaks and troughs in the comovements between the variables. This feature of our model is particularly useful for policy makers who want to act as risk managers with two targets as inflation and output growth. Moreover, even in the case in which output does not enter the objective function at all, information on conditional covariances allows to better predict the effects of monetary policy on the whole economy. As an example, our model could be useful in deepening the knowledge of the mechanisms that relate the nominal and the real sides of the economy.

6 Forecasting Volatility in Finance

The dataset we use for the second empirical application includes all the transaction prices of 89 stocks traded on the London Stock Exchange (LSE) and participating in the construction of the FTSE100 index for the whole period from 1st October 2001 to 31st July 2003 (457 working days).[10] Transaction prices have been cleaned from outliers by using the procedure described in Brownlees and Gallo (2006). Returns have been computed by using the last transaction recorded each day before the closing time of the LSE. Daily realized volatilities and covolatilities for out-of-sample evaluation are computed on a 5-minute frequency after removing the first 15 minutes of each day in order to avoid open effects (see Shephard and Barndorff-Nielsen, 2005). When computing realized covolatilities, we do not use leads and lags of intra-daily returns, as the 5-minute frequency should be low enough to avoid the nonsinchronicity bias (see Martens, 2004).

The criterion by Hallin and Liška (2007) suggests two common dynamic factors explaining 44% of the total variance. We need about 5 or 6 static factors to explain at least this same amount of variance. We use a rolling window of 350 observations and we repeat estimation and one-step-ahead volatility forecasts 100 times. At each iteration we reestimate the parameters, but we keep fixed the number of dynamic and static factors. As for the volatility proxy, we always use realized volatilities, but results are robust to other proxies as the naïve squared returns or the more sophisticated squared adjusted range.[11]

According to the ARCH test on the dynamic factors, one of the two dynamic factors is highly heteroskedastic. In particular, it is the most important (in terms of explained variance) of the two factors (see Table 9). Therefore, the largest factor determines the conditional heteroskedasticity of asset returns. This result, which holds for samples of different length, justifies our assumptions.

[10] As previous members of the Center for the Analysis of Financial Markets (CAFiM) at Sant'Anna School of Advanced Studies in Pisa, we had access to the Bloomberg financial dataset used in this section.

[11] We report only results averaged on the total number of series and only compared to realized volatility. Detailed results for all proxies and all series are available upon request.

We compare the forecasting accuracy of a univariate GARCH and of the DF-GARCH with 2 dynamic and 6 static factors estimated using both the full BEKK and the DCC specifications. We choose the univariate GARCH(1,1) model as a benchmark because it is by far the most used in practice and has usually a better performance than other models, despite being relatively simple. Moreover, to our knowledge O-GARCH and GO-GARCH applications available in the literature employ a number of series which is about ten times smaller than ours. Therefore, to have an idea of how a static model will perform with this dataset, we estimate also a static version of the DF-GARCH, i.e. a model with $r = q$.

We look at the confidence intervals at 90% significance level under the assumption of normality and show the results for four asset returns Figure 8: the univariate GARCH and the DF-GARCH have a qualitatively similar performance.[12] We also consider the 5% confidence level VaR prediction for each series and report the results averaged across the 89 series in Table 10: the DF-GARCH has a comparable and sometimes even better performance than the univariate GARCH. These are encouraging results concerning the in-sample properties of our estimation method.

To have a comparative performance evaluation for each series, we take the prediction of the different competing models and compute the one-step-ahead RMSE against the realized volatility RV_{it}. For each series i, we compute

$$RMSE_i = \sqrt{\frac{1}{100}\sum_{k=1}^{100}\left(RV_{iT+k} - \widehat{\Gamma}^x_{iiT+k|T+k-1}\right)^2} \qquad i = 1, \dots, n.$$

We then compute the ratio between the RMSE obtained with the DF-GARCH and the RMSE obtained using the univariate GARCH model. Average results across the n series are reported in Table 11 and show that the DF-GARCH in all specifications has on average a smaller RMSE than the GARCH.

Following Andersen et al. (2003), we evaluate the volatility forecasts of our model by running a Mincer-Zarnowitz regression. For each series i, we consider all the 100 one-step-ahead predictions we have computed and we regress the realized volatility on the estimated conditional variance. Table 11 reports also the average coefficient of multiple determination R^2 over all n regressions. For 71 series (i.e. 80% of the total number of series) the DF-GARCH with full BEKK specification obtains a larger R^2 coefficient than the traditional GARCH model. If we use the DCC specification the performance is slightly worse and the DF-GARCH has a larger R^2 than the univariate GARCH for 64 series (i.e. 72% of the total). The static application of DF-GARCH also performs better than the univariate GARCH with results similar to the dynamic case.

In Table 11 we report also the percentage of series for which the DF-GARCH outperforms the univariate GARCH in terms of RMSE. For the majority of the series in the sample (82%), the DF-GARCH using the full BEKK turns out to be a better predictor than the univariate

[12] Whenever we name just "DF-GARCH", we refer to the proper dynamic (i.e. $r > q$) estimation of the model independently of the chosen multivariate GARCH specification.

GARCH. When the DF-GARCH is estimated in its static representation, this percentage is slightly lower (80%).

Since the univariate GARCH could be considered as nested in the DF-GARCH, we evaluate the significance of the results described above by means of the tests of predictive accuracy proposed by Clark and West (2007). In Table 12 we show a summary of the results: at 90% significance level, the DF-GARCH with BEKK specification performs better than the traditional GARCH for 70 series (79% of total).

We build the same statistics for the off-diagonal elements of our prediction, i.e. for the predicted conditional covariances. Results are in Table 13 and report the average statistics across the $n(n-1)/2$ elements. A comparison with the traditional univariate GARCH is clearly not possible so the RMSE ratios are not defined. The coefficient of multiple determination R^2 is always higher than in the case of conditional variance prediction, being slightly better for the static case. The percentage of covariances for which we forecast the right sign is 72% for all the four DF-GARCH specifications we consider.

7 Conclusions and further research

In this paper we propose a dynamic factor model with conditionally heteroskedastic dynamic common factors (DF-GARCH), which allows us to forecast and estimate conditional covariances in large datasets. Traditionally, this issue is affected by a dimensionality problem due to the large number of parameters involved by the conditional covariance dynamics. The dynamic factor model can reduce the complexity of the problem and give room for a volatility forecast that takes into account all the cross and time relationships within the entire information set. We show the consistency of our estimation procedure and the goodness of our method in finite samples by carrying out Monte-Carlo experiments on different simulated panels. Finally, we present two empirical applications on large datasets.

The first empirical application deals with multivariate inflation forecasting. Results for the inflation level, its conditional variance and its conditional covariance with other macroeconomic variables are encouraging. The second empirical application that we present deals with the volatility of asset returns. The DF-GARCH performs always better than a static factor model and a univariate GARCH.

