Maastricht University

Hans Peters, Souvik Roy,
Ton Storcken

On the manipulability of approval voting and related scoring rules

RM/09/060

METEOR

Maastricht University School of Business and Economics
Maastricht Research School of Economics
of Technology and Organization

P.O. Box 616
NL - 6200 MD Maastricht
The Netherlands

On the manipulability of approval voting and related scoring rules[*]

Hans Peters[†]   Souvik Roy[†]   Ton Storcken[†]

November 2009

Abstract

We characterize all preference profiles at which the approval (voting) rule is manipulable, under three extensions of preferences to sets of alternatives: by comparison of worst alternatives, best alternatives, or by comparison based on stochastic dominance. We perform a similar exercise for k-approval rules, where voters approve of a fixed number k of alternatives. These results can be used to compare (k-)approval rules with respect to their manipulability. Analytical results are obtained for the case of two voters, specifically, the values of k for which the k-approval rule is minimally manipulable – has the smallest number of manipulable preference profiles – under the various preference extensions are determined. For the number of voters going to infinity, an asymptotic result is that the k-approval rule with k around half the number of alternatives is minimally manipulable among all scoring rules. Further results are obtained by simulation and indicate that k-approval rules may improve on the approval rule as far as manipulability is concerned.

JEL-Classification: D71, D72

Keywords: Approval voting rule, k-approval rules, minimal manipulability

1 Introduction

Approval voting is a well accepted voting procedure.[1] In approval voting each voter can approve of as many alternatives as he wants. It is well known (Brams and Fishburn, 1983, and the references therein) that this procedure is strategy-proof (non-manipulable) if preferences are dichotomous, that is, each voter distinguishes only between a set of good and a set of bad alternatives. With more refined preferences, however, strategy-proofness no longer holds.

In this paper we study the manipulability of the approval (voting) rule and of a related procedure called k-approval (voting) rule. In a k-approval rule each voter approves of exactly k alternatives. This procedure is less flexible than the approval rule – voters can provide less information about their preferences – but tends to be also less manipulable, as we will argue. Therefore, k-approval rules may offer a good compromise between the approval rule and scoring rules such as Borda count.

In Section 2 we introduce the approval rule and next we study its manipulability. Since the approval rule (and also each k-approval rule) is a social choice correspondence and can be multi-valued, we need to make assumptions about extending the preferences (weak orderings) of voters over alternatives to sets of alternatives. We do this in three ways: by comparing the worst alternatives of a set, or by comparing the best alternatives of a set, or by comparing sets on the basis of stochastic dominance using equal chances. In Section 3 we characterize the non-manipulable preference profiles under approval voting for worst, best, and stochastic dominance comparison. The special cases of strict preferences follow as corollaries. Strategy-proofness under dichotomous preferences follows as a special case as well.

In Section 4 we characterize the non-manipulable profiles under k-approval rules, again for worst, best, and stochastic dominance comparison. We also include a brief consideration of a lexicographic refinement of worst and best comparison. For technical reasons attention in Section 4 is restricted to strict preferences.

The main purpose of all these exercises is to compare the approval rule and k-approval rules for different values of k with respect to manipulability and under different assumptions about the voters' preferences on sets of alternatives. This comparison is based on a simple measure, namely the number of manipulable preference profiles. The implicit assumption is therefore that all profiles are equally likely. This is called 'impartial culture' in the literature. Unfortunately, a complete analytical comparison is out of the question due to the combinatorial complexity of the problem. For this reason, our comparative results are mainly based on simulations and, thus, they are conjectures and suggestions rather than theorems. A selection of the results of these simulations is presented in Section 5. They give rise to some prudent conclusions concerning the manipulability of the approval and k-approval rules under different assumptions on preference extensions. In particular, they give support to the conjecture that k-approval

---

[1]It is used, for instance, to select candidates for councils of scientific communities such as the Society for Social Choice and Welfare and the Game Theory Society.

rules for specific values of k may be less susceptible to manipulation than the approval rule.

Nevertheless, we also present some analytical comparison results. In Section 4.4 we consider the two-voter case and compute the optimal k for different preference extensions, that is, the value of k for which the k-approval rule is minimally manipulable. For $k = 1$, the k-approval rule is just plurality voting. In the two-voter case, this is non-manipulable (strategy-proof) under any reasonable preference extension, including those considered in this paper. Plurality voting, however, has a serious drawback. If (the) two voters agree on a good second-ranked alternative but disagree on the first, then under plurality voting this compromise is not chosen; it would be chosen, however, under any other k-approval rule. Therefore, for each of the three mentioned preference comparisons and for $k \neq 1$ we have established the overall optimal value of k, and the optimal value under the restriction $k \leq m/2$, where m is the total the number of alternatives. The latter restriction is justified by the desirable property of 'citizen sovereignty': for each alternative there is a preference profile resulting in that alternative as the unique outcome. For $2 \leq k \leq m/2$ we find $k = 2$ as the optimal value in case of best or stochastic dominance comparison, and $k \approx \sqrt{m}$ in case of worst comparison.

On the other extreme, in Section 4.6 we let the number of voters go to infinity and show that even among *all scoring rules* the k-approval rule with $k \in \{(m-1)/2, (m+1)/2\}$ if m is odd, and with $k = m/2$ if m is even, is minimally manipulable. Of course, this result should be interpreted with care, since the probability of manipulability by a single voter is very small anyway if the number of voters is large. The basic intuition for this result is that the (statistical) variance in scores is maximal for the mentioned value(s) of k, so that any single voter's probability of being able to change the outcome is minimal.

Related literature In most voting situations agents have the possibility to manipulate the outcome of the vote by not voting according to their true preferences. The classical theorem of Gibbard (1973) and Satterthwaite (1975) formalizes this fact for social choice functions, which assign a unique alternative to every preference profile, but it also holds for social choice correspondences under various assumptions on preference extensions to sets (e.g., Barberà, Dutta, and Sen, 2001). The present paper belongs to the strand of literature, initiated by Kelly (1988, 1989), which accepts this phenomenon as a matter of fact and looks for social choice rules which are second best in this respect, i.e., least manipulable. Other references include Fristrup and Keiding (1998) and Aleskerov and Kurbanov (1999). Maus, Peters, and Storcken (2007) contains a brief overview of this literature.

Of course, counting the non-manipulable profiles is just one way of measuring the degree of (non-)manipulability of voting rules. Many other approaches are possible (e.g., Saari, 1990, or recently Campbell and Kelly, 2008). As already mentioned, our measure of non-manipulability reflects 'impartial culture': each preference profile is implicitly regarded as equally likely. The characterizations of the sets of non-manipulable profiles derived in this paper, however, are also

3

needed when considering 'partial culture'.

Notation We denote the cardinality of a set D by $|D|$.

2 Approval voting

The set of voters is $N = \{1, ..., n\}$ with $n \geq 2$ and the (finite) set of alternatives is A with $|A| = m \geq 3$. A *preference* is a weak ordering on A, i.e., a complete, reflexive, and transitive binary relation on A. By W we denote the set of all preferences. A *preference profile* w is a function from N to W, i.e., an element of W^N. For a preference profile w, $w(i)$ is the preference of voter $i \in N$. For a non-empty subset B of A, $w(i)_{|B}$ denotes the restriction of $w(i)$ to the set B, i.e., $w(i)_{|B} = \{(x, y) \in B \times B \mid (x, y) \in w(i)\}$. Obviously, $w(i)_{|A} = w(i)$.

We next introduce some further notation. Let w be a preference profile and $i \in N$. Let $1 \leq \ell \leq m$ and suppose there exists a set of alternatives B with $|B| = \ell$, $(x, y) \in w(i)$ and $(y, x) \notin w(i)$ for all $x \in B$ and $y \in A \setminus B$. Then we denote this set by $\beta_\ell(w(i))$. Observe that $\beta_\ell(w(i))$ exists if and only if there are ℓ alternatives strictly preferred to the remaining $m - \ell$ alternatives according to $w(i)$; that is, $\beta_\ell(w(i))$ contains only full indifference classes of $w(i)$.

Also, for a subset B of A, by $\beta(w(i)_{|B})$ we denote the set of best elements of B according to $w(i)$, that is, $\beta(w(i)_{|B}) = \{x \in B \mid (x, y) \in w(i) \text{ for all } y \in B\}$. Similarly, $\omega(w(i)_{|B})$ denotes the set of worst elements of B according to $w(i)$, that is, $\omega(w(i)_{|B}) = \{x \in B \mid (y, x) \in w(i) \text{ for all } y \in B\}$. The *lower contour set* of $a \in A$ at $w(i)$ is the set $L(a, w(i)) = \{x \in A \mid (a, x) \in w(i)\}$. Observe that $a \in L(a, w(i))$ by reflexivity of $w(i)$.

In approval voting, each voter $i \in N$ approves of $k(i)$ alternatives, where $1 \leq k(i) \leq m$ is the choice of the voter. The outcome of the vote is the set of those alternatives that receive the largest number of votes. (Observe that excluding $k(i) = 0$ is without loss of generality since the option $k(i) = m$ is available.) To formalize this, a *report* of voter i is a pair $r(i) = (w(i), k(i)) \in W \times \{1, \ldots, m\}$ such that $\beta_{k(i)}(w(i))$ exists. This implies that if a voter approves of an alternative x he also has to approve of all alternatives which are indifferent or strictly preferred to x according to $w(i)$. By R we denote the set of all reports, and by R^N the set of all (report) *profiles*. We denote by

$$\text{score}(x, r) = |\{i \in N \mid x \in \beta_{k(i)}(w(i))\}|$$

the number of voters who approve of alternative $x \in A$ at profile $r = (w, k) = ((w(i), k(i)))_{i \in N} \in R^N$. The *approval rule* φ, defined by

$$\varphi(r) = \{x \in A \mid \text{score}(x, r) \geq \text{score}(y, r) \text{ for all } y \in A\}, \quad r \in R^N,$$

assigns to each profile r the subset of alternatives with maximal score.

We need a few more notations. For $r = (w, k) \in R^N$ and $i \in N$, $\varphi(r_{-i})$ denotes the set of alternatives assigned by the approval rule to the restricted profile $r_{-i} = (r_1, ..., r_{i-1}, r_{i+1}, ..., r_n) \in R^{N \setminus \{i\}}$, that is,

$$\varphi(r_{-i}) = \{x \in A \mid \text{score}(x, r_{-i}) \geq \text{score}(y, r_{-i}) \text{ for all } y \in A\},$$

where $\text{score}(x, r_{-i}) = |\{j \in N \setminus \{i\} \mid x \in \beta_{k(j)}(w(j))\}|$. Finally, for (any) $a \in \varphi(r_{-i})$,

$$\varphi^-(r_{-i}) = \{x \in A \mid \text{score}(x, r_{-i}) = \text{score}(a, r_{-i}) - 1\}$$

is the (possibly empty) set of those alternatives that have score one less than the elements of $\varphi(r_{-i})$. We call the alternatives in $\varphi(r_{-i})$ *quasi-winners* and those in $\varphi^-(r_{-i})$ *almost quasi-winners*.[2] These notations are convenient in view of the following straightforward observation, which will be used throughout the next section:

$$\varphi(r) = \begin{cases} \varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) & \text{if } \varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) \neq \emptyset, \\ \varphi(r_{-i}) \cup [\varphi^-(r_{-i}) \cap \beta_{k(i)}(w(i))] & \text{if } \varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) = \emptyset. \end{cases} \quad (1)$$

In order to define (non-)manipulability of the approval rule at particular profiles we need to be able to extend individual preferences to preferences over non-empty subsets of alternatives. For a voter i in N and a preference $w(i) \in W$, we say that a binary relation $\succeq_{w(i)}$ on $2^A \setminus \{\emptyset\}$ *extends* $w(i)$ if $\{x\} \succeq_{w(i)} \{y\} \Leftrightarrow (x, y) \in w(i)$ holds for all $x, y \in A$. We write $B \succeq_{w(i)} C$ instead of $(B, C) \in \succeq_{w(i)}$. Also, $\succ_{w(i)}$ and $\sim_{w(i)}$ denote the asymmetric and symmetric parts of $\succeq_{w(i)}$, respectively.

In this paper we will consider (three or even more) different ways to extend $w(i)$ over alternatives to a binary relation over non-empty sets of alternatives. Suppose that $\succeq_{w(i)}$ extends $w(i)$ for all $i \in N$. For $i \in N$ and $r, s \in R^N$, we say that r and s are *i-deviations* if $r_{-i} = s_{-i}$. In that case, clearly, $\varphi(r_{-i}) = \varphi(s_{-i})$ and $\varphi^-(r_{-i}) = \varphi^-(s_{-i})$. The approval rule φ is *manipulable by voter i at* $r = (w, k)$ *towards* s if r and s are i-deviations and $\varphi(s) \succ_{w(i)} \varphi(r)$. The approval rule φ is *not manipulable at r* if for all voters i there is no i-deviation s such that φ is manipulable by i at r towards s.

3 Manipulability of approval voting

The purpose of this section is to characterize the (report) profiles at which the approval rule is not manipulable, for three different preference extensions.

3.1 Worst comparison

In this subsection we extend preferences to sets by considering worst alternatives of those sets. Let $i \in N$ and $w(i) \in W$, then we define the extension $\succeq_{w(i)}$ by

$$B \succeq_{w(i)} C \Leftrightarrow (x, y) \in w(i) \text{ for every } x \in \omega\left(w(i)_{|B}\right) \text{ and } y \in \omega\left(w(i)_{|C}\right)$$

for all non-empty sets $B, C \subseteq A$. Thus, B is weakly preferred to C whenever every worst element of B is (weakly) preferred, according to $w(i)$, to every worst element of C.

---

[2]Assuming that there is no confusion about the identity of the voter whose vote is left out.

Theorem 3.1 *Let $r = (w, k) \in R^N$. The approval rule φ is not manipulable at r under worst comparison if and only if for each voter i at least one of the following two statements holds:*

(a) $\varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) \neq \emptyset$ *and* $\{x\} \sim_{w(i)} \{y\}$ *for all* $x, y \in \varphi(r_{-i}) \cap \beta_{k(i)}(w(i))$.

(b) $\{x\} \sim_{w(i)} \{y\}$ *for all* $x, y \in \varphi(r_{-i})$.

In words, condition (a) requires that if among the quasi-winners there are alternatives belonging to the $k(i)$ highest ranked alternatives of voter i, then i is indifferent between those alternatives; and (b) requires that voter i is indifferent between all quasi-winners.

Proof of Theorem 3.1. For the if-part, let s be an i-deviation of r.

In case (a), it follows by (1) that $\varphi(r) = \varphi(r_{-i}) \cap \beta_{k(i)}(w(i))$. By the assumption in (a), $\varphi(r) = \beta\left(w(i)|_{\varphi(r_{-i})}\right)$. Again by (1), $\varphi(s) \cap \varphi(r_{-i}) \neq \emptyset$, so for every $x \in \varphi(r) = \beta\left(w(i)|_{\varphi(r_{-i})}\right)$ it follows that $\{x\} \succeq_{w(i)} \varphi(s)$. So, $\varphi(r) \succeq_{w(i)} \varphi(s)$.

Now consider case (b) and assume $\varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) = \emptyset$ otherwise we are done by (a). Then $\varphi(r_{-i}) \subseteq \varphi(r)$, so we have $\omega\left(w(i)|_{\varphi(r)}\right) = \varphi(r_{-i})$. Since, by (1), $\varphi(s) \cap \varphi(r_{-i}) \neq \emptyset$, we have again $\varphi(r) \succeq_{w(i)} \varphi(s)$.

For the only if-part, suppose that there is an voter i for whom (a) nor (b) holds. It is sufficient to prove that φ is manipulable at profile r by voter i. Observe that either there exist $x, y \in \varphi(r_{-i}) \cap \beta_{k(i)}(w(i))$ such that $\{x\} \succ_{w(i)} \{y\}$, or $\varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) = \emptyset$ and there exist $x, y \in \varphi(r_{-i})$ such that $\{x\} \succ_{w(i)} \{y\}$. In both cases, by (1), $x, y \in \varphi(r)$. Now consider the report $s(i) = (w'(i), 1)$ of voter i such that $\beta(w'(i)) = \{x\}$. Then, by (1) again, $\varphi(s) = \{x\} \succ_{w(i)} \varphi(r)$. $\qquad\square$

We now consider the subclass of strict of preferences. This will enable us to compare approval voting to k-approval voting, which is studied in the next section.

3.1.1 Strict preferences

A preference $w(i)$ is *strict* (or a *linear ordering*) if it is antisymmetric, i.e., $(x, y) \in w(i)$ implies $(y, x) \notin w(i)$ for all $x, y \in A$ with $x \neq y$. Let P denote the set of all linear orderings on A, and S the set of all reports $(w(i), k(i))$ with $w(i) \in P$. The following result considers manipulability of the approval rule φ when restricted to S^N.

Corollary 3.2 *Let $r = (w, k) \in S^N$. The approval rule φ, restricted to S^N, is not manipulable at r under worst comparison if and only if for each voter i at least one of the following two statements holds:*

(a) $|\varphi(r_{-i}) \cap \beta_{k(i)}(w(i))| = 1$.

(b) $|\varphi(r_{-i})| = 1$.

Proof. For the only-if direction, note that if voter i can manipulate via a preference in W, then i can also manipulate by a strict preference, by strictifying the weak preference in any arbitrary way. Thus, the only-if direction follows from Theorem 3.1. The if-direction is immediate from Theorem 3.1. $\square$

3.2 Best comparison

In this subsection we extend preferences to sets by considering best alternatives of those sets. Let $i \in N$ and $w(i) \in W$, then we define the extension $\succeq_{w(i)}$ by[3]

$$B \succeq_{w(i)} C \Leftrightarrow (x,y) \in w(i) \text{ for every } x \in \beta\left(w(i)_{|B}\right) \text{ and } y \in \beta\left(w(i)_{|C}\right)$$

for all non-empty sets $B, C \subseteq A$. Thus, B is weakly preferred to C whenever every best element of B is (weakly) preferred, according to $w(i)$, to every best element of C.

Theorem 3.3 *Let $r = (w,k) \in R^N$. The approval rule φ is not manipulable at r under best comparison if and only if for each voter i at least one of the following two statements holds:*

(a) $(x,y) \in w(i)$ for all $x \in \beta\left(w(i)_{|\varphi(r_{-i})}\right)$ and all $y \in \varphi^-(r_{-i})$.

(b) $\varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) = \emptyset$ and $\varphi^-(r_{-i}) \cap \beta_{k(i)}(w(i)) \neq \emptyset$.

In words, condition (a) requires that any best alternative among the quasi-winners is preferred by i over all almost quasi-winners; and (b) requires that none of the quasi-winners is among his $k(i)$ highest ranked alternatives, but some of the almost quasi-winners are among his $k(i)$ highest ranked alternatives.

Proof of Theorem 3.3. For the if-part, let s be an i-deviation of r.

In case (a), it follows by (1) that there exists $x \in \varphi(r)$ with $x \in \beta\left(w(i)_{|\varphi(r_{-i})}\right)$. So by (a), $\{x\} \succeq_{w(i)} \{y\}$ for all $y \in \varphi(r_{-i}) \cup \varphi^-(r_{-i})$. This implies $\varphi(r) \succeq_{w(i)} \varphi(s)$.

In case (b), by (1), $\varphi(r) = \varphi(r_{-i}) \cup \left[\varphi^-(r_{-i}) \cap \beta_{k(i)}(w(i))\right]$. So $\varphi(r) \succeq_{w(i)} \{x\}$ for all $x \in \varphi(r_{-i}) \cup \varphi^-(r_{-i})$. This implies again $\varphi(r) \succeq_{w(i)} \varphi(s)$.

For the only if-part, suppose that there is an voter i for whom (a) nor (b) holds. It is sufficient to prove that φ is manipulable at profile r by voter i. Observe that either (i) $\varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) \neq \emptyset$ and there exists $y \in \varphi^-(r_{-i})$ such that $\{y\} \succ_{w(i)} \beta\left(w(i)_{|\varphi(r_{-i})}\right)$; or (ii) $\varphi^-(r_{-i}) \cap \beta_{k(i)}(w(i)) = \emptyset$ and there exists $y \in \varphi^-(r_{-i})$ such that $\{y\} \succ_{w(i)} \beta\left(w(i)_{|\varphi(r_{-i})}\right)$. Note that, in both cases, $\varphi(r) \subseteq \varphi(r_{-i})$. For both cases, consider the report $s(i) = (w'(i), 1)$ of voter i such that $\beta(w'(i)) = \{y\}$. Then by (1), $\varphi(s) = \varphi(r_{-i}) \cup \{y\}$, which implies $\varphi(s) \succ_{w(i)} \varphi(r)$. $\square$

For strict preferences we have the following corollary. The proof is straightforward and therefore omitted.

---

[3] In order to avoid cumbersome notation we will use the same symbols for different preference extensions in this paper.

Corollary 3.4 *Let $r = (w, k) \in S^N$. The approval rule φ, restricted to S^N, is not manipulable at r under best comparison if and only if for each voter i at least one of the following two statements holds:*

(a) $(x, y) \in w(i)$ *for all* $y \in \varphi^-(r_{-i})$*, where* $\{x\} = \beta\left(w(i)_{|\varphi(r_{-i})}\right)$.

(b) $\varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) = \emptyset$ *and* $\varphi^-(r_{-i}) \cap \beta_{k(i)}(w(i)) \neq \emptyset$.

3.3   Stochastic dominance comparison

In this subsection comparisons of sets of alternatives are based on stochastic dominance. To formalize this we need some further notions. Let u be a function from A to $\mathbb{R}$. Then u is said to be a *utility function representing preference $w(i)$* of voter i, if for all alternatives x and y in A

$$(x, y) \in w(i) \text{ if and only if } u(x) \geq u(y) \, .$$

Let B and C be two nonempty subsets of alternatives. Voter i is said to *prefer B to C according to stochastic dominance at preference $w(i)$*, denoted as $B \succeq_{w(i)} C$, if

$$\sum_{a \in B} \frac{1}{|B|} u(a) \geq \sum_{a \in C} \frac{1}{|C|} u(a) \text{ for every utility function } u \text{ representing } w(i) \, .$$

This preference extension[4] is based on the idea that, if we attach equal probabilities to the alternatives in each set, then the expected utility of the resulting lottery over B should be at least as high as the expected utility of the resulting lottery over C, for *each* utility function representing $p(i)$. Clearly, and in contrast to worst and best comparison in the preceding sections, this preference extension is not complete: many sets are incomparable. Observe that our notion of manipulability implies that a voter manages to obtain a preferred and thus comparable set.