There are many possible ways to extend this work, for example by using the DF-GARCH on data which exhibit strong comovements and high heteroskedasticity, to fully exploit its potentialities. An ideal field of application would be disaggregated price indexes, which are more dynamically correlated than asset returns and more conditionally heteroskedastic than the aggregate inflation measures. Applying our method to these series may be a good way to compute an aggregate inflation index with its confidence bands, and may also be useful in shedding light on price dynamics. Indeed, one of the first applications of dynamic factor models in the economic literature is related to the aggregation of heterogeneous microeconomic series in Forni and Lippi (1997). The issue of aggregation of economic time series in a factor model context is also considered in Zaffaroni (2004) from a general perspective and in Altissimo et al. (2007), who considers precisely the aggregation of sectoral price indexes in order to study the dynamics of the aggregate inflation indicator.

Finally, the availability of conditional covariance forecasts is an additional information that can give insights on the structural process of inflation and its interpretation in terms of relations with other macroeconomic variables (e.g. measures of economic activity). In the field of monetary policy, this road naturally leads to models of monetary rules in which not only the target variables enter, but also their conditional variances (as measures of uncertainty) and the conditional covariances between them.

Concerning estimation, we could employ the Quasi Maximum Likelihood estimator proposed by Doz et al. (2006) when estimating the parameters of the state-space form of the DF-GARCH in order to increase the asymptotic efficiency of the estimators. Concerning the assumptions of the model, the possibility of time-varying parameters should also be considered. This is particularly important when one is interested in assessing inflation uncertainty: if parameters are time-varying but are assumed to be constant, uncertainty is underestimated, and agents' forecasting ability is overestimated. One possible way to consider this source of uncertainty is to follow Motta et al. (2009), who propose a static factor model with time-varying loadings and estimate nonparametrically a time dependent covariance matrix of the common component without estimating any GARCH model. However, since this model is completely nonparametric, forecasting is not straightforward and has still to be developed.

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A Proofs

Proposition 2

The expressions for $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{\Gamma}}_0^\epsilon$ come from the usual VAR estimation. From Proposition 1 we have that, as $n, T \to \infty$,

$$||\widehat{\mathbf{\Gamma}}_k^F - \mathbf{G}'\mathbf{\Gamma}_k^F\mathbf{G}|| \xrightarrow{P} 0, \tag{A-1}$$

where $\mathbf{\Gamma}_k^F = \text{E}[\mathbf{F}_t\mathbf{F}'_{t-k}]$. Moreover, $\mathbf{A} = \text{E}[\mathbf{F}_t\mathbf{F}'_{t-1}]\left(\text{E}[\mathbf{F}_t\mathbf{F}'_t]\right)^{-1} = \mathbf{\Gamma}_1^F\left(\mathbf{\Gamma}_0^F\right)^{-1}$. Therefore, since $\mathbf{G}^{-1} = \mathbf{G}'$,

$$||\widehat{\mathbf{A}} - \mathbf{G}'\mathbf{\Gamma}_1^F\mathbf{G}\left(\mathbf{G}'\mathbf{\Gamma}_0^F\mathbf{G}\right)^{-1}|| = ||\widehat{\mathbf{A}} - \mathbf{G}'\mathbf{\Gamma}_1^F\left(\mathbf{\Gamma}_0^F\right)^{-1}\mathbf{G}|| = ||\widehat{\mathbf{A}} - \mathbf{G}'\mathbf{A}\mathbf{G}|| \xrightarrow{P} 0.$$

Analogously, from (A-1) we can prove that, as $n, T \to \infty$,

$$||\widehat{\mathbf{\Gamma}}_0^\epsilon - \mathbf{G}'\left(\mathbf{\Gamma}_0^F - \mathbf{A}\mathbf{\Gamma}_0^F\mathbf{A}'\right)\mathbf{G}|| = ||\widehat{\mathbf{\Gamma}}_0^\epsilon - \mathbf{G}'\mathbf{\Gamma}_0^\epsilon\mathbf{G}|| \xrightarrow{P} 0.$$

Notice that, unlike in VAR, we also need $n \to \infty$, as we need consistent estimates of $\mathbf{F}_t$ in order to have (A-1).□

Proposition 3

(a) We have to find a q-dimensional orthogonal linear basis of the space to which $\widehat{\boldsymbol{\epsilon}}_t = \widehat{\mathbf{F}}_t - \widehat{\mathbf{A}}\widehat{\mathbf{F}}_{t-1}$ belong. One choice is to take the q largest principal components of $\widehat{\mathbf{\Gamma}}_0^\epsilon$. Define as $\mathbf{M}$ the $r \times q$ matrix of normalized eigenvectors corresponding to the q largest eigenvalues of $\mathbf{\Gamma}_0^\epsilon$. Then, from Proposition 2 $\mathbf{G}'\mathbf{M}$ is the corresponding eigenvectors matrix for $\widehat{\mathbf{\Gamma}}_0^\epsilon$. Moreover, from the theory of principal components, we know that the estimate of this latter eigenvectors matrix is consistent only up to a $q \times q$ orthogonal transformation $\mathbf{R}$. Therefore, as $n, T \to \infty$,

$$||\widehat{\mathbf{M}} - \mathbf{G}'\mathbf{M}\mathbf{R}|| \xrightarrow{P} 0.$$

We need $n \to \infty$ as we need Proposition 2 to hold. Finally, if we define $\mathbf{u}_t = \mathbf{H}\boldsymbol{\epsilon}_t$ as the q principal components we are looking for, then $\widehat{\mathbf{H}} = \sqrt{r}\widehat{\mathbf{M}}$ and

$$||\widehat{\mathbf{H}} - \mathbf{G}'\mathbf{H}\mathbf{R}|| \xrightarrow{P} 0. \tag{A-2}$$

Moreover, from (9) $\widehat{\mathbf{u}}_t = \widehat{\mathbf{H}}(\mathbf{I}_r - \widehat{\mathbf{A}}L)\widehat{\mathbf{F}}_t$. Therefore, from Propositions 1 and 2, and (A-2)

$$||\widehat{\mathbf{u}}_t - \mathbf{R}'\mathbf{H}'\mathbf{G}(\mathbf{I}_r - \mathbf{G}'\mathbf{A}\mathbf{G}L)\mathbf{G}'\mathbf{F}_t|| = ||\widehat{\mathbf{u}}_t - \mathbf{R}'\mathbf{u}_t|| \xrightarrow{P} 0.$$

This result is analogous to the result in Proposition 1, although here we are considering exact and not approximate principal components. In practice each column of $\mathbf{H}$ must also be multiplied by the square-root of the corresponding eigenvalue in order to have $\text{E}[\mathbf{u}_t\mathbf{u}'_t] = \mathbf{I}_q$. A similar result is in Proposition 2 in Forni et al. (2009). This completes the proof of part (a).