In the following theorem we characterize the non-manipulable profiles under the stochastic dominance preference extension. To understand the proof, it is sometimes convenient to keep in mind the familiar characterization (or definition) of stochastic dominance involving only probabilities. This characterization says that a lottery ℓ is preferred over another lottery ℓ' if it can be obtained by shifting probability in ℓ' to preferred alternatives.

Theorem 3.5 *Let $r = (w, k) \in R^N$. The approval rule φ is not manipulable at r under stochastic dominance if and only if for each voter i at least one of the following three statements holds:*

(a) $\varphi(r_{-i}) \subseteq [A \backslash \beta_{k(i)}(w(i))]$ *and* $\varphi^-(r_{-i}) \cap \beta_{k(i)}(w(i)) \neq \emptyset$.

(b) $\varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) \neq \emptyset$ *and* $\{x\} \sim_{w(i)} \{y\}$ *for all* $x, y \in \varphi(r_{-i}) \cap \beta_{k(i)}(w(i))$ *and* $[A \backslash \beta_{k(i)}(w(i))] \cap \varphi(r_{-i}) \neq \emptyset$.

---

[4] The stochastic dominance criterion to compare sets has been used before, see e.g. Barberà, Dutta and Sen (2001).

(c) $\{x\} \sim_{w(i)} \{y\}$ *for all* $x, y \in \varphi(r_{-i})$ *and* $\varphi^-(r_{-i}) \subseteq L(x, w(i))$ *for some* $x \in \varphi(r_{-i})$.

In words, these three cases can be described as follows. In case (a), no quasi-winner but at least one almost quasi-winner belongs to the $k(i)$ highest ranked alternatives. In case (b) there are quasi-winners among the $k(i)$ highest ranked alternatives and voter i is indifferent between them, but there are also lower ranked quasi-winners. In case (c) voter i is indifferent between the quasi-winners, and all almost quasi-winners are lower ranked than some of the quasi-winners.

For a proof of this theorem see the Appendix.

The following corollary (proof omitted) applies to strict preferences.

Corollary 3.6 *Let* $r = (w, k) \in S^N$. *The approval rule* φ, *restricted to* S^N, *is not manipulable at* r *under stochastic dominance comparison if and only if for each voter* i *at least one of the following three statements holds:*

(a) $\varphi(r_{-i}) \subseteq [A \backslash \beta_{k(i)}(w(i))]$ *and* $\varphi^-(r_{-i}) \cap \beta_{k(i)}(w(i)) \neq \emptyset$.

(b) $\varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) = \{x\}$ *for some* $x \in A$ *and* $[A \backslash \beta_{k(i)}(w(i))] \cap \varphi(r_{-i}) \neq \emptyset$.

(c) $\varphi(r_{-i}) = \{x\}$ *for some* $x \in A$ *and* $\varphi^-(r_{-i}) \subseteq L(x, w(i))$.

3.4   Dichotomous preferences

A preference $w(i) \in W$ is *dichotomous* if it has two indifference classes, i.e., there are disjoint subsets $B_1 \neq \emptyset$ and B_2 of A such that $A = B_1 \cup B_2$, $(x, y), (y, x) \in w(i)$ for all $x, y \in B_1$ and for all $x, y \in B_2$, and $(x, y) \in w(i)$, $(y, x) \notin w(i)$ for all $x \in B_1$ and $y \in B_2$. Let $D \subseteq W$ denote the subset of all dichotomous preferences. A report $r(i) = (w(i), k(i))$ is in R_d if $w(i)$ is dichotomous and $k(i)$ is the cardinality of the higher indifference class of $w(i)$, i.e., $k(i) = |B_1|$ in the notation above[5]. A report $r(i) \in R_d$ is called *dichotomous* as well. In the following corollary we show that the approval rule is *strategy-proof* when restricted to dichotomous report profiles, under all three preference extensions considered in this paper: this means that φ is manipulable at no $r \in R_d^N$ under any of these preference extensions. This result confirms well known results on approval voting, see Brams and Fishburn (1983) and the references therein.

Corollary 3.7 *The approval rule* φ, *restricted to* R_d^N, *is strategy-proof under the worst, best, and stochastic dominance preference extensions.*

Proof. Let $r \in R_d^N$, $r(i) = (w(i), k(i))$ for all $i \in N$.

Suppose that for some $j \in N$ statement (b) in Theorem 3.1 does not hold. Then there is $x \in \varphi(r_{-j})$ with $x \in \beta_{k(j)}(w(j))$, and, clearly, $\{x\} \sim_{w(j)} \{y\}$ for

---

[5]Observe that, in this case, $k(i)$ is uniquely determined by $w(i)$.

all $x, y \in \varphi(r_{-j}) \cap \beta_{k(j)}(w(j))$. Hence, (a) holds for j. Thus, φ is strategy-proof under worst comparison.

Next, suppose that (a) in Theorem 3.3 does not hold for some $i \in N$. Then there is a $y \in \varphi^-(r_{-i})$ with $(y, x) \in w(i)$ and $(x, y) \notin w(i)$ for all $x \in \varphi(r_{-i})$. This implies that (b) holds for i. Thus, φ is strategy-proof under best comparison.

Finally, suppose (c) in Theorem 3.5 does not hold for some $i \in N$. There are two cases. If the first statement in (c) does not hold, then $\varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) \neq \emptyset$ and $\varphi(r_{-i}) \cap \left[A \setminus \beta_{k(i)}(w(i))\right] \neq \emptyset$, so that (b) holds. If the second statement in (c) does not hold, then there is $y \in \varphi^-(r_{-i})$ with $\{y\} \succ_{w(i)} \{x\}$ for some $x \in \varphi(r_{-i})$. In this case, if $\varphi(r_{-i}) \subseteq \left[A \setminus \beta_{k(i)}(w(i))\right]$ then (a) holds, and otherwise (b) holds. Thus, φ is strategy-proof under stochastic dominance comparison. $\square$

4 Manipulability of k-approval voting

A variation on approval voting is obtained by fixing the number of alternatives that has to be approved by each voter. Specifically, for a profile $p \in P^N$ of strict preferences, an alternative $x \in A$, and a number $k \in \{1, \ldots, m-1\}$, we denote by the k-score

$$\text{score}_k(x, p) = |\{i \in N \mid x \in \beta_k(p(i))\}|$$

the total number of voters for who alternative x is among the k first ranked alternatives at a profile p. The k-approval rule φ_k, defined by

$$\varphi_k(p) = \{x \in A \mid \text{score}_k(x, p) \geq \text{score}_k(y, p) \text{ for all } y \in A\}, \quad p \in P^N$$

assigns to each profile p the subset of alternatives with maximal k-score.[6]

Observe that it is, indeed, convenient to restrict attention to strict preferences, since otherwise we might have to split up indifference classes due to the fact that the number of alternatives to be approved is now fixed.

The sets $\varphi_k(p_{-i})$ and $\varphi_k^-(p_{-i})$ of quasi-winners and almost quasi-winners are defined analogously as for the approval rule. Also, we have the following useful observation:

$$\varphi_k(p) = \begin{cases} \varphi_k(p_{-i}) \cap \beta_k(p(i)) & \text{if } \varphi_k(p_{-i}) \cap \beta_k(p(i)) \neq \emptyset, \\ \varphi_k(p_{-i}) \cup \left[\varphi_k^-(p_{-i}) \cap \beta_k(p(i))\right] & \text{if } \varphi_k(p_{-i}) \cap \beta_k(p(i)) = \emptyset, \end{cases} \quad (2)$$

for all $p \in P^N$, $i \in N$, and $1 \leq k \leq m-1$.

In what follows we characterize the profiles of preferences at which the k-approval rule is not manipulable for different preference extensions, starting with worst, best, and stochastic dominance comparison. The definitions of (non)-manipulability of φ_k at a profile p are completely analogous to those for the approval rule.

---

[6]Unlike the approval rule the k-approval rule is a scoring rule, see Section 4.6. Note, further, that the case $k = m$ is uninteresting.

4.1 Worst comparison

For the definition of the worst comparison preference extension see Section 3.1.

The following theorem characterizes all profiles at which the k-approval rule is not manipulable under worst comparison.

Theorem 4.1 *Let $p \in P^N$. The k-approval rule φ_k is not manipulable at p under worst comparison if and only if for each voter i at least one of the following three statements holds:*

(a) $|\varphi_k(p_{-i}) \cap \beta_k(p(i))| = 1$.

(b) $|\varphi_k(p_{-i})| = 1$.

(c) $A \setminus \beta_k(p(i)) \subsetneq \varphi_k(p_{-i})$.

In words, condition (a) requires that exactly one of that voter i's k highest ranked alternatives is a quasi-winner; (b) requires that there is a unique quasi-winner; and (c) requires that the quasi-winners are a strict subset of the $m - k$ lowest ranked alternatives.

Proof of Theorem 4.1. For the if-part, let $i \in N$ and let q be an i-deviation of p. Assume that at least one of the cases (a), (b), and (c) holds. We show that voter i cannot manipulate from p to q.

In case (a), let $\{x\} = \varphi_k(p_{-i}) \cap \beta_k(p(i))$. By (2), $\varphi_k(p) = \{x\}$. Again by (2), either $\varphi_k(q) \subseteq \varphi_k(p_{-i})$ or $\varphi_k(p_{-i}) \subseteq \varphi_k(q)$. In the first case, if $x \in \varphi_k(q)$, then $\varphi_k(p) = \{x\} \succeq_{p(i)} \varphi_k(q)$; if $x \notin \varphi_k(q)$ then $\varphi_k(q) \subseteq A \setminus \beta_k(p(i))$ so that again $\varphi_k(p) = \{x\} \succeq_{p(i)} \varphi_k(q)$. In the second case, $\varphi_k(p) = \{x\} \subseteq \varphi_k(q)$, hence $\varphi_k(p) \succeq_{p(i)} \varphi_k(q)$.

In case (b), let $\varphi_k(p_{-i}) = \{x\}$ for some alternative x. If $x \in \beta_k(p(i))$ we are done by case (a). If $x \notin \beta_k(p(i))$ then by (2), $\varphi_k(p) = \{x\} \cup \left[\varphi_k^-(p_{-i}) \cap \beta_k(p(i))\right]$ and, thus, $\omega(p(i)_{|\varphi_k(p)}) = x$. Further, also by (2), $\varphi_k(q) = \{x\}$ or $\varphi_k(q) = \{x\} \cup \left[\varphi_k^-(p_{-i}) \cap \beta_k(q(i))\right]$; in both cases, $(x, \omega(p(i)_{|\varphi_k(q)})) \in p(i)$ and, thus, $\varphi_k(p) \succeq_{p(i)} \varphi_k(q)$.

In case (c), by (2) we have $\varphi_k(p) = \varphi_k(p_{-i}) \cap \beta_k(p(i))$ and $\varphi_k(q) = \varphi_k(q_{-i}) \cap \beta_k(q(i)) = \varphi_k(p_{-i}) \cap \beta_k(q(i))$. If $\beta_k(q(i)) = \beta_k(p(i))$ then $\varphi_k(p) = \varphi_k(q)$. Otherwise, since $A \setminus \beta_k(p(i)) \subsetneq \varphi_k(p_{-i})$, there is a $y \in [A \setminus \beta_k(p(i))] \cap \varphi_k(q)$. Hence, $\varphi_k(p) \succeq_{p(i)} \varphi_k(q)$.

For the only-if part, suppose that there is a voter $i \in N$ such that none of the three cases (a), (b), and (c) holds. It is sufficient to prove that φ_k is manipulable at profile p by voter i. For this, in turn, it is sufficient to prove that i can manipulate at profile p for the following two cases.