Part (b) is proved in Proposition 3 in Forni et al. (2009).□

Proposition 4

Part (a) and (b) would be respectively the two-step ML estimator proposed by Engle (2002) and the ML estimator proposed in Engle and Kroner (1995) if the dynamic factors were observed; part (c) would be the univariate GARCH ML estimator if the idiosyncratic component were observed. Here we plug in these usual ML estimators the previous estimates of dynamic factors and in the univariate GARCH the previous estimates of idiosyncratic components. Since approximate principal components estimates can be seen as ML estimates (see Bai, 2003) and since VAR estimates are ML estimates, we have to solve for each step of our estimation method a maximization. We therefore have four sets of M-estimators or, if we look at first order conditions, four GMM estimators. In general we have to solve the four maximizations:

1. $(\widehat{\mathbf{\Lambda}}, \widehat{\mathbf{F}}_t) = \arg\max_{\mathbf{\Lambda},\mathbf{F}_t} \boldsymbol{\ell}^{PC}(\mathbf{\Lambda}, \mathbf{F}_t|\mathcal{I}_{t-1}; \mathbf{x}_t)$,

2. $\widehat{\mathbf{A}} = \arg\max_{\mathbf{A}} \boldsymbol{\ell}^{VAR}(\mathbf{A}|\mathcal{I}_{t-1}; \widehat{\mathbf{F}}_t), \quad \widehat{\boldsymbol{\epsilon}}_t = \widehat{\mathbf{F}}_t - \widehat{\mathbf{A}}\widehat{\mathbf{F}}_{t-1}$,

3. $(\widehat{\mathbf{H}}, \widehat{\mathbf{u}}_t) = \arg\max_{\mathbf{H},\mathbf{u}_t} \boldsymbol{\ell}^{PC}(\mathbf{H}, \mathbf{u}_t|\mathcal{I}_{t-1}; \widehat{\boldsymbol{\epsilon}}_t)$,

4. $\widehat{\mathbf{Q}}_t = \arg\max_{\mathbf{Q}_t} \boldsymbol{\ell}^{MGARCH}(\mathbf{Q}_t|\mathcal{I}_{t-1}; \widehat{\mathbf{u}}_t)$,

where $\boldsymbol{\ell}^{PC}$, $\boldsymbol{\ell}^{VAR}$, and $\boldsymbol{\ell}^{MGARCH}$ are log-likelihoods for the principal components, VAR, and multivariate GARCH models respectively. Newey and McFadden (1994) provide a proof of consistency of two-step ML estimators (which can be generalized to the multi-step case) if some reasonable regularity assumptions on the objective functions and the parameter space hold. In a few words, the log-likelihoods have to be twice differentiable and must have a unique finite maximum, which is attained for values of the parameters interior to a compact set (see Section 7 in Newey and McFadden, 1994, for details). Notice that in our model we also need $n \to \infty$ as we need consistent estimates of the static factors and the idiosyncratic component.□

Proposition 5

The conditional covariance matrix of the common component is defined as

$$\mathbf{\Gamma}_t^{\chi} = \mathrm{E}\left[(\boldsymbol{\chi}_t - \mathrm{E}[\boldsymbol{\chi}_t|\mathcal{I}_{t-1}])\,(\boldsymbol{\chi}_t - \mathrm{E}[\boldsymbol{\chi}_t|\mathcal{I}_{t-1}])'\,|\mathcal{I}_{t-1}\right].$$

From (8)-(9) we know that $\mathrm{E}[\boldsymbol{\chi}_t|\mathcal{I}_{t-1}] = \mathbf{\Lambda}\mathbf{A}\mathbf{F}_{t-1}$, therefore from Assumption 6

$$\mathbf{\Gamma}_t^{\chi} = \mathbf{\Lambda}\mathbf{H}\mathrm{E}[\mathbf{u}_t\mathbf{u}_t'|\mathcal{I}_{t-1}]\mathbf{H}'\mathbf{\Lambda}' = \mathbf{\Lambda}\mathbf{H}\mathbf{Q}_t\mathbf{H}'\mathbf{\Lambda}'. \tag{A-3}$$

We estimate the conditional covariance matrix of the common component by replacing $\mathbf{\Lambda}$, $\mathbf{H}$, and $\mathbf{Q}_t$ in (A-3) with the corresponding estimates. Given the results in Propositions 1 to 4 and assuming $\mathbf{R} = \mathbf{I}_q$ we have, as $n, T \to \infty$,

$$||\widehat{\mathbf{\Lambda}}\widehat{\mathbf{H}}\widehat{\mathbf{Q}}_t\widehat{\mathbf{H}}'\widehat{\mathbf{\Lambda}}' - \mathbf{\Lambda}\mathbf{G}\mathbf{G}'\mathbf{H}\mathbf{Q}_t\mathbf{H}'\mathbf{G}\mathbf{G}'\mathbf{\Lambda}'|| \xrightarrow{P} 0.$$

Noticing that $\mathbf{G}\mathbf{G}' = \mathbf{I}_r$ completes the proof.□

B The Modified Kalman Filter

We explain here in detail the estimation of the state-space model

$$\begin{aligned} \mathbf{x}_t &= \hat{\mathbf{\Lambda}}\mathbf{F}_t + \boldsymbol{\xi}_t && \text{measurement equation ,} \\ \mathbf{F}_t &= \hat{\mathbf{A}}\mathbf{F}_{t-1} + \hat{\mathbf{H}}\mathbf{u}_t && \text{transition equation ,} \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\xi}_{t|t-1} &\sim \mathrm{N}(0, \hat{\mathbf{R}}_t) \quad \hat{\mathbf{R}}_t \text{ diagonal ,} \\ \mathbf{u}_{t|t-1} &\sim \mathrm{N}(0, \mathbf{Q}_t)\ , \\ \mathbf{Q}_t &= \hat{\mathbf{C}}_0{}'\hat{\mathbf{C}}_0 + \hat{\mathbf{C}}_1{}'\mathbf{u}_{t-1}\mathbf{u}_{t-1}'\hat{\mathbf{C}}_1 + \hat{\mathbf{C}}_2{}'\mathbf{Q}_{t-1}\hat{\mathbf{C}}_2\ . \end{aligned}$$

The multivariate GARCH representation considered here is a full BEKK, but the following procedure can be easily modified to allow for a DCC representation.

Initialization. Initial values are built as:

$$\begin{cases} \mathbf{F}_{1|1} & = \hat{\mathbf{F}}_1 \\ \mathbf{P}_{1|1} & \text{sufficiently large} \\ \mathbf{u}_{1|1} & = \hat{\mathbf{u}}_1 \\ \mathbf{Q}_{1|1} & = \hat{\mathbf{Q}}_1 \\ (\mathbf{u}_1\mathbf{u}_1')_{|1} & = \mathbf{u}_{1|1}\mathbf{u}_{1|1}' + \mathbf{Q}_{1|1} \ , \end{cases}$$

where the variables with the hat have been obtained in the estimation step presented in Section 3, $\hat{\mathbf{Q}}_1$ has been obtained by the multivariate GARCH model, and the state initial covariance matrix $\mathbf{P}_{1|1}$ must represent the high uncertainty about the initial value of the state vector.

Prediction. Hereafter, all the steps described must be repeated together for time $t = 2 \ldots T$. First we predict the unobserved state vector

$$\mathbf{F}_{t|t-1} = \hat{\mathbf{A}}\mathbf{F}_{t-1|t-1} \ ,$$

and its conditional covariance matrix

$$\mathbf{P}_{t|t-1} = \hat{\mathbf{A}}\mathbf{P}_{t-1|t-1}\hat{\mathbf{A}}' + \hat{\mathbf{H}}(\mathbf{u}_t\mathbf{u}_t')_{|t-1}\hat{\mathbf{H}}' \ ,$$

where

$$\begin{cases} (\mathbf{u}_t\mathbf{u}_t')_{|t-1} & = \mathbf{Q}_{t|t-1} \\ \mathbf{Q}_{t|t-1} & = \hat{\mathbf{C}}_0{}'\hat{\mathbf{C}}_0 + \hat{\mathbf{C}}_1{}'(\mathbf{u}_{t-1}\mathbf{u}_{t-1}')_{|t-1}\hat{\mathbf{C}}_1 + \hat{\mathbf{C}}_2{}'\mathbf{Q}_{t-1|t-1}\hat{\mathbf{C}}_2 \ . \end{cases} \tag{B-1}$$

The conditional covariance matrix for the state vector is obtained by using the GARCH estimated parameters $\hat{\mathbf{C}}_0$, $\hat{\mathbf{C}}_1$ and $\hat{\mathbf{C}}_2$; they are applied on the updated conditional covariance of the transition error $\left(\mathbf{u}_{t-1}\mathbf{u}_{t-1}'\right)$, which in turn has been obtained by the Kalman update, as we see in the next step.
The prediction error is given by

$$\boldsymbol{\eta}_{t|t-1} = \widetilde{\mathbf{x}}_t - \widetilde{\mathbf{x}}_{t|t-1} = \widetilde{\mathbf{x}}_t - \hat{\mathbf{\Lambda}}\mathbf{F}_{t|t-1} \ ,$$

whose conditional covariance is built by using the predicted conditional covariance of the static factors and the known conditional covariance of the measurement errors, as obtained previously by univariate modelling of the idiosyncratic parts:

$$\mathbf{Y}_{t|t-1} = \hat{\mathbf{\Lambda}}\mathbf{P}_{t|t-1}\hat{\mathbf{\Lambda}}' + \hat{\mathbf{R}}_t \ .$$

Update. We compute the Kalman gain

$$\mathbf{K}_t = \mathbf{P}_{t|t-1}\hat{\mathbf{\Lambda}}'\mathbf{Y}_{t|t-1}^{-1} \ ,$$

and we build more accurate inferences, exploiting information up to time t,

$$\mathbf{F}_{t|t} = \mathbf{F}_{t|t-1} + \mathbf{K}_t\boldsymbol{\eta}_{t|t-1} \ ,$$

$$\mathbf{P}_{t|t} = \mathbf{P}_{t|t-1} - \mathbf{K}_t\hat{\mathbf{\Lambda}}\mathbf{P}_{t|t-1} \ .$$

By inverting the transition equation, we get

$$\mathbf{u}_{t|t} = \mathbf{\Phi}^{-1/2}\mathbf{M}'\left(\mathbf{I}_r - \hat{\mathbf{A}}L\right)\mathbf{F}_{t|t} \ , \tag{B-2}$$

and then

$$(\mathbf{u}_t\mathbf{u}_t')_{|t} = \mathbf{u}_{t|t}\,\mathbf{u}_{t|t}' \ . \tag{B-3}$$

Equation (B-3), when put in the context of the following prediction step (B-1), is not precise. As noted by Harvey et al. (1992), a correction term should be added on the right hand side in order to take the into account the conditional variance of the dynamic factor. However, the same authors show that, when applied to the factor model by Diebold and Nerlove (1989), the effect of this correction may be empirically negligible. The differences between their estimation procedure and ours, including the update passage described in (B-2), lead us to avoid the estimation of the correction term.

Smoothing. Smoothing would be especially useful when extending our procedure to a higher number of lags in the GARCH structure of dynamic factors' conditional covariances. In any case, the smoothing procedure is recommended for getting a more precise estimate of the common and idiosyncratic components of the dataset. Following de Jong (1989) and Durbin and Koopman (2001), the following fixed interval smoother can be applied for $t = T, T-1, \ldots, 2$ in order to find more precise in-sample values of the static factors and of dynamic factors' conditional covariances. First we compute

$$\mathbf{r}_{t-1} = \mathbf{L}_t'\mathbf{r}_t + \hat{\mathbf{\Lambda}}'\mathbf{Y}_{t|t-1}^{-1}\boldsymbol{\eta}_{t|t-1} \ ,$$

$$\mathbf{F}_{t|T} = \mathbf{F}_{t|t-1} + \mathbf{P}_{t|t-1}\mathbf{r}_{t-1} \ ,$$

where $\mathbf{L}_t = \hat{\mathbf{\Lambda}}\left(\mathbf{I}_r - \mathbf{K}_t\hat{\mathbf{\Lambda}}\right)$, $\mathbf{r}_T = 0$. At each step, we also find the smoothed state variance matrix

$$\mathbf{P}_{t|T} = \mathbf{P}_{t|t-1} - \mathbf{P}_{t|t-1}\mathbf{\Theta}_{t-1}\mathbf{P}_{t|t-1} \ ,$$

where $\mathbf{\Theta}_t$ has been obtained by

$$\mathbf{\Theta}_{t-1} = \hat{\mathbf{\Lambda}}'\mathbf{Y}_t^{-1}\hat{\mathbf{\Lambda}} + \mathbf{L}_t'\mathbf{\Theta}_t\mathbf{L}_t \ ,$$

with initial value $\mathbf{\Theta}_T = 0$. At the end of each step, we get smoothed values for the dynamic factors and their conditional covariances $\mathbf{Q}_t$

$$\mathbf{u}_{t|T} = \mathbf{Q}_{t|t-1}\hat{\mathbf{H}}'\mathbf{r}_t \ ,$$

$$\mathbf{Q}_{t|T} = \mathbf{Q}_{t|t-1} - \mathbf{Q}_{t|t-1}\hat{\mathbf{H}}'\mathbf{\Theta}_t\hat{\mathbf{H}}\mathbf{Q}_{t|t-1} \ .$$