Case (i): $\varphi_k(p_{-i}) \cap \beta_k(p(i)) = \emptyset$ and $|\varphi_k(p_{-i})| \geq 2$.

Let $b = \beta\left(p(i)_{|\varphi_k(p_{-i})}\right)$. Take $q(i)$ such that the positions in $p(i)$ of b and one of the alternatives in $\beta_k(p(i))$ are swapped. Then $\varphi_k(q) = \{b\}$ and $\varphi_k(q) \succ_{p(i)} \varphi_k(p)$, hence voter i can manipulate at profile p towards q.

Case (ii): $|\varphi_k(p_{-i}) \cap \beta_k(p(i))| \geq 2$ and $[A \setminus \beta_k(p(i))] \nsubseteq \varphi_k(p_{-i})$.

Let $w = \omega\left(p(i)_{|\varphi_k(p_{-i}) \cap \beta_k(p(i))}\right)$ and $y \in A \setminus [\beta_k(p(i)) \cup \varphi_k(p_{-i})]$. Let $q(i)$ be obtained from $p(i)$ by swapping the positions of the alternatives w and y. By (2), $\varphi_k(p) = \varphi_k(p_{-i}) \cap \beta_k(p(i))$ and $\varphi_k(q) = \varphi_k(p_{-i}) \cap \beta_k(p(i)) \setminus \{w\}$ it follows that $\varphi_k(q) \succ_{p(i)} \varphi_k(p)$, proving that φ_k is manipulable by voter i at profile p towards q. $\qquad\square$

4.2 Best comparison

For the definition of the best comparison preference extension see Section 3.2.

The following theorem characterizes all profiles at which the k-approval rule is not manipulable under best comparison.

Theorem 4.2 *Let $p \in P^N$. The k-approval scoring rule φ_k is not manipulable at p under best comparison if and only if for each voter i at least one of the following three statements holds:*

(a) $\left(\beta(p(i)_{|\varphi_k(p_{-i})}), x\right) \in p(i)$ *for all $x \in \varphi_k^-(p_{-i})$.*

(b) $\varphi_k(p_{-i}) \cap \beta_k(p(i)) = \emptyset$ *and* $\varphi_k^-(p_{-i}) \cap \beta_k(p(i)) \neq \emptyset$.

(c) $|\varphi_k(p_{-i}) \cap \beta_k(p(i))| > |A \setminus [\beta_k(p(i)) \cup \varphi_k(p_{-i})]|$.

In words, condition (a) requires that the best alternative among the quasi-winners is preferred over all almost quasi-winners; (b) requires that no quasi-winner is among his k first ranked alternatives, but some of the almost quasi-winners are; and (c) requires that the number of the voter's k highest ranked alternatives among the quasi-winners is larger than the number of alternatives that are neither among his k highest ranked nor among the quasi-winners.

Proof of Theorem 4.2. For the if-part, let q be an i-deviation of p. Note that $\varphi_k(p_{-i}) = \varphi_k(q_{-i})$ and $\varphi_k^-(p_{-i}) = \varphi_k^-(q_{-i})$. Assume that at least one of the cases (a), (b), and (c) holds. We show that voter i cannot manipulate from p to q.

In case (a), for both cases occurring in (2), we obtain $\beta\left(p(i)_{|\varphi_k(p)}\right) = \beta\left(p(i)_{|\varphi_k(p_{-i})}\right)$. Since $\beta\left(p(i)_{|\varphi_k(q)}\right) \in \varphi_k(p_{-i}) \cup \varphi_k^-(p_{-i})$ and by the assumption for case (a), we conclude that $\varphi_k(p) \succeq_{p(i)} \varphi_k(q)$.

In case (b), again using (2), we have $\varphi_k(p) = \varphi_k(p_{-i}) \cup [\varphi_k^-(p_{-i}) \cap \beta_k(p(i))]$, hence $\beta\left(p(i)_{|\varphi_k(p)}\right) = \beta\left(p(i)_{|\varphi_k^-(p_{-i}) \cap \beta_k(p(i))}\right)$; and $\varphi_k(q) \in \varphi_k(p_{-i}) \cup [\varphi_k^-(p_{-i}) \cap \beta_k(q(i))]$. By the assumptions for this case, $\varphi_k(p) \succeq_{p(i)} \varphi_k(q)$.

In case (c), it is easy to see that $|A \setminus \varphi_k(p_{-i})| < |\beta_k(p(i))| = k = |\beta_k(q(i))|$, hence $\beta_k(q(i)) \cap \varphi_k(p_{-i}) \neq \emptyset$. Therefore, by (2) we have $\varphi_k(p) = \varphi_k(p_{-i}) \cap \beta_k(p(i))$ and $\varphi_k(q) = \varphi_k(p_{-i}) \cap \beta_k(q(i)) \subseteq \varphi_k(p_{-i})$. Thus, also in this case $\varphi_k(p) \succeq_{p(i)} \varphi_k(q)$.

For the only-if part, suppose that there is a voter $i \in N$ such that none of the three cases (a), (b), and (c) holds. It is sufficient to prove that φ_k is manipulable at profile p by voter i. For this, in turn, it is sufficient to prove that i can manipulate at profile p for the following two cases.

Case (i): There is an $x \in \varphi_k^-(p_{-i})$ such that $(x,b) \in p(i)$, where $b = \beta\left(\varphi_k(p_{-i}), p(i)_{|\varphi_k(p_{-i})}\right)$; $\varphi_k(p_{-i}) \cap \beta_k(p(i)) \neq \emptyset$; and $|\varphi_k(p_{-i}) \cap \beta_k(p(i))| \leq |A \setminus [\beta_k(p(i)) \cup \varphi_k(p_{-i})]|$.

For this case, note that $x \in \beta_k(p(i))$. By the assumptions for this case we can take a $q(i) \in P$ with $x \in \beta_k(q(i))$ and $\varphi_k(p_{-i}) \cap \beta_k(q(i)) = \emptyset$. Hence, $x \in \varphi_k(q) \setminus \varphi_k(p)$ and, thus, $\varphi_k(q) \succ_{p(i)} \varphi_k(p)$. So i can manipulate at profile p towards q.

Case (ii): There is an $x \in \varphi_k^-(p_{-i})$ such that $(x,b) \in p(i)$, where $b = \beta\left(p(i)_{|\varphi_k(p_{-i})}\right)$; and $\varphi_k^-(p_{-i}) \cap \beta_k(p(i)) = \emptyset$.

In this case, $\varphi_k(p) = \varphi_k(p_{-i})$. Note that the sets $\beta_k(p(i))$, $\varphi_k(p_{-i})$, and $\varphi_k^-(p_{-i})$ are pairwise disjoint. So we can take $q(i) \in P$ such that $x \in \beta_k(q(i))$ and $\varphi_k(p_{-i}) \cap \beta_k(q(i)) = \emptyset$. Then $\varphi_k(q) \supseteq \varphi_k(p_{-i}) \cup \{x\}$, so $x \in \varphi_k(q) \setminus \varphi_k(p)$, thus $\varphi_k(q) \succ_{p(i)} \varphi_k(p)$ and i can manipulate at profile p towards q. $\qquad\square$

4.3 Stochastic dominance comparison

For the definition of the stochastic dominance comparison preference extension see Section 3.3.

The following theorem characterizes all profiles at which the k-approval rule is not manipulable under stochastic dominance comparison. Its proof is placed in the Appendix.

Theorem 4.3 *Let $p \in P^N$. The k-approval scoring rule φ_k is not manipulable at p under stochastic dominance comparison if and only if for all voters i at least one of the following five statements holds:*

(a) $A \setminus \beta_k(p(i)) \subsetneq \varphi_k(p_{-i})$.

(b) $\varphi_k(p_{-i}) \subseteq [A \setminus \beta_k(p(i))]$ *and* $\varphi_k^-(p_{-i}) \cap \beta_k(p(i)) \neq \emptyset$.

(c) $\varphi_k(p_{-i}) \cap \beta_k(p(i)) = \{w\}$ *for some* $w \in A$ *and* $[A \setminus \beta_k(p(i))] \cap \varphi_k(p_{-i}) \neq \emptyset$.

(d) $\varphi_k(p_{-i}) = \{w\}$ *for some* $w \in A$ *and* $\varphi_k^-(p_{-i}) \subseteq L(w, p(i))$.

(e) $\varphi_k(p_{-i}) = \{w\}$ *for some* $w \in A$ *and* $|\varphi_k^-(p_{-i}) \cap L(w, p(i))| > m - k$.

In words, these five cases can be described as follows. In case (a), at least one of voter i's k highest ranked alternatives and all of his lower ranked alternatives are quasi-winners. In case (b), no quasi-winner but at least one almost quasi-winner is among his k highest ranked alternatives. In case (c) there is a unique quasi-winner among voter i's k highest ranked alternatives, but there are lower ranked quasi-winners as well. In case (d) there is a unique quasi-winner, which is preferred by i to all almost quasi-winners. In case (e) there is again a unique quasi-winner, and among the almost quasi-winners there are more than $m - k$ alternatives worse than the unique quasi-winner.

4.4 The two-voter case

In this subsection we concentrate on the two-voter case and consider the following question: which k-approval rule is least (or minimally) manipulable, under various assumptions on preference extensions as studied in the preceding sections?

We start with a simple theorem, which will be derived from Theorems 4.1, 4.2, and 4.3, but also easily follows directly. It states that φ_1 is *strategy-proof*, i.e., not manipulable at any profile p.

Theorem 4.4 *Let $n = 2$. Then the 1-approval rule φ_1 is strategy-proof under worst, best, and stochastic dominance comparison.*

Proof. Let $p = (p(1), p(2))$ be a preference profile and let $k = 1$. Note that (b) in Theorem 4.1 is always satisfied: this shows strategy-proofness under worst comparison. In Theorem 4.2, (a) reduces to $\beta(p(1)) = \beta(p(2))$ and (b) to $\beta(p(1)) \neq \beta(p(2))$: this shows strategy-proofness under best comparison. Finally, in Theorem 4.3, (b) reduces to $\beta(p(1)) \neq \beta(p(2))$ and (d) to $\beta(p(1)) = \beta(p(2))$: this shows strategy-proofness under stochastic dominance comparison. $\square$

This observation might make our quest for minimally manipulable rules futile, were it not the case that the 1-approval rule (i.e., plurality rule) is not unambiguously attractive. As an example, consider the case where voter 1 has preference $p(1) : xz \ldots y$ and voter 2 has preference $p(2) : yz \ldots x$ (notations obvious). Then $\varphi_1(p) = \{x, y\}$ but $\varphi_2(p) = \{z\}$. So it seems that φ_2 offers a better compromise in this case than φ_1.

Moreover, for more than two voters and apart from a few particular cases, Theorem 4.4 no longer holds.

We will now consider the three cases (worst, best, and stochastic dominance comparison) separately.

4.4.1 Worst comparison for two voters

The non-manipulable profiles for two voters under worst comparison are easily described using Theorem 4.1.