Table 1: Simulation results for $T = 250$.

n	**T**	**q**	**s**	**VR**	χ_t	**vol.**	**covol.**	**With Kalman filter** χ_t	**vol.**	**covol.**
75	250	2	2	0.3	0.9673 0.0145	0.5775 0.2169	0.5616 0.2252	0.9754 0.0113	0.6344 0.2040	0.5970 0.2194
75	250	2	2	0.5	0.9436 0.0187	0.5733 0.2049	0.5504 0.2084	0.9562 0.0158	0.6234 0.1949	0.5823 0.2032
75	250	2	4	0.3	0.9509 0.0130	0.5157 0.2292	0.4930 0.2312	0.9661 0.0136	0.5687 0.2276	0.5266 0.2330
75	250	2	4	0.5	0.9091 0.0301	0.5067 0.2078	0.4806 0.2100	0.9299 0.0326	0.5630 0.2011	0.5178 0.2081
75	250	3	2	0.3	0.9653 0.0040	0.4922 0.1763	0.4689 0.1800	0.9734 0.0038	0.5316 0.1720	0.4889 0.1815
75	250	3	2	0.5	0.9398 0.0094	0.4695 0.1707	0.4397 0.1729	0.9524 0.0088	0.5069 0.16846	0.4587 0.1756
75	250	3	4	0.3	0.9437 0.0080	0.4558 0.1681	0.4283 0.1677	0.9591 0.0102	0.4991 0.1694	0.4534 0.1713
75	250	3	4	0.5	0.8963 0.0201	0.3865 0.1668	0.3519 0.1608	0.9134 0.0246	0.4340 0.1624	0.3788 0.1602
150	250	2	2	0.3	0.9792 0.0043	0.5535 0.2443	0.5300 0.2424	0.9838 0.0035	0.5561 0.2454	0.5327 0.2435
150	250	2	2	0.5	0.9653 0.0067	0.5182 0.2342	0.4902 0.2331	0.9728 0.0056	0.5227 0.2353	0.4944 0.2340
150	250	2	4	0.3	0.9666 0.0076	0.5350 0.2384	0.5110 0.2358	0.9768 0.0057	0.5401 0.2400	0.5159 0.2373
150	250	2	4	0.5	0.9467 0.0106	0.4973 0.2308	0.4686 0.2272	0.9623 0.0081	0.5045 0.2338	0.4755 0.2297
150	250	3	2	0.3	0.9774 0.0028	0.5208 0.1811	0.4942 0.1824	0.9816 0.0027	0.5640 0.1736	0.5166 0.1803
150	250	3	2	0.5	0.9592 0.0067	0.5228 0.1556	0.4900 0.1584	0.9663 0.0057	0.5655 0.1542	0.5116 0.1601
150	250	3	4	0.3	0.9622 0.0063	0.4913 0.1678	0.4618 0.1654	0.9715 0.0069	0.5347 0.1649	0.4847 0.1651
150	250	3	4	0.5	0.9265 0.0143	0.4418 0.1571	0.4056 0.1527	0.9405 0.0146	0.4827 0.1536	0.4291 0.1528

Mean and standard errors of the R^2 coefficient of the Mincer-Zarnowitz regressions over 250 Monte Carlo simulations.

Table 2: Simulation results for $T = 500$ and $T = 750$.

n	T	q	s	VR	χ_t	vol.	covol.	With Kalman filter χ_t	With Kalman filter vol.	With Kalman filter covol.
75	500	2	2	0.3	0.9717 0.0055	0.6696 0.2152	0.6484 0.2189	0.9814 0.0039	0.7117 0.2056	0.6753 0.2151
75	500	2	2	0.5	0.9535 0.0074	0.6156 0.2368	0.5939 0.2383	0.9690 0.0058	0.6222 0.2390	0.6006 0.2402
75	500	2	4	0.3	0.9551 0.0075	0.6413 0.2131	0.6176 0.2132	0.9748 0.0049	0.6898 0.2022	0.6506 0.2101
75	500	2	4	0.5	0.9255 0.0137	0.6058 0.2094	0.5765 0.2123	0.9550 0.0120	0.6194 0.2128	0.5900 0.2154
75	500	3	2	0.3	0.9594 0.0058	0.5951 0.1651	0.5558 0.1680	0.9732 0.0043	0.6265 0.1568	0.5731 0.1653
75	500	3	2	0.5	0.9341 0.0093	0.5655 0.1658	0.5233 0.1656	0.9554 0.0073	0.6016 0.1565	0.5445 0.1627
75	500	3	4	0.3	0.9355 0.0094	0.5522 0.1760	0.5135 0.1768	0.9608 0.0076	0.5932 0.1705	0.5382 0.1773
75	500	3	4	0.5	0.8968 0.0129	0.5096 0.1646	0.4631 0.1630	0.9295 0.0137	0.5515 0.1612	0.4892 0.1639
75	750	2	2	0.3	0.9805 0.0036	0.8232 0.1453	0.7963 0.1615	0.9875 0.0028	0.8500 0.1288	0.8127 0.1525
75	750	2	2	0.5	0.9668 0.0062	0.8109 0.1539	0.7831 0.1627	0.9782 0.0051	0.8404 0.1368	0.8022 0.1534
75	750	2	4	0.3	0.9689 0.0052	0.7829 0.1729	0.7566 0.1818	0.9830 0.0037	0.8172 0.1511	0.7786 0.1706
75	750	2	4	0.5	0.9459 0.0142	0.7756 0.1500	0.7455 0.1595	0.9671 0.0149	0.8095 0.1362	0.7689 0.1525
75	750	3	2	0.3	0.9623 0.0045	0.6456 0.1577	0.6011 0.1699	0.9740 0.0037	0.6765 0.1527	0.6149 0.1705
75	750	3	2	0.5	0.9312 0.0166	0.6326 0.1503	0.5811 0.1550	0.9499 0.0196	0.6650 0.1425	0.5968 0.1523
75	750	3	4	0.3	0.9372 0.0077	0.6190 0.1565	0.5747 0.1611	0.9586 0.0107	0.6528 0.1520	0.5917 0.1602
75	750	3	4	0.5	0.8855 0.0228	0.5423 0.1626	0.4907 0.1630	0.9065 0.0381	0.5812 0.1580	0.5115 0.1627

Mean and standard errors of the R^2 coefficient of the Mincer-Zarnowitz regressions over 250 Monte Carlo simulations.

Table 3: ARCH-test on $\widehat{\mathbf{u}}_t$ for heteroskedasticity. Macro dataset.

ARCH order	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
In-sample observations from 1986:M12 to 1999:M11										
u_{1t}	2.84*	3.21	3.91	4.51	4.69	5.25	6.41	7.93	8.35	12.71
u_{2t}	0.04	0.29	4.14	4.08	4.21	4.34	15.29†	15.74†	15.58*	15.77
u_{3t}	10.38†	13.81†	13.68†	13.96†	14.35†	14.32†	14.31†	16.52†	16.56*	16.38*
u_{4t}	8.15†	11.61†	12.41†	12.40†	12.16†	15.67†	16.83†	18.33†	10.13	9.95
In-sample observations from 1992:M12 to 2005:M11										
u_{1t}	11.54†	11.79†	13.31†	15.32†	15.45†	15.54†	15.36†	16.20†	17.42†	17.36*
u_{2t}	1.32	10.34†	10.48†	12.68†	12.96†	12.83†	12.93*	13.57*	14.37	16.51*
u_{3t}	2.23	2.52	2.55	2.85	2.28	2.42	2.30	2.28	2.12	2.93
u_{4t}	0.01	7.44†	9.17†	9.22*	12.20†	13.21†	13.20*	13.21	13.03	12.94

Values of the ARCH test; † significant at 95%, * significant at 90%.

Table 4: RMSE of inflation level forecasts.

	AR-GARCH	**SW**	**GRS**	**DF-GARCH**
PCE core				
h=1	1.0133	1.0022	0.9612	0.9611
h=3	1.0193	1.0212	0.9177	0.9178
h=6	1.0033	1.0179	0.9025	0.9025
h=9	1.0005	0.9903	0.8877	0.8877
h=12	0.9959	0.9595	0.8640	0.8640
PCE				
h=1	0.9955	0.9278	0.9390	0.9389
h=3	0.9985	0.9712	0.9720	0.9720
h=6	0.9872	0.9970	0.9358	0.9358
h=9	0.9980	0.9854	0.9522	0.9522
h=12	0.9955	0.9851	0.9696	0.9696
CPI core				
h=1	0.9964	0.9732	0.9750	0.9749
h=3	0.9994	1.0109	0.9188	0.9187
h=6	0.9978	0.9633	0.8931	0.8930
h=9	1.0038	1.0497	0.8939	0.8939
h=12	1.0018	1.0665	0.8680	0.8680
CPI				
h=1	1.0089	0.8579	0.8969	0.8970
h=3	1.0044	0.9210	0.8992	0.8991
h=6	0.9917	0.9370	0.9036	0.9036
h=9	0.9943	0.9782	0.9179	0.9179
h=12	0.9980	0.9749	0.9368	0.9368

RMSEs relative to the univariate AR. A value less than one means better forecast. For a given h we compute 72 forecasts covering the period 1987:M1-2005:M12. SW = Stock and Watson (2002) model; GRS = Giannone et al. (2004) model.

Table 5: Diebold and Mariano test.

	PCE core	**PCE**	**CPI**	**CPI core**
a =**AR-GARCH**				
b =**SW**				
h=1	0.2917	1.7384*	4.0018†	0.4482
h=3	-0.0519	0.7713	2.6380†	-0.2178
h=6	-0.1955	-0.2464	1.7225*	0.7469
h=9	0.1366	0.3508	0.5284	-0.5731
h=12	0.4672	0.2985	0.6820	-0.7793
a =**AR-GARCH**				
b =**DF-GARCH**				
h=1	1.2862	1.6543*	3.1010†	0.3900
h=3	2.3088†	0.8019	3.6413†	1.6258
h=6	1.7602*	1.6900*	2.6971†	2.3772†
h=9	2.2557†	1.7509*	2.3859†	1.9752†
h=12	2.3079†	1.2065	2.4801†	2.5553†
a =**SW**				
b =**DF-GARCH**				
h=1	1.5634	-0.6049	-2.2434†	-0.0813
h=3	2.0749†	-0.0244	0.8720	2.3486†
h=6	1.7472*	1.7775*	1.2507	1.9028*
h=9	1.6994*	1.1171	1.7711*	2.7254†
h=12	1.8361*	0.6211	1.6279	3.4923†

Values of the statistics $d = (Forecast^a - Observed)^2 - (Forecast^b - Observed)^2$; † significant at 95%, * significant at 90%. Model b performs better than model a when values are positive and significant.

Table 6: Clark and West test.

	PCE core	**PCE**	**CPI**	**CPI core**
h=1	3.2869†	3.2891†	4.3859†	2.0263†
h=3	3.7389†	2.1075†	4.7057†	3.0699†
h=6	2.5872†	2.6981†	3.4964†	4.0254†
h=9	3.6867†	2.8715†	3.2062†	3.3972†
h=12	3.4734†	1.9306*	3.2125†	4.0327†

Values of the statistics $f = (Forecast^{GARCH} - Observed)^2 - [(Forecast^{DF-GARCH} - Observed)^2 - (Forecast^{GARCH} - Forecast^{DF-GARCH})^2]$; † significant at 95%, * significant at 90%. DF-GARCH performs better than AR-GARCH when values are positive and significant.

Table 7: Kupiec test for CPI and CPI core inflation.

	Lower tail		**Upper tail**	
	DF-GARCH	**GARCH**	**DF-GARCH**	**GARCH**
CPI core				
h=1	2.81†	0.13†	0.13†	0.03†
h=3	0.84†	0.09†	0.09†	0.56†
h=6	2.44†	0.05†	0.11†	0.05†
h=9	6.67	0.58†	0.58†	0.58†
h=12	2.01†	0.00†	2.01†	0.47†
CPI				
h=1	0.03†	0.47†	0.13†	0.03†
h=3	0.56†	0.56†	4.40	0.56†
h=6	0.11†	0.11†	4.83	0.11†
h=9	1.98†	1.98†	1.98†	0.02†
h=12	1.04†	0.00†	5.79	1.04†

Values of the LR statistics; † significant at 95%, i.e. we accpet the null hypothesis of a correct model specification.

Table 8: RMSE of in-sample inflation volatility estimates.

	PCE core	**PCE**	**CPI core**	**CPI**
AR lags				
3	2.00	0.83	2.56	0.72
6	2.00	0.97	0.46	0.86
12	0.69	1.71	0.22	1.64

RMSEs are relative to univariate AR-GARCH, when considering as a benchmark the proxy suggested in Engle (1983) built using 3, 6 or 12 lags. Values smaller than one indicate a better performance of DF-GARCH.

Table 9: ARCH-test on $\widehat{\mathbf{u}}_t$ for heteroskedasticity. Financial dataset.