Corollary 4.5 *Let $n = 2$ and $2 \leq k < m$. Let $p \in P$ and consider worst comparison.*

(a) *If $k \leq (m+1)/2$, then φ_k is not manipulable at p if and only if $|\varphi_k(p)| = 1$, or equivalently,*

$$|\beta_k(p(i)) \cap \beta_k(p(2))| = 1 \ .$$

(b) *If $k > (m+1)/2$, then φ_k is not manipulable at p if and only if $|\varphi_k(p)| = 2k - m$, or equivalently,*

$$|\beta_k(p(i)) \cap \beta_k(p(2))| = 2k - m \ .$$

14

Proof. Case (b) in Theorem 4.1 does not apply. If case (a) in Theorem 4.1 applies then we have $|\beta_k(p(1)) \cap \beta_k(p(2))| = 1$ (or, equivalently, $|\varphi_k(p)| = 1$), but this is possible if and only if $k \le (m+1)/2$. If case (c) in Theorem 4.1 applies then we have $|\beta_k(p(1)) \cap \beta_k(p(2))| = 2k - m$ (or, equivalently, $|\varphi_k(p)| = 2k - m$), but this is possible if and only if $k \ge (m+1)/2$; but for $k = (m+1)/2$ we have $2k - m = 1$, so that we are back in case (a). □

Denote by $\eta(m, k)$ the number of profiles (for two voters) at which φ_k is not manipulable. By straightforward counting we obtain the following result for the number of manipulable profiles for two voters under worst comparison.

Theorem 4.6 *Let $n = 2$ and $2 \le k < m$. Consider worst comparison. Then*

$$
\eta(m, k) = \begin{cases} m! \, k \begin{pmatrix} m - k \\ k - 1 \end{pmatrix} k! \, (m - k)! & \text{if } k \le (m + 1)/2 \\[2ex] m! \begin{pmatrix} k \\ 2k - m \end{pmatrix} k! \, (m - k)! & \text{if } k > (m + 1)/2 \,. \end{cases}
$$

From this theorem we derive the following corollary (see the Appendix for a proof), which states some facts about k as far as non-manipulability is concerned. (The exact meaning of k^* being close to $\sqrt{m}$ in part (a) is explained in the proof.)

Corollary 4.7

(a) $\eta(m, k)$ *increases in k between 2 and an integer k^*, which is close to $\sqrt{m}$, and decreases between k^* and $\frac{1}{2}(m - 1)$.*

(b) $\eta(m, k)$ *increases between $\frac{1}{2}(m - 1)$ and $(m - 1)$.*

(c) *The $(m - 1)$-approval scoring rule is second best since $\eta(m, (m - 1)) > \eta(m, k)$ for all $m - 1 > k \ge 2$.*

The first-best value of k is $k = 1$ (Theorem 4.4), but φ_1 has the drawback that it does not give much opportunity for compromises. Among other values of k, the value $k = m - 1$ is best. We might, however, prefer to have $k \le (m+1)/2$, for the following reason. Call φ_k *citizen-sovereign* if for every alternative $x \in A$ there is a profile $p \in P$ with $\varphi_k(p) = \{x\}$. It is not difficult to see that φ_k is citizen-sovereign for any number of voters $n \ge 2$ if $k \le (m+1)/2$. For $n = 2$ and $k > (m+1)/2$, however, φ_k is not citizen-sovereign. Hence, if we restrict ourselves to citizen-sovereign rules with $k \ge 2$, then the best rule is φ_{k^*}, where k^* is close to $\sqrt{m}$.

4.4.2 Best comparison for two voters

The non-manipulable profiles for two voters under best comparison can be derived from Theorem 4.2.

Corollary 4.8 *Let $n = 2$ and $2 \leq k < m$. Let $p \in P$ and consider best comparison.*

(a) *If $k \leq m/2$ then φ_k is not manipulable at $p \in P$ if and only if either*

$$\beta(p(1)) \in \beta_k(p(2)) \ \text{ and } \ \beta(p(2)) \in \beta_k(p(1))$$

or

$$\beta_k(p(1)) \cap \beta_k(p(2)) = \emptyset \ .$$

(b) *If $k > m/2$ then φ_k is not manipulable at any $p \in P$.*

Proof. If $k > m/2$ then case (c) in Theorem 4.2 applies to all $p \in P$, and if $k \leq m/2$ then case (c) applies to no $p \in P$. This implies part (b) of the corollary, and it also implies that for $k \leq m/2$ we only have to consider cases (a) and (b) in Theorem 4.2. It is easily seen that these cases result in the two cases in part (a) of the corollary. $\qquad\square$

The number of non-manipulable profiles $\eta(m, k)$ if $k \leq m/2$ is computed in the following theorem.

Theorem 4.9 *Let $n = 2$ and $2 \leq k \leq m/2$. Consider best comparison. Then*

$$\eta(m, k) = m! \, (m - 2)! \, (k - 1)^2 + m! \, (m - 1)! + m! \, [(m - k)!]^2 / (m - 2k)! \ .$$

Proof. The first case in (a) in Corollary 4.8 with $\beta(p(1)) \neq \beta(p(2))$ results in

$$m! \, (k - 1) \binom{m - 2}{k - 2} (k - 1)! \, (m - k)!$$

different non-manipulable profiles. This yields the first term of $\eta(m, k)$ in the theorem. If $\beta(p(1)) = \beta(p(2))$ then this number is simply equal to $m! \, (m - 1)!$, which yields the second term. The second case in (a) in Corollary 4.8 results in

$$m! \binom{m - k}{k} k! \, (m - k)!$$

different non-manipulable profiles, which simplifies to the third term for $\eta(m, k)$ in the theorem. $\qquad\square$

If we require $k \neq 1$ and citizen-sovereignty, i.e., $k \leq m/2$, then the optimal value of k with respect to non-manipulability, i.e., the value of k that maximizes $\eta(m, k)$, is equal to 2.

To see this, note that by Theorem 4.9 and some elementary calculations we have for $2 < k \leq \frac{m}{2}$:

$$\eta(m, 2) > \eta(m, k)$$

$$\Leftrightarrow (m - 2)(m - 3) > k(k - 2) + \frac{\overbrace{(m - k)(m - k - 1) \cdot \ldots \cdot (m - 2k + 1)}^{k \text{ factors}}}{\underbrace{(m - 2)(m - 3) \cdot \ldots \cdot (m - k + 1)}_{k - 2 \text{ factors}}} \ .$$

16

Since $k > 2$ it is therefore sufficient to prove that

$$(m-2)(m-3) > k(k-2) + (m-2k+2)(m-2k+1) .$$

This simplifies to $(4k-8)m > 5k^2 - 8k - 4$. Since $m \geq 2k$, it is sufficient to show that $3k^2 - 8k + 4 > 0$, which indeed holds for $k > 2$.

4.4.3 Stochastic dominance comparison for two voters

The non-manipulable profiles for two voters under stochastic dominance comparison can be derived from Theorem 4.3.

Corollary 4.10 *Let $n = 2$ and $2 \leq k < m$. Let $p \in P$ and consider stochastic dominance comparison. Then φ_k is not manipulable at p if and only if at least one of the following holds.*

(a) $\beta_k(p(1)) \cap \beta_k(p(2)) = \emptyset$.

(b) $|\beta_k(p(1)) \cap \beta_k(p(2))| = 1$.

(c) $\beta_k(p(1)) \cap \beta_k(p(2)) \neq \emptyset$ and $[A \setminus \beta_k(p(1))] \cap [A \setminus \beta_k(p(2))] = \emptyset$.

Proof. For $n = 2$ and $k \geq 2$ cases (d) and (e) in Theorem 4.3 are not possible. Case (c) in Theorem 4.3 reduces to case (b) above, and case (a) in Theorem 4.3 reduces to case (c) above. Finally, case (b) in the theorem reduces to case (a) above. $\qquad\square$

From this description we can again derive the number of manipulable profiles $\eta(m, k)$.

Theorem 4.11 *Let $n = 2$, $k \geq 2$, and consider stochastic dominance comparison.*

(a) If $k \leq m/2$ then

$$\eta(m,k) = m! \, [(m-k)!]^2 / (m-2k)! + m! \, k^2 \, [(m-k)!]^2 / (m-2k+1)! .$$

(b) If $k > m/2$ then
$$\eta(m,k) = m! \, [k!]^2 / (2k-m)! .$$

Proof. If $k \leq m/2$ then (c) in Corollary 4.10 is not possible, and cases (a) and (b) in the corollary are mutually exclusive. In case (a) of Corollary 4.10 there are

$$m! \binom{m-k}{k} k! \, (m-k)!$$

non-manipulable profiles, resulting in the first term for $\eta(m, k)$, and in case (b) of the corollary there are

$$m! \, k \binom{m-k}{k-1} k! \, (m-k)!$$

non-manipulable profiles, resulting in the second term for $\eta(m,k)$.

If $k > m/2$ then case (a) of Corollary 4.10 is not possible, and (b) is a special case of (c). For the latter case, we just have to count the number of profiles for which $[A \setminus \beta_k(p(1))] \cap [A \setminus \beta_k(p(2))] = \emptyset$, since the other condition is always fulfilled. This number is equal to

$$m! \binom{k}{m-k} k!\,(m-k)!$$

which is equal to $m!\,[k!]^2/(2k-m)!$. $\qquad\square$

About the value of k that maximizes $\eta(m,k)$, so the value of k that is optimal with respect to non-manipulability, we can say the following.

1. For $2 \le k \le \frac{m}{2}$, the number of non-manipulable profiles decreases with k, and thus $k = 2$ is optimal.

2. For $\frac{m}{2} < k \le m-1$, the number of non-manipulable profiles increases with k, and thus $k = m-1$ is optimal.

3. $\eta(m,2) > \eta(m,m-1)$ for $m \ge 4$, so $k = 2$ is the overall optimal value between 2 and $m-1$.

To prove these statements, first assume $k \le \frac{m}{2}$. Then, using Theorem 4.11(a) and simplifying, we derive

$$\eta(m,k+1) < \eta(m,k) \Leftrightarrow 3k^2 - 2km - 1 < 0 ,$$

and it is easily seen that the right hand side holds for all $2 \le k \le \frac{m}{2}$. Next, assume $\frac{m}{2} < k \le m-1$. Then, using Theorem 4.11(b) and simplifying, we derive

$$\eta(m,k+1) > \eta(m,k) \Leftrightarrow 3k^2 + k(4-4m) + m^2 - 3m + 1 < 0 .$$

The roots of the quadratic expression in k at the right hand side are $\frac{2}{3}(m-1) \pm \frac{1}{3}\sqrt{m^2 + m + 1}$; the smaller root is smaller than $\frac{m}{2}$, whereas the larger root is larger than $m-1$. Thus, the right hand side holds for all $\frac{m}{2} < k \le m-1$. Finally, by Theorem 4.11 again,

$$\eta(m,2) > \eta(m,m-1) \Leftrightarrow m > 3 ,$$

so that $k = 2$ is the overall optimal value of k for $2 \le k \le m-1$.

4.5 Lexicographic worst and best comparison

In this subsection we briefly consider a natural extension of worst and best comparison, namely lexicographic worst and best comparison. These preference extensions to sets are given by the following recursive definition. For two subsets B and C of alternatives, we say that B is (weakly) *preferred to C under lexicographic worst comparison* by voter i with preference $p(i)$ if

1. $C = \emptyset$, or

2. B and C are non-empty and $\big(\omega(p(i)_{|B}), \omega(p(i)_{|C})\big) \in p(i)$, or

3. $\omega(p(i)_{|B}) = \omega(p(i)_{|C}) =: w$ and $B \setminus \{w\}$ is preferred to $C \setminus \{w\}$ under lexicographic worst comparison by voter i with preference $p(i)$.

The definition for lexicographic best comparison is obtained simply by replacing the worst alternative by the best alternative, i.e., by replacing $\omega(\cdot)$ by $\beta(\cdot)$. Thus, under lexicographic worst comparison a voter first considers the worst elements of B and C. If these are different, then he prefers the set with the better worst element. Otherwise, the voter considers the second worst elements. If these are different, then he prefers the set with the better second worst element. Otherwise, he considers the third worst elements, etc. Similarly, of course, for lexicographic best comparison.

Complete characterizations of the non-manipulable profiles for both lexicographic worst and lexicographic best comparison can be given but are rather technical (even more so than for stochastic dominance comparison) and therefore not included.

Note that any profile that is manipulable under worst [best] comparison is also manipulable under lexicographic worst [best] comparison. Hence, the set of non-manipulable profiles under lexicographic worst [best] comparison is always a subset of the set of non-manipulable profiles under worst [best] comparison. It is not very difficult to check (we omit the proof for the sake of briefness) that all the profiles listed in Corollary 4.5, that is, all two-voter profiles that are non-manipulable under worst comparison, are also non-manipulable under lexicographic worst comparison, so that in this case considering lexicographic worst comparison instead of just worst comparison does not make any difference. The non-manipulable profiles coincide, and the optimal value of k as far as non-manipulability is concerned, is the same as in Section 4.4.1.

For two voters and lexicographic best comparison the situation is different and the set of non-manipulable profiles is a strict subset of the set of non-manipulable profiles under best comparison, that is, the set of profiles described in Corollary 4.8. To be precise, we have the following result, which can be derived from Corollary 4.8 (the proof is again left to the reader).

Corollary 4.12 *Let $n = 2$ and $2 \leq k < m$. Let $p \in P$ and consider lexicographic best comparison.*

(a) If $k \leq m/2$ then φ_k is not manipulable at $p \in P$ if and only if either

$$\{\beta(p(1))\} = \{\beta(p(2))\} = \beta_k(p(1)) \cap \beta_k(p(1))$$

or

$$\beta_k(p(1)) \cap \beta_k(p(2)) = \emptyset \ .$$

(b) If $k > m/2$ then φ_k is not manipulable at any $p \in P$.

In this case, the total number of non-manipulable profiles for $2 \leq k \leq m/2$ is equal to

$$\eta(m, k) = \frac{m! \, [(m-k)!]^2 \, (m - 2k + 2)}{(m - 2k + 1)!}$$

and this number is decreasing in k, so that $k = 2$ is the value of k that minimizes manipulability subject to $2 \leq k \leq m/2$, just as in the best comparison case. The proofs of these facts are somewhat simpler than for the best comparison case. For the sake of briefness we omit them.

4.6 An asymptotic result

We start with defining the class of all scoring rules. A (normalized) *scoring vector* is a vector $s = (s_1, s_2, \ldots, s_m) \in \mathbb{R}^m$ with $1 = s_1 \geq s_2 \geq \ldots \geq s_m = 0$. For a preference $\pi \in P$ and an alternative $x \in A$ let $t(\pi, x)$ denote the *rank* of x in the preference π, i.e., $t(\pi, x) = k$ where $k = 1$ if $\{x\} = \beta(\pi)$ and $\{x\} = \beta_k(\pi) \setminus \beta_{k-1}(\pi)$ otherwise.

For a scoring vector s, a profile $p \in P^N$, and an alternative $x \in A$, we denote by the *s-score*

$$\text{score}_s(x, p) = \sum_{i \in N} s_{t(p(i), x)}$$

the total score that x obtains under profile p and score vector s. The *scoring rule with scoring vector s* is defined by

$$\varphi_s(p) = \{x \in A \mid \text{score}_s(x, p) \geq \text{score}_s(y, p) \text{ for all } y \in A\}, \quad p \in P^N .$$

Clearly, a k-approval rule is a scoring rule with scoring vector s such that $s_1 = \ldots = s_k = 1$ and $s_{k+1} = \ldots = s_m = 0$.

Now, in what follows, we fix the number of alternatives m and let the number of voters go to infinity. We will show, formally, that then any two scoring rules lead to the same expected values of the highest score, second highest score, and so on, up to a multiplicative constant proportional to the standard deviations of the scoring vectors: the higher this standard deviation the larger the differences between the expected scores. Since the standard deviation is maximal for k-approval rules with k around $m/2$, we can conclude by the law of large numbers that the proportion of manipulable profiles is smallest for this rule.[7]

In order to derive the announced result, assume that voter preferences are drawn from the uniform distribution over P – that is, according to 'impartial culture'. Let $Y = (Y_\pi)_{\pi \in P}$ denote the random vector giving the numbers of voters for each preference (so $\sum_{\pi \in P} Y_\pi = n$). Then Y has a multinomial distribution with mean $(n/m!)\mathbf{1}$, where $\mathbf{1}$ is a vector with all entries equal to 1. Write $A = \{x_1, \ldots, x_m\}$, then for a scoring vector s the random vector Y gives rise to

---

[7]We thank Eric Beutner (Maastricht University) for helpful discussions on this topic.

a random vector of scores $X^s = (X_1^s, \ldots, X_m^s)$ where $X_j^s = \sum_{\pi \in P} Y_\pi s_{t(\pi, x_j)}$ for $j = 1, \ldots, m$. Let

$$\sigma(s) = \sqrt{\frac{s_1^2 + \ldots + s_m^2}{m} - \bar{s}^2}$$

denote the standard deviation of the scoring vector s, where $\bar{s} = (s_1 + \ldots + s_m)/m$ is the mean of s. Proposition 2 in Pritchard and Wilson (2009) asserts that $(X^s - n\bar{s}\mathbf{1})/\sqrt{n}$ converges in distribution to $Z^s := \sigma(s) \left(\frac{m}{m-1}\right)^{1/2} (Z - \bar{Z}\mathbf{1})$, where $Z = (Z_1, \ldots, Z_m)$ is a vector of independent standard normal random variables and $\bar{Z} = (1/m) \sum_{j=1}^m Z_j$. In words, this means that the limit distributions of the vectors of normalized random variables X^s differ only in a multiplicative constant, namely the standard deviation $\sigma(s)$. This implies $Z^s = (\sigma(s)/\sigma(s'))Z^{s'}$ for any two scoring vectors s and s'. In particular, this also holds for the associated order statistics $(Z_{(1)}^s, \ldots, Z_{(m)}^s)$ and $(Z_{(1)}^{s'}, \ldots, Z_{(m)}^{s'})$, with $Z_{(1)}^s$ and $Z_{(1)}^{s'}$ being the (limit) distributions of the highest scores. As a consequence we obtain the following proposition.

Proposition 4.13 *For all scoring vectors* s *and* s',

$$E[Z_{(j)}^s] - E[Z_{(j+1)}^s] = (\sigma(s)/\sigma(s')) \left(E[Z_{(j)}^{s'}] - E[Z_{(j+1)}^{s'}]\right)$$

for all $j = 1, \ldots, m - 1$, *where* E *denotes the expectation operator.*

Proposition 4.13 implies that the difference in expected value between any two consecutive scores is largest for rules based on scoring vectors with maximal standard deviation. Since the vectors of random variables Z^s have the same distributions up to these standard deviations of the score vectors, and taking into account that the probability of all alternatives having distinct scores converges to 1 if the number of voters goes to infinity[8], we have by the law of large numbers that scoring rules φ_s with maximal standard deviation $\sigma(s)$ have the smallest proportion of manipulable profiles. The following result, of which for completeness a proof is given in the Appendix, then implies that the k-approval rule with k around $m/2$ is least manipulable if the number of voters becomes large.

Proposition 4.14 *Among all scoring vectors* s, $\sigma(s)$ *is maximal if and only if* $s_1 = \ldots = s_k = 1$ *and* $s_{k+1} = \ldots = s_m = 0$, *where* $k = m/2$ *if* m *is even and* $k \in \{(m - 1)/2, (m + 1)/2\}$ *if* m *is odd.*

For ease of reference we formulate the main result of this subsection as a corollary.

Corollary 4.15 *Let* k^* *denote the value(s) of* k *in Proposition 4.14 and let* s *be an arbitrary scoring vector unequal to the scoring vector associated with* k^*. *Then for* n *sufficiently large the proportion of manipulable profiles under* φ_{k^*} *is smaller than the proportion of manipulable profiles under* φ_s.

---

[8] This is Proposition 3 in Pritchard and Wilson, 2009.

This asymptotic result should be taken with some care, since the probability of being able to manipulate becomes very small if the number of voters grows, and so we are comparing small numbers. On the other hand there is some evidence that already for a relatively small number of voters the k-approval rule with k close to $m/2$ performs best, at least among the k-approval rules. See Table 3 in the next section.

5 Some simulation results

Since general comparisons between the approval rule and k-approval rules are complex and hard to obtain, we present here some results of simulations.[9]

Table 1 gives the approximate percentages of non-manipulable profiles for the approval rule with 3–10 alternatives and 2, 3, 6, and 10 voters, based on 1,00,000 trials. While the number of trials is relatively low, we nevertheless think that the numbers in the table give reliable impressions.

	m	3	4	5	6	7	8	9	10
$n = 2$	worst comp	56	45	38	33	30	27	25	23
	best comp	65	59	55	52	49	48	46	45
	stoch comp	41	37	34	31	30	28	27	25
$n = 3$	worst comp	57	49	44	41	38	36	34	32
	best comp	56	47	41	38	34	32	30	28
	stoch comp	25	20	17	16	15	14	14	13
$n = 6$	worst comp	67	61	57	54	52	50	48	47
	best comp	51	40	33	29	25	22	20	18
	stoch comp	28	20	16	13	11	10	09	08
$n = 10$	worst comp	73	69	65	62	60	59	57	56
	best comp	54	43	37	32	28	25	23	21
	stoch comp	35	27	22	19	16	15	13	12

Table 1: Approximate percentages of non-manipulable preference profiles for the approval rule, based on 1,00,000 trials.

Some conclusions can be drawn from this table. Clearly, the possibility of manipulation increases with the number of alternatives. For more than two voters manipulability also increases from worst comparison to best comparison and from best comparison to stochastic dominance comparison. This is not entirely intuitive at first glance. One might expect that many profiles that are manipulable under stochastic comparison are also manipulable under worst and best comparison, since in order to improve under stochastic comparison a necessary condition is that the worst and best alternatives of a set should not decrease

---

[9]We thank Bram Driesen for doing these simulations (with Matlab). They are based on randomly drawing profiles and checking for non-manipulability using the characterizations in Sections 3 and 4.

in preference. Thus, to explain the results in Table 1, it seems to be the case that manipulation under stochastic comparison is often performed by improving intermediate alternatives. Moreover, apparently this kind of manipulation can often lead to comparable sets, in spite of the fact that stochastic dominance preference is not complete.

As a final comment on Table 1, under worst comparison manipulability seems to decrease with the number of voters, but under the other two preference extensions manipulability first seems to increase and then to decrease again.

In Table 2 we present the results for k-approval rules for 6 alternatives; 2, 3, 6, or 10 voters; and again based on 1,00,000 trials. For comparison the corresponding results for the approval rule from Table 1 are copied in Table 2.