ARCH order	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Case q = 2										
u_{1t}	0.71	47.67*	72.44*	72.86*	81.43*	87.68*	88.04*	92.71*	93.18*	93.06*
u_{2t}	2.54	2.53	3.44	8.13	9.03	11.26	11.59	12.78	12.41	12.62
Case q = 1										
u_{1t}	0.24	47.26*	75.35*	75.74*	85.02*	88.60*	88.77*	92.75*	96.22*	95.98*

* significant at 99%. Observations from $t = 1$ to $t = 350$, i.e. first in-sample window.

Table 10: Confidence interval in-sample estimation.

Model	num.obs./T
GARCH	0.0458
DF-GARCH (BEKK)	0.0507
DF-GARCH (DCC))	0.0499

Average across the n series of the fraction of observations below the 5-th percentile of a Normal distribution.

Table 11: Volatility forecasts.

Model	RMSE	$\mathbf{R^2}$	P
GARCH	1.0000	0.0780	n.a.
DF-GARCH (BEKK)(static)	0.9728	0.1084	80.899
DF-GARCH (BEKK)	0.9713	0.1100	82.022
DF-GARCH (DCC)(static)	0.9757	0.0981	77.528
DF-GARCH (DCC)	0.9738	0.1000	79.755

Average results across the n series. RMSEs relative to univariate GARCH; R^2 of Mincer-Zarnowitz regressions; P is the percentage of series for which a given model has a RMSE lower than the GARCH.

Table 12: Clark and West test.

Signif. level	DF-GARCH (BEKK)	DF-GARCH (BEKK)(static)	DF-GARCH (DCC)	DF-GARCH (DCC)(static)
90%	79%	79%	76%	75%
95%	65%	67%	62%	63%

Percentage of series for which the DF-GARCH outperforms the univariate GARCH.

Table 13: Covolatility forecasts.

Model	RMSE	$\mathbf{R^2}$	P
DF-GARCH (BEKK)(static)	1.0000	0.1426	72
DF-GARCH (BEKK)	0.9624	0.1417	72
DF-GARCH (DCC)(static)	0.9044	0.1204	72
DF-GARCH (DCC)	0.9036	0.1160	72

Average results across the $n(n-1)/2$ couples. RMSEs relative to the static version of DF-GARCH (BEKK); R^2 of Mincer-Zarnowitz regressions; P is the percentage of couples for which we forecast the right sign.

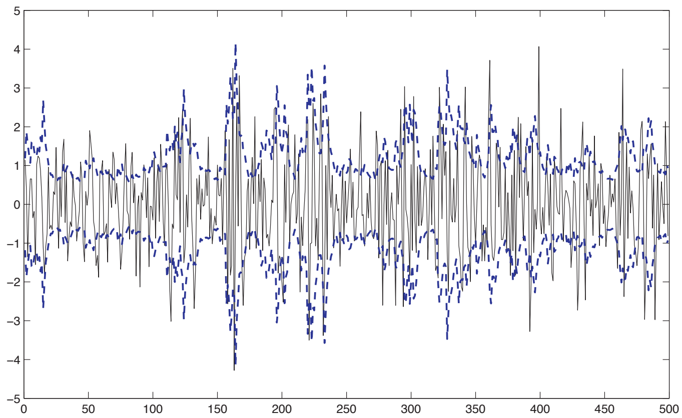
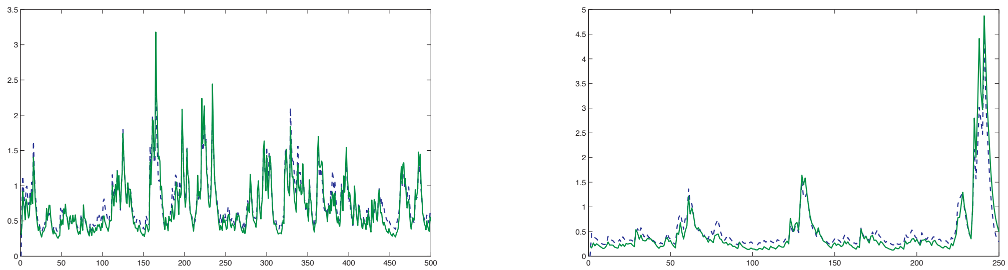
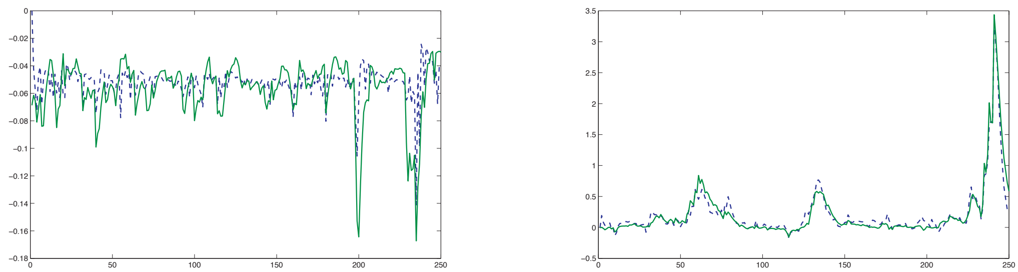


Figure 1: Estimated confidence interval for $n = 150$, $T = 500$, $q = 2$, $s = 4$, and VR$= 0.3$. Simulated χ_t: solid line. Estimated 5^{th} and 95^{th} conditional percentiles: dashed line.



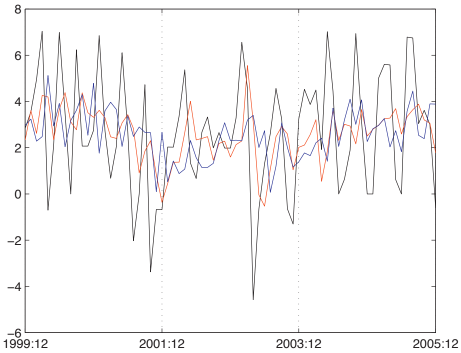
2.1: $n = 150$, $T = 500$, $q = 2$, $s = 4$, VR$= 0.3$. 2.2: $n = 150$, $T = 250$, $q = 3$, $s = 2$, VR$= 0.3$.

Figure 2: Conditional variances. Simulated: solid line. Estimated: dashed line.

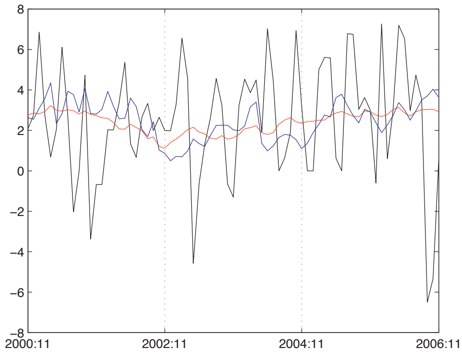


3.1: $n = 150$, $T = 500$, $q = 2$, $s = 4$, VR$= 0.3$. 3.2: $n = 150$, $T = 250$, $q = 3$, $s = 2$, VR$= 0.3$.