$(m = 6)$	k	1	2	3	4	5	Approval rule
$n = 2$	worst comp	100	54	45	40	83	33
	best comp	100	60	35	100	100	52
	stoch comp	100	93	50	40	83	31
$n = 3$	worst comp	44	57	61	32	56	41
	best comp	100	63	51	48	100	38
	stoch comp	100	30	54	35	56	16
$n = 6$	worst comp	60	64	60	51	25	54
	best comp	50	59	50	46	78	29
	stoch comp	34	41	24	16	16	13
$n = 10$	worst comp	69	70	68	64	56	62
	best comp	45	56	52	40	51	32
	stoch comp	29	36	31	27	07	19

Table 2: Approximate percentages of non-manipulable preference profiles for k-approval rules and the approval rule, $m = 6$, based on 1,00,000 trials. (The percentages equal to 100 are exact and reflect strategy-proofness in the involved cases.)

Again we see that for relatively high numbers of voters manipulability seems to increase from worst to best and from best to stochastic dominance comparison. Further, for more than two voters the approval rule is outperformed by (at least) the 3-approval rule as far as non-manipulability is concerned.

The final simulation results we present are collected in Table 3. This table gives the percentages of non-manipulable preference profiles for k-approval rules with k odd and the approval rule for 10 alternatives, 25 agents, based on 1,000,000 trials. We give more accurate numbers than in the other tables since some differences are very small. A prudent observation is that the 5-approval rule performs best with respect to (non-)manipulability among the odd values of k (except for the case $k = 9$ and best comparison) – in line with the asymptotic result in Section 4.6. Also, it performs better than the approval rule.

23

$m = 10$	k	1	3	5	7	9	Approval rule
$n = 25$	worst comp	67.1	74.4	74.5	70.2	48.8	69.5
	best comp	33.5	49.9	50.2	40.9	54.8	37.6
	stoch comp	26.6	37.6	37.8	29.7	03.7	29.1

Table 3: Percentages of non-manipulable preference profiles for k-approval rules and the approval rule, $m = 10$, $n = 25$, based on 1,000,000 trials.

6   Concluding remarks

We have characterized all (non-)manipulable preference profiles for the approval rule and for k-approval rules. Our simulation results indicate that k-approval voting may be a good substitute for approval voting. It preserves the simplicity of the voting procedure characteristic for approval voting but tends to be less manipulable. Asymptotically, the k-approval rule with k close to half of the number of alternatives is even best among all scoring rules in this respect. This result is in line with Pritchard and Wilson (2009), although their context is somewhat different.

Our characterizations are also useful for studying 'partial culture' voting, where uniform distribution of preference profiles is not assumed.

References

Aleskerov F, Kurbanov E (1999) Degree of manipulability of social choice procedures. In: *Proceedings of the Third International Meeting of the Society for the Advancement of Economic Theory.* Springer, Berlin Heidelberg New York

Barberà S, Dutta B, Sen A (2001) Strategy-proof social choice correspondences. Journal of Economic Theory 101:374–394

Brams SJ, Fishburn PC (1983) *Approval voting.* Birkhauser, Boston MA

Campbell DE, Kelly JS (2008) Gains from manipulation of social choice rules. Economic Theory (forthcoming)

Fristrup P, Keiding H (1998) Minimal manipulability and interjacency for two-person social choice functions. Social Choice and Welfare 15:455–467

Gibbard A (1973) Manipulation of voting schemes: a general result. Econometrica 41: 587–602

Kelly JS (1988) Minimal manipulability and local strategy-proofness. Social Choice and Welfare 5: 81–85

Kelly JS (1989) Interjacency. Social Choice and Welfare 6: 331–335

Maus S, Peters H, Storcken T (2007) Minimal manipulability: unanimity and nondictatorship. Journal of Mathematical Economics 43:675–691.

Pritchard G, Wilson MC (2009) Asymptotics of the minimum manipulating coalition size for positional voting rules under impartial culture behaviour. Mathematical Social Sciences 58:35–57

Saari D (1990) Susceptibility to manipulation. Public Choice 64:21–41

Satterthwaite M (1975) Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory 10: 187–217

Appendix: remaining proofs

Proof of Theorem 3.5. For the if-part, let s be an i-deviation of r, with $s(i) = (q(i), \ell(i))$.

In case (a), $\varphi(r) = \varphi(r_{-i}) \cup [\varphi^-(r_{-i}) \cap \beta_{k(i)}(w(i))]$. If $\varphi(r_{-i}) \cap \beta_{\ell(i)}(q(i)) = \emptyset$ then $\varphi(s) = \varphi(r_{-i}) \cup [\varphi^-(r_{-i}) \cap \beta_{\ell(i)}(q(i))]$ but this is not preferred to $\varphi(r)$ according to $\succeq_{w(i)}$. If $\varphi(r_{-i}) \cap \beta_{\ell(i)}(q(i)) \neq \emptyset$ then $\varphi(s) = \varphi(r_{-i}) \cap \beta_{\ell(i)}(q(i))$ which again is not preferred to $\varphi(r)$ according to $\succeq_{w(i)}$.

In case (b), $\varphi(r) = \varphi(r_{-i}) \cap \beta_{k(i)}(w(i))$ and $\{x\} \sim_{w(i)} \{y\}$ for all $x, y \in \varphi(r)$. If $\varphi(s) \subseteq \varphi(r)$ then $\varphi(s)$ is not preferred to $\varphi(r)$. Otherwise, there are $z \in \varphi(s) \cap \varphi(r_{-i})$ with $(x, z) \in w(i)$ for all $x \in \varphi(r)$. In that case too, $\varphi(s)$ is not preferred to $\varphi(r)$ according to $\succeq_{w(i)}$.

For case (c), note that $\varphi(r) \subseteq \varphi(r_{-i})$ and thus, $\{x\} \sim_{w(i)} \{y\}$ for all $x, y \in \varphi(r)$. Also, $\{x\} \succeq_{w(i)} \{y\}$ for all $x \in \varphi(r_{-i})$ and $y \in \varphi^-(r_{-i})$. So $\varphi(s) \not\succ_{w(i)} \varphi(r)$.

For the only-if part, suppose that there is a voter $i \in N$ such that none of the three cases (a)–(c) holds. It is sufficient to prove that $\varphi(r)$ is manipulable at profile r by voter i. By taking negations it follows that none of (a)–(c) holding is equivalent to *all* of the following three statements holding for r and i.

1. (i) $\varphi(r_{-i}) \not\subseteq A \backslash \beta_{k(i)}(w(i))$ or (ii) $\varphi^-(r_{-i}) \cap \beta_{k(i)}(w(i)) = \emptyset$.

2. (i) $\varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) = \emptyset$ or (ii) for some $x, y \in \varphi(r_{-i}) \cap \beta_{k(i)}(w(i))$ $\{x\} \succ_{w(i)} \{y\}$ or (iii) $A \backslash \beta_{k(i)}(w(i)) \cap \varphi(r_{-i}) = \emptyset$.

3. (i) $\{x\} \succ_{w(i)} \{y\}$ for some $x, y \in \varphi(r_{-i})$ or (ii) there are $x \in \varphi^-(r_{-i})$ such that $\{x\} \succ_{w(i)} \varphi(r_{-i})$.

Requiring (1)–(3) to hold implies twelve cases to consider. However, 1(i) and 2(i) are contradictory, so that ten cases are left. Now cases 1(i) & 2(ii) & 3(i); 1(i) & 2(ii) & 3(ii); 1(i) & 2(iii) & 3(i); 1(ii) & 2(ii) & 3(i); 1(ii) & 2(ii) & 3(ii); and 1(ii) & 2(iii) & 3(i) are covered by case I below. Case 1(ii) & 2(i) & 3(i) is covered by case II. Cases 1(i) & 2(iii) & 3(ii) and 1(ii) & 2(iii) & 3(ii) are covered by case III. Lastly, case 1(ii) & 2(i) & 3(ii) is covered by case IV.

I. $\varphi(r_{-i}) \cap \beta_{k(i)}(w(i)) \neq \emptyset$ and there exist $x, y \in \varphi(r_{-i}) \cap \beta_{k(i)}(w(i))$ such that $\{x\} \succ_{w(i)} \{y\}$.

II. $\varphi(r_{-i}) \subseteq [A \backslash \beta_{k(i)}(w(i))]$, $\varphi^-(r_{-i}) \subseteq [A \backslash \beta_{k(i)}(w(i))]$ and there exist $x, y \in \varphi(r_{-i})$ such that $\{x\} \succ_{w(i)} \{y\}$.

III. $\varphi(r_{-i}) \subseteq \beta_{k(i)}(w(i))$ and there exist $x \in \varphi^-(r_{-i})$ such that $\{x\} \succ_{w(i)} \{y\}$ for all $y \in \varphi(r_{-i})$.

IV. $\varphi(r_{-i}) \subseteq [A \backslash \beta_{k(i)}(w(i))]$, $\varphi^-(r_{-i}) \subseteq [A \backslash \beta_{k(i)}(w(i))]$ and there exist $x \in \varphi^-(r_{-i})$ such that $\{x\} \succ_{w(i)} \{y\}$ for all $y \in \varphi(r_{-i})$.

In cases I and II, take $b \in \beta\left(w(i)_{|\varphi(r_{-i})}\right)$ and consider the report $s(i) = (w'(i), 1)$ of voter i such that $\beta(w'(i)) = \{b\}$. Then by (1), $\varphi(s) = \{b\}$. In case I, $\varphi(r) = \varphi(r_{-i}) \cap \beta_{k(i)}(w(i))$, so by the second condition in I, $\varphi(s) = \{b\} \succ_{w(i)} \varphi(r)$. In case II, $\varphi(r) = \varphi(r_{-i})$, and by the last condition in case II, again $\varphi(s) = \{b\} \succ_{w(i)} \varphi(r)$.

In both cases III and IV, $\varphi(r) = \varphi(r_{-i})$. Take $x \in \varphi^-(r_{-i})$ such that $\{x\} \succ_{w(i)} \{y\}$ for all $y \in \varphi(r_{-i})$. Consider the report $s(i) = (w'(i), 1)$ of voter i such that $\beta(w'(i)) = \{x\}$. Then by (1), $\varphi(s) = \{x\} \cup \varphi(r_{-i})$. This implies $\varphi(s) = \{b\} \succ_{w(i)} \varphi(r)$. $\hfill\square$

Proof of Theorem 4.3. For the if-part, let q be an i-deviation of p. Note that $\varphi_k(p_{-i}) = \varphi_k(q_{-i})$ and $\varphi_k^-(p_{-i}) = \varphi_k^-(q_{-i})$. Assume that at least one of the cases (a)–(e) holds. We show that voter i cannot manipulate from p to q.

In case (a), we have $\varphi_k(p) = \varphi_k(p_{-i}) \cap \beta_k(p(i))$ and $\varphi_k(q) = \varphi_k(p)$ or $\varphi_k(q) \cap [A \setminus \beta_k(p(i))] \neq \emptyset$. In both cases, it is easy to see that $\varphi_k(p) \succeq_{p(i)} \varphi_k(q)$.

In case (b), $\varphi_k(p) = \varphi_k(p_{-i}) \cup [\varphi_k^-(p_{-i}) \cap \beta_k(p(i))]$. If $\beta_k(q(i)) \cap \varphi_k(p_{-i}) = \emptyset$ then $\varphi_k(q) = \varphi_k(p_{-i}) \cup [\varphi_k^-(p_{-i}) \cap \beta_k(q(i))]$ but this is never preferred to $\varphi_k(p)$ according to $\succeq_{p(i)}$. If $\beta_k(q(i)) \cap \varphi_k(p_{-i}) \neq \emptyset$ then $\varphi_k(q) = \beta_k(q(i)) \cap \varphi_k(p_{-i})$, which again is never preferred to $\varphi_k(p)$ according to $\succeq_{p(i)}$.

In case (c), (d), and (e), $\varphi_k(p) = \{w\}$. If $\varphi_k(q) \neq \{w\}$ then $x \in \varphi_k(q)$ for some $x \in L(w, p(i)) \setminus \{w\}$. In that case $\varphi_k(q)$ is never preferred to $\varphi_k(p)$ according to $\succeq_{p(i)}$.

For the only-if part, suppose that there is a voter $i \in N$ such that none of the five cases (a)–(e) holds. It is sufficient to prove that φ_k is manipulable at profile p by voter i.

By taking negations it follows that none of (a)–(e) holding is equivalent to all of the following four statements holding for p and i.

  I   (i) $A \setminus \beta_k(p(i)) \not\subseteq \varphi_k(p_{-i})$ or (ii) $A \setminus \beta_k(p(i)) = \varphi_k(p_{-i})$.

  II   (i) $\varphi_k(p_{-i}) \not\subseteq [A \setminus \beta_k(p(i))]$ or (ii) $\varphi_k^-(p_{-i}) \cap \beta_k(p(i)) = \emptyset$.

  III   (i) $|\varphi_k(p_{-i}) \cap \beta_k(p(i))| \geq 2$ or (ii) $\varphi_k(p_{-i}) \cap \beta_k(p(i)) = \emptyset$ or (iii) $\varphi_k(p_{-i}) \subseteq \beta_k(p(i))$.

  IV   (i) $|\varphi_k(p_{-i})| > 1$ or (ii) $[\,\varphi_k(p_{-i}) = \{w\}$ for some $w \in A$ and $\varphi_k^-(p_{-i}) \not\subseteq L(w, p(i))$ and $|\varphi_k^-(p_{-i}) \cap L(w, p(i))| \leq m - k\,]$.

Requiring (I)–(IV) to hold implies 24 cases to consider. However, the following combinations are contradictory: I(ii) and III(i); I(ii) and III(iii); II(i) and III(ii); and III(i) and IV(ii). Moreover, III(iii) and IV(i) together imply III(i), so that we do not have the case with IV(i) and III(iii) separately. This leaves eight cases, which we will consider two at a time.

A. Cases I(i) & II(i) & III(i) & IV(i) and I(i) & II(ii) & III(i) & IV(i). These two cases are covered by the following assumptions: $|\varphi_k(p_{-i}) \cap \beta_k(p(i))| \geq 2$ and $A \setminus \beta_k(p(i)) \not\subseteq \varphi_k(p_{-i})$.

In this case we have $\varphi_k(p) = \varphi_k(p_{-i}) \cap \beta_k(p(i))$. Let $w = \omega\big(p(i)_{|\varphi_k(p_{-i}) \cap \beta_k(p(i))}\big)$. Then we can construct an i-deviation q of p such that $\varphi_k(p_{-i}) \cap \beta_k(q(i)) = [\varphi_k(p_{-i}) \cap \beta_k(p(i))] \setminus \{w\}$. Then $\varphi_k(q) = [\varphi_k(p_{-i}) \cap \beta_k(p(i))] \setminus \{w\}$ and, thus, $\varphi_k(q) \succeq_{p(i)} \varphi_k(p)$.

B. Cases I(i) & II(ii) & III(ii) & IV(i) and I(ii) & II(ii) & III(ii) & IV(i). These two cases are covered by the following assumptions: $|\varphi_k(p_{-i})| > 1$ and $\varphi_k(p_{-i}) \cap \beta_k(p(i)) = \emptyset$ and $\varphi_k^-(p_{-i}) \cap \beta_k(p(i)) = \emptyset$.

In this case we have $\varphi_k(p) = \varphi_k(p_{-i})$. Let $b = \beta\big(p(i)_{|\varphi_k(p_{-i})}\big)$. We can construct an i-deviation q of p such that $\varphi_k(p_{-i}) \cap \beta_k(q(i)) = \{b\}$. Then $\varphi_k(q) = \{b\} \succ_{p(i)} \varphi_k(p_{-i}) = \varphi_k(p)$.

C. Cases I(i) & II(ii) & III(ii) & IV(ii) and I(ii) & II(ii) & III(ii) & IV(ii). These two cases are covered by the following assumptions: $[\varphi_k(p_{-i}) = \{w\}$ for some $w \in A$ and $\varphi_k^-(p_{-i}) \not\subseteq L(w, p(i))$ and $|\varphi_k^-(p_{-i}) \cap L(w, p(i))| \le m - k]$ and $\varphi_k(p_{-i}) \cap \beta_k(p(i)) = \emptyset$ and $\varphi_k^-(p_{-i}) \cap \beta_k(p(i)) = \emptyset$.

In this case $\varphi_k(p) = \{w\}$. Let $b = \beta\left(p(i)_{|\varphi_k^-(p_{-i})}\right)$. Then $(b, w) \in p(i)$ since $\varphi_k^-(p_{-i}) \not\subseteq L(w, p(i))$. Construct an i-deviation q of p such that $\varphi_k^-(p_{-i}) \cap \beta_k(q(i)) = \{b\}$. Then $\varphi_k(q) = \{b, w\} \succ_{p(i)} \{b\} = \varphi_k(p)$.

D. Cases I(i) & II(i) & III(iii) & IV(ii) and I(i) & II(ii) & III(iii) & IV(ii). These cases are covered by the assumptions: $[\varphi_k(p_{-i}) = \{w\}$ for some $w \in A$ and $\varphi_k^-(p_{-i}) \not\subseteq L(w, p(i))$ and $|\varphi_k^-(p_{-i}) \cap L(w, p(i))| \le m - k]$ and $\varphi_k(p_{-i}) \subseteq \beta_k(p(i))$.

In this case, $\varphi_k(p) = \{w\}$. We can construct an i-deviation q of p such that $\varphi_k^-(p_{-i}) \cap L(w, p(i)) \subseteq A \setminus \beta_k(q(i))$. Then $\varphi_k(q) = \{w\} \cup [\varphi_k^-(p_{-i}) \cap \beta_k(q(i))]$. The set $\varphi_k^-(p_{-i}) \cap \beta_k(q(i))$ is non-empty and contains only points x with $(x, w) \in p(i)$. Hence, $\varphi_k(q) \succ_{p(i)} \varphi_k(p)$. $\qquad\square$

Proof of Corollary 4.7. For $2 \le k \le (m+1)/2$ we have

$$\eta(m, k) = \frac{m! \, k^2 \, [(m-k)!]^2}{(m-2k+1)!}$$

by Theorem 4.6. By a few elementary computations we obtain for $2 \le k \le (m+1)/2 - 1$:

$$\eta(m, k) < [>] \, \eta(m, k+1) \Leftrightarrow (m-k)^2 k^2 < [>] \, (k+1)^2(m-2k+1)(m-2k) . \; (*)$$

Denote $A := (k+1)^2(m-2k+1)(m-2k)$, then $A > (k+1)^2(m-2k)^2$ and this latter expression at least as large as $(m-k)^2 k^2$ if and only if $k^2 + 2k - m \le 0$, which, in turn, holds if and only if $k \le \sqrt{m+1} - 1$. Thus, by $(*)$,

$$k \le \underline{k} := \sqrt{m+1} - 1 \Rightarrow \eta(m, k) < \eta(m, k+1) .$$

Similarly, $A < (k+1)^2(m-2k+1)^2$ and this latter expression is not larger than $(m-k)^2 k^2$ if and only if $k^2 + k - (m+1) \ge 0$, which, in turn, holds if and only if $k \ge \frac{1}{2}\sqrt{1 + 4(m+1)} - \frac{1}{2}$. Thus, by $(*)$,

$$k \ge \overline{k} := \frac{1}{2}\sqrt{1 + 4(m+1)} - \frac{1}{2} \Rightarrow \eta(m, k) > \eta(m, k+1) .$$

It is straightforward to derive that $\underline{k} < \overline{k} < (m+1)/2$ and $\overline{k} - \underline{k} < 1$. Now statement (a) follows by taking $k^* = \lfloor \underline{k} \rfloor + 1$ or $k^* \in \{\lfloor \underline{k} \rfloor + 1, \lfloor \underline{k} \rfloor + 2\}$, depending on the exact values of $\underline{k}$ and $\overline{k}$.

For $m - 1 \geq k > (m+1)/2$ we have

$$\eta(m, k) = \frac{m! \, k! \, k!}{(2k - m)!}$$

by Theorem 4.6. To go from $\eta(m, k)$ to $\eta(m, k+1)$ we multiply by a factor $(k+1)^2/(2k-m+2)(2k-m+1)$, which is larger than 1 since $k+1 \geq 2k-m+2$. This proves statement (b).

To show statement (c) we have to show $\alpha(m, k) < 1$ for all $2 \leq k \leq (m+1)/2$, where $\alpha(m, k) = \eta(m, k)/\eta(m, m-1)$. By a simple computation we derive

$$\alpha(m+1, k) = \alpha(m, k) \cdot \frac{(m+1-k)^2}{m^2 \, (m-2k+2)} \; .$$

Since $(m+1-k)^2 < m^2$ and $m - 2k + 2 \geq 1$, this implies $\alpha(m+1, k) < \alpha(m, k)$. Hence, to show $\alpha(m, k) < 1$ for all $2 \leq k \leq (m+1)/2$, it is sufficient to show $\alpha(2k+1, k) < 1$ for all $k \geq 2$. We show this by induction on k. For $k = 2$ we have $\alpha(5, 2) = 3/4 < 1$. Now assume $\alpha(2k+1, k) < 1$ then it is sufficient to show $\alpha(2k+3, k+1) < 1$. To show this, by straightforward computation we have

$$\alpha(2k+3, k+1) = \alpha(2k+1, k) \cdot \frac{(k+2)^2}{2k \cdot (2k+1)} \; .$$

By the induction hypothesis and since $(k+2)^2 < 2k \cdot (2k+1)$ for $k \geq 2$, we obtain $\alpha(2k+3, k+1) < 1$. $\qquad\square$

Proof of Proposition 4.14. Let s be an arbitrary scoring vector. We will prove the proposition for the variance $\sigma^2(s)$, and first show that for $\sigma^2(s)$ to be maximal every coordinate of s has to be equal to 0 or 1. Let $j \in \{2, \ldots, m-1\}$ be arbitrary and write

$$\sigma^2(s) = \frac{1}{m} \left[\sum_{i \neq j} s_i^2 + s_j^2 - m \left(\frac{\sum_{i \neq j} s_i + s_j}{m} \right)^2 \right] \; .$$

Differentiating with respect to s_j, we obtain after simplification:

$$\frac{d\sigma^2(s)}{ds_j} = \frac{2(m-1)}{m^2} \left[s_j - \frac{\sum_{i \neq j} s_i}{m-1} \right] \; .$$

This implies $\frac{d\sigma^2(s)}{ds_j} > (=, <) \; 0$ exactly when $s_j > (=, <) \; \frac{\sum_{i \neq j} s_i}{m-1}$. In turn, this implies that with respect to s_j the variance $\sigma^2(s)$ is maximal for $s_j = 0$ or $s_j = 1$.

Second, let $\sigma^2(k)$ denote the variance of the scoring vector associated with the k-approval rule. Then it is simple to verify that $\sigma^2(k) = (1/m)[k - k^2/m]$, which implies that $\sigma^2(k)$ is maximal exactly if $k \in \{(m-1)/2, (m+1)/2\}$ for m odd and $k = m/2$ for m even.

The proof is complete by combining the two arguments. $\qquad\square$