Figure 3: Conditional covariances. Simulated: solid line. Estimated: dashed line.

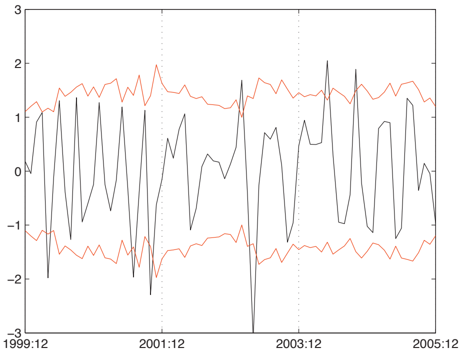


4.1: $h = 1$.

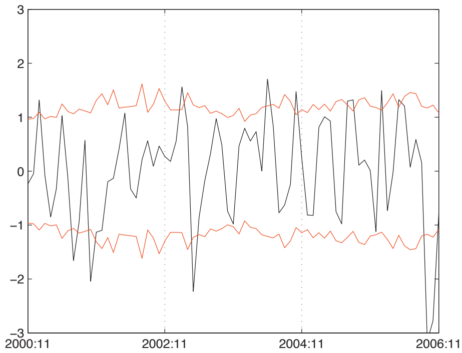


4.2: $h = 12$.

Figure 4: CPI inflation forecasts: levels. Observed series: black. DF-GARCH forecast: red. AR(p)-GARCH(1,1) forecast: blue.



5.1: $h = 1$.



5.2: $h = 12$.

Figure 5: CPI inflation forecasts: confidence intervals. Observed series (standardized): black. DF-GARCH 90% confidence interval forecasts: red.

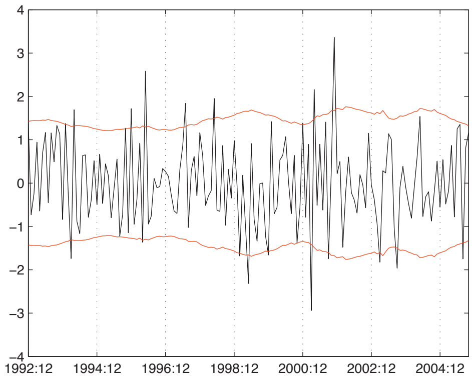
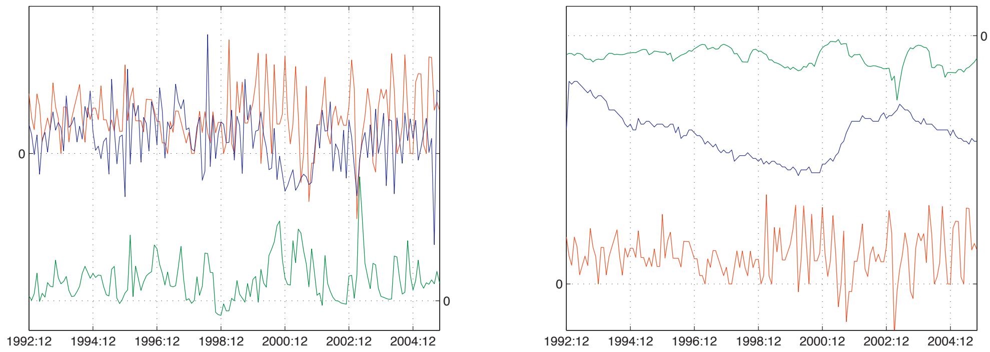
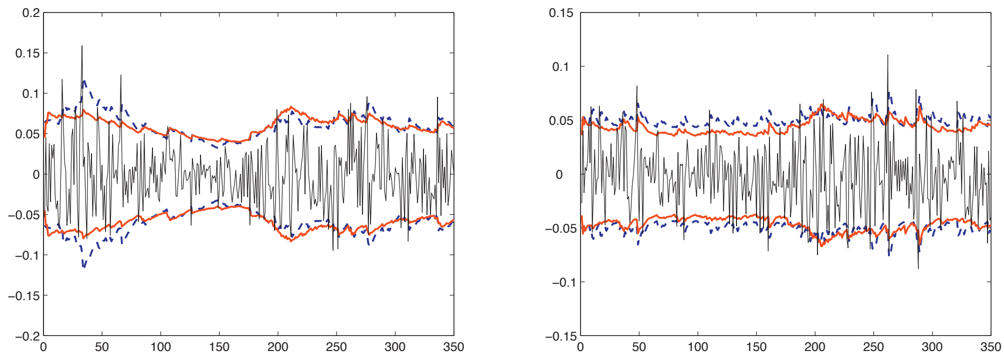


Figure 6: CPI inflation in-sample estimate: confidence intervals. Observed series (standardized): black. DF-GARCH 90% confidence interval estimate: red.

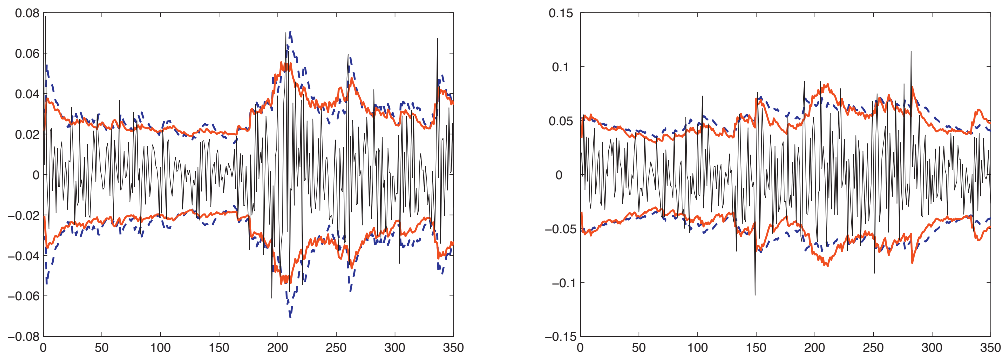


7.1: $\mathrm{cov}(\Delta y_t, \pi_t | \mathcal{I}_{t-1})$. 7.2: $\mathrm{cov}(u_t, \pi_t | \mathcal{I}_{t-1})$.

Figure 7: In-sample estimated conditional covariances. CPI: red. Industrial production growth rate (first panel) or unemployment (second panel): blue. Estimated conditional covariance: green (scale on the right hand side).



8.1: British Airways. 8.2: EMI.



8.3: Northern Rock. 8.4: Vodafone.

Figure 8: Confidence intervals for asset return series. DF-GARCH (BEKK) confidence interval: solid line. Univariate GARCH confidence interval: dashed line.

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